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A NEW PROOF OF THE VON NEUMANN MINIMAX THEOREM

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Abstract. In this paper, we establish a new proof to the Von Neumann minimax Theorem in Euclidean spaces. Our key argument relies on standard arguments of convex optimization as well as elementary Euclidean geometric properties.

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1. Introduction

In this paper, we present a short and simple new proof to the well-known Von Neumann minimax Theorem, which reads as follows.

Theorem 1.1. (Von Neumann, [18, 1928]). *Let A be a linear mapping between Euclidean spaces E and F. Let $C \subset E$ and $K \subset F$ be nonempty compact and convex sets. Then,*

$$\max_{x\in C}\min_{y\in K}\langle Ax, y\rangle = \min_{y\in K}\max_{x\in C}\langle Ax, y\rangle. \tag{1.1}$$

Indeed, (1.1) was proved by Von Neumann [18, 1928] which is qualified since then as a starting point of game theory. Precisely, (1.1) found its original motivation firstly in n-player zero-sum game theory and successively in more complex games, decision theory in the presence of uncertainty, scenario analysis in philosophy, statistics, artificial intelligence, and many other applicable scientific disciplines. Therefore, several generalizations and alternative versions of the Von Neumann's original minimax theorem appeared in the literature such as the Sion's Theorem [17] and its recent developments.

If one puts $f(x,y) = \langle Ax, y\rangle$ or considers a more general function $f : C\times K \longrightarrow \mathbb{R}$, one easily observes that the following inequality always holds

$$\sup_{y\in K}\inf_{x\in C} f(x,y) \leq \inf_{x\in C}\sup_{y\in K} f(x,y).$$

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In this regard, we refer to Rockafellar [16, Lemma 36.1] for an example. Accordingly, the Von Neumann's (1.1) (and/or its extended Sion's format) equality is, in reality, equivalent to the other minimax inequality:

$$\inf_{x\in C}\sup_{y\in K} f(x,y) \leq \sup_{y\in K}\inf_{x\in C} f(x,y). \tag{1.2}$$

If $C = K$ with $E = F$, then it is a simple matter to check that (1.2) implies Ky Fan's minimax result [8, 1972]:

$$\inf_{x\in C} f(x,x) \leq \sup_{y\in C}\inf_{x\in C} f(x,y), \tag{1.3}$$

which was proved along times to be a cornerstone inequality of nonlinear analysis. Indeed, the Ky Fan inequality is intimately related to the existence theory of equilibria in various games studied in economics. Observe that Ky Fan's minimax inequality does not imply the minimax inequality (1.2). For a counterexample, we refer to Galan [6]. To continue our discussion, we remark that whenever $f(x,x) \geq 0$ for all $x \in C$, (1.3) clearly implies, under semicontinuity conditions, the existence of a point $\bar{x} \in C$ such that

$$f(\bar{x},y) \geq 0, \ \forall y \in C. \tag{1.4}$$

(1.4) is exactly the equilibrium problem in the sense of Blum and Oettli [4], which in turns unifies many mathematical models, such as optimization, fixed points problems, variational inequalities, hemi-variational inequalities and so forth. For theories, algorithms, applications, and nonlinear programming techniques for equilibria, we refer the reader to [1, 2, 3, 7, 9, 12] and the references therein. Note that minimax Theorems have been extended to the nonconvex setting in [11, 13, 14, 15]. To come back to the Von Neumann minimax Theorem, we know several that mathematicians devoted a great attention to establish new proofs to this celebrated minimax result; see, e.g., [5] for more information, useful details, and various proof techniques.

In this paper, we present a new constructive scheme to prove Theorem 1.1 without using the previous well-known and sophisticated arguments such as the Brouwer fixed point Theorem, Kakutani Theorem or Hahn Banach Theorem, Riesz's representation Theorem, KKM techniques or else duality theory. Our method here relies directly on standard convex optimization and elementary Euclidean geometric properties. We would like to mention that most of Von Neumann's minmax theorem proofs usually make recourse to strong results of functional analysis with complicated proofs; see, e.g., [5, 10]. Our proof here is based on the simple geometric notion that is the projection of a point on a closed and convex set and Weierstrass theorem beside the achievement of the minimum of a continuous function over a convex and compact set.

