Appl. Set-Valued Anal. Optim. 7 (2025), No. 3, pp. 313-334
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.7.2025.3.03

SOME OF THE LATEST APPLICATIONS OF CERTAIN MINIMAX THEOREMS

BIAGIO RICCERI

Department of Mathematics and Informatics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy

Abstract. In this paper, we present some of the latest applications of our minimax theorems established recently. These applications concern: The exact computation of the infimum of certain functionals on L^p spaces; the multiplicity of global minima under a non-convexity condition; the multiplicity of periodic solutions for Lagrangian systems of relativistic oscillators; and a new property of strictly convex functions. In addition, a challenging conjecture is proposed in this paper.
Keywords. Functionals on L^p spaces; Minimax equality; Multiple global minima; Periodic solutions; Relativistic oscillators.
2020 Mathematics Subject Classification. 49J35, 49K35, 90C47, 35J92.

1. Introduction

Let X,Y be two non-empty sets and $f : X \times Y \to \mathbf{R}$ be a given function. It is obvious that $\sup_Y \inf_X f \leq \inf_X \sup_Y f$. The classical minimax problem, in the sense of von Neumann, is to provide suitable conditions under which the equality

$$\sup_Y \inf_X f = \inf_X \sup_Y f. \tag{1.1}$$

holds. The ancestor of the minimax theorems was the one proved by von Neumann in [7]. This result was then extended and improved by Ky Fan in [4]. In turn, Fan's result was improved by Sion in [28] who established the following result.

Theorem 1.A. *Let X,Y be two convex sets, each in a topological vector space. Assume that one of them is compact. Let $f : X \times Y \to \mathbf{R}$ be lower semicontinuous and quasi-convex in X, and upper semicontinuous and quasi-concave in Y. Then, equality (1.1) holds.*

There is no doubt that Theorem 1.A, out of the circle of specialists, is considered as the standard reference minimax theorem. Notice that Theorem 1.A fully lies in a convex setting. But, what about minimax theorems in non-convex contexts? We provided several answers to this question, and the first one was given in [9]. Here, in particular, we proved the following result.

Theorem 1.1. *Let X be a topological space and Y be a real interval. Let $f : X \times Y \to \mathbf{R}$ be lower semicontinuous in X, and upper semicontinous and quasi-concave in Y. Moreover, assume that there is a set $D \subseteq Y$, dense in Y, such that for each $y \in D$ and each $r \in \mathbf{R}$, $\{x \in X : f(x,y) < r\}$*

E-mail address: biagio.ricceri@unict.it
Received 25 May 2024; Accepted 20 July 2024; Published online 11 April 2025.

is connected. Finally, assume that either Y is compact, or there is some $y_0 \in Y$ such that $f(\cdot, y_0)$ is inf-compact in X. Then, equality (1.1) holds.

More precisely, Theorem 1.1 was obtained in [9] for $D = Y$. The current statement was proved in [15]. Since its appearance, wide applications of Theorem 1.1 to integral functionals in L^p spaces were made possible thanks to the following very interesting result obtained in ([27]).

Theorem 1.B. *Let $(T, \mathscr{F}, \mu)$ be a σ-finite non-atomic measure space, let E be a Banach space, and let $\varphi : T \times E \to \mathbf{R}$ be a function such that $\varphi(\cdot, u(\cdot))$ is μ-measurable for every μ-strongly measurable function $u : T \to E$. Let $p \geq 1$ and let $A \subseteq L^p(T, E)$ be a decomposable set. Then, for every $r \in \mathbf{R}$, $\left\{u \in A : \varphi(\cdot, u(\cdot)) \in L^1(T), \int_T \varphi(t, u(t))d\mu \leq r\right\}$ is connected in the norm-topology.*

For instance, a joint application of Theorem 1.1 and Theorem 1.B gives the following remarkable variational property (see [10, 12, 13]):

Theorem 1.C. *Let $(T, \mathscr{F}, \mu)$ be a σ-finite non-atomic measure space, let E be a Banach space, and let $\varphi : T \times E \to \mathbf{R}$ be a function such that $\varphi(\cdot, u(\cdot))$ is μ-measurable for every μ-strongly measurable function $u : T \to E$. Let $p \geq 1$ and assume that there exist $\alpha \in L^1(T)$, $\gamma_i \in]0,1[$, and $\beta_i \in L^{\frac{p}{p-\gamma_i}}$, $i = 1, \ldots, k$, such that $-\alpha(t) \leq \varphi(t,x) \leq \alpha(t) + \sum_{i=1}^k \beta_i(t)\|x\|^{\gamma_i}$ for μ-a.e. $t \in T$ and for all $x \in E$. Then, for every decomposable linear subspace $F \subseteq L^p(T, E)$, for every $\psi \in F^* \setminus \{0\}$, and for every $r \in \mathbf{R}$, $\inf_{u \in L^p(T,E)} \int_T \varphi(t, u(t))d\mu = \inf_{u \in \psi^{-1}(r)} \int_T \varphi(t, u(t))d\mu$.*

Other related papers are [11] and [6]. Consider now the following proposition.

Proposition 1.A. *Let X be a Hausdorff topological space and let $\varphi : X \to \mathbf{R}$ be a lower semicontinuous function such that, for some $r > \inf_X \varphi$, the set $\varphi^{-1}(]-\infty, r])$ is compact and disconnected. Then, φ has at least two local minima.*

In view of Proposition 1.A, a direct consequence of Theorem 1.1 is as follows.

Theorem 1.D. *Let X be a Hausdorff topological space and Y be a real interval. Let $f : X \times Y \to \mathbf{R}$ be lower semicontinuous and inf-compact in X, and upper semicontinous and quasi-concave in Y. Also, assume that*

$$\sup_Y \inf_X f < \inf_X \sup_Y f \ . \tag{1.2}$$

Then, there exists a non-empty open set $A \subset Y$ such that, for each $y \in A$, the function $f(\cdot, y)$ has at least two local minima in X.

Theorem 1.D has an enormous impact for the multiplicity of solutions of nonlinear equations of variational nature. Actually, applying it jointly with [8, Corollary 1], we have the following.

Theorem 1.E. *Let X be a reflexive real Banach space and Y be a real interval. Let $f : X \times Y \to \mathbf{R}$ be sequentially weakly lower semicontinuous, coercive, continuously differentiable and satisfying the Palais-Smale condition in X, and upper semicontinuous and quasi-concave in Y. Also, assume that (1.2) holds. Then, there exists a non-empty open set $A \subset Y$ such that, for each $y \in A$, the function $f(\cdot, y)$ has at least three critical points in X.*

Theorem 1.E is from [14]. Other related results were established in [16, 17, 19]. As an easy inspection to the relevant literature shows, such three critical point theorems have been proven to be very powerful and flexible tools, having been successfully applied in several hundred papers dealing with the multiplicity of solutions of nonlinear equations; see [2, 18]. In view of Theorems 1.D and 1.E, the following remark becomes a must. Starting from [7], mathematicians working in this area studied solely the problem of ensuring the validity of equality (1.1). After [14], to the contrary, it became likewise important to know when (1.2) holds. In [20], we built a theory devoted to (1.2). Recently, in [23], we obtained the following variant of Theorem 1.1.

Theorem 1.2. *Let X be a topological space, Y be a compact real interval and $f : X \times Y \to \mathbf{R}$ be an upper semicontinuous function which is continuous in X. Assume that there exists a set $D \subseteq Y$, dense in Y, such that, for each $y \in D$ and each $r \in \mathbf{R}$, $\{x \in X : f(x,y) < r\}$ is connected. Moreover, assume that, for each $x \in X$, the set of all global maxima of the function $f(x,\cdot)$ is connected. Then, equality (1.1) holds.*

A common feature of Theorems 1.1 and 1.2 is that the space Y must be a real interval. As simple examples show that these results are no longer true when Y is a convex set of dimension at least two. In [22], we established another very general minimax theorem (for non-convex functions in X), where Y is an arbitrary convex set. It reads as follows.

Theorem 1.3. *Let X be a topological space, let Y be a non-empty convex set in a real topological vector space and let $f : X \times Y \to \mathbf{R}$ be lower semicontinuous and inf-compact in X, and quasi-concave and continuous in Y. Then, at least one of the following assertions holds:*

(a) $\sup_Y \inf_X f = \inf_X \sup_Y f$;

(b) there exists $\tilde{y} \in Y$ such $f(\cdot,\tilde{y})$ has at least two global minima.

The aim of this paper is to provide an account of some of the latest applications of Theorems 1.2 and 1.3.

2. Exact Computation of the Infimum of Certain Functionals on L^p Spaces

As we mentioned in Introduction, the connectedness result of Saint Raymond permits applications of Theorems 1.1 and 1.2 to integral functionals on L^p spaces under very general assumptions. In this section, we report some recent results on the subject obtained in [1] and [5].

Throughout this section, $(T,\mathscr{F},\mu)$ is used to denote a measure space, with $\mu(T) < +\infty$, E is used to denote a real Banach space and $p \geq 1$. We denote by $L^p(T,E)$ the space of all equivalence classes of strongly μ-measurable functions u with $\int_T \|u(t)\|^p d\mu < +\infty$, equipped with the norm $\|u\|_{L^p(T,E)} = \left(\int_T \|u(t)\|^p d\mu\right)^{\frac{1}{p}}$.

We write $L^p(T)$ instead of $L^p(T,\mathbf{R})$. A set $D \subseteq L^p(T,E)$ is said to be decomposable if, for every $u,v \in D$ and every $A \in \mathscr{F}$, $t \to \chi_A(t)u(t) + (1-\chi_A(t))v(t)$ is an element of D, where χ_A is the characteristic function of A. A function $f : T \times E \to \mathbf{R}$ is said to be a Carathéodory function if it is measurable in T and continuous in E.

First, joint applications of Theorem 1.1 and Theorem 1.B give the following.

