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HÖLDER STABILITY FOR SOLUTIONS TO VECTOR EQUILIBRIUM PROBLEMS VIA SCALARIZATION

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Abstract. In this paper, we are concerned with vector equilibrium problems when both objective functions and constraints are subjected to perturbations. By using the scalarization method, we obtain sufficient conditions for the Hölder continuity of solution maps to parametric vector equilibrium problems. Our results are new or improve the ones documented in the literature.
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1. Introduction

The equilibrium problem provides a general framework that includes numerous optimization-related problems, such as variational inequality problems, complementarity problems, Nash equilibrium problems, minimax problems, fixed-point and coincidence-point problems, and traffic network problems [1], as special cases. This indicates that the results in the theory of equilibrium problems can be applied to a multitude of important areas, such as economics, physics (particularly mechanics), engineering, transportation, sociology, chemistry, biology, and more; see, e.g., [2, 3].

The topic of existence conditions for solutions of the equilibrium problem and its generalizations has received considerable attention and is the first and most extensively explored area of research recently. Numerous studies in the literature focused on this particular subject; see, e.g., [4, 5, 6, 7, 8, 9, 10] and the references therein. Another important topic of research is the stability property of solutions. Numerous works were done for investigating the upper and lower semicontinuity of solution maps; see, e.g., [11, 12, 13, 14, 15, 16, 17, 18] and the references therein.

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In recent years, the investigation of Hölder continuity of solution maps became an important and actively explored area in the field; see, e.g., [19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. Observe that these papers are importance on the assumptions related to strong monotonicity, strong pseudomonotonicity, or strong convexity. For convenience to the reader, let us briefly review some of them. Under strong monotonicity assumptions, [19, 20, 21] established the Hölder continuity of solution maps for scalar equilibrium problems. Similarly, [22] obtained sufficient conditions for the Hölder continuity of solution maps to scalar equilibrium problems by incorporating strong pseudomonotonicity and quasimonotonicity assumptions. The results presented in [22] were extended to vector cases in [29], and subsequently improved significantly by [23], which also enhanced the results in [30] when applied to variational inequality problems. In [31], the results from [19, 29, 23] were generalized to general quasiequilibrium problems. The authors of [21] investigated the Hölder continuity and uniqueness of solutions for quasiequilibrium problems and further sharpened the Hölder degree from [19, 20, 22, 29, 23, 31]. In [28, 24], assumptions related to strong monotonicity and strong pseudomonotonicity were replaced by strong convexity, leading to sufficient conditions for the Hölder continuity of solution maps in scalar equilibrium problems. Analogously, the authors explored the vector case in [25] by using the same approach. In [26, 27], assumptions requiring the information of solutions were employed to establish the Hölder property, which is difficult to apply to practical situations.

It is worth noting that, due to the technical difficulties, the authors in [25] initially established the Hölder continuity of the solution map for a parametric weak vector equilibrium problem by primarily relying on the strong convexity of the objective map. However, strong convexity is a restrictive condition in optimization theory, limiting its applications. Therefore, in this paper, we aim to relax this requirement by using a weaker condition to broaden the applicability. More precisely, we employ the strong quasiconvexity condition to study sufficient conditions for the Hölder continuity of solution maps of parametric weak vector equilibrium problems. The results achieved in this paper, along with those in [32], will significantly contribute to the advancement of research on Hölder continuity for solution maps of both strong and weak vector equilibrium problems.

The remainder of this paper is organized as follows. Section 2 serves as a review, introducing definitions and properties needed in what follows. Section 3 focuses on establishing sufficient conditions for the Hölder continuity of the solution map to the equilibrium problem, employing a scalarization method. In this section, we also provide examples to demonstrate the essentialness of the imposed assumptions and compare our results with those of previous studies. Finally, in Section 4, we present our concluding remarks.

2. Preliminaries

Our notations in this paper are standard. We use $\|\cdot\|$ for the norm in any normed space. We denote by X^* the topological dual space of the normed space X and by $\mathbb{R}_+$ the set of nonnegative real numbers. $\mathbb{B}(x,r)$ denotes the closed ball centered at x with radius $r \geq 0$. int$A$ and conv(A) stand for the interior and convex hull, respectively (resp.), of a subset A. The diameter of A is $\text{diam}A := \sup_{x,y\in A}\|x-y\|$. For a set-valued map $G : X \rightrightarrows Y$, $\text{graph}G := \{(x,y) \in X \times Y \mid y \in G(x)\}$ is the graph of G. $\mathscr{L}(X,Y)$ is the collection of all continuous linear mappings of X into Y. For two subsets A,B of the normed space X, the excess of A beyond B is defined by $\text{ex}(A,B) := \sup_{a\in A} d(a,B)$, with $d(a,B) := \inf_{b\in B}\|a-b\|$. The Pompeiu-Hausdorff distance is

$\mathscr{H}(A,B) := \max\{\mathrm{ex}(A,B), \mathrm{ex}(B,A)\}$. Equivalently,

$$\mathrm{ex}(A,B) = \inf\{\tau \geq 0 \mid A \subset B + \tau\mathbb{B}\}$$

and

$$\mathscr{H}(A,B) = \inf\{\tau \geq 0 \mid A \subset B + \tau\mathbb{B}, B \subset A + \tau\mathbb{B}\}.$$

The distance with respect to ρ is defined by $\rho(A,B) := \sup_{a\in A, b\in B} \|a-b\|$.

2.1. **Statement of problems.** From now on, unless explicitly stated otherwise, let X,Y,Z,W be normed spaces, and A be nonempty and convex subset of X, Λ, M be nonempty subsets of Z and W, respectively. Let $C \subset Y$ be a pointed closed and convex cone with nonempty interiors. Let $K : \Lambda \rightrightarrows A$ be a set-valued map with nonempty, convex, and compact values, and let $f : A \times A \times M \to Y$ be a vector-valued map satisfying $f(x,x,\mu) = 0_Y$ for all $x \in A$ and $\mu \in M$. For each $(\lambda,\mu) \in \Lambda \times M$, we consider the following parametric weak vector equilibrium problem:

(WEP) Find $\bar{x} \in K(\lambda)$ such that $f(\bar{x},y,\mu) \notin -\mathrm{int}C$ for all $y \in K(\lambda)$.

