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ON RANDOM HEMIVARIATIONAL INEQUALITIES: SOME SOLVABILITY AND STABILITY RESULTS

JOACHIM GWINNER

Department of Aerospace Engineering, Institute of Applied Mathematics, Universität der Bundeswehr München, 85577 Neubiberg/München, Germany

Dedicated to Prof. H. Walk on the occasion of his 85th birthday

Abstract. This paper is concerned with various classes of parameter dependent hemivariational inequalities. First we study mixed random hemivariational inequalities and give solvability results in the Bochner-Lebesgue space $L^\infty(\Omega,\mu,H)$, built on a finite measure space (Ω,μ) and a real separable Hilbert space H. Next, we specialize Ω to a finite interval with Lebesgue measure and prove solvability and regularity results for extended real-valued time-dependent hemivariational inequalities. Then we focus on a probability space (Ω,P) and derive a stability result for mixed random hemivariational inequalities from a recent fundamental stability theorem. As an application, we investigate a nonsmooth boundary value problem with unilateral, friction-like, and nonmonotone boundary conditions under uncertainty and present a concrete stability result.
Keywords. Carathéodory operator; Extended real-valued equilibrium problem; Mixed hemivariational inequality; Monotone bifunction; Nonsmooth boundary value problem; Uncertainty.
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1. Introduction

The purpose of this paper is two-fold. Firstly, we build on the recent abstract existence theory in [1] and extend the measurability and solvability results in [2], see also [3, sect. 6.2], from random monotone variational inequalities to novel mixed random hemivariational inequalities of the following type (pathwisely formulated): For each $\omega\in\Omega$, find $x^*_\omega\in K$ such that

$$\langle\Phi(\omega,x^*_\omega),x-x^*_\omega\rangle+J^0(\gamma x^*_\omega;\gamma x-\gamma x^*_\omega)+\varphi(x)-\varphi(x^*_\omega)\geq\lambda(\omega,x-x^*_\omega)\quad\forall x\in K,$$

where $(\Omega,\mathscr{A},\mu)$ is a complete σ-finite measure space and $(H,\langle\cdot,\cdot\rangle,\|\cdot\|)$ is a real separable Hilbert space identified with the dual space H^*.

Next, we have a Carathéodory operator $\Phi:\Omega\times H\to H$, i.e., for each fixed $x\in H$, $\Phi(\cdot,x)$ is measurable with respect to $\mathscr{A}$ and to the Borel algebra $\mathscr{B}(H)$, and for every $\omega\in\Omega$, $\Phi(\omega,\cdot)$ is continuous. Moreover, for each $\omega\in\Omega$, $\Phi(\omega,\cdot)$ is a monotone operator on H, i.e., $\langle\Phi(\omega,x)-\Phi(\omega,x'),x-x'\rangle\geq 0$ for all $x,x'\in H$. Further, the right hand side $\lambda:\Omega\times H\to\mathbb{R}$ is a Carathéodory function such that, for every $\omega\in\Omega$, $\lambda(\omega,\cdot)$ is linear and continuous, also - by abuse of notation - $\lambda:\omega\in\Omega\mapsto\lambda(\omega)\in H$ is in $L^\infty(\Omega;H)$. For simplicity, we content ourselves with a

E-mail address: joachim.gwinner@unibw-muenchen.de
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fixed closed, convex, and nonempty subset $K \subset H$ in contrast to [2], where the constraint set is random, too. In addition, we have the deterministic data: $\varphi : H \to \mathbb{R}$ is a convex continuous function, $\gamma := \gamma_{H\to Z}$ denotes a linear continuous operator into a real Banach space Z, and J is a real-valued locally Lipschitz functional defined on Z, giving rise to the Clarke generalized directional derivative J^0 as made precise below.

Specializing Ω to a time interval, we obtain as a byproduct of the abstract existence theory in [1] an existence result, complemented by a regularity result for time-dependent extended real-valued hemivariational inequalities that extend an existence-regularity result for time-dependent extended real-valued variational inequalities in [4].