2. NOTATION AND PRELIMINARIES

Throughout this paper, E and F are used to denote Euclidean spaces, $C \subset E$ and $K \subset F$ are assumed to be two nonempty, compact, and convex sets. The inner product and its associated norm in E or F ae denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. We consider a linear mapping A between Euclidean spaces E and F and write for every $(x,y) \in E \times F$,

$$f(x,y) = \langle Ax, y \rangle. \tag{2.1}$$

We denote the adjoint operator of A by A^*. The norm of a linear map between E and F in the space of linear maps $L(E,F)$ is denoted again by $\|.\|$. For a point x in E or F and $r>0$, $\overline{B}(x,r)$ (resp. $B(x,r)$) stands for the closed (resp. open) ball centred at x with radius r.

Next, for a later use in our proof of Theorem 1.1, we fix some observations from elementary optimization and Euclidean geometric properties as follows.

Lemma 2.1. *Let $C\subset E$ and $K\subset F$ be nonempty, compact, and convex sets. Then the following assertions hold:*

a) For any vector $u\in E$, there exists $\bar{x}\in C$ such that

$$\langle u, x-\bar{x}\rangle \geq 0, \forall x\in C; \tag{2.2}$$

b) For any vector $v\in F$, there exists $\bar{z}\in K$ such that

$$\langle v, y-\bar{z}\rangle \geq 0, \forall y\in K. \tag{2.3}$$

Proof. For a), let $u\in E$. Let $x^*\in C$ be a minimum point to the continuous function $x\longmapsto\langle u,x\rangle$ over the convex and compact set C. Clearly, for all $x\in C$, we have $\langle u,x\rangle\geq\langle u,x^*\rangle$, which implies (2.2). The assertion b) is similar. This ends the proof. □

Remark 2.1. Observe that the elementary variational inequality in (2.2) is equivalent to the minimization problem of $u\longmapsto\langle u,x\rangle$ over $x\in C$. Besides, a point $\bar{x}\in C$ is a solution to (2.2) if and only if $-u\in N_C(\bar{x})$, where $N_C(\bar{x})$ is the normal cone to the compact convex subset C at $\bar{x}$ given by $N_C(\bar{x})=\{v\in E:\langle v,x-\bar{x}\rangle\leq 0,\ \forall x\in C\}$.

Remark 2.2. As a consequence of Lemma 2.1, for the linear operator A introduced above between Euclidean spaces E and F, and for the already fixed nonempty compact and convex sets $C\subset E$ and $K\subset F$, the following assertions hold:

c) For any vector $y\in F$, there exists $\bar{x}\in C$ such that

$$\langle A\bar{x}-Ax, y\rangle \geq 0, \forall x\in C; \tag{2.4}$$

d) For any vector $x\in E$, there exists $\bar{z}\in K$ such that

$$\langle Ax, y-\bar{z}\rangle \geq 0, \forall y\in K. \tag{2.5}$$

Indeed, for c), for a given $y\in F$, it suffices to take $u=-A^*y$ in (2.2). For d), for a given $x\in E$, we take $v=Ax$ in (2.3).

Now, for all $y\in K$, put $\varphi(y)=\max\limits_{x\in C}f(x,y)$ and for all $x\in C$, we set $\psi(x)=\min\limits_{y\in K}f(x,y)$. Then, with notation (2.1), we clearly see that

$$\psi(x)\leq f(x,y)\leq\varphi(y),\ \forall x\in C, \forall y\in K. \tag{2.6}$$

The following Lemma provides the existence of remarkable points in C and K that provide more information than (2.6).

Lemma 2.2. *Let A be a linear mapping between E and F. Let $C\subset E$ and $K\subset F$ be nonempty, compact, and convex sets. Then, there exist $(\bar{x},\bar{y},\bar{z})\in C\times K\times K$ such that*

$$f(\bar{x},\bar{y})=\varphi(\bar{y}) \quad \textit{and} \quad f(\bar{x},\bar{z})=\psi(\bar{x}). \tag{2.7}$$