Listed below are some special cases of (WEP):

(i) If $g : A \times M \to Y$ and $f(x,y,\mu) := g(y,\mu) - g(x,\mu)$, then (WEP) reduces to the following vector optimization problem.

(VOP) Find $\bar{x} \in K(\lambda)$ such that $g(y,\mu) - g(\bar{x},\mu) \notin -\mathrm{int}C$ for all $y \in K(\lambda)$.

When $Y = \mathbb{R}$, (VOP) is the scalar optimization problem (OP).

(OP) Find $\bar{x} \in K(\lambda)$ such that $g(\bar{x},\mu) = \min_{y\in K(\lambda)} g(y,\mu)$.

(ii) If $\phi : A \times M \to \mathscr{L}(X,Y)$ and $f(x,y,\mu) := \langle \phi(x,\mu), y-x\rangle$, then (WEP) reduces to the vector variational inequality.

(VVI) Find $\bar{x} \in K(\lambda)$ such that $\langle \phi(\bar{x},\mu), y-\bar{x}\rangle \notin -\mathrm{int}C$ for all $y \in K(\lambda)$.

If $Y = \mathbb{R}$, then (VVI) is the following variational inequality.

(VI) Find $\bar{x} \in K(\lambda)$ such that $\langle \phi(\bar{x},\mu), y-\bar{x}\rangle \geq 0$ for all $y \in K(\lambda)$.

(iii) When X,Y,M are Hilbert spaces with $X \equiv Y$, let $\varphi : A \times M \to A$. The fixed point problem is

(FP) Find $\bar{x} \in A$ such that $\varphi(\bar{x},\mu) = \bar{x}$. This problem is equivalent to the following special case of (WEP).

(WEP′) Find $\bar{x} \in A$ such that $\langle x - \varphi(x,\mu), y-x\rangle \notin -\mathrm{int}C$ for all $y \in A$.

Indeed, if $\bar{x}$ is a solution of (FP), then $\langle \bar{x} - \varphi(\bar{x},\mu), y-\bar{x}\rangle = 0$ for all $y \in A$, and hence $\bar{x}$ solves (WEP′). Conversely, let $\bar{x}$ is a solution of (WEP′), i.e., $\langle \bar{x} - \varphi(\bar{x},\mu), y-\bar{x}\rangle \notin -\mathrm{int}C$ for all $y \in A$. Taking $y = \varphi(\bar{x},\mu)$, we have $\langle \bar{x} - \varphi(\bar{x},\mu), \varphi(\bar{x},\mu)-\bar{x}\rangle \notin -\mathrm{int}C$ and thus we must have the equality, that is, $\bar{x}$ is a solution of (FP).

We denote the solution set of (WEP) at $(\lambda,\mu) \in \Lambda \times M$ by $S(\lambda,\mu)$, i.e.,

$$S(\lambda,\mu) := \{\bar{x} \in K(\lambda) \mid f(\bar{x},y,\mu) \notin -\mathrm{int}C, \ \forall y \in K(\lambda)\}.$$

Let $C^* := \{\xi \in Y^* \mid \xi(y) \geq 0, \forall y \in C\}$ be the dual cone of C. Since the interior of C is nonempty, the dual cone C^* has a weak* compact base. Let $e \in \mathrm{int}C$ be given. Then $\mathscr{B}_e^* := \{\xi \in C^* \mid \xi(e) = 1\}$ is a weak* compact base of C^*.

Lemma 2.1. (See [33]) *If Y is a real topological linear space and C is a convex cone with nonempty interior, then* $\mathrm{int}C = \{y \in Y \mid \xi(y) > 0, \forall \xi \in C^* \setminus \{0_Y\}\}$.

For every $\xi \in \mathscr{B}_e^*$ and $(\lambda,\mu) \in \Lambda \times M$, we denote the set of all ξ-efficient solutions of (WEP) at $(\lambda,\mu) \in \Lambda \times M$ by

$$\Pi_\xi(\lambda,\mu) := \{x \in K(\lambda) \mid \xi\big(f(x,y,\mu)\big) \geq 0, \forall y \in K(\lambda)\}.$$

2.2. **Convexity and related concepts.**

Definition 2.1. (Classical) Let $g : X \to \mathbb{R}$ be a function and $\Omega \subset X$.

(a) g is convex on a convex subset Ω iff, for all $x_1, x_2 \in \Omega$ and $t \in [0,1]$,

$$g\big((1-t)x_1 + tx_2\big) \leq (1-t)g(x_1) + tg(x_2).$$

(b) g is quasiconvex on a convex subset Ω iff, for all $x_1, x_2 \in \Omega$ and $t \in [0,1]$,

$$g\big((1-t)x_1 + tx_2\big) \leq \max\{g(x_1), g(x_2)\}.$$

(c) g is convex-like on Ω (Ω is not necessarily convex) iff, for all $x_1, x_2 \in \Omega$ and $t \in [0,1]$, there is $z \in \Omega$ such that

$$g(z) \leq (1-t)g(x_1) + tg(x_2).$$

(d) g is quasiconvex-like on Ω (Ω is not necessarily convex) iff, for all $x_1, x_2 \in \Omega$ and $t \in [0,1]$, there is $z \in \Omega$ such that

$$g(z) \leq \max\{g(x_1), g(x_2)\}.$$

Next, the notion of strong convexity of a function on a convex subset is demonstrated in the following definition.

Definition 2.2. (See [24, 28]) Let $g : X \to \mathbb{R}$, $\Omega \subset X$ be a convex subset and h, β be positive.