Secondly, we derive a novel well-posedness result for the considered mixed random hemivariational inequalities from a recent fundamental stability theorem in [5], where we handle perturbations not only with respect to the right hand side given by linear random forms, but also with respect to convex functions by using the concept of Mosco convergence.

As an application we treat a nonsmooth boundary value problem with unilateral, friction-like, and nonmonotone boundary conditions under uncertainty and present a concrete stability result, where we give explicit conditions on the given functions and random variables that are involved in the perturbed convex function and in the perturbed linear random form.

The theory of hemivariational inequalities (HVIs) was introduced and has been investigated since 1980s by Panagiotopoulos [6], as a generalization of variational inequalities (VIs) with the aim to model many problems coming from mechanics when the energy functionals are nonconvex, but locally Lipschitz, so the Clarke generalized differentiation calculus [7] can be used; see, e.g., [8, 9, 10]. For more recent monographs on HVIs with application to contact problems, we refer to [11, 12].

Let us discuss the present paper in comparison with the recent papers [5, 13, 14, 15]. The fundamental stability theorem for extended real-valued HVIs established in [5, 13] can be applied in different directions to various partial differential equation (PDE) problems and variational problems. [13] investigates a nonlinear scalar interface problem on an unbounded domain with nonmonotone set-valued transmission conditions involving a nonlinear monotone PDE in the interior domain and the Laplacian in the exterior domain. Using boundary integral methods from singular operator theory, this interface problem can be reduced to a HVI on the coupling boundary. Based on the above-mentioned stability theorem various stability results for the interface problem, as well as stability of a related bilateral obstacle interface problem with respect to the obstacles are obtained. [5] goes a step further and employs the above-mentioned stability theorem to arrive at the existence of optimal controls for four kinds of optimal control problems: Distributed control on the bounded domain, boundary control, simultaneous boundary-distributed control governed by the interface problem, as well as control of the obstacle driven by a related bilateral obstacle interface problem. While the papers [5, 13] are concerned with deterministic problems, [14] investigates mixed random VIs in the setting of a separable Hilbert space and a complete σ -finite measure space and, based on a stability result for linear extended real-valued VIs, derives a well-posed result for such random VIs. The survey paper [15] presents both stability results for linear extended real-valued VIs and for extended real-valued HVIs, respectively, and provides applications of these stability results to various variational problems, namely for a class of random mixed variational inequalities, and for a scalar bilateral obstacle problem with unilateral and nonmonotone boundary conditions, and moreover discusses the stability in a frictionless unilateral contact problems with locking material in linear elasticity with respect to the locking constraint. Thus a main novelty of the present paper lies in a well-posedness result for a new class of random HVIs, besides as a byproduct, a new existence-regularity result for time-dependent extended real-valued HVIs.

The outline of the paper is as follows. The subsequent section, Section 2, first collects the basic notions of Clarke's generalized differential calculus, and then recalls the fundamental concept of Mosco convergence for extended real-valued convex lower semicontinuous proper functions in the framework of a real Banach space. In addition, a preliminary result is provided for handling the Mosco convergence of the sum of two convex lower semicontinuous proper functions. Here we also give reference to the general result on the sum of maximally monotone operators in [16]. Then we present the above-mentioned stability result (Theorem 3) from [5] for extended real-valued hemivariational inequalities; see Theorem 2.2. In Section 3, we study the above mixed random hemivariational inequality (random HVI). First we are concerned with solvability of the random HVI in its pathwise formulation in a general complete σ-finite measure space. In particular, we show that, under specific assumptions on the data, the unique solution $x^* : \omega \in \Omega \mapsto x^*(\omega) \in K$ lies in an appropriate Bochner-Lebesgue space [17, section 4.2]; see Theorem 3.2. Next, in Section 4, as a byproduct of the abstract existence theory in [1], we provide a solvability result, complemented by a regularity result for extended real-valued time-dependent HVIs; see Theorem 4.1. Then we focus to a probability space $(\Omega, \mathbb{A}, P)$, and derive from Theorem 2.2 a stability result for random HVIs with respect to perturbations in the random right hand side and in the convex continuous function with respect to Mosco convergence; see Corollary 5.1. The next section, Section 6, applies the above general stability theory to a nonsmooth boundary value problem with unilateral, friction-like, and nonmonotone boundary conditions under uncertainty and presents a concrete stability result with explicit conditions on the involved functions and random variables; see Theorem 6.1. The final section, Section 7, gives some concluding remarks and sketches some directions of further research.

