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SOME GENERALIZATIONS OF REVERSE POWER INEQUALITIES FOR LOG-CONCAVE FUNCTIONS

MOHAMED AMINE IGHACHANE[1,*], MOHAMMED BOUCHANGOUR[2], MOHAMMAD W. ALOMARI[3]

[1]*Sciences and Technologies Team (ESTE), Higher School of Education and Training of El Jadida, Chouaib Doukkali University, El Jadida, Morocco*
[2]*Department of Mathematics, LIABM Laboratory, Faculty of Sciences, Mohammed First University, B.P. 717, 60000 Oujda, Morocco*
[3]*Department of Mathematics, Faculty of Science and Information Technology, Jadara University, P.O. Box 733, Irbid, P.C. 21110, Jordan*

Abstract. The main goal of this article is to develop a general method to improve some new power inequalities for log-concave functions, which extends and unifies some recent results. As a consequence, we prove a generalized multiple term reverses refinements of the difference between the arithmetic and power means inequality for scalars and operators.
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1. INTRODUCTION

Concave functions have played a key role in different fields, including mathematical inequalities, optimization, functional analysis, applied mathematics and mathematical physics, to mention a few. We recall that a function $f: J \to \mathbb{R}$ is said to be concave on the interval J if

$$f((1-\alpha)a+\alpha b) \geq (1-\alpha)f(a)+\alpha f(b); a,b \in J, 0 \leq \alpha \leq 1.$$

If this inequality is reversed, then f is said to be convex. On the other hand, if $\log f$ is a concave function, then f is called log-concave, and it is called log-convex if $\log f$ is convex. Consequently, a log-concave (log-convex) function is necessarily a positive function. If $f: J \to (0,\infty)$ is log-concave function, then it satisfies

$$f((1-\alpha)a+\alpha b) \geq f^{1-\alpha}(a) f^{\alpha}(b); a,b \in J, 0 \leq \alpha \leq 1,$$

while a log-convex function satisfies the reversed inequality.

Consequently, it is readily seen that a concave function is necessarily log-concave, while a log-convex function is convex. The opposite of these statements are not valid. We observe that

*Corresponding author.
E-mail address: mohamedamineighachane@gmail.com (M. A. Ighachane), m.bouchangour@ump.ac.ma (M. Bouchangour), mwomath@gmail.com (M. W. Alomari).
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when f is a log-concave, three quantities appear to be significant. Namely,

$$f((1-\alpha)a+\alpha b), (1-\alpha)f(a)+\alpha f(b) \text{ and } f^{1-\alpha}(a)f^{\alpha}(b).$$

If $a=0$ and $b=1$, then these quantities reduce to

$$f(\alpha), (1-\alpha)f(0)+\alpha f(1) \text{ and } f^{1-\alpha}(0)f^{\alpha}(1).$$

The primary objective of this paper is to find new explicit delicate mixed relations involving the three quantities.

We emphasize that the following inequality is not true for a log-concave function

$$(1-\alpha)f(0)+\alpha f(1) \le f(\alpha).$$

Indeed, on one side the function $f(\alpha) := ((1-\alpha)a^p+\alpha b^p)^{\frac{1}{p}}$ for $a,b>0$, and $p\in(0,1)$ is log-concave. On the other side, via the increasing property of the power mean, we have

$$(1-\alpha)f(0)+\alpha f(1) \ge f(\alpha).$$

The second primary purpose of this paper is to find a positive term that we can add or subtract from each side of this inequality and make it true.

The structure of this paper is outlined in the following manner. In the next section, we provide a brief overview of the motivation behind this work, followed by the demonstration of several results for log-concave functions that contribute to achieving our objective. After completing the basic form, we proceed to establish a more complicated version of the desired inequalities for log-concave functions. Applications that include scalar inequalities and operator inequalities will be discussed then.

2. Motivation

In this section, we describe the motivation behind this work. An essential and significant inequality in functional analysis is the well-known Young's inequality, which states that

$$a^{\alpha}b^{1-\alpha} \le \alpha a+(1-\alpha)b,\ a,b>0,\ 0\le\alpha\le 1$$

with equality if and only if $a=b$. This inequality has many proofs. The easiest proof is done by using the notion of convexity. Namely, the function $f(\alpha)=a^{\alpha}b^{1-\alpha}$ is convex. Hence, $f(\alpha)\le(1-\alpha)f(0)+\alpha f(1)$, which is equivalent to Young's inequality.

Kittaneh and Manasrah [10] refined Young's inequality as follows

$$a^{\alpha}b^{1-\alpha}+r_0\left(\sqrt{a}-\sqrt{b}\right)^2 \le \alpha a+(1-\alpha)b, \tag{2.1}$$

where $r_0=\min\{\alpha,1-\alpha\}$ and $0\le\alpha\le 1$.

Earlier, the following squared version was shown in [1]

$$(a^{\alpha}b^{1-\alpha})^2+r_0^2(a-b)^2 \le (\alpha a+(1-\alpha)b)^2, \tag{2.2}$$

where $r_0=\min\{\alpha,1-\alpha\}$.

Reverses of (2.1) and (2.2) were shown in [11], as follows

$$(\alpha a+(1-\alpha)b)^2 \le (a^{\alpha}b^{1-\alpha})^2+R_0^2(a-b)^2,$$

and

$$\alpha a+(1-\alpha)b \le a^{\alpha}b^{1-\alpha}+R_0\left(\sqrt{a}-\sqrt{b}\right)^2,$$

respectively, where $R_0 = \max\{\alpha, 1-\alpha\}$ and $0 \le \alpha \le 1$.

Ighachane and Akkouchi in [3] obtained a generalization of inequalities (2.1) and (2.2) as follows:

Theorem 2.1. *Let $a, b > 0$ and $0 < \alpha < 1$. Then, for all positive integers m,*

$$\left(a^\alpha b^{1-\alpha}\right)^m + r_0^m\left(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\right) \le \left(\alpha a + (1-\alpha)b\right)^m. \tag{2.3}$$

Recently, Zhao [17] obtained a reverse of inequality (2.3) in the following manner:

Theorem 2.2 ([17])**.** *Let $a, b > 0$ and $0 < \alpha < 1$. Then, all positive integers m,*

$$\left(\alpha a + (1-\alpha)b\right)^m \le \left(a^\alpha b^{1-\alpha}\right)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\right), \tag{2.4}$$

where $\binom{m}{k}$ is the binomial coefficient and $R_m := \max\{\binom{m}{k}\alpha^k(1-\alpha)^{m-k},\ k = 0, \dots, m\}$.

Afterwards, Ighachane [2] obtained the following refinement of (2.4).

Theorem 2.3. *Let $a, b > 0, 0 < \alpha < 1$ and let m be a positive integer such that $R_m = \max\{\binom{m}{k}$ $\alpha^k(1-\alpha)^{m-k},\ k = 0, \dots, m\}$.*

(1) If $0 < \alpha < \frac{1}{4}$, then

$$\left(\alpha a + (1-\alpha)b\right)^m \le \left(a^\alpha b^{1-\alpha}\right)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\right)$$
$$- 2\alpha^m\left((ab)^{\frac{m}{4}} - a^{\frac{m}{2}}\right)^2 - r_m\left(a^{\frac{m}{2}} - (a^3b)^{\frac{m}{8}}\right)^2,$$

where $r_m = \min\{4\alpha^m, R_m - 3\alpha^m\}$.