(a) g is $h.\beta$-strongly convex on Ω iff, for all $x_1, x_2 \in \Omega$ and $t \in (0,1)$,

$$g\big((1-t)x_1 + tx_2\big) \leq (1-t)g(x_1) + tg(x_2) - ht(1-t)\|x_1 - x_2\|^\beta.$$

(b) g is $h.\beta$-strongly quasiconvex on $\Omega \subset X$ iff, for all $x_1, x_2 \in \Omega$ and $t \in (0,1)$,

$$g\big((1-t)x_1 + tx_2\big) \leq \max\{g(x_1), g(x_2)\} - ht(1-t)\|x_1 - x_2\|^\beta.$$

(c) g is $h.\beta$-strongly convex-like on Ω (Ω is not necessarily convex) iff, for all $x_1, x_2 \in \Omega$ and $t \in (0,1)$, there is $z \in \Omega$ such that,

$$g(z) \leq (1-t)g(x_1) + tg(x_2) - ht(1-t)\|x_1 - x_2\|^\beta.$$

We can summarize relationships as being among the above notions in the following figure.

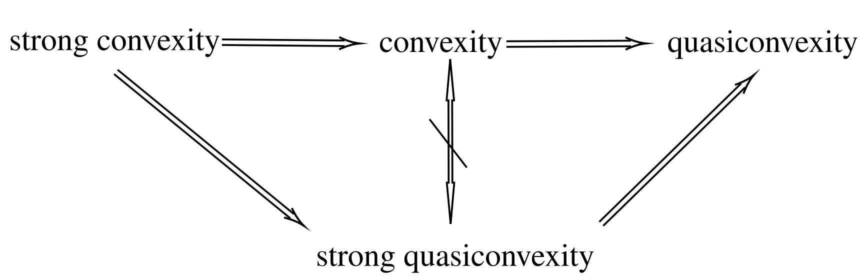

FIGURE 1. Relationships among notions related to convexity

The implications in Figure 1 are obvious from the definitions. We now provide some examples to demonstrate that there is no relation between the convexity and the strong quasiconvexity.

Example 2.1. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = -x^2 - 2x$ for all $x \in [0,1]$. Then, g is 1.2-strongly quasiconvex on $[0,1]$ but, it is not convex on $[0,1]$ (and hence it is not strongly convex on $[0,1]$).

Example 2.2. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 0, & x = 0 \\ -1, & x \neq 0. \end{cases}$$

for all $x \in [0,1]$. Then, g is convex on $\mathbb{R}$, but not strongly quasiconvex on $\mathbb{R}$.

In [32], the author extended the notion of strong quasiconvexity of a function to the case of vector-valued one as follows.

Definition 2.3. (See [32]) Let $g : X \to Y, \Omega \subset X$ be a convex subset and h, β be positive. Then

(a) g is $h.\beta$-strongly quasiconvex with respect to (wrt, shortly) e on a convex set Ω iff, for all $x_1, x_2 \in \Omega$ and $t \in (0,1)$, either

$$g(x_1) - g\big((1-t)x_1 + tx_2\big) - ht(1-t)\|x_1 - x_2\|^{\beta} e \in C,$$

or

$$g(x_2) - g\big((1-t)x_1 + tx_2\big) - ht(1-t)\|x_1 - x_2\|^{\beta} e \in C.$$

(b) g is $h.\beta$-strongly quasiconvex-like wrt e on Ω (Ω is not necessarily convex) iff, for all $x_1, x_2 \in \Omega$ and $t \in (0,1)$, there exists $z \in \Omega$ such that, either

$$g(x_1) - g(z) - ht(1-t)\|x_1 - x_2\|^{\beta} e \in C,$$

or

$$g(x_2) - g(z) - ht(1-t)\|x_1 - x_2\|^{\beta} e \in C.$$

2.3. **Hölder continuity and related concepts.**

Definition 2.4. (See [25]) Let $n, \gamma > 0$, and $\theta \geq 0$.

(a) A vector-valued map $g : X \to Y$ is $n.\gamma$-Hölder continuous wrt e at $\bar{x} \in X$ iff there exists a neighborhood U of $\bar{x}$ such that, for all $x_1, x_2 \in U$,

$$g(x_1) - g(x_2) + n\|x_1 - x_2\|^{\gamma} e \in C.$$

(b) A set-valued $K : \Lambda \rightrightarrows X$ is $n.\gamma$-Hölder continuous at $\bar{\lambda} \in \Lambda$ if there exists a neighborhood N of $\bar{\lambda}$ such that, for all $\lambda_1, \lambda_2 \in N$,

$$H(K(\lambda_1), K(\lambda_2)) \leq n\|\lambda_1 - \lambda_2\|^{\gamma}.$$

Definition 2.5. A vector-valued map $g : X \times X \times M \to Y$ is $n.\gamma$-Hölder continuous at $\bar{\mu} \in M$ wrt e, θ-uniformly over $\Omega \subset X$ iff there exists a neighborhood U of $\bar{\mu}$ such that, for all $\mu_1, \mu_2 \in U$ and $x, y \in X$ with $x \neq y$,

$$g(x, y, \mu_1) - g(x, y, \mu_2) + n\|\mu_1 - \mu_2\|^{\gamma}\|x - y\|^{\theta} e \in C.$$

We say that a certain property is satisfied on a subset $\Omega \subset X$ if and only if it is satisfied at every point of Ω.

3. Hölder Continuity of the Solution Maps

Before stating the main results of the paper, we first start with some important lemmas.