Proof. The equalities in (2.7) dispose actually at multiple solutions. Indeed, the conclusion in (2.7) can be obtained by standard arguments of variational inequalities or fixed point theorems or else by basic equilibrium theory as in Blum and Oettli [4] or else in Göpfert, Riahi, Tammer, and Zălinescu [7]. Here, we adopt a simplest way from classic convex optimization Rockafellar [16]. Take an arbitrary $\bar{y} \in K$. From assertion c) of Remark 2.2, we see that there exists $\bar{x} \in C$ verifying (2.4), which means $f(\bar{x},\bar{y}) \geq f(x,\bar{y})$ for all $x \in C$. This implies that

$$f(\bar{x},\bar{y}) = \max_{x\in C} f(x,\bar{y}) = \varphi(\bar{y}),$$

which proves the first part of (2.7). Now, from the assertion d) of the Remark 2.2), with $x = \bar{x}$, we see that there exists $\bar{z} \in K$ satisfying (2.5), which is equivalently expressed as $f(\bar{x},\bar{z}) \leq f(\bar{x},y)$ for all $y \in K$. Then, $f(\bar{x},\bar{z}) = \min_{y\in K} f(\bar{x},y) = \psi(\bar{x})$. This ends the proof. □

We need equally to fix two real numbers $R, R' > 0$ such that

$$C \subset B(0,R) \quad \text{and} \quad K \subset B(0,R').$$

Notation: In what follows, we adopt the following notation for any real-valued bifunction Ψ over a product of compact sets, say $C \times K$. We write

$$\alpha(\Psi) := \max_{x\in C} \min_{y\in K} \Psi(x,y) \quad \text{and} \quad \beta(\Psi) := \min_{y\in K} \max_{x\in C} \Psi(x,y).$$

3. The Proof to Theorem 1.1

Now we are in a position to present our proof for Theorem 1.1 as follows.

Proof. Since the inequality $\alpha(f) \leq \beta(f)$ is always true (as well-known in minimax theory), it suffices to prove that $\beta(f) \leq \alpha(f)$. Let $\bar{y} \in K$. By Lemma 2.2, we see that there exist $(\bar{x},\bar{y},\bar{z}) \in C \times K \times K$ such that

$$f(\bar{x},\bar{y}) = \varphi(\bar{y}) = \max_{x\in C} f(x,\bar{y}) \quad \text{and} \quad f(\bar{x},\bar{z}) = \psi(\bar{x}) = \min_{y\in K} f(\bar{x},y). \tag{3.1}$$

Without loss of Generality, we may assume that $A\bar{x} \neq 0$, otherwise from (3.2) it immediately follows that

$$\beta(f) \leq f(\bar{x},\bar{y}) = \langle A\bar{x},\bar{y}\rangle = 0 = \langle A\bar{x},\bar{z}\rangle = f(\bar{x},\bar{z}) \leq \alpha(f).$$

This ensures that $\beta(f) \leq \alpha(f)$, which ends the proof of the Theorem.

Now, from (3.1), we see that

$$f(\bar{x},\bar{y}) = \varphi(\bar{y}) \geq \min_{y\in K} \varphi(y) = \min_{y\in K} \max_{x\in C} f(x,y) = \beta(f)$$

and

$$f(\bar{x},\bar{z}) = \psi(\bar{x}) \leq \max_{x\in C} \psi(x) = \max_{x\in C} \min_{y\in K} f(x,y) = \alpha(f).$$

Thus

$$\beta(f) \leq f(\bar{x},\bar{y}) \quad \text{and} \quad f(\bar{x},\bar{z}) \leq \alpha(f). \tag{3.2}$$

We need to consider two sequences $(r_n)_n$ and $(\varepsilon_n)_n$ of real numbers decreasing to 0 (i.e., $r_n \searrow 0$ and $\varepsilon_n \searrow 0$) such that $\varepsilon_n < r_n$ and define for $n > 1$ a set $K_n := \overline{K \cap B(\bar{y},r_n)}$. Let $n > 1$. By the continuity of f in y, we see that the function $y \longmapsto f(\bar{x},y)$ attains its minimum over $y \in K_n$ at a point $\tilde{z}_n \in K_n$. Then, for all $y \in K_n$, we have

$$f(\bar{x},\tilde{z}_n) \leq f(\bar{x},y). \tag{3.3}$$

Since $\tilde{z}_n \in K_n = \overline{K \cap B(\bar{y}, r_n)}$, $\tilde{z}_n$ is a limit of a sequence $(u_p^n)_p$ such that

$$u_p^n \in K \cap B(\bar{y}, r_n), \ \forall p \in \mathbb{N}, \tag{3.4}$$

for p large enough, $u_p^n \in B(\tilde{z}_n, r_n - \varepsilon_n)$, so $K \cap B(\bar{y}, r_n) \cap B(\tilde{z}_n, r_n - \varepsilon_n) \neq \emptyset$. Moreover, one has

$$K \cap B(\bar{y}, r_n) \cap B(\tilde{z}_n, r_n - \varepsilon_n) \subset K_n. \tag{3.5}$$