Theorem 2.A. *Let $X \subseteq L^p(T,E)$ a decomposable set, $[a,b]$ a compact real interval, and $\gamma : [a,b] \to \mathbf{R}$ a convex (resp. concave) and continuous function. Moreover, let $\varphi, \psi, \omega : T \times E \to \mathbf{R}$*

be three Carathéodory functions such that, for some $M \in L^1(T)$, $k \in \mathbf{R}$,

$$\max\{\|\varphi(t,x)\|, \|\psi(t,x)\|, \|\omega(t,x)\|\} \le M(t) + k\|x\|^p$$

for all $(t,x) \in T \times E$ and

$$\gamma(a)\int_T \psi(t,u(t))d\mu + a\int_T \omega(t,u(t))d\mu \ne \gamma(b)\int_T \psi(t,u(t))d\mu + b\int_T \omega(t,u(t))d\mu$$

for all $u \in X$ such that $\int_T \psi(t,u(t))d\mu > 0$ (resp. $\int_T \psi(t,u(t))d\mu < 0$). Then,

$$\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(\int_T \varphi(t,u(t))d\mu + \gamma(\lambda)\int_T \psi(t,u(t))d\mu + \lambda\int_T \omega(t,u(t))d\mu\right)$$
$$= \inf_{u\in X}\sup_{\lambda\in[a,b]}\left(\int_T \varphi(t,u(t))d\mu + \gamma(\lambda)\int_T \psi(t,u(t))d\mu + \lambda\int_T \omega(t,u(t))d\mu\right).$$

Theorem 2.B. *Let $X \subseteq L^p(T,E)$ be a decomposable set, $[a,b]$ a compact real interval and $\gamma, \delta \in C^0([a,b]) \cap C^1(]a,b[)$ two functions such that $\gamma'(\lambda) \ne 0$ for all $\lambda \in [a,b]$ and $\frac{\delta'}{\gamma'}$ is strictly monotone in $]a,b[$. Moreover, let $\varphi, \psi, \omega : T \times E \to \mathbf{R}$ be three Carathéodory functions such that, for some $M \in L^1(T)$, $k \in \mathbf{R}$, $\max\{\|\varphi(t,x)\|, \|\psi(t,x)\|, \|\omega(t,x)\|\} \le M(t) + k\|x\|^p$ for all $(t,x) \in T \times E$ and*

$$\gamma(a)\int_T \psi(t,u(t))d\mu + \delta(a)\int_T \omega(t,u(t))d\mu \ne \gamma(b)\int_T \psi(t,u(t))d\mu + \delta(b)\int_T \omega(t,u(t))$$

in each of the two following cases:
- *$\frac{\delta'}{\gamma'}$ is strictly increasing, $u \in X$ and $\gamma'(\lambda)\int_T \omega(t,u(t))d\mu > 0$ for all $\lambda \in]a,b[$;*
- *$\frac{\delta'}{\gamma'}$ is strictly decreasing, $u \in X$ and $\gamma'(\lambda)\int_T \omega(t,u(t))d\mu < 0$ for all $\lambda \in]a,b[$.*

Then

$$\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(\int_T \varphi(t,u(t))d\mu + \gamma(\lambda)\int_T \psi(t,u(t))d\mu + \delta(\lambda)\int_T \omega(t,u(t))d\mu\right)$$
$$= \inf_{u\in X}\sup_{\lambda\in[a,b]}\left(\int_T \varphi(t,u(t))d\mu + \gamma(\lambda)\int_T \psi(t,u(t))d\mu + \delta(\lambda)\int_T \omega(t,u(t))d\mu\right).$$

Let $I \subseteq E$ be a non-empty set. We denote by $\mathscr{A}_I$ the class of all pairs of continuous functions $\omega, \psi : E \to \mathbf{R}$, with $\omega(x) \ge 0$ and $\psi(x) > 0$ for all $x \in I$, such that

$$\sup_{x\in E}\frac{|\omega(x)| + |\psi(x)|}{1 + \|x\|^p} < +\infty$$

and

$$\sup_{x\in I}\frac{\omega(x)}{\psi(x)} < +\infty .$$

Moreover, we denote by $\mathscr{B}_I$ the family of all decomposable subsets X of $L^p(T,E)$ such that $u(T) \subseteq I$ for all $u \in X$, and it contains each constant function taking its value in I.

Remark 2.1. If $(\omega, \psi) \in \mathscr{A}_I$ and $X \in \mathscr{B}_I$, then

$$\inf_{x\in I}\frac{\omega(x)}{\psi(x)} \le \frac{\int_T \omega(u(t))d\mu}{\int_T \psi(u(t))d\mu} \le \sup_{x\in I}\frac{\omega(x)}{\psi(x)}$$

for all $u \in X$.

The following six results are the consequences of Theorem 2.A. Precisely, Theorems 2.1, 2.2, 2.3, and 2.6 were obtained by Giandinoto, while Theorems 2.4 and 2.5 were obtained by Ait Mansour, Lahrache, and Ziane.

Theorem 2.1. *Let $(\omega, \psi) \in \mathscr{A}_I$, $X \in \mathscr{B}_I$, and let $r > 1$. Set $a := \left(\frac{1}{r}\inf_{x\in I}\frac{\omega(x)}{\psi(x)}\right)^{\frac{1}{r-1}}$ and $b := \left(\frac{1}{r}\sup_{x\in I}\frac{\omega(x)}{\psi(x)}\right)^{\frac{1}{r-1}}$. Then,*

$$\inf_{u\in X}\frac{\left(\int_T \omega(u(t))d\mu\right)^r}{\int_T \psi(u(t))d\mu} = \left(\mu(T)\frac{r^{\frac{r}{r-1}}}{r-1}\sup_{\lambda\in[a,b]}\inf_{x\in I}\Big(\lambda\omega(x) - \lambda^r\psi(x)\Big)\right)^{r-1}. \tag{2.1}$$

Proof. By Remark 2.1, we have

$$\left\{\left(\frac{\int_T \omega(u(t))d\mu}{r\int_T \psi(u(t))d\mu}\right)^{\frac{1}{r-1}} : u \in X\right\} \subseteq [a,b] .$$

Since X contains each constant function taking its value in I, we clearly have

$$\inf_{u\in X}\left(\int_T \Big(\lambda\omega(u(t)) - \lambda^r\psi(u(t))\Big)d\mu\right) = \mu(T)\inf_{x\in I}\Big(\lambda\omega(x) - \lambda^r\psi(x)\Big)$$

for all $\lambda \in [a,b]$. Hence,

$$\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(\int_T \Big(\lambda\omega(u(t)) - \lambda^r\psi(u(t))\Big)d\mu\right) = \mu(T)\sup_{\lambda\in[a,b]}\inf_{x\in I}\Big(\lambda\omega(x) - \lambda^r\psi(x)\Big). \tag{2.2}$$

Now, since $\int_T \psi(u(t))d\mu > 0$ for all $u \in X$, we find by using Theorem 2.A with $\gamma(\lambda) = -\lambda^r$ and $\varphi = 0$ that

$$\begin{aligned} &\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(\int_T \Big(\lambda\omega(u(t)) - \lambda^r\psi(u(t))\Big)d\mu\right) \\ &= \inf_{u\in X}\sup_{\lambda\in[a,b]}\left(\lambda\int_T \omega(u(t))d\mu - \lambda^r\int_T \psi(u(t))d\mu\right). \end{aligned} \tag{2.3}$$

Fixing $u \in X$, one sees that $\lambda \to \lambda\int_T \omega(u(t))d\mu - \lambda^r\int_T \psi(u(t))d\mu$ is concave in $[0,+\infty[$ and its derivative vanishes at $\left(\frac{\int_T \omega(u(t))d\mu}{r\int_T \psi(u(t))d\mu}\right)^{\frac{1}{r-1}}$, which lies in $[a,b]$. Consequently, we have

$$\begin{aligned} &\inf_{u\in X}\sup_{\lambda\in[a,b]}\left(\lambda\int_T \omega(u(t))d\mu - \lambda^r\int_T \psi(u(t))d\mu\right) \\ &= \inf_{u\in X}\left(\left(\frac{\int_T \omega(u(t))d\mu}{r\int_T \psi(u(t))d\mu}\right)^{\frac{1}{r-1}}\int_T \omega(u(t))d\mu - \left(\frac{\int_T \omega(u(t))d\mu}{r\int_T \psi(u(t))d\mu}\right)^{\frac{r}{r-1}}\int_T \psi(u(t))d\mu\right) \\ &= \inf_{u\in X}\frac{r-1}{r^{\frac{r}{r-1}}}\left(\frac{\left(\int_T \omega(u(t))d\mu\right)^r}{\int_T \psi(u(t))d\mu}\right)^{\frac{1}{r-1}}. \end{aligned}$$

In view of (2.2) and (2.3), one has

$$\inf_{u\in X}\frac{r-1}{r^{\frac{r}{r-1}}}\left(\frac{\left(\int_T \omega(u(t))d\mu\right)^r}{\int_T \psi(u(t))d\mu}\right)^{\frac{1}{r-1}}=\mu(T)\sup_{\lambda\in[a,b]}\inf_{x\in I}\Big(\lambda\omega(x)-\lambda^r\psi(x)\Big),$$

which is equivalent to (2.1). □

In addition, we have the following result from Theorem 2.1.

Theorem 2.2. *Let* $(\omega,\psi)\in\mathscr{A}_I$, $X\in\mathscr{B}_I$ *and let* $r>1$. *Let* $\inf_{x\in I}(\omega(x)-\lambda\psi(x))=-\infty$ *for all* $\lambda>0$. *Then,* $\inf_{u\in X}\frac{\left(\int_T \omega(u(t))d\mu\right)^r}{\int_T \psi(u(t))d\mu}=0$.

Proof. Writing

$$\omega(x)-\lambda\psi(x)=\psi(x)\left(\frac{\omega(x)}{\psi(x)}-\lambda\right),$$

we infer that $\inf_{x\in I}\frac{\omega(x)}{\psi(x)}=0$. Thus(2.1) holds with $a=0$ and the right-hand side of (2.1) is 0, as claimed. □

In turn, a particular case of Theorem 2.2 is as follows.

Theorem 2.3. *Let I be an unbounded set whose closure does not contain* 0, *and let q,r,s be three positive numbers such that $s<q\le p$ and $r>1$. Then, for each* $X\in\mathscr{B}_I$, $\inf_{u\in X}\frac{\left(\int_T \|u(t)\|^s d\mu\right)^r}{\int_T \|u(t)\|^q d\mu}=0$.

Proof. It is sufficient to notice that $(\|\cdot\|^s,\|\cdot\|^q)$ belongs to $\mathscr{A}_I$ and that $\inf_{x\in I}(\omega(x)-\lambda\psi(x))=-\infty$ for all $\lambda>0$ is satisfied. □

Theorem 2.4. *Let* $(\omega,\psi)\in\mathscr{A}_I$, $X\in\mathscr{B}_I$, *and* $r>1$, *and set* $a:=\inf_{x\in I}\left(\frac{\omega(x)}{\psi(x)}\right)^{\frac{1}{r-1}}$ *and* $b:=\sup_{x\in I}\left(\frac{\omega(x)}{\psi(x)}\right)^{\frac{1}{r-1}}$. *If* $b<+\infty$, *then*

$$\inf_{u\in X}\frac{(r-1)(\int_T \omega(u(t))d\mu)^{\frac{r}{r-1}}+(\int_T \psi(u(t)d\mu)^{\frac{r}{r-1}}}{\left(\int_T \psi(u(t)d\mu\right)^{\frac{1}{r-1}}}=\mu(T)\sup_{\lambda\in[a,b]}\inf_{x\in I}(r\lambda\omega(x)+(1-\lambda^r)\psi(x)).$$

Proof. By Remark 2.1, we have

$$a\le\left(\frac{\int_T \omega(u(t))d\mu}{\int_T \psi(u(t))d\mu}\right)^{\frac{1}{r-1}}\le b$$

for all $u\in X$. Since X contains each constant function taking its value in I, we clearly have, for all $\lambda\in[a,b]$,

$$\inf_{u\in X}\left(r\lambda\int_T \omega(u(t))d\mu-(\lambda^r-1)\int_T \psi(u(t))d\mu\right)=\mu(T)\inf_{x\in I}(r\lambda\omega(x)+(1-\lambda^r)\psi(x)).$$