Lemma 3.1. *For each $\xi \in \mathscr{B}_e^*, x \in A$ and $\mu \in M$, if $f(x,\cdot,\mu)$ is $h.\beta$-strongly quasiconvex on A, then $\varphi(x,\cdot,\mu) := \xi\big(f(x,\cdot,\mu)\big)$ is also $h.\beta$-strongly quasiconvex on A.*

Proof. Let $x \in A$, $\mu \in U$, $y_1, y_2 \in A$ and $t \in (0,1)$ be arbitrary. By the strong quasiconvexity of $f(x,\cdot,\mu)$, there exist two cases to consider. The first is that

$$f(x,y_1,\mu) - f\big(x,(1-t)y_1 + ty_2,\mu\big) - ht(1-t)\|y_1 - y_2\|^\beta e \in C.$$

Then,

$$\xi\big(f(x,y_1,\mu) - f\big(x,(1-t)y_1 + ty_2,\mu\big) - ht(1-t)\|y_1 - y_2\|^\beta e\big) \geq 0.$$

By the linearity of ξ, one has

$$\xi\big(f(x,y_1,\mu)\big) - \xi\big(f(x,(1-t)y_1 + ty_2,\mu)\big) - ht(1-t)\|y_1 - y_2\|^\beta \geq 0,$$

which leads to

$$\varphi\big(x,(1-t)y_1 + ty_2,\mu\big) \leq \varphi(x,y_1,\mu) - ht(1-t)\|y_1 - y_2\|^\beta. \tag{3.1}$$

Passing to the second case, one has

$$f(x,y_2,\mu) - f\big(x,(1-t)y_1 + ty_2,\mu\big) - ht(1-t)\|y_1 - y_2\|^\beta e \in C.$$

Using the same arguments, one sees that

$$\varphi\big(x,(1-t)y_1 + ty_2,\mu\big) \leq \varphi(x,y_2,\mu) - ht(1-t)\|y_1 - y_2\|^\beta. \tag{3.2}$$

Equations (3.1) and (3.2) allow us to conclude the validity of the strong quasiconvexity of φ. □

Lemma 3.2. *For every $\xi \in \mathscr{B}_e^*$ and $x \in A, \mu \in M$, if $f(x,\cdot,\mu)$ is $m.\delta$-Hölder continuous on A, then $\varphi(x,\cdot,\mu) := \xi\big(f(x,\cdot,\mu)\big)$ is also $m.\delta$-Hölder continuous on A.*

Proof. For each $y_1, y_2 \in A$, we have $f(x,y_1,\mu) - f(x,y_2,\mu) + m\|y_1 - y_2\|^\delta e \in C$, which yields

$$\xi\Big(f(x,y_1,\mu) - f(x,y_2,\mu) + m\|y_1 - y_2\|^\delta e\Big) \geq 0.$$

Utilizing by the linearity of ξ, we arrive at

$$\varphi(x,y_2,\mu) \leq \varphi(x,y_1,\mu) + m\|y_1 - y_2\|^\delta.$$

Changing the roles of y_1 and y_2, we also have $\varphi(x,y_1,\mu) \leq \varphi(x,y_2,\mu)) + m\|y_1 - y_2\|^\delta$. Thus

$$|\varphi(x,y_1,\mu) - \varphi(x,y_2,\mu))| \leq m\|y_1 - y_2\|^\delta,$$

i.e., the Hölder continuity of φ is checked. □

Definition 3.1. (See [25]) A vector-valued map $g : X \times X \to Y$ is said to be monotone on $\Omega \subset X$ if, for all $x, y \in \Omega$, $g(x,y) + g(y,x) \in -C$.

Lemma 3.3. *For every $\xi \in \mathscr{B}_e^*$ and $\mu \in M$, if $f(\cdot,\cdot,\mu)$ is monotone on A, then $\varphi(\cdot,\cdot,\mu) := \xi\big(f(\cdot,\cdot,\mu)\big)$ is also monotone on A.*

Proof. We omit the proof since it is trivial. □

The following lemma can be deduced from [18, Lemma 3.2].

Lemma 3.4. *If, for each $\lambda \in \Lambda$, $x \in K(\lambda)$ and $\mu \in M$, $f(x,\cdot,\mu)$ is C-convex-like on $K(\lambda)$, then*

$$S(\lambda,\mu) = \bigcup_{\xi \in \mathscr{B}_e^*} \Pi_\xi(\lambda,\mu).$$

Theorem 3.1. *For* (WEP)*, assume that, for each $\xi \in \mathscr{B}_e^*$, the ξ-efficient solutions exist in a neighborhood of the considered point $(\lambda_0,\mu_0) \in \Lambda \times M$. Furthermore, assume that the following conditions are satisfied:*

(i) *K is $l.\alpha$-Hölder continuous on a neighborhood N of λ_0;*
(ii) *there exists a neighborhood U of μ_0 such that, for all $x \in K(N)$ and $\mu \in U$, $f(x,\cdot,\mu)$ is $h.\beta$-strongly quasiconvex as well as $m.\delta$-Hölder continuous on* $\operatorname{conv}(K(N))$;
(iii) *for each $\mu \in U$, $f(\cdot,\cdot,\mu)$ is monotone on $K(N)$;*
(iv) *f is $n.\gamma$-Hölder continuous on U, θ-uniformly over $K(N)$ with $\theta < \beta$.*

Then, for any $\bar{\xi} \in \mathscr{B}_e^*$, there exist open neighborhoods $N(\bar{\xi})$ of $\bar{\xi}$, $N_{\bar{\xi}}(\lambda_0)$ of λ_0 and $N_{\bar{\xi}}(\mu_0)$ of μ_0 such that the ξ-efficient solution mapping Π_ξ satisfies the following Hölder property on $N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0)$ i.e., for all (λ_1,μ_1), $(\lambda_2,\mu_2) \in N \times U$,

$$\begin{aligned}&\rho(\Pi_\xi(\lambda_1,\mu_1),\Pi_\xi(\lambda_2,\mu_2))\\ &\leq \left(\frac{4ml^\delta(1+2^{-\delta})}{h}\right)^{\frac{1}{\beta}} \|\lambda_1-\lambda_2\|^{\frac{\alpha\delta}{\beta}} + \left(\frac{4n}{h}\right)^{\frac{1}{\beta-\theta}} \|\mu_1-\mu_2\|^{\frac{\gamma}{\beta-\theta}}.\end{aligned}$$

Proof. Let $\bar{\xi} \in \mathscr{B}_e^*$ and $N(\bar{\xi}) \times N_{\bar{\xi}}(\lambda_0) \times N_{\bar{\xi}}(\mu_0) \subset \mathscr{B}_e^* \times U \times V$ be open. The proof is divided into three steps.