2. Some Preliminaries

2.1. **Some preliminaries from Clarke's generalized differential calculus.** From Clarke's generalized differential calculus [7], we need the concept of the *generalized directional derivative* of a locally Lipschitz function $\phi : X \to \mathbb{R}$ on a real Banach space X at $x \in X$ in the direction $z \in X$ defined by

$$\phi^0(x;z) := \limsup_{y \to x; t \downarrow 0} \frac{\phi(y+tz) - \phi(y)}{t}.$$

Note that the function $z \in X \mapsto \phi^0(x;z)$ is finite, sublinear, and hence convex and continuous; further, the function $(x,z) \mapsto \phi^0(x;z)$ is upper semicontinuous. The *generalized gradient* of the function ϕ at x, denoted by (simply) $\partial\phi(x)$, is the unique nonempty weak* compact convex subset of the dual space X', whose support function is $\phi^0(x;.)$. Thus

$$\xi \in \partial\phi(x) \Leftrightarrow \phi^0(x;z) \geq \langle \xi, z \rangle, \forall z \in X,$$
$$\phi^0(x;z) = \max\{\langle \xi, z \rangle \; : \; \xi \in \partial\phi(x)\}, \forall z \in X.$$

In the case $X = \mathbb{R}^n$, according to Rademacher's theorem, ϕ is differentiable almost everywhere, and the generalized gradient of ϕ at a point $x \in \mathbb{R}^n$ can be characterized by

$$\partial\phi(x) = \text{co}\{\xi \in \mathbb{R}^n : \xi = \lim_{k\to\infty} \nabla\phi(x_k),\ x_k \to x,\ \phi \text{ is differentiable at } x_k\},$$

where "co" denotes the convex hull.

2.2. **The concept of Mosco convergence.** In this article, we use the concept of epi-convergence of extended real-valued convex lower semicontinuous proper functions in the sense of Mosco [18, 19] ("Mosco convergence") in the framework of a real Banach space V. Note that such a convex lower semicontinuous function $F : V \to \mathbb{R} \cup \{+\infty\}$ is said to be proper iff $F \not\equiv \infty$ on V. This means that the effective domain of F in the sense of convex analysis ([20]),

$$\text{dom}\, F := \{v \in V : F(v) < +\infty\}$$

is nonempty and convex.

Definition 2.1. Let $F_n\ (n \in \mathbb{N}), F : V \to \mathbb{R} \cup \{+\infty\}$ be convex lower semicontinuous proper functions. Then F_n is said to be *Mosco convergent* to F, written $F_n \xrightarrow{\text{M}} F$, if and only if the subsequent two hypotheses hold:

(M1) If $v_n \in V\ (n \in \mathbb{N})$ weakly converges to v ($v_n \rightharpoonup v$) for $n \to \infty$, then $F(v) \leq \liminf_{n\to\infty} F_n(v_n)$.
(M2) For any $v \in V$ with $F(v) < \infty$, there exist $v_n \in V\ (n \in \mathbb{N})$ strongly converging to v for $n \to \infty$ such that $F(v) = \lim_{n\to\infty} F_n(v_n)$.

For later use we need the following result on Mosco convergence of the sum of two convex lower semicontinuous proper functions.