Hence,

$$\begin{aligned}&\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(r\lambda\int_T \omega(u(t))d\mu-(\lambda^r-1)\int_T \psi(u(t))d\mu\right)\\&=\mu(T)\sup_{\lambda\in[a,b]}\inf_{x\in I}(r\lambda\omega(x)+(1-\lambda^r)\psi(x)).\end{aligned}\tag{2.4}$$

Using Theorem 2.A with $\varphi = 0$, $\gamma(\lambda) = 1 - \lambda^r$ (and $r\omega$ instead of ω), one has

$$\begin{aligned} &\sup_{\lambda\in[a,b]}\inf_{u\in X}\left(r\lambda\int_T \omega(u(t)d\mu + (1-\lambda^r)\int_T \psi(u(t)d\mu\right) \\ &= \inf_{u\in X}\sup_{\lambda\in[a,b}\left(r\lambda\int_T \omega(u(t)d\mu + (1-\lambda^r)\int_T \psi(u(t)d\mu\right). \end{aligned} \tag{2.5}$$

Fixing $u \in X$, one sees that $F : \lambda \to r\lambda \int_T \omega(u(t)d\mu + (1-\lambda^r)\int_T \psi(u(t)d\mu$ is concave in $[0,+\infty[$ and its derivative is given by $F'(\lambda) = r\int_T \omega(u(t)d\mu - r\lambda^{r-1}\int_T \psi(u(t)d\mu$, which vanishes at $\left(\frac{\int_T \omega(u(t))d\mu}{\int_T \psi(u(t))d\mu}\right)^{\frac{1}{r-1}}$, which lies in $[a,b]$. Consequently,

$$\begin{aligned} &\inf_{u\in X}\sup_{\lambda\in[a,b]}\left(r\lambda\int_T \omega(u(t))d\mu + (1-\lambda^r)\int_T \psi(u(t))d\mu\right) \\ &= \inf_{u\in X}\frac{(r-1)(\int_T \omega(u(t))d\mu)^{\frac{r}{r-1}} + (\int_T \psi(u(t)d\mu)^{\frac{r}{r-1}}}{(\int_T \psi(u(t)d\mu)^{\frac{1}{r-1}}}, \end{aligned}$$

which, jointly with (2.4) and (2.5), gives the conclusion immediately. □

Now, from Theorem 2.4, we have the following result.

Theorem 2.5. *Let $E = \mathbf{R}$, $I =]c,d[$, and let $(\omega,\psi) \in \mathscr{A}_I$. Let ω,ψ be continuous and concave in $[c,d]$. Assume that $\omega(d) = 0$, $\psi(c) < \psi(d)$, and $\sup_{x\in I}\frac{\omega(x)}{\psi(x)} = 1$. Set $\delta := \frac{\omega(c)}{\psi(d)-\psi(c)}$. Assume that $\sqrt{\delta^2+1} - \delta \le \frac{\omega(c)}{\psi(c)}$ provided $\psi(c) > 0$, Then, for every $X \in \mathscr{B}_I$,*

$$\inf_{u\in X}\frac{(\int_T \omega(u(t))d\mu)^2 + (\int_T \psi(u(t)d\mu)^2}{\int_T \psi(u(t)d\mu} = 2\mu(T)\delta(\sqrt{\delta^2+1}-\delta)\omega(c)\ .$$

Proof. Fix $\lambda \in [0,1]$. Since $2\lambda\omega + (1-\lambda^2)\psi$ is concave in $[c,d]$, its infimum is attained either at c or at d. That is, (recalling that $\omega(d) = \psi(c) = 0$)

$$\inf_{x\in I}(2\lambda\omega(x) + (1-\lambda^2)\psi(x)) = \min\{2\lambda\omega(c),(1-\lambda^2)\psi(d)\}\ .$$

On the other hand, we have $2\lambda\omega(c) \le (1-\lambda^2)\psi(d)$ if and only if $\lambda \le -\delta + \sqrt{\delta^2+1}$. Consequently

$$\inf_{x\in I}(2\lambda\omega(x) + (1-\lambda^2)\psi(x)) = \begin{cases} 2\lambda\omega(c) + (1-\lambda^2)\psi(x) & \text{if } \lambda \in \left[0, -\delta+\sqrt{\delta^2+1}\right] \\ (1-\lambda^2)\psi(d) & \text{if } \lambda \in \left[-\delta+\sqrt{\delta^2+1}, 1\right] \end{cases}$$

From this, it clearly follows that

$$\sup_{\lambda\in[0,1]}\inf_{x\in I}(2\lambda\omega(x) + (1-\lambda^2)\psi(x)) = 2\delta(\sqrt{\delta^2+1}-\delta)\psi(d)\ .$$

Now, the conclusion follows directly from Theorem 2.4 with $r = 2$. □

Remark 2.2. Concerning Theorem 2.5, it is important to observe that the infimum of the restriction of functional

$$u \to \frac{(\int_T \omega(u(t))d\mu)^2 + (\int_T \psi(u(t)d\mu)^2}{\int_T \psi(u(t)d\mu}$$

to the set of all constant functions taking their values in $]c,d[$ (say $\tilde{X}$) can be strictly larger than $2\mu(T)\delta(\sqrt{\delta^2+1}-\delta)\psi(d)$. To see this, it is enough to consider the following setting: $[c,d]=[0,1]$, $\omega(x)=1-x$, and $\psi(x)=x+1$. Indeed, in this case, we have $\delta=1$ and

$$\inf_{u\in\tilde{X}} \frac{(\int_T(1-u(t))d\mu)^2 + (\int_T(u(t)+1)d\mu)^2}{\int_T(u(t)+1)d\mu} = \mu(T)\frac{50}{27} > 4\mu(T)(\sqrt{2}-1).$$

Applying Theorem 2.B, Giandinoto obtained the following result.

Theorem 2.6. *Let $I\subseteq E$ be a non-empty set, $X\in\mathscr{B}_I$ and $\omega,\psi:\mathbf{R}\to\mathbf{R}$ two continuous functions such that $\omega(x)>0$ for all $x\in I$ and $\sup_{x\in E}\frac{\omega(x)+|\psi(x)|}{1+\|x\|^p}<+\infty$. Then,*

$$\begin{aligned}&\inf_{u\in X}\sqrt{\left(\int_T \psi(u(t))d\mu\right)^2+\left(\int_T \omega(u(t))d\mu\right)^2}\\ &=\mu(T)\sup_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]}\inf_{x\in I}(\psi(x)\sin\lambda+\omega(x)\cos\lambda).\end{aligned} \tag{2.6}$$

Proof. We need to apply Theorem 2.B by taking $[a,b]=[-\frac{\pi}{2},\frac{\pi}{2}]$, $\gamma(\lambda)=\sin\lambda$, and $\delta(\lambda)=\cos\lambda$. Since $\frac{\delta'}{\gamma'}$ is strictly decreasing and $\gamma'(\lambda)\int_T\omega(u(t))d\mu>0$ for all $\lambda\in]a,b[$, $u\in X$, no other condition has to be satisfied. Consequently, we have

$$\begin{aligned}&\inf_{u\in X}\sup_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]}\left(\int_T\psi(u(t))d\mu\sin\lambda+\int_T\omega(u(t))d\mu\cos\lambda\right)\\ &=\sup_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]}\inf_{u\in X}\left(\int_T\psi(u(t))d\mu\sin\lambda+\int_T\omega(u(t))d\mu\cos\lambda\right).\end{aligned} \tag{2.7}$$

On the other hand, since $X\in\mathscr{B}_I$, we have

$$\begin{aligned}&\sup_{\lambda\in-\frac{\pi}{2},\frac{\pi}{2}]}\inf_{u\in X}\left(\int_T\psi(u(t))d\mu\sin\lambda+\int_T\omega(u(t))d\mu\cos\lambda\right)\\ &=\mu(T)\sup_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]}\inf_{x\in I}(\psi(x)\sin\lambda+\omega(x)\cos\lambda).\end{aligned} \tag{2.8}$$

Fix $u \in X$. An easy checking shows that $\lambda \to \int_T \psi(u(t))d\mu \sin\lambda + \int_T \omega(u(t))d\mu \cos\lambda$ reaches its maximum at $\arctan\left(\frac{\int_T \psi(u(t))d\mu}{\int_T \omega(u(t))d\mu}\right)$. Thus

$$\begin{aligned}
&\sup_{\lambda\in[-\frac{\pi}{2},\frac{\pi}{2}]}\left(\int_T \psi(u(t))d\mu \sin\lambda + \int_T \omega(u(t))d\mu \cos\lambda\right) \\
&= \sin\left(\arctan\left(\frac{\int_T \psi(u(t))d\mu}{\int_T \omega(u(t))d\mu}\right)\right)\int_T \psi(u(t))d\mu \\
&\quad + \cos\left(\arctan\left(\frac{\int_T \psi(u(t))d\mu}{\int_T \omega(u(t))d\mu}\right)\right)\int_T \omega(u(t))d\mu \\
&= \frac{\int_T \psi(u(t))d\mu}{\int_T \omega(u(t))d\mu\sqrt{1+\left(\frac{\int_T \psi(u(t))d\mu}{\int_T \omega(u(t))d\mu}\right)^2}}\int_T \psi(u(t))d\mu \\
&\quad + \frac{1}{\sqrt{1+\left(\int_T \psi(u(t))d\mu \int_T \omega(u(t))d\mu\right)^2}}\int_T \omega(u(t))d\mu \\
&= \sqrt{\left(\int_T \psi(u(t))d\mu\right)^2+\left(\int_T \omega(u(t))d\mu\right)^2}.
\end{aligned}$$

Now (2.6) follows directly from (2.7) and (2.8) immediately. □

Remark 2.3. Let X, ω and ψ be as in Theorem 2.6. Consider the set

$$K = \left\{\left(\int_T \omega(u(t))d\mu, \int_T \psi(u(t))d\mu\right) : u \in X\right\} \subseteq \mathbf{R}^2 .$$

Theorem 2.6 gives us the exact value of the distance of 0 from K. Since, by the Lyapunov convexity theorem, K is convex, this information is very useful in applying Theorem 1 of [21] to $\overline{K}$.

3. Multiple Global Minima under a Non-convexity Condition

In this section, as an application of Theorem 1.3, we present the following general multiplicity result ([24]).

Theorem 3.1. *Let X be a topological space, E be a real normed space, $I : X \to \mathbf{R}$, $\psi : X \to E$, and $S \subseteq E^*$ be a convex set weakly-star dense in E^*. Assume that $\psi(X)$ is not convex and that $I+\eta\circ\psi$ is lower semicontinuous and inf-compact for all $\eta \in S$. Then, there exists $\tilde{\eta} \in S$ such that the function $I+\tilde{\eta}\circ\psi$ has at least two global minima in X.*

First, we prove the following result.