Step 1. We prove that, for all $x_{11} \in \Pi_\xi(\lambda_1,\mu_1)$ and $x_{21} \in \Pi_\xi(\lambda_2,\mu_1)$, the following inequality holds

$$\|x_{11}-x_{21}\| \leq \left(\frac{4ml^\delta(1+2^{-\delta})}{h}\right)^{\frac{1}{\beta}} \|\lambda_1-\lambda_2\|^{\frac{\alpha\delta}{\beta}}. \tag{3.3}$$

From the Hölder continuity of K in (i), we find $x_1 \in K(\lambda_1)$ and $x_2 \in K(\lambda_2)$ such that

$$\max\{\|x_{11}-x_2\|,\|x_{21}-x_1\|\} \leq l\|\lambda_1-\lambda_2\|^\alpha. \tag{3.4}$$

Since K has convex values, one has $\frac{x_1+x_{11}}{2} \in K(\lambda_1)$. The definition of the solution set gives us

$$\min\left\{\xi\left(f\left(x_{11},\frac{x_1+x_{11}}{2},\mu_1\right)\right),\xi\left(f(x_{21},x_2,\mu_1)\right)\right\} \geq 0. \tag{3.5}$$

Using the monotonicity of f given in (iv) together with Lemma 3.3, we obtain

$$\xi\left(f(x_{11},x_{21},\mu_1)\right) \leq -\xi\left(f(x_{21},x_{11},\mu_1)\right). \tag{3.6}$$

It follows from the the strong quasiconvexity of f in (ii) that

$$\xi\left(f\left(x_{11},\frac{x_{11}+x_{21}}{2},\mu_1\right)\right) \leq \max\left\{\xi\left(f(x_{11},x_{11},\mu_1)\right),\xi\left(f(x_{11},x_{21},\mu_1)\right)\right\} - \frac{h}{4}\|x_{11}-x_{21}\|^\beta,$$

which is equivalent to

$$\frac{h}{4}\|x_{11}-x_{21}\|^\beta \leq \max\left\{0,\xi\left(f(x_{11},x_{21},\mu_1)\right)\right\} - \xi\left(f\left(x_{11},\frac{x_{11}+x_{21}}{2},\mu_1\right)\right). \tag{3.7}$$

Case 1: If $\xi\left(f(x_{11},x_{21},\mu_1)\right)\geq 0$, then (3.7) becomes

$$\frac{h}{4}\|x_{11}-x_{21}\|^{\beta}\leq \xi\left(f(x_{11},x_{21},\mu_1)\right)-\xi\left(f\left(x_{11},\frac{x_{11}+x_{21}}{2},\mu_1\right)\right).$$

Combining this with (3.5) and (3.6), we see that

$$\begin{aligned}\frac{h}{4}\|x_{11}-x_{21}\|^{\beta}&\leq -\xi\left(f(x_{21},x_{11},\mu_1)\right)-\xi\left(f\left(x_{11},\frac{x_{11}+x_{21}}{2},\mu_1\right)\right)\\&\quad+\xi\left(f(x_{21},x_2,\mu_1)\right)+\xi\left(f\left(x_{11},\frac{x_1+x_{11}}{2},\mu_1\right)\right)\\&\leq\left[\xi\left(f\left(x_{11},\frac{x_1+x_{11}}{2},\mu_1\right)\right)-\xi\left(f\left(x_{11},\frac{x_{11}+x_{21}}{2},\mu_1\right)\right)\right]\\&\quad+\left[\xi\left(f(x_{21},x_2,\mu_1)\right)-\xi\left(f(x_{21},x_{11},\mu_1)\right)\right].\end{aligned}$$

Utilizing the Hölder continuity assumed in (ii) together with Lemma 3.2, we have

$$\begin{aligned}\frac{h}{4}\|x_{11}-x_{21}\|^{\beta}&\leq m\left\|\frac{x_1+x_{11}}{2}-\frac{x_{11}+x_{21}}{2}\right\|^{\delta}+m\|x_{11}-x_2\|^{\delta}\\&\leq \frac{m}{2^{\delta}}\|x_{21}-x_1\|^{\delta}+m\|x_{11}-x_2\|^{\delta}.\end{aligned}$$

This inequality together with (3.4) implies that

$$\frac{h}{4}\|x_{11}-x_{21}\|^{\beta}\leq \frac{ml^{\delta}}{2^{\delta}}\|\lambda_1-\lambda_2\|^{\alpha\delta}+ml^{\delta}\|\lambda_1-\lambda_2\|^{\alpha\delta}.$$

That is,

$$\frac{h}{4}\|x_{11}-x_{21}\|^{\beta}\leq ml^{\delta}(1+2^{-\delta})\|\lambda_1-\lambda_2\|^{\alpha\delta}.$$

Consequently, inequality (3.3) holds.

Case 2: If $\xi\left(f(x_{11},x_{21},\mu_1)\right)\leq 0$, then (3.7) can be rewritten as

$$\frac{h}{4}\|x_{11}-x_{21}\|^{\beta}\leq -\xi\left(f\left(x_{11},\frac{x_{11}+x_{21}}{2},\mu_1\right)\right).$$

From this and (3.5), we see that

$$\frac{h}{4}\|x_{11}-x_{21}\|^{\beta}\leq \xi\left(f\left(x_{11},\frac{x_1+x_{11}}{2},\mu_1\right)\right)-\xi\left(f\left(x_{11},\frac{x_{11}+x_{21}}{2},\mu_1\right)\right).$$

Utilizing the Hölder continuity in (ii) together with Lemma 3.2 and (3.4) again, one obtains

$$\frac{h}{4}\|x_{11}-x_{21}\|^{\beta}\leq \frac{m}{2^{\delta}}\|x_{21}-x_1\|^{\delta}\leq \frac{ml^{\delta}}{2^{\delta}}\|\lambda_1-\lambda_2\|^{\alpha\delta},$$

which evidently leads to the validity of inequality (3.3). The proof of Step 1 is complete.