(2) If $\frac{1}{4} \le \alpha < \frac{1}{2}$, then

$$\left(\alpha a + (1-\alpha)b\right)^m \le \left(a^\alpha b^{1-\alpha}\right)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\right)$$
$$- (1-2\alpha)^m\left((ab)^{\frac{m}{4}} - a^{\frac{m}{2}}\right)^2 - 2(1-2\alpha)^m\left((ab)^{\frac{m}{4}} - (a^3b)^{\frac{m}{8}}\right)^2.$$

(3) If $\frac{1}{2} \le \alpha < \frac{3}{4}$, then

$$\left(\alpha a + (1-\alpha)b\right)^m \le \left(a^\alpha b^{1-\alpha}\right)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\right)$$
$$- (2\alpha-1)^m\left((ab)^{\frac{m}{4}} - b^{\frac{m}{2}}\right)^2 - 2(2\alpha-1)^m\left((ab)^{\frac{m}{4}} - (ab^3)^{\frac{m}{8}}\right)^2.$$

(4) If $\frac{3}{4} \le \alpha < 1$, then

$$\Big(\alpha a+(1-\alpha)b\Big)^m \le \Big(a^\alpha b^{1-\alpha}\Big)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a}-(m+1)(ab)^{\frac{m}{2}}\right)$$
$$-2R_m\Big((ab)^{\frac{m}{4}}-b^{\frac{m}{2}}\Big)^2 - r_m\Big(b^{\frac{m}{2}}-(ab^3)^{\frac{m}{8}}\Big)^2,$$

where $r_m = \min\{4(1-\alpha)^m, R_m - 3(1-\alpha)^m\}$.

For recent progress in this direction, we refer to [4, 5, 6, 8, 9, 13, 14, 15] and the references therein. The main motivation of our paper is to discuss some results for log-concave functions. Those results are used to obtain a general inequality that extends and unifies Theorems 2.2 and 2.3. At this stage, we encourage the reader to see Section 5 below to be able to oversee the purpose of this introduction. In the following, we list some lemmas that we will need in our analysis.

Lemma 2.1 ([7])**.** *Let m be a positive integer and let α a positive number such that $0 \le \alpha \le 1$. Then $\sum_{k=1}^{m}\binom{m}{k}k\alpha^k(1-\alpha)^{m-k} = m\alpha$, $\sum_{k=0}^{m-1}\binom{m}{k}(m-k)\alpha^k(1-\alpha)^{m-k} = m(1-\alpha)$, and $\sum_{k=1}^{m}\binom{m}{k}k = \sum_{k=0}^{m-1}\binom{m}{k}(m-k) = m2^{m-1}$.*

Lemma 2.2 ([7])**.** *Let α be a positive number such that $0 \le \alpha \le 1$ and m be a positive integer such that $R_m = \max\left\{\binom{m}{k}\alpha^k(1-\alpha)^{m-k},\ k = 0,\dots,m\right\}$. (1) If $0 \le \alpha \le \frac{1}{4}$, then $R_m - 3\alpha^m \ge (1-\alpha)^m - 3\alpha^m \ge 0$. (2) If $\frac{1}{4} \le \alpha \le \frac{1}{2}$, then $R_m - \alpha^m - (1-2\alpha)^m \ge (1-\alpha)^m - \alpha^m - (1-2\alpha)^m \ge 0$.*

The following lemma can be easily established by some elementary calculus.

Lemma 2.3. *Let $f : [0,1] \longrightarrow (0,\infty)$ be a given function, and let m,n be two positive integers with $n \ge 2$, and $0 \le \alpha \le 1$. Then*

$$M_n := \sum_{k=2}^{n} 2^{k-1}\left(\sqrt{f^m(1)} - \sqrt[2^k]{f^m(0)f^{m(2^{k-1}-1)}(1)}\right)^2$$
$$= (2^n-2)f^m(1) + 2(f(0)f(1))^{\frac{m}{2}} - 2^n f^{\frac{m}{2}}(1)\sqrt[2^n]{f^m(0)f^{m(2^{n-1}-1)}(1)},$$

and

$$N_n := (f(0)f(1))^{\frac{m}{2}}\sum_{k=2}^{n} 2^{k-2}\left(\sqrt[2^k]{\frac{f^m(1)}{f^m(0)}} - 1\right)^2$$
$$= (2^{n-1}-1)(f(0)f(1))^{\frac{m}{2}} + f^m(1) - 2^{n-1}f^{\frac{m(2^{n-1}-1)}{2^n}}(0)f^{\frac{m(2^{n-1}+1)}{2^n}}(1).$$

Lemma 2.4 ([7])**.** *Let n,m be two positive integers such that $n \ge 2$ and α be a positive number.*

(1) If $\alpha \in [0, \frac{1}{2^n}]$, then $R_m - \alpha^m - (2^n-2)\alpha^m \ge (1-\alpha)^m - \alpha^m - (2^n-2)\alpha^m \ge 0$.
(2) If $\alpha \in [0, \frac{1}{2}]$, then $R_m - \alpha^m - (1-2\alpha)^m \ge (1-\alpha)^m - \alpha^m - (1-2\alpha)^m \ge 0$.

3. The Simple Form

Now, we are ready to state and prove our first main result about log-concave functions.

Theorem 3.1. *Let $f:[0,1]\longrightarrow(0,\infty)$ be a log-concave function, $0\le\alpha\le 1$, and m be a positive integer such that $R_m=\max\left\{\binom{m}{k}\alpha^k(1-\alpha)^{m-k},\ k=0,\dots,m\right\}$.*

(1) If $0<\alpha<\frac{1}{4}$, then

$$\left(\alpha f(1)+(1-\alpha)f(0)\right)^m\le f^m(\alpha)+R_m\left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-(m+1)(f(0)f(1))^{\frac{m}{2}}\right)$$
$$-2\alpha^m\left((f(1)f(0))^{\frac{m}{4}}-f^{\frac{m}{2}}(1)\right)^2-r_m\left(f^{\frac{m}{2}}(1)-\left(f^3(1)f(0)\right)^{\frac{m}{8}}\right)^2, \tag{3.1}$$

where $r_m=\min\{4\alpha^m,R_m-3\alpha^m\}$.

(2) If $\frac{1}{4}\le\alpha<\frac{1}{2}$, then

$$\left(\alpha f(1)+(1-\alpha)f(0)\right)^m\le f^m(\alpha)+R_m\left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-(m+1)(f(0)f(1))^{\frac{m}{2}}\right)$$
$$-(1-2\alpha)^m\left((f(1)f(0))^{\frac{m}{4}}-f^{\frac{m}{2}}(1)\right)^2-2(1-2\alpha)^m\left((f(1)f(0))^{\frac{m}{4}}-\left(f^3(1)f(0)\right)^{\frac{m}{8}}\right)^2. \tag{3.2}$$

(3) If $\frac{1}{2}\le\alpha<\frac{3}{4}$, then

$$\left(\alpha f(1)+(1-\alpha)f(0)\right)^m\le f^m(\alpha)+R_m\left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-(m+1)(f(0)f(1))^{\frac{m}{2}}\right)$$
$$-(2\alpha-1)^m\left((f(1)f(0))^{\frac{m}{4}}-f^{\frac{m}{2}}(0)\right)^2-2(2\alpha-1)^m\left((f(1)f(0))^{\frac{m}{4}}-\left(f(1)f^3(0)\right)^{\frac{m}{8}}\right)^2. \tag{3.3}$$

(4) If $\frac{3}{4}\le\alpha<1$, then

$$\left(\alpha f(1)+(1-\alpha)f(0)\right)^m\le f^m(\alpha)+R_m\left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-(m+1)(f(0)f(1))^{\frac{m}{2}}\right)$$
$$-2R_m\left((f(1)f(0))^{\frac{m}{4}}-f^{\frac{m}{2}}(0)\right)^2-r_m\left(f^{\frac{m}{2}}(0)-(f^3(0)f(1))^{\frac{m}{8}}\right)^2, \tag{3.4}$$

where $r_m=\min\{4R_m,\alpha^m-3R_m\}$.