Proposition 3.1. *Let X be a non-empty set, E be a real vector space, $I : X \to \mathbf{R}$, and $\psi : X \to E$. Let $x_1,\dots,x_n \in X$, $\lambda_1,\dots,\lambda_n \in [0,1]$, with $\sum_{i=1}^n \lambda_i = 1$. Then,*

$$\sup_{\eta\in E'}\inf_{x\in X}\left(I(x)+\eta\left(\psi(x)-\sum_{i=1}^n \lambda_i\psi(x_i)\right)\right) \le \max_{1\le i\le n} I(x_i) .$$

Proof. Fix $\eta \in E'$. Clearly, for some $j' \in \{1,...,n\}$, we have

$$\eta\left(\psi(x_{j'}) - \sum_{i=1}^{n} \lambda_i \psi(x_i)\right) \leq 0. \tag{3.1}$$

Indeed, if not, then $\eta(\psi(x_j)) > \sum_{i=1}^{n} \lambda_i \eta(\psi(x_i))$ for each $j \in \{1,...,n\}$. By multiplying by λ_j and summing, we see that $\sum_{j=1}^{n} \lambda_j \eta(\psi(x_j)) > \sum_{i=1}^{n} \lambda_i \eta(\psi(x_i))$, a contradiction. In view of (3.1), we have

$$\inf_{x\in X}\left(I(x) + \eta\left(\psi(x) - \sum_{i=1}^{n} \lambda_i \psi(x_i)\right)\right) \leq I(x_{j'}) + \eta\left(\psi(x_{j'}) - \sum_{i=1}^{n} \lambda_i \psi(x_i)\right)$$
$$\leq I(x_{j'}) \leq \max_{1\leq i\leq n} I(x_i).$$

From the arbitrariness of η, we obtain the desired conclusion immediately. □

Proof of Theorem 3.1. Fix $u_0 \in \operatorname{conv}(\psi(X)) \setminus \psi(X)$ and consider the function $g : X \times E^* \to \mathbf{R}$ defined by $g(x,\eta) = I(x) + \eta(\psi(x) - u_0)$ for all $(x,\eta) \in X \times E^*$. By Proposition 3.1, we know that $\sup_{E^*} \inf_X g < +\infty$. On the other hand, for each $x \in X$, since $\psi(x) \neq u_0$, we have $\sup_{\eta\in E^*} \eta(\psi(x) - u_0) = +\infty$. Since S is weakly-star dense in E^* and $g(x,\cdot)$ is weakly-star continuous, we have $\sup_{\eta\in S} g(x,\eta) = +\infty$. Thus $\sup_S \inf_X g < \inf_X \sup_S g$. Now, we can apply Theorem 1.3 to $g_{|X\times S}$. We see there exists $\tilde{\eta} \in S$ such that $g(\cdot,\tilde{\eta})$ (and so $I + \tilde{\eta} \circ \psi$) has at least two global minima in X, as claimed.

Based on Theorem 3.1, we also have the following result.

Theorem 3.2. *Let E be a real normed space, V be a reflexive real Banach space, $x_0 \in V$, $r > 0$, X be the open ball in V with radius r, centered at x_0, $\gamma : [0,r[\to \mathbf{R}$, with $\lim_{\xi\to r^-} \gamma(\xi) = +\infty$, and $I : X \to \mathbf{R}$ and $\psi : X \to E$ be two Gâteaux differentiable functions. Moreover, let I be sequentially weakly lower semicontinous, ψ be sequentially weakly continuous, be $\psi(X)$ is bounded and non-convex, and $\gamma(\|x - x_0\|) \leq I(x)$ for all $x \in X$. Then, for every convex set $S \subseteq E^*$ weakly-star dense in E^*, there exists $\tilde{\eta} \in S$ such that the equation $I'(x) + (\tilde{\eta} \circ \psi)'(x) = 0$ has at least two solutions in X.*

Proof. We apply Theorem 3.1 by considering X equipped with the relative weak topology. Let $\eta \in E^*$. Since $\psi(X)$ is bounded, we have $c := \inf_{x\in X} \eta(\psi(x)) > -\infty$. Letting $s \in \mathbf{R}$, we see that

$$\{x \in X : I(x) + \eta(\psi(x)) \leq s\} \subseteq \{x \in X : I(x) \leq s - c\} \subseteq \{x \in X : \gamma(\|x - x_0\|) \leq s - c\}. \tag{3.2}$$

Since $\lim_{\xi\to r^-} \gamma(\xi) = +\infty$, one sees that there exist $\delta \in]0,r[$ such that $\gamma(\xi) > s - c$ for all $\xi \in]\delta,r[$. Consequently, from (3.2), we obtain

$$\{x \in X : I(x) + \eta(\psi(x)) \leq s\} \subseteq \{x \in V : \|x - x_0\| \leq \delta\}. \tag{3.3}$$

From the assumptions, it follows that $I + \eta \circ \psi$ is sequentially weakly lower semicontinuous in X. Since $\delta < r$ and V is reflexive, we infer from (3.3) that $\{x \in X : I(x) + \eta(\psi(x)) \leq s\}$ is sequentially weakly compact and hence weakly compact, by the Eberlein-Smulyan theorem. Therefore, we can apply Theorem 3.1. Accordingly, there exists $\tilde{\eta} \in S$ such that $I + \tilde{\eta} \circ \psi$ has at least two global minima in X, which are critical points of it since X is open. □

We now present an application of Theorem 3.2 to a class of singular Kirchhoff-type problems.

Theorem 3.3. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, let $\rho > 0$, and let $\omega : [0,\rho[\to [0,+\infty[$ be a continuous increasing function such that $\lim_{\xi\to\rho^-} \int_0^\xi \omega(x)dx = +\infty$. Consider $C^0([0,1]) \times C^0([0,1])$ endowed with the norm $\|(\alpha,\beta)\| = \int_0^1 |\alpha(t)|dt + \int_0^1 |\beta(t)|dt$. Then, the following assertions are equivalent:*

(a) the restriction of f to $\left[-\frac{\sqrt{\rho}}{2}, \frac{\sqrt{\rho}}{2}\right]$ is not constant;

(b) for every convex set $S \subseteq C^0([0,1]) \times C^0([0,1])$ dense in $C^0([0,1]) \times C^0([0,1])$, there exists $(\alpha,\beta) \in S$ such that

$$\begin{cases} -\omega\left(\int_0^1 |u'(t)|^2dt\right) u'' = \beta(t)f(u) + \alpha(t) \quad \text{in } [0,1] \\ u(0) = u(1) = 0 \\ \int_0^1 |u'(t)|^2dt < \rho \end{cases}$$

has at least two classical solutions.

Proof. Consider the Sobolev space $H_0^1(]0,1[)$ with the usual scalar product $\langle u,v\rangle = \int_0^1 u'(t)v'(t)dt$. Let $B_{\sqrt{\rho}}$ be the open ball in $H_0^1(]0,1[$, of radius $\sqrt{\rho}$, centered at 0. Let $g : [0,1]\times\mathbf{R} \to \mathbf{R}$ be a continuous function. Consider the functionals $I, J_g : B_{\sqrt{\rho}} \to \mathbf{R}$ defined by

$$I(u) = \frac{1}{2}\tilde{\omega}\left(\int_0^1 |u'(t)|^2dt\right),\ J_g(u) = \int_0^1 \tilde{g}(t,u(t))dt$$

for all $u \in B_{\sqrt{\rho}}$, where $\tilde{\omega}(\xi) = \int_0^\xi \omega(x)dx$ and $\tilde{g}(t,\xi) = \int_0^\xi g(t,x)dx$. Baking into account that if $\omega(x) = 0$, then $x = 0$, it follows that the classical solutions of

$$\begin{cases} -\omega\left(\int_0^1 |u'(t)|^2dt\right) u'' = g(t,u) \quad \text{in } [0,1] \\ u(0) = u(1) = 0 \\ \int_0^1 |u'(t)|^2dt < \rho \end{cases}$$

are exactly the critical points in $B_{\sqrt{\rho}}$ of $I - J_g$. Let us prove that $(a) \to (b)$.

We next apply Theorem 3.2 by taking $V = H_0^1(]0,1[)$, $x_0 = 0$, $r = \sqrt{\rho}$, I as above, $\gamma(\xi) = \frac{1}{2}\tilde{\omega}(\xi^2)$, $E = C^0([0,1]) \times C^0([0,1])$, and $\psi : B_{\sqrt{\rho}} \to E$ defined by $\psi(u)(\cdot) = (u(\cdot), \tilde{f}(u(\cdot)))$ for all $u \in B_{\sqrt{\rho}}$, where $\tilde{f}(\xi) = \int_0^\xi f(x)dx$. Clearly, I is continuous and strictly convex (and so weakly lower semicontinuous), while ψ is Gâteaux differentiable and sequentially weakly continuous due to the compact embedding of $H_0^1(]0,1[)$ into $C^0([0,1])$. Recall that $\max_{[0,1]}|u| \leq \frac{1}{2}\sqrt{\int_0^1 |u'(t)|^2dt}$ for all $u \in H_0^1(]0,1[)$. Thus $\psi\left(B_{\sqrt{\rho}}\right)$ is bounded and non-convex, due to (a). Hence, each assumption of Theorem 3.2 is satisfied. Now, we consider the operator $T : E \to E^*$ defined by

$$T(\alpha,\beta)(u,v) = \int_0^1 \alpha(t)u(t)dt + \int_0^1 \beta(t)v(t)dt$$

for all $(\alpha,\beta), (u,v) \in E$. It is obvious that T is linear and the linear subspace $T(E)$ is total over E. Hence, $T(E)$ is weakly-star dense in E^*. Moreover, notice that T is continuous with respect to the weak-star topology of E^*. Indeed, let $\{(\alpha_n,\beta_n)\}$ be a sequence in E converging to some $(0,0)$. Fix $(u,v) \in E$. We have to show that

$$\lim_{n\to\infty} T(\alpha_n,\beta_n)(u,v) = 0. \tag{3.4}$$

Notice that

$$\lim_{n\to\infty}\left(\int_0^1|\alpha_n(t)|dt+\int_0^1|\beta_n(t)|dt\right)=0. \tag{3.5}$$

On the other hand, we have

$$|T(\alpha_n,\beta_n)(u,v)|=\left|\int_0^1\alpha_n(t)u(t)dt+\int_0^1\beta_n(t)v(t)dt\right|$$
$$\leq\left(\int_0^1|\alpha_n(t)|dt+\int_0^1|\beta_n(t)|dt\right)\max\left\{\max_{[0,1]}|u|,\max_{[0,1]}|v|\right\}.$$

Hence (3.4) follows in view of (3.5).

Finally, fix a convex set $S\subseteq C^0([0,1])\times C^0([0,1])$ dense in $C^0([0,1])\times C^0([0,1])$. Then, by the kind of continuity of T just now proved, the convex set $T(-S)$ is weakly-star dense in E^*. Thanks to Theorem 3.2, we see that there exists $(\alpha_0,\beta_0)\in -S$ such that, if we put $g(t,\xi)=\alpha_0(t)+\beta_0(t)f(\xi)$, then functional $I-J_g$ has at least two critical points in $B_{\sqrt{\rho}}$, which are the claimed solutions to the problem in (b) with $\alpha=-\alpha_0$ and $\beta=-\beta_0$.