In what follows, we use the notation $\tilde{V} = B(\bar{y}, r_n) \cap B(\tilde{z}_n, r_n - \varepsilon_n)$, which is a neighborhood of u_n^p for any p large enough. We claim that the element $\tilde{z}_n$ of K_n is also a global minimum of $f(\bar{x}, .)$ over K. Let us prove this claim. To this end, we fix an arbitrary element $y \in K$ such that

$$y \in K \cap \tilde{V}. \tag{3.6}$$

We further consider y^* defined by

$$y^* := y - A\bar{x}. \tag{3.7}$$

<u>If</u> $y^* \in K_n$**,** then, for p large enough, u_p^n is a local minimum of $f(\bar{x}, .)$. Precisely, for p large enough, we assert that

$$f(\bar{x}, y) \geq f(\bar{x}, u_p^n). \tag{3.8}$$

Indeed, from (3.3), $f(\bar{x}, y^*) \geq f(\bar{x}, \tilde{z}_n)$, which means that

$$f(\bar{x}, y) \quad \geq \quad f(\bar{x}, A\bar{x}) + f(\bar{x}, \tilde{z}_n) = \|A\bar{x}\|^2 + f(\bar{x}, \tilde{z}_n). \tag{3.9}$$

In view of the assumption, $A\bar{x} \neq 0$, we have $\eta = \|A\bar{x}\|^2 > 0$. As $\lim_{p \to +\infty} u_p^n = \tilde{z}_n$, by the continuity of f in y, for p large enough, it results that $f(\bar{x}, \tilde{z}_n) + \eta \geq f(\bar{x}, u_p^n)$. This combined with (3.9) leads to

$$f(\bar{x}, y) \geq f(\bar{x}, u_p^n). \tag{3.10}$$

Now, for a given p large enough, (3.10) is true for the arbitrary element $y \in K \cap \tilde{V}$ which is a neighborhood for u_n^p. Then, u_p^n is a local minimum of $f(\bar{x}, .)$ over K, which proves the assertion in (3.8). Thanks to the convexity of f in y, we have hat u_p^n is a global minimum of $f(\bar{x}, .)$ over K. Hence, $f(\bar{x}, \tilde{z}_n) \geq f(\bar{x}, u_p^n)$. But at the same time we have $u_p^n \in K_n$ (see (3.4)), so it follows from (3.3) that $f(\bar{x}, u_p^n) \geq f(\bar{x}, \tilde{z}_n)$, which conducts to $f(\bar{x}, \tilde{z}_n) = f(\bar{x}, u_p^n)$. This means that $\tilde{z}_n$ is in turn a global minimum of $f(\bar{x}, .)$ over K, which completes the proof of the claim under the assumption $y^* \in K_n$. In the sequel, we have to suppose that $\mathbf{y}^* \notin \mathbf{K_n}$. Set $w_n = P_{K_n}(y^*)$, where P_{K_n} denotes the metric projection onto K_n. Since K_n is convex, we are able to involve the characterization of $P_{K_n}(y^*)$ to see that $\langle v - w_n, y^* - w_n \rangle \leq 0$ for all $v \in K_n$. Then, for all v in K_n, we see that

$$\langle v - w_n, y^* - w_n \rangle = \langle v - w_n, y^* \rangle - \langle v - w_n, w_n \rangle = \langle y^*, v - w_n \rangle - \langle v, w_n \rangle + \|w_n\|^2 \leq 0.$$

This implies $\langle y^*, w_n - v \rangle \geq \|w_n\|^2 - \langle v, w_n \rangle$ for all $v \in K_n$, which, together with the definition of y^* in (3.7), gives us $\langle y - A\bar{x}, w_n - v \rangle \geq \|w_n\|^2 - \langle v, w_n \rangle$ for all $v \in K_n$. This means that

$$\langle y, w_n - v \rangle - \langle A\bar{x}, w_n - v \rangle \geq \|w_n\|^2 - \langle v, w_n \rangle, \ \forall v \in K_n.$$

Then, for all $v \in K_n$,

$$-\langle A\bar{x}, w_n - v \rangle = f(\bar{x}, v) - f(\bar{x}, w_n) \geq \|w_n\|^2 - \langle v, w_n \rangle - \langle y, w_n - v \rangle,$$

which shows that

$$f(\bar{x},v) \geq f(\bar{x},w_n) + \|w_n\|^2 - \langle v,w_n\rangle - \langle y,w_n - v\rangle, \ \forall v \in K_n. \tag{3.11}$$