Proof. (1) Let $0<\alpha<\frac{1}{4}$. We claim that

$$R_m\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-\left(\alpha f(1)+(1-\alpha)f(0)\right)^m+f^m(\alpha)$$
$$-2\alpha^m\left((f(1)f(0))^{\frac{m}{4}}-f^{\frac{m}{2}}(1)\right)^2-r_m\left(f^{\frac{m}{2}}(1)-(f^3(1)f(0))^{\frac{m}{8}}\right)^2 \tag{3.5}$$
$$\ge(m+1)R_m(f(0)f(1))^{\frac{m}{2}}.$$

We have the following identity

$$
\begin{aligned}
&R_m \frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)} - \Big(\alpha f(1)+(1-\alpha)f(0)\Big)^m + f^m(\alpha) \\
&-2\alpha^m\Big((f(1)f(0))^{\frac{m}{4}} - f^{\frac{m}{2}}(1)\Big)^2 - r_m\Big(f^{\frac{m}{2}}(1)-(f^3(1)f(0))^{\frac{m}{8}}\Big)^2 \\
&=\sum_{k=0}^{m-1}\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right) f^k(1)f^{m-k}(0) + f^m(\alpha) \\
&+\Big(R_m - 3\alpha^m - r_m\Big) f^m(1) - 2\alpha^m (f(0)f(1))^{\frac{m}{2}} \\
&+\Big(4\alpha^m - r_m\Big)\Big(f(0)f^3(1)\Big)^{\frac{m}{4}} + 2r_m\Big(f^7(1)f(0)\Big)^{\frac{m}{8}}.
\end{aligned}
$$

Thus inequality (3.5) is equivalent to

$$
\begin{aligned}
&((m+1)R_m+2\alpha^m)^{-1}\left[\sum_{k=0}^{m-1}\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right) f^k(1)f^{m-k}(0)\right. \\
&+f^m(\alpha) + \Big(R_m - 3\alpha^m - r_m\Big) f^m(1) \\
&\left.+\Big(4\alpha^m - r_m\Big)(f(0)f^3(1))^{\frac{m}{4}} + 2r_m\Big(f^7(1)f(0)\Big)^{\frac{m}{8}}\right] \geq (f(0)f(1))^{\frac{m}{2}}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&((m+1)R_m+2\alpha^m)^{-1}\left[\sum_{k=0}^{m-1}\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right) f^k(1)f^{m-k}(0)\right. \\
&f^m(\alpha) + \Big(R_m - 3\alpha^m - r_m\Big) f^m(1) \\
&\left.+\Big(4\alpha^m - r_m\Big)(f(0)f^3(1))^{\frac{m}{4}} + 2r_m(f^7(1)f(0))^{\frac{m}{8}}\right] \\
&=\sum_{k=0}^{m+3}\alpha_k((m+1)R_m+2\alpha^m)^{-1}x_k,
\end{aligned}
$$

where

$$
x_k = \begin{cases}
f^k(1)f^{m-k}(0), & 0\leq k\leq m-1 \\
f^m(1), & k=m \\
f^m(\alpha), & k=m+1 \\
(f(0)f^3(1))^{\frac{m}{4}}, & k=m+2 \\
(f(0)f^7(1))^{\frac{m}{8}}, & k=m+3,
\end{cases}
$$

and

$$\alpha_k = \begin{cases} \left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right), & 0 \le k \le m-1 \\ R_m - 3\alpha^m - r_m, & k = m \\ 1, & k = m+1 \\ 4\alpha^m - r_m, & k = m+2 \\ 2r_m, & k = m+3. \end{cases}$$

By Lemma 2.2, we have

(a) $x_k > 0$ for all $k \in \{0, 1, \dots, m+3\}$,

(b) $\alpha_k \ge 0$ for all $k \in \{0, 1, \dots, m+3\}$, with $\sum_{k=0}^{m+3}((m+1)R_m + 2\alpha^m)^{-1}\alpha_k = 1$.

The arithmetic-geometric mean inequality yields

$$\sum_{k=0}^{m+3}((m+1)R_m + 2\alpha^m)^{-1}\alpha_k x_k \ge \prod_{k=0}^{m+3} x_k^{\alpha_k} = f^{\alpha(m)}(1) f^{\beta(m)}(0) f^{\gamma(m)}(\alpha),$$

where

$$\begin{aligned} \alpha(m) &= ((m+1)R_m + 2\alpha^m)^{-1}\left[\sum_{k=1}^{m-1} k\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)\right. \\ &+ \left. m\left(R_m - 3\alpha^m - r_m\right) + \frac{3m}{4}\left(4\alpha^m - r_m\right) + \frac{7m}{8}2r_m\right] \\ &= \frac{\frac{m(m+1)}{2}R_m - m\alpha + m\alpha^m}{(m+1)R_m + 2\alpha^m}, \quad \text{(by Lemma 2.1)} \end{aligned}$$

$$\begin{aligned} \beta(m) &= ((m+1)R_m + 2\alpha^m)^{-1}\left[\sum_{k=0}^{m-1}(m-k)\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)\right. \\ &+ \left. \frac{m}{4}\left(4\alpha^m - r_m\right) + \frac{m}{8}2r_m\right] \\ &= \frac{\frac{m(m+1)}{2}R_m - m(1-\alpha) + m\alpha^m}{(m+1)R_m + 2\alpha^m}, \quad \text{(by Lemma 2.1)} \end{aligned}$$

and

$$\gamma(m) = \frac{m}{(m+1)R_m + 2\alpha^m}.$$

Applying the log-concavity of the function f, we arrive at

$$f^{\alpha(m)}(1) f^{\beta(m)}(0) f^{\gamma(m)}(\alpha) \ge f^{\alpha(m)}(1) f^{\beta(m)}(0) f^{\alpha\gamma(m)}(1) f^{(1-\alpha)\gamma(m)}(0) = (f(0)f(1))^{\frac{m}{2}}.$$

This completes the proof of (3.1).

(2) Let $\frac{1}{4} \leq \alpha < \frac{1}{2}$. We claim that

$$R_m \frac{f^{m+1}(0) - f^{m+1}(1)}{f(0) - f(1)} - \Big(\alpha f(1) + (1-\alpha) f(0)\Big)^m + f^m(\alpha)$$
$$- (1-2\alpha)^m \Big((f(1)f(0))^{\frac{m}{4}} - f^{\frac{m}{2}}(1) \Big)^2 - 2(1-2\alpha)^m \Big((f(1)f(0))^{\frac{m}{4}} - (f^3(1)f(0))^{\frac{m}{8}} \Big)^2$$
$$\geq (m+1) R_m (f(0)f(1))^{\frac{m}{2}}. \tag{3.6}$$

Furthermore, we have

$$R_m \frac{f^{m+1}(0) - f^{m+1}(1)}{f(0) - f(1)} - \Big(\alpha f(1) + (1-\alpha) f(0)\Big)^m + f^m(\alpha)$$
$$(1-2\alpha)^m \Big((f(1)f(0))^{\frac{m}{4}} - f^{\frac{m}{2}}(1) \Big)^2 - 2(1-2\alpha)^m \Big((f(1)f(0))^{\frac{m}{4}} - (f^3(1)f(0))^{\frac{m}{8}} \Big)^2$$
$$= \sum_{k=0}^{m-1} \left(R_m - \binom{m}{k} \alpha^k (1-\alpha)^{m-k} \right) f^k(1) f^{m-k}(0) + f^m(\alpha)$$
$$+ \Big(R_m - \alpha^m - (1-2\alpha)^m \Big) f^m(1) - 3(1-2\alpha)^m (f(0)f(1))^{\frac{m}{2}} + 4(1-2\alpha)^m (f^5(1)f^3(0))^{\frac{m}{8}}.$$

Therefore, (3.6) is equivalent to

$$((m+1)R_m + 3(1-2\alpha)^m)^{-1} \left[\sum_{k=0}^{m-1} \left(R_m - \binom{m}{k} \alpha^k (1-\alpha)^{m-k} \right) f^k(1) f^{m-k}(0) \right.$$
$$\left. + f^m(\alpha) + \Big(R_m - \alpha^m - (1-2\alpha)^m \Big) f^m(1) + 4(1-2\alpha)^m (f^5(1)f^3(0))^{\frac{m}{8}} \right] \geq (f(0)f(1))^{\frac{m}{2}}.$$