Now, let us prove that $(b)\to(a)$. Assume that the restriction of f to $\left[-\frac{\sqrt{\rho}}{2},\frac{\sqrt{\rho}}{2}\right]$ is constant. Let c be such a value. So, the classical solutions of

$$\begin{cases} -\omega\left(\int_0^1|u'(t)|^2dt\right)u''=c\beta(t)+\alpha(t) & in\ [0,1]\\ u(0)=u(1)=0 \\ \int_0^1|u'(t)|^2dt<\rho \end{cases}$$

are the critical points in $B_{\sqrt{\rho}}$ of $u\to\frac{1}{2}\tilde{\omega}\left(\int_0^1|u'(t)|^2dt\right)-\int_0^1(c\alpha(t)+\beta(t))u(t)dt$. But, since ω is increasing and non-negative, this functional is strictly convex and so it possesses a unique critical point. The proof is complete. □

4. Multiplicity of Periodic Solutions for Lagrangian Systems of Relativistic Oscillators

In this section, we present an application of Theorem 3.1 to Lagrangian systems of relativistic oscillators ([25]). In what follows, L,T are assumed to be two fixed positive numbers. For each $r>0$, we set $B_r=\{x\in\mathbf{R}^n:|x|<r\}$ ($|\cdot|$ being the Euclidean norm on $\mathbf{R}^n$) and $\overline{B_r}$ is the closure of B_r. The scalar product on $\mathbf{R}^n$ is denoted by $\langle\cdot,\cdot\rangle$. We denote by $\mathscr{A}$ the family of all homeomorphisms ϕ from B_L onto $\mathbf{R}^n$ such that $\phi(0)=0$ and $\phi=\nabla\Phi$, where the function $\Phi:\overline{B_L}\to]-\infty,0]$ is continuous and strictly convex in $\overline{B_L}$, and of class C^1 in B_L. Notice that 0 is the unique global minimum of Φ in $\overline{B_L}$. We denote by $\mathscr{B}$ the family of all functions $F:[0,T]\times\mathbf{R}^n\to\mathbf{R}$ which are measurable in $[0,T]$, of class C^1 in $\mathbf{R}^n$ and such that ∇_xF is measurable in $[0,T]$ and, for each $r>0$, one has $\sup_{x\in B_r}|\nabla_xF(\cdot,x)|\in L^1([0,T])$, with $F(\cdot,0)\in L^1([0,T])$. Clearly, $\mathscr{B}$ is a linear subspace of $\mathbf{R}^{[0,T]\times\mathbf{R}^n}$.

Given $\phi\in\mathscr{A}$ and $F\in\mathscr{B}$, we consider the problem

$$\begin{cases} (\phi(u'))'=\nabla_xF(t,u) & in\ [0,T]\\ u(0)=u(T)\ ,\ u'(0)=u'(T)\ . \end{cases} \tag{$P_{\phi,F}$}$$

A solution of this problem is any function $u : [0,T] \to \mathbf{R}^n$ of class C^1, with $u'([0,T]) \subset B_L$, $u(0) = u(T)$, $u'(0) = u'(T)$, such that the composite function $\phi \circ u'$ is absolutely continuous in $[0,T]$ and one has $(\phi \circ u')'(t) = \nabla_x F(t,u(t))$ for a.e. $t \in [0,T]$.

Now, we set

$$K = \{u \in \mathrm{Lip}([0,T],\mathbf{R}^n) : |u'(t)| \le L \ \mathit{for\ a.e.}\ t \in [0,T], u(0) = u(T)\},$$

where $\mathrm{Lip}([0,T],\mathbf{R}^n)$ is the space of all Lipschitzian functions from $[0,T]$ into $\mathbf{R}^n$.

Clearly, one has

$$\sup_{[0,T]} |u| \le LT + \inf_{[0,T]} |u| \tag{4.1}$$

for all $u \in K$.

Next, we consider the functional $I : K \to \mathbf{R}$ defined by $I(u) = \int_0^T (\Phi(u'(t)) + F(t,u(t)))dt$ for all $u \in K$. In [3], Brezis and Mawhin proved the following result.

Theorem 4.A. *Any global minimum of I in K is a solution to problem $(P_{\phi,F})$.*

Here is our result.

Theorem 4.1. *Let $\phi \in \mathscr{A}$, $F,G \in \mathscr{B}$ and $H \in C^1(\mathbf{R}^n)$. Assume that*

(a_1) *there exists $q > 0$ such that*

$$\lim_{|x|\to+\infty} \frac{\inf_{t\in[0,T]} F(t,x)}{|x|^q} = +\infty$$

and

$$\limsup_{|x|\to+\infty} \frac{\sup_{t\in[0,T]} |G(t,x)| + |H(x)|}{|x|^q} < +\infty\,;$$

(a_2) *there exist $\gamma \in \{\inf_{\mathbf{R}^n} H, \sup_{\mathbf{R}^n} H\}$, with $H^{-1}(\gamma)$ at most countable, and $v,w \in H^{-1}(\gamma)$ such that $\int_0^T G(t,v)dt \neq \int_0^T G(t,w)dt$.*

Then, for each $\alpha \in L^\infty([0,T])$ with a constant sign and $\mathrm{meas}(\alpha^{-1}(0)) = 0$, there exists $(\tilde{\lambda},\tilde{\mu}) \in \mathbf{R}^2$ such that

$$\begin{cases} (\phi(u'))' = \nabla_x \Big(F(t,u) + \tilde{\lambda} G(t,u) + \tilde{\mu}\alpha(t)H(u)\Big) & \mathit{in}\ [0,T] \\ u(0) = u(T)\ ,\ u'(0) = u'(T) \end{cases} \tag{P}$$

has at least two solutions which are global minima in K to

$$u \to \int_0^T (\Phi(u'(t)) + F(t,u(t)) + \tilde{\lambda} G(t,u(t)) + \tilde{\mu}\alpha(t)H(u(t)))dt.$$

Proof. Fix $\alpha \in L^\infty([0,T])$ with a constant sign and $\mathrm{meas}(\alpha^{-1}(0)) = 0$. Let $C^0([0,T],\mathbf{R}^n)$ be the space of all continuous functions from $[0,T]$ into $\mathbf{R}^n$, with the norm $\sup_{[0,T]} |u|$. We are going to apply Theorem 3.1 by taking $X = K$, regarded as a subset of $C^0([0,T],\mathbf{R}^n)$ with the relative topology, $E = \mathbf{R}^2$ and $I : K \to \mathbf{R}$, $\psi : K \to \mathbf{R}^2$ defined by

$$I(u) = \int_0^T (\Phi(u'(t)) + F(t,u(t)))dt,\ \psi(u) = \left(\int_0^T G(t,u(t))dt, \int_0^T \alpha(t)H(u(t))dt\right)$$

for all $u \in K$. Fix $(\lambda,\mu) \in \mathbf{R}^2$. By Lemma 4.1 of [3], we see that $I(\cdot)+\langle\psi(\cdot),(\lambda,\mu)\rangle$ is lower semicontinuous in K. Let us show that it is inf-compact too. First, observe that if $P \in \mathscr{B}$ then, for each $r>0$, there exists $M \in L^1([0,T])$ such that

$$\sup_{x\in B_r}|P(t,x)| \leq M(t) \tag{4.2}$$

for all $t \in [0,T]$. Indeed, by the mean value theorem, we have $P(t,x)-P(t,0)=\langle\nabla_x P(t,\xi),x\rangle$ for some ξ in the segment joining 0 and x. Consequently, for all $t\in[0,T]$ and $x \in B_r$, by the Cauchy-Schwarz inequality, we clearly have $|P(t,x)| \leq r\sup_{y\in B_r}|\nabla_x P(t,y)|+|P(t,0)|$. To reach (4.2), we can choose $M(t) := r\sup_{y\in B_r}|\nabla_x P(t,y)|+|P(t,0)|$ which is in $L^1([0,T])$ since $P \in \mathscr{B}$. Now, by (a_1), we can fix $c_1,\delta>0$ so that

$$|G(t,x)|+|H(x)| \leq c_1|x|^q \tag{4.3}$$

for all $(t,x)\in[0,T]\times(\mathbf{R}^n\setminus B_\delta)$. We now set $c_2 := c_1\max\left\{|\lambda|,|\mu|\|\alpha\|_{L^\infty([0,T])}\right\}$. By (a_1) again, we fix $c_3>c_2$ and $\delta_1>\delta$ so that

$$F(t,x) \geq c_3|x|^q \tag{4.4}$$

for all $(t,x)\in[0,T]\times(\mathbf{R}^n\setminus B_{\delta_1})$.

On the other hand, for what remarked above, there exists $M \in L^1([0,T])$ such that

$$\sup_{x\in B_{\delta_1}}(|F(t,x)|+|\lambda G(t,x)|+|\mu\alpha(t)H(x)|) \leq M(t) \tag{4.5}$$

for all $t\in[0,T]$. From (4.3), (4.4), and (4.5), we infer that

$$F(t,x) \geq c_3|x|^q - M(t) \tag{4.6}$$

and

$$|\lambda G(t,x)|+|\mu\alpha(t)H(x)| \leq c_2|x|^q+M(t) \tag{4.7}$$

for all $(t,x)\in[0,T]\times\mathbf{R}^n$. Set $b := T\Phi(0)-2\int_0^T M(t)dt$. For each $u\in K$, with $\sup_{[0,T]}|u| \geq LT$, taking (4.1), (4.6), and (4.7) into account, we have

$$\begin{aligned}
&I(u)+\langle\psi(u),(\lambda,\mu)\rangle\\
&\geq T\Phi(0)+\int_0^T F(t,u(t))dt-\int_0^T|\lambda G(t,u(t))|dt-\int_0^T|\mu\alpha(t)H(u(t))|dt\\
&\geq T\Phi(0)+c_3\int_0^T|u(t)|^q dt-\int_0^T M(t)dt-c_2\int_0^T|u(t)|^q dt-\int_0^T M(t)dt\\
&\geq (c_3-c_2)T\inf_{[0,T]}|u|^q+b \geq (c_3-c_2)T\left(\sup_{[0,T]}|u|-LT\right)^q+b.
\end{aligned}$$

Consequently

$$\sup_{[0,T]}|u| \leq \left(\frac{I(u)+\langle\psi(u),(\lambda,\mu)\rangle-b}{(c_3-c_2)T}\right)^{\frac{1}{q}}+LT. \tag{4.8}$$