Step 2. For all $x_{21}\in\Pi_{\xi}(\lambda_2,\mu_1)$ and $x_{22}\in\Pi_{\xi}(\lambda_2,\mu_2)$, we show that

$$\|x_{21}-x_{22}\|\leq\left(\frac{4n}{h}\right)^{\frac{1}{\beta-\theta}}\|\mu_1-\mu_2\|^{\frac{\gamma}{\beta-\theta}}. \tag{3.8}$$

Since $x_{21}\in\Pi_{\xi}(\lambda_2,\mu_1)$ and $x_{22}\in\Pi_{\xi}(\lambda_2,\mu_2)$, we have

$$\min\left\{\xi\left(f(x_{21},x_{22},\mu_1)\right),\xi\left(f(x_{22},x_{21},\mu_2)\right)\right\}\geq 0. \tag{3.9}$$

It follows from the convexity of $K(\lambda_2)$ that $\frac{x_{21}+x_{22}}{2} \in K(\lambda_2)$. Therefore,

$$\xi\left(f\left(x_{21}, \frac{x_{21}+x_{22}}{2}, \mu_1\right)\right) \geq 0.$$

This together with the strong quasiconvexity of f yields that

$$\max\left\{\xi\left(f(x_{21},x_{21},\mu_1)\right), \xi\left(f(x_{21},x_{22},\mu_1)\right)\right\} - \frac{h}{4}\|x_{21}-x_{22}\|^{\beta} \geq 0.$$

From the facts $\xi\left(f(x_{21},x_{21},\mu_1)\right) = 0$ and (3.9), we see that

$$\xi\left(f(x_{21},x_{22},\mu_1)\right) - \frac{h}{4}\|x_{21}-x_{22}\|^{\beta} \geq 0. \tag{3.10}$$

The monotonicity of f and Lemma 3.3 imply that $\xi\left(f(x_{21},x_{22},\mu_1)\right) \leq -\xi\left(f(x_{22},x_{21},\mu_1)\right)$. Combining this with (3.9) and (3.10), we have

$$\frac{h}{4}\|x_{21}-x_{22}\|^{\beta} \leq \xi\left(f(x_{22},x_{21},\mu_2)\right) - \xi\left(f(x_{22},x_{21},\mu_1)\right).$$

Employing the Hölder continuity of f in (iv), we obtain

$$\frac{h}{4}\|x_{21}-x_{22}\|^{\beta} \leq n\|\mu_1-\mu_2\|^{\gamma}.\|x_{21}-x_{22}\|^{\theta},$$

which implies inequality (3.8).

Step 3. We are in a position to complete the proof. For all $x_{11} \in \Pi_{\xi}(\lambda_1,\mu_1)$ and $x_{22} \in \Pi_{\xi}(\lambda_2,\mu_2)$, from the triangular inequality, we have, for every $x_{21} \in K(N)$,

$$\|x_{11}-x_{22}\| \leq \|x_{11}-x_{21}\| + \|x_{21}-x_{22}\|.$$

Taking into account (3.3) and (3.8), we obtain

$$\begin{aligned}&\rho\left(\Pi_{\xi}(\lambda_1,\mu_1), \Pi_{\xi}(\lambda_2,\mu_2)\right)\\ &\leq \left(\frac{4ml^{\delta}(1+2^{-\delta})}{h}\right)^{\frac{1}{\beta}} \|\lambda_1-\lambda_2\|^{\frac{\alpha\delta}{\beta}} + \left(\frac{4n}{h}\right)^{\frac{1}{\beta-\theta}} \|\mu_1-\mu_2\|^{\frac{\gamma}{\beta-\theta}}.\end{aligned}$$

We finish the proof. □

Theorem 3.2. *For* (WEP)*, assume that, for each $\xi \in \mathscr{B}_e^*$, the ξ-efficient solutions exist in a neighborhood of the considered point $(\lambda_0,\mu_0) \in \Lambda \times M$. Furthermore, assume that the following conditions are satisfied:*

(i) *K is $l.\alpha$-Hölder continuous on a neighborhood N of λ_0;*
(ii) *there is a neighborhood U of μ_0 such that, for all $x \in K(N)$ and $\mu \in U$, $f(x,\cdot,\mu)$ is $h.\beta$-strongly quasiconvex, convex-like as well as $m.\delta$-Hölder continuous on* $\mathrm{conv}(K(N))$;
(iii) *for each $\mu \in U$, $f(\cdot,\cdot,\mu)$ is monotone on $K(N)$;*
(iv) *f is $n.\gamma$-Hölder continuous on U, θ-uniformly over $K(N)$ with $\theta < \beta$.*

Then, there exist open neighborhoods $N'(\lambda_0)$ of λ_0 and $N'(\mu_0)$ of μ_0 such that the solution mapping S satisfies the following Hölder property in $N'(\lambda_0) \times N'(\mu_0)$, i.e., for all (λ_1,μ_1), $(\lambda_2,\mu_2) \in N'(\lambda_0) \times N'(\mu_0)$,

$$H(S(\lambda_1,\mu_1),S(\lambda_2,\mu_2)) \leq \left(\frac{4ml^{\delta}(1+2^{-\delta})}{h}\right)^{\frac{1}{\beta}} \|\lambda_1-\lambda_2\|^{\frac{\alpha\delta}{\beta}} + \left(\frac{4n}{h}\right)^{\frac{1}{\beta-\theta}} \|\mu_1-\mu_2\|^{\frac{\gamma}{\beta-\theta}}.$$