Thus

$$((m+1)R_m + 3(1-2\alpha)^m)^{-1} \left[\sum_{k=0}^{m-1} \left(R_m - \binom{m}{k} \alpha^k (1-\alpha)^{m-k} \right) f^k(1) f^{m-k}(0) \right.$$
$$\left. + f^m(\alpha) + \Big(R_m - \alpha^m - (1-2\alpha)^m \Big) f^m(1) + 4(1-2\alpha)^m \Big(f^5(1)f^3(0) \Big)^{\frac{m}{8}} \right]$$
$$= \sum_{k=0}^{m+2} \alpha_k ((m+1)R_m + 3(1-2\alpha)^m)^{-1} x_k,$$

where

$$x_k = \begin{cases} f^k(1) f^{m-k}(0), & 0 \leq k \leq m-1 \\ f^m(1), & k = m \\ f^m(\alpha), & k = m+1 \\ (f^5(1)f^3(0))^{\frac{m}{8}}, & k = m+2, \end{cases}$$

and

$$\alpha_k = \begin{cases} \left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right), & 0 \le k \le m-1 \\ R_m - \alpha^m - (1-2\alpha)^m, & k = m \\ 1, & k = m+1 \\ 4(1-2\alpha)^m, & k = m+2. \end{cases}$$

Lemma 2.2 implies that

(a) $x_k > 0$ for all $k \in \{0,1,\dots,m+2\}$,

(b) $\alpha_k \ge 0$ for all $k \in \{0,1,\dots,m+2\}$, with

$$\sum_{k=0}^{m+2}((m+1)R_m + 3(1-2\alpha)^m)^{-1}\alpha_k = 1.$$

By the arithmetic-geometric mean inequality, we may write

$$\begin{aligned} \sum_{k=0}^{m+2}((m+1)R_m + 3(1-2\alpha)^m)^{-1}\alpha_k x_k &\ge \prod_{k=0}^{m+2} x_k^{\alpha_k} \\ &= f^{\alpha(m)}(1) f^{\beta(m)}(0) f^{\gamma(m)}(\alpha), \end{aligned}$$

where

$$\begin{aligned} \alpha(m) &= ((m+1)R_m + 3(1-2\alpha)^m)^{-1}\left[\sum_{k=1}^{m-1} k\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)\right. \\ &+ \left. m\left(R_m - \alpha^m - (1-2\alpha)^m\right) + \frac{5m}{8}4(1-2\alpha)^m\right] \\ &= \frac{\frac{m(m+1)}{2}R_m - m\alpha + \frac{3m}{2}(1-2\alpha)^m}{(m+1)R_m + 3(1-2\alpha)^m}, \quad \text{(by Lemma 2.1)} \end{aligned}$$

$$\begin{aligned} \beta(m) &= ((m+1)R_m + 3(1-2\alpha)^m)^{-1}\left[\sum_{k=0}^{m-1}(m-k)\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)\right. \\ &+ \left. \frac{3m}{8}4(1-2\alpha)^m\right] \\ &= \frac{\frac{m(m+1)}{2}R_m - m(1-\alpha) + \frac{3m}{2}(1-2\alpha)^m}{(m+1)R_m + 3(1-2\alpha)^m}, \quad \text{(by Lemma 2.1)} \end{aligned}$$

and

$$\gamma(m) = \frac{m}{(m+1)R_m + 3(1-2\alpha)^m}.$$

Applying the log-concavity of the function f, we see that

$$f^{\alpha(m)}(1)f^{\beta(m)}(0)f^{\gamma(m)}(\alpha) \ge f^{\alpha(m)}(1)f^{\beta(m)}(0)f^{\alpha\gamma(m)}(1)f^{(1-\alpha)\gamma(m)}(0) = (f(0)f(1))^{\frac{m}{2}}.$$

(3) If $\frac{1}{2} \le \beta \le \frac{3}{4}$, then $\frac{1}{4} \le 1-\beta \le \frac{1}{2}$. By changing $f(x)$ and β by $f(1-x)$ and $1-\beta$ respectively in inequality (3.2), we see that inequality (3.3) is obtained.

(4) If $\frac{3}{4} \le \beta \le 1$, then $0 \le 1-\beta \le \frac{1}{4}$. So by changing $f(x)$ and β by $f(1-x)$ and $f(1-x)$ respectively in inequality (3.1), we see that inequality (3.4) is obtained.

This completes the proof. □

In the following remark, we describe the inequalities in Theorem 3.1. In particular, we shed some light on (3.1).

Remark 3.1. Inequality (3.1) states that

$$\Big(\alpha f(1)+(1-\alpha)f(0)\Big)^m \le f^m(\alpha)+R_m\left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-(m+1)(f(0)f(1))^{\frac{m}{2}}\right)$$
$$-2\alpha^m\left((f(1)f(0))^{\frac{m}{4}}-f^{\frac{m}{2}}(1)\right)^2-r_m\left(f^{\frac{m}{2}}(1)-\Big(f^3(1)f(0)\Big)^{\frac{m}{8}}\right)^2,$$

for $0<\alpha<\frac{1}{4}$ and for any positive integer m, where $r_m=\min\{4\alpha^m, R_m-3\alpha^m\}$.

(1) Since

$$2\alpha^m\left((f(1)f(0))^{\frac{m}{4}}-f^{\frac{m}{2}}(1)\right)^2+r_m\left(f^{\frac{m}{2}}(1)-(f^3(1)f(0))^{\frac{m}{8}}\right)^2 \ge 0,$$

we obtain the following weaker inequality

$$\Big(\alpha f(1)+(1-\alpha)f(0)\Big)^m \le f^m(\alpha)+R_m\left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-(m+1)(f(0)f(1))^{\frac{m}{2}}\right). \tag{3.7}$$

If $m=1$, then the previous inequality becomes

$$\alpha f(1)+(1-\alpha)f(0) \le f(\alpha)+2(1-\alpha)\left(\frac{f(1)+f(0)}{2}-\sqrt{f(1)f(0)}\right).$$

This is an interesting inequality concerning log-concave functions, as we discussed in the introduction. It should be emphasized that

$$\alpha f(1)+(1-\alpha)f(0) \le f(\alpha)$$

is not true for a general log-concave function. This illustrates the significance of (3.7), when $m=1$.

4. More General Discussion

Now, we are ready to state and prove our second main result about log-concave functions.

Theorem 4.1. *Let $f:[0,1]\longrightarrow(0,+\infty)$ be a log-concave function and $0\le\alpha\le1$. Let m and n be two positive integers such that $n\ge2$ and $R_m=\max\{\binom{m}{k}\alpha^k(1-\alpha)^{m-k},\ k=0,\dots,m\}$.*

(1) If $\alpha \in [0, \frac{1}{2^n}]$, then

$$\Big(\alpha f(1)+(1-\alpha)f(0)\Big)^m \leq f^m(\alpha)+R_m\left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-(m+1)(f(0)f(1))^{\frac{m}{2}}\right)$$
$$-\sum_{k=2}^{n}2^{k-1}\alpha^m\left(\sqrt{f^m(1)}-\sqrt[2^k]{(f(0)f^{2^{k-1}-1}(1))^m}\right)^2. \tag{4.1}$$

(2) If $\alpha \in [\frac{1}{2^n}, \frac{1}{2}]$, then

$$\Big(\alpha f(1)+(1-\alpha)f(0)\Big)^m \leq f^m(\alpha)+R_m\left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-(m+1)(f(0)f(1))^{\frac{m}{2}}\right)$$
$$-(1-2\alpha)^m\sqrt{(f(1)f(0))^m}\sum_{k=2}^{n}2^{k-2}\left(\sqrt[2^k]{\frac{f^m(1)}{f^m(0)}}-1\right)^2. \tag{4.2}$$

(3) If $\alpha \in [\frac{1}{2}, \frac{2^n-1}{2^n}]$, then

$$\Big(\alpha f(1)+(1-\alpha)f(0)\Big)^m \leq f^m(\alpha)+R_m\left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-(m+1)(f(0)f(1))^{\frac{m}{2}}\right)$$
$$-(2\alpha-1)^m\sqrt{(f(1)f(0))^m}\sum_{k=2}^{n}2^{k-2}\left(\sqrt[2^k]{\frac{f^m(0)}{f^m(1)}}-1\right)^2. \tag{4.3}$$

(4) If $\alpha \in [\frac{2^n-1}{2^n}, 1]$, then

$$\Big(\alpha f(1)+(1-\alpha)f(0)\Big)^m \leq f^m(\alpha)+R_m\left(\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-(m+1)(f(0)f(1))^{\frac{m}{2}}\right)$$
$$-\sum_{k=2}^{n}2^{k-1}R_m\left(\sqrt{f^m(0)}-\sqrt[2^k]{(f(1)f^{2^{k-1}-1}(0))^m}\right)^2. \tag{4.4}$$

Proof. It suffices to prove the first and second inequality. The proofs of the other two inequalities are similar.