Fix $\rho\in\mathbf{R}$. By (4.8), we see that $C_\rho := \{u\in K : I(u)+\langle\psi(u),(\lambda,\mu)\rangle \leq \rho\}$ turns out to be bounded. Moreover, the functions belonging to C_ρ are equi-continuous since they lie in K. As a consequence, by the Ascoli-Arzelà theorem, C_ρ is relatively compact in $C^0([0,T],\mathbf{R}^n)$. By lower semicontinuity, C_ρ is closed in K. But K is closed in $C^0([0,T],\mathbf{R}^n)$. Thus C_ρ is compact. The inf-compactness of $I(\cdot)+\langle\psi(\cdot),(\lambda,\mu)\rangle$ is obtained. Now, we are going to prove that $\psi(K)$ is not convex. By (a_2), the set $\left\{\int_0^T G(t,x)dt : x\in H^{-1}(\gamma)\right\}$ is at most countable since $H^{-1}(\gamma)$ is so. Hence, since $\int_0^T G(t,v)dt \neq \int_0^T G(t,w)dt$, we can fix $\lambda \in]0,1[$ so that

$$\int_0^T G(t,x)dt \neq \int_0^T G(t,w)dt+\lambda\left(\int_0^T G(t,v)dt-\int_0^T G(t,w)dt\right) \tag{4.9}$$

for all $x\in H^{-1}(\gamma)$. Since K contains the constant functions, the points

$$\left(\int_0^T G(t,v)dt,\gamma\int_0^T \alpha(t)dt\right)$$

and

$$\left(\int_0^T G(t,w)dt,\gamma\int_0^T \alpha(t)dt\right)$$

belong to $\psi(K)$. To show that $\psi(K)$ is not convex, it is enough to check that the point

$$\left(\int_0^T G(t,w)dt+\lambda\left(\int_0^T G(t,v)dt-\int_0^T G(t,w)dt\right),\gamma\int_0^T \alpha(t)dt\right)$$

does not belong to $\psi(K)$. Arguing by contradiction, we suppose that there exists $u\in K$ such that

$$\int_0^T G(t,u(t))dt = \int_0^T G(t,w)dt+\lambda\left(\int_0^T G(t,v)dt-\int_0^T G(t,w)dt\right), \tag{4.10}$$

and

$$\int_0^T \alpha(t)H(u(t))dt = \gamma\int_0^T \alpha(t)dt. \tag{4.11}$$

Since α and $H\circ u-\gamma$ do not change sign, (4.11) implies that $\alpha(t)(H(u(t))-\gamma)=0$ a.e. in $[0,T]$. Consequently, since $\text{meas}(\alpha^{-1}(0))=0$, we have $H(u(t))=\gamma$ a.e. in $[0,T]$. Hence, $H(u(t))=\gamma$ for all $t\in[0,T]$ since $H\circ u$ is continuous. In other words, the connected set $u([0,T])$ is contained in $H^{-1}(\gamma)$ which is at most countable. This implies that u must be constant and so (4.10) contradicts (4.9). Therefore, I and ψ satisfy the assumptions of Theorem 3.1 and hence there exists $(\tilde\lambda,\tilde\mu)\in\mathbf{R}^2$ such that $I(\cdot)+\langle\psi(\cdot),(\tilde\lambda,\tilde\mu)\rangle$ has at least two global minima in K. Thanks to Theorem 4.A, they are solutions to Problem (P), and the proof is complete. □

Remark 4.1. It is obvious that (a_2) is the leading assumption of Theorem 4.1. The request that $H^{-1}(\gamma)$ must be at most countable cannot be removed. Indeed, if we remove such a request, we could take $H=0$, $G(t,x)=\langle x,\omega\rangle$, with $\omega\in\mathbf{R}^n\setminus\{0\}$ and $F(t,x)=\frac{1}{p}|x|^p$, with $p>1$. Now, we observe that, by [3, Proposition 3.2], for all $\lambda\in\mathbf{R}$,

$$\begin{cases} (\phi(u'))' = |u|^{p-2}u+\lambda\omega \quad in\ [0,T] \\ u(0)=u(T)\ ,\ u'(0)=u'(T) \end{cases}$$

has a unique solution. To the contrary, the question of whether $\int_0^T G(t,v)dt \neq \int_0^T G(t,w)dt$ (keeping $v\neq w$) can be dropped remains open at present. We feel, however, that it cannot be removed. In this connection, we propose the following conjecture.

Conjecture 4.1. There exist $\phi \in \mathscr{A}$, $F \in \mathscr{B}$, $H \in C^1(\mathbf{R}^n)$, $\alpha \in L^\infty([0,T])$, with $\alpha \geq 0$ and $\mathrm{meas}(\alpha^{-1}(0)) = 0$, and $q > 0$ for which the following assertions hold:

(b_1)

$$\lim_{|x|\to+\infty} \frac{\inf_{t\in[0,T]} F(t,x)}{|x|^q} = +\infty$$

and

$$\limsup_{|x|\to+\infty} \frac{|H(x)|}{|x|^q} < +\infty;$$

(b_2) H has exactly two global minima;

(b_3) for each $\mu \in \mathbf{R}$, $u \to \int_0^T (\Phi(u'(t)) + F(t,u(t)) + \mu\alpha(t)H(u(t)))dt$ has a unique global minimum in K.

5. A Property of Strictly Convex Functions

In this section, we apply Theorem 1.3 to detect a new property of strictly convex functions expressed in Theorem 5.1 below ([26]).

When E is a real vector space and $A \subseteq E$, a point $x_0 \in A$ is said to be an algebraically interior point of A (with respect to E) if, for every $y \in E$, there exists $\delta > 0$ such that $x_0 + \lambda y \in A$ for all $\lambda \in [0,\delta]$. The algebraic interior of A is the set of all its algebraically interior points.

Theorem 5.1. *Let E be a reflexive real Banach space and let $X \subset E$ be a closed convex set, with non-empty interior, whose boundary is sequentially weakly closed and non-convex. Then, for every function $\varphi : \partial X \to \mathbf{R}$ and for every convex set $S \subseteq E^*$ dense in E^*, there exists $\tilde{\psi} \in S$ with the following property, for every strictly convex lower semicontinuous function $J : X \to \mathbf{R}$, Gâteaux differentiable in $\mathrm{int}(X)$, such that $J_{|\partial X} - \varphi$ is constant in ∂X and $\lim_{\|x\|\to+\infty} \frac{J(x)}{\|x\|} = +\infty$ if X is unbounded, $\tilde{\psi}$ is an algebraically interior point of $J'(int(X))$ (with respect to E^*)*

Now, some comments are in order. The main feature of Theorem 5.1 is the fact that $\tilde{\psi}$ does not depend on J. But, for a moment, we consider simply the following by-product of Theorem 5.1.

If X is as before, then, for every stricly convex lower semicontinuous function $J : X \to \mathbf{R}$, Gâteaux differentiable in $int(X)$ and with $\lim_{\|x\|\to+\infty} \frac{J(x)}{\|x\|} = +\infty$ if X is unbounded, the algebraic interior of $J'(int(X))$ (with respect to E^*) is non-empty.

As far as we know, such a corollary itself is new when E is infinite-dimensional. To the contrary, if E is finite-dimensional, due to the strict convexity of J, J' is injective and continuous in $int(X)$, so $J'(int(X))$ turns out to be open, thanks to the invariance of domain theorem. The facts that ∂X is sequentially weakly closed and that $\lim_{\|x\|\to+\infty} \frac{J(x)}{\|x\|} = +\infty$ if X is unbounded are both necessary. Actually, consider the following situation. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous, increasing and bounded function. Define the functional $J : L^2([0,1]) \to \mathbf{R}$ by

$$J(u) = \int_0^1 \left(\int_0^{u(x)} f(t)dt \right) dx$$

for all $u \in L^2([0,1])$. Clearly, J is stricly convex and C^1, with $J'(u) = f \circ u$ for all $u \in L^2([0,1])$ (after identifying $(L^2([0,1])^*$ to $L^2([0,1])$). Notice that, since $J'(L^2([0,1])) \subseteq L^\infty([0,1])$, for each $A \subseteq L^2([0,1])$, the algebraic interior of $J'(A)$ (with respect to $L^2([0,1])$) is empty. Now, let

X be any closed ball in $L^2([0,1])$. Thus the restriction of J to X is weakly inf-compact. In this case, the conclusion of the corollary fails since ∂X is not sequentially weakly closed. On the other hand, when X is as in the corollary, the conclusion fails since the restriction of J to X is not weakly inf-compact.

Now, come back to the full statement of Theorem 5.1. We observe that the non-convexity of ∂X is necessary. In this connection, assume that E is a Hilbert space. Fix $w \in E$, with $\|w\| = 1$, and consider the set $X := \{x \in E : \langle w,x\rangle \geq 0\}$. So, X is a closed convex set with non-empty interior and convex boundary. For such set X the conclusion of Theorem 5.1 does not hold. Indeed, suppose the contrary. Then, in particular, there would be some $\tilde{\psi} \in E$ such that, for each strictly convex function lower semicontinous functional $J : X \to \mathbf{R}$, Gâteaux differentiable in $\operatorname{int}(X)$, with $J(x) = \frac{1}{2}\|x\|^2$ for all $x \in \partial X$ and $\lim_{\|x\|\to+\infty} \frac{J(x)}{\|x\|} = +\infty$, we have $\tilde{\psi} \in J'(X)$. So, in particular, for each $\lambda \in \mathbf{R}$, we have $\tilde{\psi} \in J'_\lambda(X)$, where $J_\lambda(x) := \frac{1}{2}\|x\|^2 + \lambda\langle w,x\rangle$. But $J'_\lambda(x) = x + \lambda w$. Thus we have $\tilde{\psi} \in X + \lambda w$ for all $\lambda \in \mathbf{R}$. This means $\langle w, \tilde{\psi} - \lambda w\rangle \geq 0$, so $\langle w, \tilde{\psi}\rangle \geq \lambda$ for each $\lambda \in \mathbf{R}$, which is absurd.

We next obtain Theorem 5.1 as a consequence of an abstract result (Theorem 5.2 below) whose proof is fully based on the use of Theorem 1.3. In what follows, E is a topological space and Y is a convex set in a topological vector space. Let us introduce the two main classes of functions we deal with. Let $X \subseteq C \subseteq E$ and let $\varphi : X \to \mathbf{R}$ be a given function. We denote by $\mathscr{B}(X,C,\varphi)$ the class of all functions $J : C \to \mathbf{R}$ such that $J_{|X} - \varphi$ is constant in X. Let $C \subseteq E$ and $S \subseteq Y$. Let $f : C \times S \to \mathbf{R}$ be a given function. We denote by $\mathscr{C}(C,S,f)$ the class of all functions $J : C \to \mathbf{R}$ such that, for each $y \in S$, $f(\cdot,y) + J(\cdot)$ has at most one global minimum in C. Our main abstract result is as follows.

Theorem 5.2. *Let $X \subseteq C \subseteq E$, let $S \subseteq Y$ be a convex set dense in Y, let $f : C \times Y \to \mathbf{R}$, and let $\varphi : X \to \mathbf{R}$. Assume that*

(a) for each $y \in S$, $f(\cdot,y) + \varphi(\cdot)$ is lower semicontinuous and inf-compact in X;

(b) for each $x \in X$, $f(x,\cdot)$ is quasi-concave and continuous in Y;

(c) $\sup_{y\in Y}\inf_{x\in X}(f(x,y) + \varphi(x)) < \inf_{x\in X}\sup_{y\in Y}(f(x,y) + \varphi(x))$.