Proof. We see that $\mathscr{B}_e^*$ is a weak* compact set and in the Theorem 3.1, the system of $\{N(\bar{\xi})\}_{\bar{\xi}\in\mathscr{B}_e^*}$ is an open covering of $\mathscr{B}_e^*$. Consequently, there exists a finite number of points $\xi_i \in \mathscr{B}_e^*$ with $i = 1,...,n)$ satisfying

$$\mathscr{B}_e^* \subset \bigcup_{i=1}^{n} N(\xi_i). \tag{3.11}$$

Let $N'(\mu_0) = \bigcap_{i=1}^n N_{\xi_i}(\mu_0)$ and $N'(\lambda_0) = \bigcap_{i=1}^n N_{\xi_i}(\lambda_0)$. Take arbitrarily $(\lambda,\mu) \in N'(\lambda_0) \times N'(\mu_0)$. Thanks to (3.11), for any $\xi \in \mathscr{B}_e^*$, there exists $i_0 \in \{1,...,n\}$ such that $\xi \in N(\xi_{i_0})$. From the construction of the neighborhoods $N'(\lambda_0)$ and $N'(\mu_0)$, we have $(\lambda,\mu) \in N_{\xi_{i_0}}(\lambda_0)\times N_{\xi_{i_0}}(\mu_0)$. So, $N'(\lambda_0)$ and $N'(\mu_0)$ are desired neighborhoods of λ_0 and μ_0, respectively.

Due to the convex-likeness of f, applying Lemma 3.4, one has

$$S(\lambda,\mu) = \bigcup_{\xi\in\mathscr{B}_e^*} \Pi_\xi(\lambda,\mu).$$

For all $(\lambda_1,\mu_1),(\lambda_2,\mu_2) \in N'(\lambda_0)\times N'(\mu_0)$, we now claim that

$$\begin{aligned} &H\big(S(\lambda_1,\mu_1),S(\lambda_2,\mu_2)\big)\\ &\le \left(\frac{4ml^\delta(1+2^{-\delta})}{h}\right)^{\frac{1}{\beta}} \|\lambda_1-\lambda_2\|^{\frac{\alpha\delta}{\beta}} + \left(\frac{4n}{h}\right)^{\frac{1}{\beta-\theta}} \|\mu_1-\mu_2\|^{\frac{\gamma}{\beta-\theta}}. \end{aligned} \tag{3.12}$$

Indeed, for each $x_1 \in S(\lambda_1,\mu_1) = \bigcup_{\xi\in\mathscr{B}_e^*}\Pi_\xi(\lambda_1,\mu_1)$, there exists $\hat{\xi} \in \mathscr{B}_e^*$ such that $x_1 \in \Pi_{\hat{\xi}}(\lambda_1,\mu_1)$. In view of $\Pi_{\hat{\xi}}(\lambda_2,\mu_2) \subseteq S(\lambda_2,\mu_2)$ and Theorem 3.1, one has

$$\begin{aligned} d\big(x_1,S(\lambda_2,\mu_2)\big) &\le d\big(x_1,\Pi_{\hat{\xi}}(\lambda_2,\mu_2)\big)\\ &\le \mathrm{ex}\left(\Pi_{\hat{\xi}}(\lambda_1,\mu_1),\Pi_{\hat{\xi}}(\lambda_2,\mu_2)\right)\\ &\le H\big(\Pi_{\hat{\xi}}(\lambda_1,\mu_1),\Pi_{\hat{\xi}}(\lambda_2,\mu_2)\big)\\ &\le \rho\big(\Pi_{\hat{\xi}}(\lambda_1,\mu_1),\Pi_{\hat{\xi}}(\lambda_2,\mu_2)\big)\\ &\le \left(\frac{4ml^\delta(1+2^{-\delta})}{h}\right)^{\frac{1}{\beta}} \|\lambda_1-\lambda_2\|^{\frac{\alpha\delta}{\beta}} + \left(\frac{4n}{h}\right)^{\frac{1}{\beta-\theta}} \|\mu_1-\mu_2\|^{\frac{\gamma}{\beta-\theta}}. \end{aligned}$$

Thus

$$\mathrm{ex}\left(S(\lambda_1,\mu_1),S(\lambda_2,\mu_2)\right) \le \left(\frac{4ml^\delta(1+2^{-\delta})}{h}\right)^{\frac{1}{\beta}} \|\lambda_1-\lambda_2\|^{\frac{\alpha\delta}{\beta}} + \left(\frac{4n}{h}\right)^{\frac{1}{\beta-\theta}} \|\mu_1-\mu_2\|^{\frac{\gamma}{\beta-\theta}}. \tag{3.13}$$

Similarly, we also have

$$\mathrm{ex}\left(S(\lambda_2,\mu_2),S(\lambda_1,\mu_1)\right) \le \left(\frac{4ml^\delta(1+2^{-\delta})}{h}\right)^{\frac{1}{\beta}} \|\lambda_1-\lambda_2\|^{\frac{\alpha\delta}{\beta}} + \left(\frac{4n}{h}\right)^{\frac{1}{\beta-\theta}} \|\mu_1-\mu_2\|^{\frac{\gamma}{\beta-\theta}}. \tag{3.14}$$

From (3.13) and (3.14), we obtain (3.12). Hence, the proof is complete. □

Remark 3.1. Studies on the Hölder property, especially for vector equilibrium problems are very interesting and have received much attention from researchers, as mentioned in the Introduction. In [25], using a direct approach together with the strong convexity condition, the authors established the Hölder property for strong vector equilibrium problems. However, the approach in [25] cannot be applied to weak vector equilibrium problems because the complement of the convex cone is not convex. Fortunately, with linear scalarization, we have successfully formulated the Hölder property for weak vector equilibrium problems. Therefore, the results obtained in this paper differ from those in [25] not only in terms of the approach and problem setting but also in applications to scalar equilibrium problems. In other words, when strong vector equilibrium problems and weak vector equilibrium problems coincide, Theorem 3.2 improves upon the main results in [25] by relaxing strong convexity to strong quasiconvexity.