(1) Suppose that $\alpha \in \left[0, \frac{1}{2^n}\right]$. We claim that

$$\begin{aligned} &R_m\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-\Big(\alpha f(1)+(1-\alpha)f(0)\Big)^m+f^m(\alpha)\\ &-\sum_{k=2}^{n}2^{k-1}\alpha^m\left(\sqrt{f^m(1)}-\sqrt[2^k]{(f(0)f^{2^{k-1}-1}(1))^m}\right)^2\\ &\geq (m+1)R_m(f(0)f(1))^{\frac{m}{2}}. \end{aligned} \tag{4.5}$$

We have the following identity

$$
\begin{aligned}
&R_m \frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)} - \Big(\alpha f(1)+(1-\alpha)f(0)\Big)^m + f^m(\alpha) \\
&-\sum_{k=2}^{n} 2^{k-1}\alpha^m\Big(\sqrt{f^m(1)} - \sqrt[2^k]{(f(0)f^{2^{k-1}-1}(1))^m}\Big)^2 \\
&=\sum_{k=0}^{m-1}\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right) f^k(1)f^{m-k}(0) + f^m(\alpha) \\
&+\Big(R_m-\alpha^m-(2^n-2)\alpha^m\Big)f^m(1) - 2\alpha^m(f(0)f(1))^{\frac{m}{2}} \\
&+2^n\alpha^m f^{\frac{m}{2}}(1)\sqrt[2^n]{(f(0)f^{2^{n-1}-1}(1))^m}.
\end{aligned}
$$

It remains to show the following inequality, which is equivalent to (4.5):

$$
\begin{aligned}
&((m+1)R_m+2\alpha^m)^{-1}\Bigg[\sum_{k=0}^{m-1}\left(R_m-\binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)f^k(1)f^{m-k}(0) \\
&+f^m(\alpha)+\Big(R_m-\alpha^m-(2^n-2)\alpha^m\Big)f^m(1) \\
&+2^n\alpha^m f^{\frac{m}{2}}(1)\sqrt[2^n]{(f(0)f^{2^{n-1}-1}(1))^m}\Bigg] \geq (f(0)f(1))^{\frac{m}{2}}. \qquad (4.6)
\end{aligned}
$$

To prove this, we notice that

$$
\begin{aligned}
&((m+1)R_m+2\alpha^m)^{-1}\Bigg[\sum_{k=0}^{m-1}\left(R_m-\binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)f^k(1)f^{m-k}(0) \\
&+f^m(\alpha)+\Big(R_m-\alpha^m-(2^n-2)\alpha^m\Big)f^m(1) \\
&+2^n\alpha^m f^{\frac{m}{2}}(1)\sqrt[2^n]{(f(0)f^{2^{n-1}-1}(1))^m}\Bigg] \\
&=\sum_{k=0}^{m+2}\alpha_k((m+1)R_m+2\alpha^m)^{-1}x_k,
\end{aligned}
$$

where

$$
x_k = \begin{cases}
f^k(1)f^{m-k}(0), & 0\leq k\leq m-1 \\
f^m(1), & k=m \\
f^m(\alpha), & k=m+1 \\
f^{\frac{m}{2}}(1)\sqrt[2^n]{(f(0)f^{2^{n-1}-1}(1))^m}, & k=m+2,
\end{cases}
$$

and

$$\alpha_k = \begin{cases} \left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right), & 0 \le k \le m-1 \\ R_m - \alpha^m - (2^n-2)\alpha^m, & k = m \\ 1, & k = m+1 \\ 2^n\alpha^m, & k = m+2. \end{cases}$$

Using Lemma 2.2, we have

(a) $x_k > 0$ for all $k \in \{0,1,\ldots,m+2\}$,

(b) $\alpha_k \ge 0$ for all $k \in \{0,1,\ldots,m+2\}$, and $\sum_{k=0}^{m+2}((m+1)R_m + 2\alpha^m)^{-1}\alpha_k = 1$.

Applying the arithmetic-geometric mean inequality, we have

$$\sum_{k=0}^{m+2}((m+1)R_m + 2\alpha^m)^{-1}\alpha_k x_k \ \ge\ f^{\alpha(m)}(1)f^{\beta(m)}(0)f^{\gamma(m)}(\alpha),$$

where

$$\begin{aligned} \alpha(m) &= ((m+1)R_m + 2\alpha^m)^{-1}\left[\sum_{k=1}^{m-1} k\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)\right. \\ &+ \left. m\left(R_m - \alpha^m - (2^n-2)\alpha^m\right) + m2^n\alpha^m\left(\frac{1}{2} + \frac{(2^{n-1}-1)}{2^n}\right)\right] \\ &= \frac{\frac{m(m+1)}{2}R_m - m\alpha + m\alpha^m}{(m+1)R_m + 2\alpha^m}, \ \text{(by Lemma 2.1)} \end{aligned}$$

$$\begin{aligned} \beta(m) &= ((m+1)R_m + 2\alpha^m)^{-1}\left[\sum_{k=0}^{m-1}(m-k)\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right) + m2^n\alpha^m\frac{1}{2^n}\right] \\ &= \frac{\frac{m(m+1)}{2}R_m - m(1-\alpha) + m\alpha^m}{(m+1)R_m + 2\alpha^m}, \ \text{(by Lemma 2.1)} \end{aligned}$$

and

$$\gamma(m) = \frac{m}{(m+1)R_m + 2\alpha^m}.$$

Applying the log-concavity of the function f, we see that

$$f^{\alpha(m)}(1)f^{\beta(m)}(0)f^{\gamma(m)}(\alpha) \ge f^{\alpha(m)}(1)f^{\beta(m)}(0)f^{\alpha\gamma(m)}(1)f^{(1-\alpha)\gamma(m)}(0) = (f(0)f(1))^{\frac{m}{2}}.$$

This proves (4.6), and hence the first desired inequality is obtained.

(2) Suppose that $\alpha \in [\frac{1}{2^n}, \frac{1}{2}]$. We claim that

$$\begin{aligned} & R_m\frac{f^{m+1}(0) - f^{m+1}(1)}{f(0) - f(1)} - \left(\alpha f(1) + (1-\alpha)f(0)\right)^m + f^m(\alpha) \\ & - (1-2\alpha)^m\sqrt{(f(1)f(0))^m}\sum_{k=2}^{n} 2^{k-2}\left(\sqrt[2^k]{\frac{f^m(1)}{f^m(0)}} - 1\right)^2 \\ & \ge (m+1)R_m(f(0)f(1))^{\frac{m}{2}}. \end{aligned} \tag{4.7}$$

We have

$$
\begin{aligned}
&R_m\frac{f^{m+1}(0)-f^{m+1}(1)}{f(0)-f(1)}-\Big(\alpha f(1)+(1-\alpha)f(0)\Big)^m+f^m(\alpha)\\
&-(1-2\alpha)^m\sqrt{(f(1)f(0))^m}\sum_{k=2}^{n}2^{k-2}\left(\sqrt[2^k]{\frac{f^m(1)}{f^m(0)}}-1\right)^2\\
&=\sum_{k=0}^{m-1}\left(R_m-\binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)f^k(1)f^{m-k}(0)+f^m(\alpha)\\
&+\left(R_m-\alpha^m-(1-2\alpha)^m\right)f^m(1)-(2^{n-1}-1)(1-2\alpha)^m(f(0)f(1))^{\frac{m}{2}}\\
&+2^{n-1}(1-2\alpha)^m f^{\frac{m(2^{n-1}-1)}{2^n}}(0)f^{\frac{m(2^{n-1}+1)}{2^n}}(1).
\end{aligned}
$$