Then, there exists a point $y^* \in S$ such that $\inf_{x\in C}(f(x,y^*) + J(x)) < \inf_{x\in X}(f(x,y^*) + J(x))$ for every $J \in \mathscr{B}(X,C,\varphi) \cap \mathscr{C}(C,S,f)$.

Proof. Consider the function $g : X \times S \to \mathbf{R}$ defined by $g(x,y) = f(x,y) + \varphi(x)$ for all $(x,y) \in X \times S$. For each $x \in X$, by continuity of $f(x,\cdot)$ and density of S, we have $\sup_{y\in S} f(x,y) = \sup_{y\in Y} f(x,y)$, so $\sup_{y\in S} g(x,y) = \sup_{y\in Y} g(x,y)$. In view of (c), we have

$$\sup_S \inf_X g \leq \sup_Y \inf_X g < \inf_X \sup_Y g = \inf_X \sup_S g.$$

The function g is lower semicontinuous and inf-compact in X, and quasi-concave and continuous in S. Thanks to Theorem 1.3, there exists $y^* \in S$ such that $(f(\cdot,y^*))_{|X} + \varphi(\cdot)$ has at least two global minima in X. Now, fix $J \in \mathscr{B}(X,C,\varphi) \cap \mathscr{C}(C,S,f)$. Since $J_{|X} - \varphi$ is constant in X, $(f(\cdot,y^*) + J(\cdot))_{|X}$ and $(f(\cdot,y^*))_{|X} + \varphi(\cdot)$ have the same global minima in X. Arguing by contradiction, assume that

$$\inf_{x\in C}(f(x,y^*) + J(x)) = \inf_{x\in X}(f(x,y^*) + J(x)). \tag{5.1}$$

We know that $(f(\cdot,y^*)+J(\cdot))_{|X}$ has at least two global minima. In view of (5.1), they turn out to be global minima of $f(\cdot,y^*)+J(\cdot)$ in C, against the fact that $J\in\mathscr{C}(C,S,f)$. The proof is complete. □

Next, we present a remarkable corollary of Theorem 5.2.

Theorem 5.3. *Let $X\subseteq C\subseteq E$, let $S\subseteq Y$ be a convex set dense in Y, let $f:C\times Y\to\mathbf{R}$, and let $\varphi:X\to\mathbf{R}$. Assume that*

(i) for each $y\in S$, $f(\cdot,y)+\varphi(\cdot)$ is lower semicontinuous and inf-compact in X;

(ii) for each $x\in X$, $f(x,\cdot)$ is quasi-concave and continuous in Y;

(iii) $\inf_X\sup_Y f=+\infty$ and there exists a finite set $A\subset X$ such that $\sup_Y\inf_A f<+\infty$.

Then, there exists a point $y^*\in S$ such that $\inf_{x\in C}(f(x,y^*)+J(x))<\inf_{x\in X}(f(x,y^*)+J(x))$ for every $J\in\mathscr{B}(X,C,\varphi)\cap\mathscr{C}(C,S,f)$.

Proof. Observe that

$$\sup_{y\in Y}\inf_{x\in X}(f(x,y)+\varphi(x))\leq\sup_Y\inf_A+\sup_A\varphi<+\infty=\inf_X\sup_Y f=\inf_{x\in X}\sup_{y\in Y}(f(x,y)+\varphi(x)).$$

Thus the conclusion follows directly from Theorem 5.2. □

We now present a first consequence of Theorem 5.3. In the next two results, E is also a real vector space (and the topology on E is still arbitrary).

Theorem 5.4. *Let $X\subseteq E$, let F be a real normed space, let $I:\operatorname{conv}(X)\to\mathbf{R}$, and let $\psi:\operatorname{conv}(X)\to F$ be such that $\psi(X)$ is not convex. Then, for every convex set $S\subseteq F^*$ weakly-star dense in F^* and for every $\varphi:X\to\mathbf{R}$ such that $(I+\eta\circ\psi)_{|X}+\varphi$ is lower semicontinuous and inf-compact in X for all $\eta\in S$, there exists $\tilde{\eta}\in S$ with the following property: for every function $J:\operatorname{conv}(X)\to\mathbf{R}$ such that $J_{|X}-\varphi$ is constant in X and $I+J+\eta\circ\psi$ is strictly convex for all $\eta\in S$,*

$$\inf_{x\in\operatorname{conv}(X)}(I(x)+J(x)+\tilde{\eta}(\psi(x)))<\inf_{x\in X}(I(x)+J(x)+\tilde{\eta}(\psi(x)))\,.$$

Proof. Fix $y_0\in\operatorname{conv}(\psi(X))\setminus\psi(X)$. We apply Theorem 5.3 with $C=\operatorname{conv}(X)$ and $Y=F^*$. Consider F^* equipped with the weak-star topology and take $f(x,\eta)=I(x)+\eta(\psi(x)-y_0)$ for all $(x,\eta)\in C\times Y$. Clearly, f satisfies conditions (i) and (ii). Moreover, if $y_0=\sum_{i=1}^n\lambda_i\psi(x_i)$, where $x_i\in X$, $\lambda_i\geq 0$, $\sum_{i=1}^n\lambda_i=1$, by Proposition 3.1, we know that

$$\sup_{\eta\in Y}\inf_{1\leq i\leq n}f(x_i,\eta)\leq\max_{1\leq i\leq n}I(x_i)<+\infty.$$

On the other hand, for each $x\in X$, since $\psi(x)\neq y_0$, we have $\sup_{\eta\in Y}\eta(\psi(x)-y_0)=+\infty$, so condition (*iii*) is satisfied too. Now, Theorem 5.3 ensures the existence of $\tilde{\eta}\in S$ such that, for every $J\in\mathscr{B}(X,C,\varphi)\cap\mathscr{C}(C,S,f)$, one has $\inf_{x\in\operatorname{conv}(X)}(f(x,\tilde{\eta})+J(x))<\inf_{x\in X}(f(x,\tilde{\eta})+J(x))$, which means

$$\inf_{x\in\operatorname{conv}(X)}(I(x)+J(x)+\tilde{\eta}(\psi(x)))<\inf_{x\in X}(I(x)+J(x)+\tilde{\eta}(\psi(x))).$$

To finish the proof, it is suffice to remark that if $J:\operatorname{conv}(X)\to\mathbf{R}$ is such that $I+J+\eta\circ\psi$ is strictly convex for all $\eta\in S$, then $J\in\mathscr{C}(C,S,f)$. □

The following result is a particularly simple consequence of Theorem 5.4.

Theorem 5.5. *Let $X \subset E$ be a compact set, let F be a real normed space, and let $\psi : \mathrm{conv}(X) \to F$ be an affine operator, continuous with respect to the weak topology on F, such that $\psi(X)$ is not convex. Then, for every convex set $S \subseteq F^*$ weakly-star dense in F^* and for every lower semicontinuous function $\varphi : X \to \mathbf{R}$, there exists $\tilde{\eta} \in S$ with the following property: for every strictly convex function $J : \mathrm{conv}(X) \to \mathbf{R}$ such that $J_{|X} - \varphi$ is constant in X,*

$$\inf_{x\in \mathrm{conv}(X)}(\tilde{\eta}(\psi(x)) + J(x)) < \inf_{x\in X}(\tilde{\eta}(\psi(x)) + J(x)) .$$

Proof. For each $\eta \in F^*$, $\eta \circ \psi$ is continuous since η is weakly continuous. Thus $\eta \circ \psi + \varphi$ is lower semicontinuous and inf-compact, since X is compact. Hence, the assumptions of Theorem 5.4 are satisfied, with $I = 0$. Now, our conclusion follows from that of Theorem 5.4 by taking into account that if $\eta \in F^*$ and $J : \mathrm{conv}(X) \to \mathbf{R}$ is a strictly convex function, then $\eta \circ \psi + J$ is since ψ is affine. □

The next consequence of Theorem 5.4 can be considered as the central result. Actually, Theorem 5.1 is its corollary.

Theorem 5.6. *Let E be a reflexive real Banach space, let $C \subset E$ be a proper closed convex set, with non-empty interior, such that ∂C is sequentially weakly closed and non-convex, and let $I : C \to \mathbf{R}$ be Gâteaux differentiable in $int(C)$. Then, for every convex set $S \subseteq E^*$ dense in E^* and for every function $\varphi : \partial C \to \mathbf{R}$, there exists $\tilde{\eta} \in S$ with the following property: for every function $J : C \to \mathbf{R}$, Gâteaux differentiable in $\mathrm{int}(C)$, such that $J_{|\partial C} - \varphi$ is constant in ∂C and $I + J$ is lower semicontinuous and strictly convex, with $\lim_{\|x\|\to+\infty} \frac{I(x)+J(x)}{\|x\|} = +\infty$ if C is unbounded, and for every sequentially weakly lower semicontinuous function $\beta : C \to \mathbf{R}$, Gâteaux differentiable in $\mathrm{int}(C)$, there exists $\varepsilon > 0$ such that, for each $\lambda \in [0, \varepsilon]$, the equation $I'(x) + J'(x) + \lambda\beta'(x) = \tilde{\eta}$ has at least one solution in $int(C)$.*

Proof. Fix a convex set $S \subseteq E^*$ dense in E^* and a function $\varphi : \partial C \to \mathbf{R}$. We apply Theorem 5.4 by considering E equipped with the weak topology (but the interior of C is referred to the strong topology) and taking $X = \partial C$, $F = E$ and $\psi(x) = x$ for all $x \in C$. Of course, it is implicitly understood that there are functions $J : C \to \mathbf{R}$ such that $J_{|\partial C} - \varphi$ is constant in ∂C, and $I + J$ is lower semicontinuous and strictly convex, with $\lim_{\|x\|\to+\infty} \frac{I(x)+J(x)}{\|x\|} = +\infty$ if C is unbounded. If J is such a function, it follows that, for each $\eta \in E^*$, $I + J + \eta$ is weakly inf-compact in C, so $(I + \eta \circ \psi)_{\partial C} + \varphi$ is weakly inf-compact in ∂C since ∂C is sequentially weakly closed (use also the Eberlein-Smulyan theorem). Therefore, the assumptions of Theorem 5.4 are satisfied. Consequently, there exists $\tilde{\eta} \in S$, with the following property: for every function $J : C \to \mathbf{R}$ such that $J_{|\partial C} - \varphi$ is constant in ∂C and $I + J + \eta \circ \psi$ is strictly convex for all $\eta \in S$, one has

$$\inf_{x\in C}(I(x) + J(x) + \tilde{\eta}(\psi(x))) < \inf_{x\in\partial C}(I(x) + J(x) + \tilde{\eta}(\psi(x))).$$