Lemma 2.1. *Let $F_{i,n}\ (n \in \mathbb{N}), F_i : V \to \mathbb{R} \cup \{+\infty\}\ (i = 1,2)$ be convex lower semicontinuous proper functions. Suppose that for $n \to \infty$, $F_{1,n} \xrightarrow{\text{M}} F_1$; $(F_{2,n}; F_2)$ satisfies* (M1) *and there holds*

(C) *If $F_2(w) < \infty$ and $w_n \in V\ (n \in \mathbb{N})$ strongly converges to w for $n \to \infty$, then $F_2(w) = \lim_{n\to\infty} F_{2,n}(w_n)$.*

Then $F_n := F_{1,n} + F_{2,n} \xrightarrow{\text{M}} F := F_1 + F_2$.

Proof. To show (M1) for $(F_n; F)$, let $v_n \rightharpoonup v$ (weak convergence) in V. By (M1) for $(F_{1,n}; F_1)$ and $(F_{2,n}; F_2)$, one has

$$\begin{aligned} F(v) = F_1(v) + F_2(v) &\leq \liminf_{n\to\infty} F_{1,n}(v_n) + \liminf_{n\to\infty} F_{2,n}(v_n) \\ &\leq \liminf_{n\to\infty} [F_{1,n}(v_n) + F_{2,n}(v_n)] \\ &= \liminf_{n\to\infty} F_n(v_n)\,. \end{aligned}$$

To show (M2) for $(F_n; F)$, let $v \in V$ with $F(v) < \infty$. Then $F_i(v) < \infty; i = 1,2$. Since $(F_{1,n}; F_1)$ satisfies (M2), one has that there exist $v_n \in V$ such that $v_n \to v$ (strong convergence) and $F_1(v) = \lim_{n\to\infty} F_{1,n}(v_n)$. By (C), $F_2(v) = \lim_{n\to\infty} F_{2,n}(v_n)$ and the conclusion follows. □

In virtue of [16, Theorems 6,7], if $F_{i,n} \xrightarrow{\text{M}} F_i$ holds for $i = 1$ and $i = 2$ both, then one can conclude the Mosco convergence of the sum under an extra condition related to the Brézis-Crandall-Pazy condition.

Needless to say, for linear functionals $\lambda, \lambda_n\ (n \in \mathbb{N})$ in dual V^*, Mosco convergence of $F_n := \lambda_n \xrightarrow{\text{M}} F := \lambda$ follows from strong convergence $\lambda_n \to \lambda$ for $n \to \infty$.

2.3. **Well-posedness of extended real-valued hemivariational inequalities.** In this subsection, we deal with well-posedness for extended real-valued HVIs of the following type: Find $\hat{v} \in \mathrm{dom}\, F$ such that

$$\mathsf{A}(\hat{v})(v-\hat{v}) + J^0(\gamma\hat{v}; \gamma v - \gamma\hat{v}) + F(v) - F(\hat{v}) \geq 0 \quad \forall v \in V\,. \tag{2.1}$$

Here V is a real reflexive Banach space. Next the nonlinear monotone continuous operator $\mathsf{A} : V \to V^*$ is strongly monotone with some monotonicity constant $c_{\mathsf{A}} > 0$, that is, $\langle \mathsf{A}(v) - \mathsf{A}(v'), v - v' \rangle_{V^*\times V} \geq c_{\mathsf{A}} \|v - v'\|_V^2$ for all $v, v' \in V$. Further, $\gamma := \gamma_{V\to X}$ denotes a linear continuous operator into a real Banach space X, J^0 stands for the generalized directional derivative of a real-valued locally Lipschitz functional J defined on X, and in addition, $F : V \to \mathbb{R}\cup\{+\infty\}$ is a convex lower semicontinuous proper function.