Thus we need to prove the following inequality, which is equivalent to (4.7):

$$
\begin{aligned}
&((m+1)R_m+(2^{n-1}-1)(1-2\alpha)^m)^{-1}\\
&\times\Bigg[\sum_{k=0}^{m-1}\left(R_m-\binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)f^k(1)f^{m-k}(0)\\
&+f^m(\alpha)+\Big(R_m-\alpha^m-(1-2\alpha)^m\Big)f^m(1)\\
&+2^{n-1}(1-2\alpha)^m f^{\frac{m(2^{n-1}-1)}{2^n}}(0)f^{\frac{m(2^{n-1}+1)}{2^n}}(1)\Bigg]\geq(f(0)f(1))^{\frac{m}{2}}.
\end{aligned}
\tag{4.8}
$$

Observe that

$$
\begin{aligned}
&\Big((m+1)R_m+(2^{n-1}-1)(1-2\alpha)^m\Big)^{-1}\\
&\times\Bigg[\sum_{k=0}^{m-1}\left(R_m-\binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)f^k(1)f^{m-k}(0)\\
&+f^m(\alpha)+\Big(R_m-\alpha^m-(1-2\alpha)^m\Big)f^m(1)\\
&+2^{n-1}(1-2\alpha)^m f^{\frac{m(2^{n-1}-1)}{2^n}}(0)f^{\frac{m(2^{n-1}+1)}{2^n}}(1)\Bigg]\\
&=\sum_{k=0}^{m+2}\alpha_k\Big((m+1)R_m+(2^{n-1}-1)(1-2\alpha)^m\Big)^{-1}x_k,
\end{aligned}
$$

where

$$
x_k=\begin{cases}
f^k(1)f^{m-k}(0), & 0\leq k\leq m-1\\
f^m(1), & k=m\\
f^m(\alpha), & k=m+1\\
f^{\frac{m(2^{n-1}-1)}{2^n}}(0)f^{\frac{m(2^{n-1}+1)}{2^n}}(1), & k=m+2,
\end{cases}
$$

and

$$\alpha_k = \begin{cases} \left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right), & 0 \le k \le m-1 \\ R_m - \alpha^m - (1-2\alpha)^m, & k = m \\ 1, & k = m+1 \\ 2^{n-1}(1-2\alpha)^m, & k = m+2. \end{cases}$$

Lemma 2.2 again implies

(a) $x_k > 0$ for all $k \in \{0,1,\dots,m+2\}$,

(b) $\alpha_k \ge 0$ for all $k \in \{0,1,\dots,m+2\}$, with

$$\sum_{k=0}^{m+2}((m+1)R_m + (2^{n-1}-1)(1-2\alpha)^m)^{-1}\alpha_k = 1.$$

Then the arithmetic-geometric mean inequality implies

$$\sum_{k=0}^{m+2}\left((m+1)R_m + (2^{n-1}-1)(1-2\alpha)\right)^m)^{-1}\alpha_k x_k \ge f^{\alpha(m)}(1)f^{\beta(m)}(0)f^{\gamma(m)}(\alpha),$$

where

$$\begin{aligned} \alpha(m) &= \left((m+1)R_m + (2^{n-1}-1)(1-2\alpha)^m\right)^{-1} \\ &\times \left[\sum_{k=1}^{m-1} k\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)\right. \\ &+ \left. m\left(R_m - \alpha^m - (1-2\alpha)^m\right) + 2^{n-1}(1-2\alpha)^m\frac{m(2^{n-1}+1)}{2^n}\right] \\ &= \frac{\frac{m(m+1)}{2}R_m - m\alpha + \frac{m}{2}(2^{n-1}-1)(1-2\alpha)^m}{(m+1)R_m + (2^{n-1}-1)(1-2\alpha)^m}, \quad \text{(by Lemma 2.1)} \end{aligned}$$

$$\begin{aligned} \beta(m) &= \left((m+1)R_m + (2^{n-1}-1)(1-2\alpha)^m\right)^{-1} \\ &\times \left[\sum_{k=0}^{m-1}(m-k)\left(R_m - \binom{m}{k}\alpha^k(1-\alpha)^{m-k}\right)\right. \\ &+ \left. 2^{n-1}(1-2\alpha)^m\frac{m(2^{n-1}-1)}{2^n}\right] \\ &= \frac{\frac{m(m+1)}{2}R_m - m(1-\alpha) + \frac{m}{2}(2^{n-1}-1)(1-2\alpha)^m}{(m+1)R_m + (2^{n-1}-1)(1-2\alpha)^m}, \quad \text{(by Lemma 2.1)} \end{aligned}$$

and

$$\gamma(m) = \frac{m}{(m+1)R_m + (2^{n-1}-1)(1-2\alpha)^m}.$$

Applying the log-concavity of the function f, we obtain

$$f^{\alpha(m)}(1)f^{\beta(m)}(0)f^{\gamma(m)}(\alpha) \ge f^{\alpha(m)}(1)f^{\beta(m)}(0)f^{\alpha\gamma(m)}(1)f^{(1-\alpha)\gamma(m)}(0) = (f(0)f(1))^{\frac{m}{2}}.$$

This completes the proof of (4.8), and hence the second desired inequality has been shown.

(3) If $\beta \in [\frac{1}{2}, \frac{2^n-1}{2^n}]$, then $1-\beta \in [\frac{1}{2^n}, \frac{1}{2}]$. By changing $f(x)$, and β by $f(1-x)$ and $1-\beta$ respectively in (4.2), the desired inequality (4.3) is obtained immediately.

(4) If $\beta \in [\frac{2^n-1}{2^n}, 1]$, then $1-\beta \in [0, \frac{1}{2^n}]$. Changing $f(x)$, and β by $f(1-x)$ and $1-\beta$ respectively in y (4.1), the desired inequality (4.4) is obtained immediately.

□

5. Applications to Scalar Means

In this section, we present some scalar applications by using our main results, which present new refinement of some classical inequalities between the difference of arithmetic-power and arithmetic-geometric means for scalars.

Let $a, b > 0$, $x \in [0,1]$ and $p \in \mathbb{R} \setminus \{0\}$. It is widely known that

$$f(x) = a\sharp_{p,x}b := (xa^p + (1-x)b^p)^{\frac{1}{p}}$$

is increasing on $\mathbb{R} \setminus \{0\}$. In particular, $a\nabla_\alpha b \le a\sharp_{p,\alpha}b$, for every $p \in (1, +\infty)$, where $a\nabla_\alpha b = \alpha a + (1-\alpha)b$. Furthermore, it is known that $\lim_{\substack{p\to 0\\ p\neq 0}} a\sharp_{p,\alpha}b = a\sharp_\alpha b = a^\alpha b^{1-\alpha}$.

On the other hand, we can easily show that, for every $p \in (0, +\infty)$, $\lambda \mapsto a\sharp_{p,\lambda}b$ is a log-concave function on $[0,1]$. By applying Theorems 3.1 and 4.1, we obtain the following new reverse for the difference between the arithmetic and power means.

Theorem 5.1. *Let $a, b > 0, 0 < \alpha < 1$, $R_m = \max\{\binom{m}{k}\alpha^k(1-\alpha)^{m-k},\ k = 0, \ldots, m\}$, and let m be a positive integer.*

(1) If $0 < \alpha < \frac{1}{4}$, then

$$\begin{aligned}\Big(\alpha a + (1-\alpha)b\Big)^m &\le \Big(a\sharp_{p,\alpha}b\Big)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\right)\\ &- 2\alpha^m\Big((ab)^{\frac{m}{4}} - a^{\frac{m}{2}}\Big)^2 - r_m\Big(a^{\frac{m}{2}} - (a^3b)^{\frac{m}{8}}\Big)^2,\end{aligned} \tag{5.1}$$

where $r_m = \min\{4\alpha^m, R_m - 3\alpha^m\}$.