In the special case that $Y = \mathbb{R}, C = \mathbb{R}_+$ and $e = 1$, we have the following result.

Corollary 3.1. *Assume that the following conditions are satisfied:*

(i) *K is $l.\alpha$-Hölder continuous on a neighborhood N of λ_0;*
(ii) *there exists a neighborhood U of μ_0 such that, for all $x \in K(N)$ and $\mu \in U$, $f(x,\cdot,\mu)$ is $h.\beta$-strongly quasiconvex as well as $m.\delta$-Hölder continuous on* $\operatorname{conv}(K(N))$;
(iii) *for each $\mu \in U$, $f(\cdot,\cdot,\mu)$ is monotone on $K(N)$;*
(iv) *f is $n.\gamma$-Hölder continuous on U, θ-uniformly over $K(N)$ with $\theta < \beta$.*

Then, the solution mapping S satisfies the following Hölder property in $N \times U$, i.e., for all (λ_1,μ_1), $(\lambda_2,\mu_2) \in N \times U$,

$$H(S(\lambda_1,\mu_1),S(\lambda_2,\mu_2)) \le \left(\frac{4ml^{\delta}(1+2^{-\delta})}{h}\right)^{\frac{1}{\beta}} \|\lambda_1-\lambda_2\|^{\frac{\alpha\delta}{\beta}} + \left(\frac{4n}{h}\right)^{\frac{1}{\beta-\theta}} \|\mu_1-\mu_2\|^{\frac{\gamma}{\beta-\theta}}.$$

Remark 3.2. In this special case, (WEP) reduces to (PKFI) in [28] and to (KF) in [24]. It is worth noting that Corollary 3.1 improves Theorem 3.2 in [28] and Theorem 3.1 in [24] in the following aspects:

(a) The strong convexity of the objective function f in the second component is relaxed to the strong quasiconvexity. It is known that the strong convexity is a stringent condition, making it challenging to apply in practical situations. Therefore, this relaxation holds great significance.
(b) The Lipschitz continuity of the objective function f in the second component has been relaxed to the Hölder continuity.

The following simple example indicates the case in which Theorem 3.1 can be applied, whereas while results in the literature cannot.

Example 3.1. Let $X = A = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, e = (1,1) \in \operatorname{int} C, \Lambda = M = [0,1], K(\lambda) = [0,\lambda]$, and $f(x,y,\lambda) = (\lambda+1)(y-x,y-x)$. Then, we see that all the assumptions of Theorem 3.1 are satisfied with $l = 1, \alpha = 1, h = 2, \beta = 1, m = 8, \delta = 1, n = 2, \gamma = 1$, and $\theta = 0$. The solution set is $S(\lambda) = \{0\}$ for all λ, which is Hölder continuous.

It should be noted that the strong convexity condition of f is not fulfilled. That is, the results in [24, 28] are not applicable in this case. Furthermore, due to the fact that $f(1,1,\lambda) = 0_{\mathbb{R}^2}$ for all $\lambda \in \Lambda$, the strong monotonicity as required in [19, 20, 21], the strong pseudo-monotonicity as described in [22, 29, 34], and the conditions related to strong monotonicity discussed in [23] are not satisfied. Hence, the corresponding results in these papers cannot be applied.

We now provide some examples to show that the assumptions in Theorem 3.1 are essential.

Example 3.2. (the strong quasiconvexity is crucial) Let $X = A = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, \Lambda = M = [0,1], K(\lambda) = [\lambda, 2], \lambda_0 = 0$, and $f(x,y,\lambda) = \lambda^2(x^2 - y^2, x^2 - y^2)$. Then, (i) is fulfilled with $l = 1$, $\alpha = 1$, and the Hölder continuity in (ii) is satisfied with $m = 4, \delta = 1$. Assumption (iii) is clear, and (iv) holds with $n = 8, \gamma = 1$ and $\theta = 1$. Some direct computations gives us the solution

$$S(\lambda) = \begin{cases} [0,2], & \lambda = 0, \\ \{2\}, & \lambda \neq 0. \end{cases}$$

Hence, the solution set $S(0)$ is not a singleton and S is not even lower semicontinuous at $\lambda_0 = 0$. The reason is that the strong quasiconvexity of f imposed in (ii) is violated.

Example 3.3. (the monotonicity is essential) Let $X = A = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, \Lambda = M = [0,1]$, $K(\lambda) = [0,1]$, $\lambda_0 = 0$, and $f(x,y,\lambda) = (y - \lambda x, y - \lambda x)$. Then, (i) holds with $l = 0, \alpha = 1$ and (ii) is satisfied with $h = 1$, $\beta = 1$, and $m = 2, \delta = 1$. Condition (iv) holds with $n = 1, \gamma = 1, \theta = 0$. One has

$$S(\lambda) = \begin{cases} [0,1], & \lambda = 0, \\ \{0\}, & \lambda \neq 0. \end{cases}$$

Thus, the solution set $S(0)$ is not a singleton and S is not even lower semicontinuous at $\lambda_0 = 0$. The reason is that the monotonicity of f is not satisfied. Indeed, one has $f(x,y,0) + f(y,x,0) = (x+y, x+y) \notin -\mathbb{R}^2_-$ for many $x, y \in [0,1]$.

4. Concluding Remarks

By employing relaxed assumptions on convexity, this paper establishes sufficient conditions for the Hölder continuity of a solution map to a vector equilibrium problem via the scalarization method. The applicability as well as sophistication of the imposed assumptions are illustrated through numerous examples and counterexamples. Our results in this study have significant implications for various optimization-related models mentioned in Section 1, as equilibrium problems encompass them as special cases. Furthermore, we believe that this approach holds potential applications in generalized models of equilibrium problems, such as variational relation problems, variational inclusion problems, and more.

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