Further, similar to [12], we suppose the one-sided Lipschitz condition for the generalized directional derivative J^0: There exists $c_J > 0$ such that

$$J^0(y_1; y_2 - y_1) + J^0(y_2; y_1 - y_2) \leq c_J \|y_1 - y_2\|_X^2 \quad \forall y_1, y_2 \in X\,, \tag{2.2}$$

and in addition the smallness condition

$$c_J \|\gamma\|^2_{V\to X} < c_{\mathsf{A}}\,. \tag{2.3}$$

Next, we define the bifunction

$$\Phi(v, w) := \mathsf{A}(v)(w - v) + J^0(\gamma v; \gamma w - \gamma v).$$

Thus, under assumptions (2.2) and (2.3), by [5, Proposition 1], Φ is strongly monotone and HVI (2.1) falls into the framework of an *extended real-valued equilibrium problem of monotone type* in the sense of [1]. Since the convex proper lower semicontinuous function F is conically minorized, that is, it enjoys the estimate $F(v) \geq -c_F(1 + \|v\|)$, $v \in V$ with some $c_F > 0$, strong monotonicity implies the asymptotic coercivity condition in [1], too. Thus the existence result [1, Theorem 5.2] applies to the HVI (2.1) to conclude the following result.

Theorem 2.1. *Suppose (2.2) and (2.3). Then the HVI (2.1) is uniquely solvable.*

By this solvability result, we can introduce the solution map $\mathscr{S}$ by $\mathscr{S}(F) := \hat{v}$, the solution of (2.1). Next, we deal with the stability of the solution map $\mathscr{S}$ with respect to the extendend real-valued function F. In view of our later applications, it is not hard to require that F_n are uniformly conically minorized, that is, there holds the estimate

$$F_n(v) \geq -d_0(1 + \|v\|),\ \forall n \in \mathbb{N}, v \in V \tag{2.4}$$

with some $d_0 \geq 0$. Moreover, similar to [12], in addition to the one-sided Lipschitz continuity (2.2), we assume that the locally Lipschitz function J satisfies the following growth condition

$$\|\zeta\|_{X^*} \leq d_J(1 + \|z\|_X), \quad \forall z \in X,\ \zeta \in \partial J(z) \tag{2.5}$$

for some $d_J > 0$, what - as seen later - for an integral functional J immediately follows from the growth condition for the associated integrand function

Now we are in the position to state the following general stability theorem; see [5, Theorem 3].

Theorem 2.2. *Suppose that the operator* A *is continuous and strongly monotone with monotonicity constant* $c_{\mathsf{A}} > 0$*, the linear operator* γ *is compact, and the generalized directional derivative* J^0 *satisfies the one-sided Lipschitz condition (2.2) and the growth condition (2.5). Moreover, suppose the smallness condition (2.3) holds. Let* $F, F_n : V \to \mathbb{R} \cup \{+\infty\}$ $(n \in \mathbb{N})$ *be convex lower semicontinuous proper functions that satisfy lower estimate (2.4). Suppose* $F_n \xrightarrow{\mathrm{M}} F$*. Then strong convergence* $\mathscr{S}(F_n) \to \mathscr{S}(F)$ *holds.*

3. Random Hemivariational Inequalities - Measurability and Solvability

In this section, we study measurability and solvability of the class of random hemivariational inequalities described in the pathwise formulation. For convenience, we recall the general framework. Let $(\Omega, \mathscr{A}, \mu)$ be a complete σ-finite measure space and $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ a separable Hilbert space. Let K be a closed, convex, and nonempty subset of H. Further we have a Carathéodory operator $\Phi : \Omega \times H \to H$, i.e., for each fixed $x \in H$, $\Phi(\cdot, x)$ is measurable with respect to $\mathscr{A}$ and to the Borel algebra $\mathscr{B}(H)$, and for every $\omega \in \Omega$, $\Phi(\omega, \cdot)$ is continuous. Moreover, for each $\omega \in \Omega$, $\Phi(\omega,\cdot)$ is a monotone operator on H, i.e., $\langle \Phi(\omega,x) - \Phi(\omega,x'), x - x'\rangle \geq 0$ for all $x, x' \in H$. Here let us simply write $\Phi(\omega) := \Phi(\omega,\cdot)$. Further, the right hand side $\lambda : \Omega \times H \to \mathbb{R}$ is Carathéodory such that - by abuse of notation - $\lambda : \omega \in \Omega \mapsto \lambda(\omega) \in H$ is in $L^\infty(\Omega; H)$. Moreover, we have the deterministic data: $\varphi : H \to \mathbb{R}$ is a convex continuous function, $\gamma := \gamma_{H\to Z}$ is a linear continuous operator into a real Banach space Z, and J is a real-valued locally Lipschitz functional defined on Z giving rise to the Clarke generalized directional derivative J^0 as made precise above.