(2) If $\frac{1}{4} \le \alpha < \frac{1}{2}$, then

$$\begin{aligned}\Big(\alpha a + (1-\alpha)b\Big)^m &\le \Big(a\sharp_{p,\alpha}b\Big)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\right)\\ &- (1-2\alpha)^m\Big((ab)^{\frac{m}{4}} - a^{\frac{m}{2}}\Big)^2 - 2(1-2\alpha)^m\Big((ab)^{\frac{m}{4}} - (a^3b)^{\frac{m}{8}}\Big)^2.\end{aligned} \tag{5.2}$$

(3) If $\frac{1}{2} \le \alpha < \frac{3}{4}$, then

$$\begin{aligned}\Big(\alpha a + (1-\alpha)b\Big)^m &\le \Big(a\sharp_{p,\alpha}b\Big)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a} - (m+1)(ab)^{\frac{m}{2}}\right)\\ &- (2\alpha-1)^m\Big((ab)^{\frac{m}{4}} - b^{\frac{m}{2}}\Big)^2 - 2(2\alpha-1)^m\Big((ab)^{\frac{m}{4}} - (ab^3)^{\frac{m}{8}}\Big)^2.\end{aligned} \tag{5.3}$$

(4) If $\frac{3}{4} \leq \alpha < 1$, then

$$\begin{aligned}\Big(\alpha a+(1-\alpha)b\Big)^m &\leq \Big(a\sharp_{p,\alpha}b\Big)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a}-(m+1)(ab)^{\frac{m}{2}}\right)\\ &-2R_m\left((ab)^{\frac{m}{4}}-b^{\frac{m}{2}}\right)^2 - r_m\left(b^{\frac{m}{2}}-(ab^3)^{\frac{m}{8}}\right)^2,\end{aligned} \tag{5.4}$$

where $r_m = \min\{4R_m, \alpha^m - 3R_m\}$.

Theorem 5.2. *Let $a,b > 0, 0 < \alpha < 1$ and m,n be two positive integers such that $n \geq 2$ and $R_m = \max\{\binom{m}{k}\alpha^k(1-\alpha)^{m-k},\ k = 0,\dots,m\}$.*

(1) If $\alpha \in [0, \frac{1}{2^n}]$, then

$$\begin{aligned}\Big(\alpha a+(1-\alpha)b\Big)^m &\leq \Big(a\sharp_{p,\alpha}b\Big)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a}-(m+1)(ab)^{\frac{m}{2}}\right)\\ &-\sum_{k=2}^{n} 2^{k-1}\alpha^m\left(\sqrt{a^m}-\sqrt[2^k]{(ba^{2^{k-1}-1})^m}\right)^2.\end{aligned} \tag{5.5}$$

(2) If $\alpha \in [\frac{1}{2^n}, \frac{1}{2}]$, then

$$\begin{aligned}\Big(\alpha a+(1-\alpha)b\Big)^m &\leq \Big(a\sharp_{p,\alpha}b\Big)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a}-(m+1)(ab)^{\frac{m}{2}}\right)\\ &-(1-2\alpha)^m\sqrt{(ab)^m}\sum_{k=2}^{n} 2^{k-2}\left(\sqrt[2^k]{\frac{a^m}{b^m}}-1\right)^2.\end{aligned} \tag{5.6}$$

(3) If $\alpha \in [\frac{1}{2}, \frac{2^n-1}{2^n}]$, then

$$\begin{aligned}\Big(\alpha a+(1-\alpha)b\Big)^m &\leq \Big(a\sharp_{p,\alpha}b\Big)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a}-(m+1)(ab)^{\frac{m}{2}}\right)\\ &-(2\alpha-1)^m\sqrt{(ab)^m}\sum_{k=2}^{n} 2^{k-2}\left(\sqrt[2^k]{\frac{b^m}{a^m}}-1\right)^2.\end{aligned} \tag{5.7}$$

(4) If $\alpha \in [\frac{2^n-1}{2^n}, 1]$, then

$$\begin{aligned}\Big(\alpha a+(1-\alpha)b\Big)^m &\leq \Big(a\sharp_{p,\alpha}b\Big)^m + R_m\left(\frac{b^{m+1}-a^{m+1}}{b-a}-(m+1)(ab)^{\frac{m}{2}}\right)\\ &-\sum_{k=2}^{n} 2^{k-1}R_m\left(\sqrt{b^m}-\sqrt[2^k]{(ab^{2^{k-1}-1})^m}\right)^2.\end{aligned} \tag{5.8}$$

Remark 5.1. Let $p \longrightarrow 0$ in Theorem 5.1 and Theorem 5.2, we obtain Theorem 3.1 and Theorem 3.2 presented in [2], respectively.

6. Applications to Operators

Let $B(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. An operator $A \in B(\mathscr{H})$ is called positive, denoted by $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$. The set of all positive operators is denoted by $B(\mathscr{H})^+$. The set of all invertible operators in $B(\mathscr{H})^+$ is denoted by $B(\mathscr{H})^{++}$. When $\mathscr{H}$ is finite dimensional, we identify $B(\mathscr{H})$ with the algebra $\mathbf{M}_n$ of all $n \times n$ complex matrices. Among the most important operator means are the arithmetic, geometric, harmonic and power means defined, respectively, for $A, B \in B(\mathscr{H})^{++}$ and $\alpha \in [0,1]$, as follows:

$$A\nabla_\alpha B := (1-\alpha)A + \alpha B, A\sharp_\alpha B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\alpha A^{1/2}, A!_\alpha B := ((1-\alpha)A^{-1} + \alpha B^{-1})^{-1},$$

and

$$A\sharp_{p,\alpha} B := A^{1/2}\left((1-\alpha)I + \alpha(A^{-1/2}BA^{-1/2})^p\right)^{\frac{1}{p}} A^{1/2};\ p \in \mathbb{R}\backslash\{0\}.$$

If $p \longrightarrow 0$, then

$$A\sharp_{0,\alpha} B = A\sharp_\alpha B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\alpha A^{1/2}.$$

Furthermore, the values $p = 1, -1$ give the arithmetic and harmonic means, respectively.

The following lemma is crucial for establishing operator inequalities from their corresponding scalar counterparts.

Lemma 6.1 ([12, p. 3]). *Let $A \in B(\mathscr{H})$ be self-adjoint. If f and g are both continuous real valued functions with $f(t) \geq g(t)$ for $t \in Sp(A)$ (where the sign $Sp(A)$ denotes the spectrum of A,), then $f(A) \geq g(A)$.*

The following theorem presents the operator version of Theorem 5.1.

Theorem 6.1. *Let $A, B \in B(\mathscr{H})^{++}$, $p \in (0, +\infty)$ and $0 \leq \alpha \leq 1$. Let m be a positive integer such that $R_m = \max\left\{\binom{m}{k}\alpha^k(1-\alpha)^{m-k},\ k = 0, \ldots, m\right\}$.*

(1) If $0 < \alpha < \frac{1}{4}$, then

$$\begin{aligned}A\sharp_m(A\nabla_\alpha B) &\leq A\sharp_m(A\sharp_{p,\alpha}B) + R_m\left(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\right)\\ &- 2\alpha^m\left(A\sharp_{\frac{m}{2}}B + A\sharp_m B - 2A\sharp_{\frac{3m}{4}}B\right) - r_m\left(A\sharp_{\frac{3m}{4}}B + A\sharp_m B - 2A\sharp_{\frac{7m}{8}}B\right),\end{aligned}$$

where $r_m = \min\{4\alpha^m, R_m - 3\alpha^m\}$.