In particular, for any $y \in K \cap \tilde{V}$ with $v = y$ in (3.11) (note that $y \in K_n$ due to (3.5) and (3.6)), we derive

$$f(\bar{x},y) \geq f(\bar{x},w_n) + \|w_n\|^2 - 2\langle y,w_n\rangle + \|y\|^2 = f(\bar{x},w_n) + \|w_n - y\|^2.$$

Hence, $f(\bar{x},y) \geq f(\bar{x},w_n) + \|w_n - y\|^2$. Given that $w_n \in K_n$, by taking into account (3.3), we see that

$$f(\bar{x},y) \ \ \geq \ \ f(\bar{x},w_n) + \|w_n - y\|^2 \geq f(\bar{x},\tilde{z}_n) + \|w_n - y\|^2. \tag{3.12}$$

At this stage, we discuss two cases:
Case 1: $w_n \neq y$, i.e., $\|w_n - y\| > 0$. By the same argument used in the proof of (3.8) above, with (3.12), for p large enough, we obtain that u_p^n is a local minimum of $f(\bar{x},.)$. Then, it is a global minimum of $f(\bar{x},.)$ (thanks to convexity of $f(\bar{x},.)$), and so is $\tilde{z}_n$. This is exactly the required conclusion for this case.
Case 2: $w_n = y$. Using (3.11) and taking into account the assumption of the case ($y = w_n$), we deduce that, for all $v \in K_n$,

$$f(\bar{x},v) \geq f(\bar{x},y), \ \forall v \in K_n. \tag{3.13}$$

We see that, in view of (3.5), the inequality in (3.13) is true for all $v \in K \cap \tilde{V}$. Then, as y is taken in $K \cap \tilde{V}$, which is of course a neighborhood of y relatively to K, it follows that y is a local minimum of $f(\bar{x},.)$, so as in Case 1. The convexity of $f(\bar{x},.)$ permits y to be a global minimum of $f(\bar{x},.)$ over K. In addition, with $v = \tilde{z}_n$ in (3.13), we deduce that $f(\bar{x},\tilde{z}_n) \geq f(\bar{x},y)$, but $y = w_n \in K_n$ and $\tilde{z}_n$ minimizes $f(\bar{x},.)$ over K_n. Thus

$$f(\bar{x},\tilde{z}_n) = f(\bar{x},y). \tag{3.14}$$

Therefore, from (3.14), we have that $\tilde{z}_n$ is a global minimum of $f(\bar{x},.)$ too. This ends the proof of the claim. Since $\bar{z}$ is another global minimum of $f(\bar{x},.)$ over K (see (3.1)), we conclude that

$$f(\bar{x},\tilde{z}_n) = f(\bar{x},\bar{z}). \tag{3.15}$$

Note that the case that $\tilde{z}_n$ is constant, that is, $\tilde{z}_n = a$ for n large enough. By the convexity of f in y, a is a global minimum of $f(\bar{x},.)$ over K, which implies that $f(\bar{x},a) = f(\bar{x},\bar{z})$.

Finally, we are in a position to draw our desired conclusion. By using (3.2) and (3.15), we have

$$\begin{aligned}
\beta(f) &\leq f(\bar{x},\bar{z}) + f(\bar{x},\bar{y}) - f(\bar{x},\bar{z}) \\
&= f(\bar{x},\bar{z}) + f(\bar{x},\bar{y}) - f(\bar{x},\tilde{z}_n) \\
&\leq \alpha(f) + f(\bar{x},\bar{y} - \tilde{z}_n) \\
&\leq \alpha(f) + \|A\bar{x}\|\|\bar{y} - \tilde{z}_n\| \\
&\leq \alpha(f) + \|A\|\|\bar{x}\|\|\bar{y} - \tilde{z}_n\| \\
&\leq \alpha(f) + Rr_n\|A\|,
\end{aligned}$$

which is of course satisfied for all $n > 1$. Consequently, as n goes to 0, we immediately obtain $\beta(f) \leq \alpha(f)$. As $\alpha(f) \leq \beta(f)$ is always true as mentioned above, we deduce that $\beta(f) = \alpha(f)$. This completes the proof. □

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