In addition, assume that J is lower semicontinuous in C, Gâteaux differentiable in $int(C)$ and the sub-level sets of $I + J + \tilde{\eta} \circ \psi$ are bounded. Fix σ such that

$$\inf_{x\in C}(I(x) + J(x) + \tilde{\eta}(\psi(x))) < \sigma < \inf_{x\in\partial C}(I(x) + J(x) + \tilde{\eta}(\psi(x))). \tag{5.2}$$

Observe that $I + J + \tilde{\eta} \circ \psi$ is sequentially weakly lower semicontinuous (recall that the assumptions imply that it is lower semicontinuous) and that $\{x \in C : I(x) + J(x) + \tilde{\eta}(\psi(x)) \le \sigma\}$ is sequentially weakly compact, since E is reflexive. Now, in view of Theorem 2.1 of [18], there exists $\varepsilon > 0$ such that, for every $\lambda \in [0,\varepsilon]$, the restriction of the function $I + J + \tilde{\eta} \circ \psi + \lambda\beta$ to the set $\{x \in C : I(x) + J(x) + \tilde{\eta}(\psi(x)) < \sigma\}$ has a global minimum, say $\tilde{x}$. But, due to (5.2), we have $\{x \in C : I(x) + J(x) + \tilde{\eta}(\psi(x)) < \sigma\} \subseteq \mathrm{int}(C)$, which implies that

$$I'(\tilde{x}) + J'(\tilde{x}) + \lambda\beta(\tilde{x}) + (\tilde{\eta} \circ \psi)'(\tilde{x}) = 0,$$

as claimed. □

The proof of Theorem 5.1 Let $S \subseteq E^*$ be a convex set dense in E^*, and let $\varphi : \partial X \to \mathbf{R}$. Apply Theorem 5.6 with $I = 0$ and let $\tilde{\gamma} \in S$ be as in the conclusion of Theorem 5.6. Fix any lower semicontinuous and strictly convex function $J : X \to \mathbf{R}$, Gâteaux differentiable in *int*(X), such that $J_{|\partial X} - \varphi$ is constant in ∂X, with $\lim_{\|x\|_E \to +\infty} \frac{J(x)}{\|x\|} = +\infty$ if X is unbounded. Now, fix any $G \in E^*$. Then, there exists $\varepsilon > 0$ such that $J'(x) - \lambda G = \tilde{\gamma}$ has a solution in *int*(X) for all $\lambda \in [0,\varepsilon]$, and this means exactly that the set $\tilde{\gamma}$ is an algebraically interior point of $J'(\mathrm{int}(X))$ (with respect to E^*).

In a finite-dimensional setting, another consequence of Theorem 5.6 is as follows.

Theorem 5.7. *Let E be a finite-dimensional real Banach space and let $C \subset E$ be a compact convex set with non-empty interior. Then, for every function $\varphi : \partial C \to \mathbf{R}$, there exists $\tilde{\gamma} \in E^*$ having the following property: for every lower semicontinuous stricly convex function $J : C \to \mathbf{R}$, Gâteaux differentiable in* $\mathrm{int}(C)$, *such that $J_{|\partial C} - \varphi$ is constant in ∂C, and for every lower semicontinuous, bounded below and Gâteaux differentiable function $H : \mathrm{int}(C) \to \mathbf{R}$, there exists $\varepsilon > 0$ such that, for each $\lambda \in [0,\varepsilon]$, the equation $J'(x) + \lambda H'(x) = \tilde{\gamma}$ has at least one solution in* $\mathrm{int}(C)$.

Proof. Since C is compact, we have that ∂C is not convex. Fix $\varphi : \partial C \to \mathbf{R}$ and apply Theorem 5.6, with $I = 0$. Let $\tilde{\gamma} \in E^*$ be as in the conclusion of Theorem 5.6. Let $J : C \to \mathbf{R}$ be a lower semicontinuous strictly convex function, Gâteaux differentiable in $\mathrm{int}(C)$, such that $J_{|\partial C} - \varphi$ is constant in ∂C, and let $H : \mathrm{int}(C) \to \mathbf{R}$ be lower semicontinuous, bounded below, and Gâteaux differentiable. Now, consider the function $G : C \to \mathbf{R}$ defined by

$$G(x) = \begin{cases} H(x) & \text{if } x \in \mathrm{int}(C) \\ a & \text{if } x \in \partial C, \end{cases}$$

where $a = \inf_C H$. Clearly, G is lower semicontinuous in C. Consequently, there exists $\varepsilon > 0$ such that, for each $\lambda \in [0,\varepsilon]$, $J'(x) + \lambda G'(x) = \tilde{\gamma}$ has at least one solution in $\mathrm{int}(C)$, and the conclusion holds since $H = G$ in $\mathrm{int}(C)$. □

Finally, we highlight the following consequence of Theorem 5.7.

Theorem 5.8. *Let E be a finite-dimensional real Hilbert space and let $C \subset E$ be a closed ball, centered at* 0. *Then, for every function $\varphi : \partial C \to \mathbf{R}$, there exists $\tilde{\gamma} \in E^*$, having the following property: for every $P \in C^1(C)$ such that P' is Lipschitzian with Lipschitz constant $L \geq 0$, for every $\mu > L$, for every lower semicontinuous convex function $Q : C \to \mathbf{R}$, Gâteaux differentiable in* $\mathrm{int}(C)$, *such that $(P+Q)_{|\partial C} - \varphi$ is constant in ∂C, and for every lower semicontinuous, bounded below and Gâteaux differentiable function $H : \mathrm{int}(C) \to \mathbf{R}$, there exists $\varepsilon > 0$ such that, for each $\lambda \in [0,\varepsilon]$, the equation*

$$\mu x + P'(x) + Q'(x) + \lambda H'(x) = \tilde{\gamma}$$

has at least one solution in $\mathrm{int}(C)$.

Proof. Fix $\varphi : \partial C \to \mathbf{R}$. Let $\tilde{\gamma}$ be as in the conclusion of Theorem 5.7. Observe that, since $\mu > L$, $\frac{\mu}{2}\|\cdot\|^2 + P(\cdot)$ turns out to be strictly convex since E is a Hilbert space. Consequently, $J(\cdot) := \frac{\mu}{2}\|\cdot\|^2 + P(\cdot) + Q(\cdot)$ is lower semicontinuous, strictly convex, Gâteaux differentiable in $int(C)$, and $J_{|\partial C} - \varphi$ is constant in ∂C. Hence, the conclusion of Theorem 5.7 applies with such a function J and we are done. □

Acknowledgments

This work was funded by the European Union - NextGenerationEU Mission 4 - Component 2 - Investment 1.1 under the Italian Ministry of University and Research (MUR) programme "PRIN 2022" - grant number 2022BCFHN2 - Advanced theoretical aspects in PDEs and their applications - CUP: E53D23005650006. The author has also been supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by the Università degli Studi di Catania, PIACERI 2020-2022, Linea di intervento 2, Progetto "MAFANE".

REFERENCES

[1] M. Ait Mansour, J. Lahrache, N. Ziane, Some applications of two minimax theorems, Matematiche (Catania) 78 (2023), 405-414.
[2] V. Ambrosio, F. Essebei, Multiple solutions for double phase problems in $\mathbf{R}^n$ via Ricceri's principle, J. Math. Anal. Appl. 528 (2023), 127513.
[3] H. Brezis, J. Mawhin, Periodic solutions of Lagrangian systems of relativistic oscillators, Commun. Appl. Anal. 15 (2011), 235-250.
[4] K. Fan, Minimax theorems, Proc. Nat. Acad. Sci. U.S.A. 39 (1953), 42-47.
[5] D. Giandinoto, Further applications of two minimax theorems, Matematiche (Catania) 77 (2022), 449-463.
[6] O. Naselli, On the solution set of an equation of the type $f(t,(u)(t)) = 0$, Set-Valued Anal. 4 (1996), 399-405.
[7] J. V. Neumann, Zur Theorie der Gesellschaftsspiele, Math. Ann. 100 (1928), 295-320.
[8] P. Pucci, J. Serrin, A mountain pass theorem, J. Differential Equations, 60 (1985), 142-149.
[9] B. Ricceri, Some topological mini-max theorems via an alternative principle for multifunctions, Arch. Math. (Basel) 60 (1993), 367-377.
[10] B. Ricceri, A variational property of integral functionals on L^p-spaces of vector-valued functions, C. R. Acad. Sci. Paris, Série I, 318 (1994), 337-342.
[11] B. Ricceri, On the integrable selections of certain multifunctions, Set-Valued Anal. 4 (1996), 91-99.
[12] B. Ricceri, More on a variational property of integral functionals, J. Optim. Theory Appl. 94 (1997), 757-763.
[13] B. Ricceri, Further considerations on a variational property of integral functionals, J. Optim. Theory Appl. 106 (2000), 677-681.
[14] B. Ricceri, On a three critical points theorem, Arch. Math. (Basel), 75 (2000), 220-226.
[15] B. Ricceri, Recent advances in minimax theory and applications, In: Pareto Optimality, Game Theory and Equilibria, A. Chinchuluun, P.M. Pardalos, A. Migdalas, L. Pitsoulis (eds.), pp. 23-52, Springer, 2008.
[16] B. Ricceri, A three critical points theorem revisited, Nonlinear Anal. 70 (2009), 3084-3089.
[17] B. Ricceri, A further three critical points theorem, Nonlinear Anal. 71 (2009), 4151-4157.
[18] B. Ricceri, Nonlinear eigenvalue problems, In: Handbook of Nonconvex Analysis and Applications, D. Y. Gao and D. Motreanu (eds.), pp. 543-595, International Press, 2010.
[19] B. Ricceri, A further refinement of a three critical points theorem, Nonlinear Anal. 74 (2011), 7446-7454.
[20] B. Ricceri, A strict minimax inequality criterion and some of its consequences, Positivity, 16 (2012), 455-470.
[21] B. Ricceri, More on the metric projection onto a closed convex set in a Hilbert space, In: Contributions in Mathematics and Engineering - In Honor of Constantin Carathéodory, P. M. Pardalos and Th. M. Rassias (eds.), pp. 529-534, Springer, 2016.

[22] B. Ricceri, On a minimax theorem: an improvement, a new proof and an overview of its applications, Minimax Theory Appl. 2 (2017), 99-152.
[23] B. Ricceri, Minimax theorems in a fully non-convex setting, J. Nonlinear Var. Anal. 3 (2019), 45-52.
[24] B. Ricceri, Multiplicity theorems involving functions with non-convex range, Stud. Univ. Babeş-Bolyai Math. 68 (2023), 125-137.
[25] B. Ricceri, A further multiplicty result for Lagrangian systems of relativistic oscillators, Appl. Anal. Optim. 7 (2023), 135-140.
[26] B. Ricceri, A property of strictly convex functions which differ from each other by a constant on the boundary of their domain, J. Convex Anal. 31 (2024), 779-786.
[27] J. Saint Raymond, Connexité des sous-niveaux des fonctionnelles intégrales, Rend. Circ. Mat. Palermo 44 (1995), 162-168.
[28] M. Sion, On general minimax theorems, Pacific J. Math. 8 (1958), 171-176.