(2) If $\frac{1}{4} \leq \alpha < \frac{1}{2}$, then

$$\begin{aligned}A\sharp_m(A\nabla_\alpha B) &\leq A\sharp_m(A\sharp_{p,\alpha}B) + R_m\left(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\right)\\ &- (1-2\alpha)^m\left(A\sharp_{\frac{m}{2}}B + A\sharp_m B - 2A\sharp_{\frac{3m}{4}}B\right) - 2(1-2\alpha)^m\left(A\sharp_{\frac{m}{2}}B + A\sharp_{\frac{3m}{4}}B - 2A\sharp_{\frac{5m}{8}}B\right).\end{aligned}$$

(3) If $\frac{1}{2} \le \alpha < \frac{3}{4}$, *then*

$$A\sharp_m(A\nabla_\alpha B) \le A\sharp_m(A\sharp_{p,\alpha}B) + \alpha^m\Big(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\Big)$$
$$-(2\alpha-1)^m\Big(A + A\sharp_{\frac{m}{2}}B - 2A\sharp_{\frac{m}{4}}B\Big) - 2(1-2\alpha)^m\Big(A\sharp_{\frac{m}{2}}B + A\sharp_{\frac{m}{4}}B - 2A\sharp_{\frac{3m}{8}}B\Big).$$

(4) If $\frac{3}{4} \le \alpha < 1$, *then*

$$A\sharp_m(A\nabla_\alpha B) \le A\sharp_m(A\sharp_{p,\alpha}B) + \alpha^m\Big(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\Big)$$
$$-2R_m\Big(A + A\sharp_{\frac{m}{2}}B - 2A\sharp_{\frac{m}{4}}B\Big) - r_m\Big(A + A\sharp_{\frac{m}{4}}B - 2A\sharp_{\frac{m}{8}}B\Big),$$

where $r_m = \min\{4R_m, \alpha^m - 3R_m\}$.

Proof. We prove the first inequality. The other inequalities are shown similarly. Let $b = 1$ in inequality (5.1). Then

$$(\alpha a + (1-\alpha))^m \le (\alpha a^p + (1-\alpha))^{\frac{m}{p}} + R_m((a+1)^m - 2^m a^{\frac{m}{2}})$$
$$-2\alpha^m\Big(a^m + a^{\frac{m}{2}} - 2a^{\frac{3m}{4}}\Big) - r_m\Big(a^m + a^{\frac{3m}{4}} - 2a^{\frac{7m}{8}}\Big). \tag{6.1}$$

Since r $C = A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ has a positive spectrum, Lemma 6.1 and inequality (6.1) imply

$$(\alpha C + (1-\alpha)I)^m \le (\alpha C^p + (1-\alpha)I)^{\frac{m}{p}} + R_m((C+I)^m - 2^m C^{\frac{m}{2}})$$
$$-2\alpha^m\Big(C^m + C^{\frac{m}{2}} - 2C^{\frac{3m}{4}}\Big) - r_m\Big(C^m + C^{\frac{3m}{4}} - 2C^{\frac{7m}{8}}\Big). \tag{6.2}$$

Finally, multiplying (6.2) by $A^{\frac{1}{2}}$ on the left and the right side, we see the following inequality

$$A\sharp_m(A\nabla_\alpha B) \le A\sharp_m(A\sharp_{p,\alpha}B) + \alpha^m\Big(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\Big)$$
$$-2R_m\Big(A + A\sharp_{\frac{m}{2}}B - 2A\sharp_{\frac{m}{4}}B\Big) - r_m\Big(A + A\sharp_{\frac{m}{4}}B - 2A\sharp_{\frac{m}{8}}B\Big).$$

□

Theorem 6.2. *Let* $A, B \in B(\mathscr{H})^{++}$, $p \in (0,+\infty)$, $0 \le \alpha \le 1$. *Let* m *and* n *be two positive integers such that* $n \ge 2$ *and* $R_m = \max\{\binom{m}{k}\alpha^k(1-\alpha)^{m-k},\ k = 0,\dots,m\}$.

(1) If $\alpha \in [0, \frac{1}{2^n}]$, *then*

$$A\sharp_m(A\nabla_\alpha B) \le A\sharp_m(A\sharp_{p,\alpha}B) + R_m\Big(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\Big)$$
$$-\sum_{k=2}^{n} 2^{k-1}\alpha^m\left(A\sharp_m B + A\sharp_{\frac{m(2^{k-1}-1)}{2^{k-1}}}B - 2A\sharp_{\frac{m(2^k-1)}{2^k}}B\right).$$

(2) If $\alpha \in [\frac{1}{2^n}, \frac{1}{2}]$, *then*

$$A\sharp_m(A\nabla_\alpha B) \le A\sharp_m(A\sharp_{p,\alpha}B) + R_m\left(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\right)$$
$$-\sum_{k=2}^{n} 2^{k-2}(1-2\alpha)^m\left(A\sharp_{\frac{m}{2}}B + A\sharp_{\frac{m(2^{k-2}+1)}{2^{k-1}}}B - 2A\sharp_{\frac{m(2^{k-1}+1)}{2^k}}B\right).$$

(3) If $\alpha \in [\frac{1}{2}, \frac{2^n-1}{2^n}]$, *then*

$$A\sharp_m(A\nabla_\alpha B) \le A\sharp_m(A\sharp_{p,\alpha}B) + \alpha^m\left(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\right)$$
$$-\sum_{k=2}^{n} 2^{k-2}(1-2\alpha)^m\left(A\sharp_{\frac{m}{2}}B + A\sharp_{\frac{m(2^{k-2}+1)}{2^{k-1}}}B - 2A\sharp_{\frac{m(2^{k-1}+1)}{2^k}}B\right).$$

(4) If $\alpha \in [\frac{2^n-1}{2^n}, 1]$, *then*

$$A\sharp_m(A\nabla_\alpha B) \le A\sharp_m(A\sharp_{p,\alpha}B) + \alpha^m\left(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\right)$$
$$-\sum_{k=2}^{n} 2^{k-1}R_m\left(A + A\sharp_{\frac{m}{2^{k-1}}}B - 2A\sharp_{\frac{m}{2^k}}B\right).$$

Proof. Here, we only prove the first inequality. For the other inequalities, they can be demonstrated similarly. Letting $b = 1$ in inequality (5.5), we have

$$(\alpha a + (1-\alpha))^m \le (\alpha a^p + (1-\alpha))^{\frac{m}{p}} + R_m\left((a+1)^m - 2^m a^{\frac{m}{2}}\right)$$
$$-\sum_{k=2}^{n} 2^{k-1}\alpha^m\left(a^m + a^{\frac{m(2^{k-1}-1)}{2^{k-1}}} - 2a^{\frac{m(2^k-1)}{2^k}}\right). \tag{6.3}$$

Since $C = A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ has a positive spectrum, we find by Lemma 6.1 and inequality (6.3) that

$$(\alpha C + (1-\alpha)I)^m \le (\alpha C^p + (1-\alpha)I)^{\frac{m}{p}} + R_m((C+I)^m - 2^m C^{\frac{m}{2}})$$
$$-\sum_{k=2}^{n} 2^{k-1}\alpha^m\left(C^m + C^{\frac{m(2^{k-1}-1)}{2^{k-1}}} - 2C^{\frac{m(2^k-1)}{2^k}}\right). \tag{6.4}$$

Finally, as before by multiplying (6.4) by $A^{\frac{1}{2}}$ on the left and right side, one obtains that

$$A\sharp_m(A\nabla_\alpha B) \le A\sharp_m(A\sharp_{p,\alpha}B) + R_m\left(A\sharp_m(2A\nabla B) - 2^m A\sharp_{\frac{m}{2}}B\right)$$
$$-\sum_{k=2}^{n} 2^{k-1}\alpha^m\left(A\sharp_m B + A\sharp_{\frac{m(2^{k-1}-1)}{2^{k-1}}}B - 2A\sharp_{\frac{m(2^k-1)}{2^k}}B\right).$$

□

Lemma 6.2 ([16])**.** *Let* $A, B \in B(\mathscr{H})^{++}$ *and let* α, β *be two real numbers. Then* $A\sharp_\alpha(A\sharp_\beta B) = A\sharp_{\alpha\beta}B$.

At the end of this paper, we mention that by using the previous lemma and passing to the limit $p \to 0$ in Theorem 6.1 and Theorem 6.2, we can obtain Theorem 4.1 and Theorem 4.2 presented in [2], respectively.

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