Then we consider the following problem: For each $\omega \in \Omega$, find $x^*_\omega \in K$ such that

$$\langle \Phi(\omega, x^*_\omega), x - x^*_\omega\rangle + J^0(\gamma x^*_\omega; \gamma x - \gamma x^*_\omega) + \varphi(x) - \varphi(x^*_\omega) \geq \langle \lambda(\omega), x - x^*_\omega\rangle, \quad \forall x \in K. \tag{3.1}$$

Throughout we assume that the operator Φ is uniformly strongly monotone in the sense that there exists a constant $c_\Phi > 0$ such that

$$c_\Phi \, \|x-y\|^2 \leq \langle \Phi(\omega,x) - \Phi(\omega,y), x-y\rangle, \quad \text{for all } \omega \in \Omega, \forall x,y \in H. \tag{3.2}$$

We again assume that the locally Lipschitz function J satisfies the linear growth condition

$$\|\zeta\|_{X^*} \leq d_J(1 + \|z\|_X), \quad \forall z \in X, \zeta \in \partial J(z) \tag{3.3}$$

for some $d_J > 0$, which implies the estimate

$$J^0(z; w) \leq d_j(1 + \|z\|_X)\|w\|_X \qquad \forall z, w \in X. \tag{3.4}$$

Further, suppose the one-sided Lipschitz condition for the generalized directional derivative J^0: There exists $c_J > 0$ such that

$$J^0(y_1; y_2 - y_1) + J^0(y_2; y_1 - y_2) \leq c_J \|y_1 - y_2\|^2_X \qquad \forall y_1, y_2 \in X, \tag{3.5}$$

and the smallness condition

$$c_J \|\gamma\|^2_{H\to X} < c_\Phi. \tag{3.6}$$

Next, we define the bifunction

$$\Psi_\omega(x,y) := \langle \Phi(\omega,x), y - x\rangle + J^0(\gamma x; \gamma y - \gamma x) - \langle \lambda(\omega), y - x\rangle. \tag{3.7}$$

Thus, under assumptions (3.2), (3.5), and (3.6), by [5, Proposition 1], Ψ_ω is strongly monotone and the above HVI (3.1) in the pathwise formulation falls into the framework of an *equilibrium problem of monotone type* in the sense of [1]. Therefore we can use Minty's lemma (see, e.g., [1, Prop. 3.2]). Moreover, since the convex continuous function φ is conically minorized, that is, it enjoys the estimate

$$\varphi(x) \geq -c_\varphi(1+\|x\|)\,, x \in H \tag{3.8}$$

with some $c_\varphi > 0$, and the asymptotic coercivity condition in [1] holds. Therefore, the solution $x^*_\omega \in K$ of (3.1) exists uniquely; see [1, Theorem 6.2]. Thus we arrive at the following measurability result for the solution mapping $\Sigma : \Omega \to H$ given by $\Sigma(\omega) := x^*_\omega \in K$, the solution of (3.1), with respect to the σ-algebra $\mathbb{B}(H)$ of the Borel subsets of H.

Theorem 3.1. *Suppose (3.2), (3.5), and (3.6). Then the solution map Σ is measurable.*

Proof. Since H is separable, then metric subspace K is also separable. Let $\{z_\nu\}_{\nu\in\mathbb{N}}$ be dense in K. By Minty's lemma and by continuity, $x^*_\omega = \Sigma(\omega)$ if and only if

$$x^*_\omega \in K, \Psi_\omega(z_\nu, x^*_\omega) + \varphi(x^*_\omega) \leq \varphi(z_\nu) \quad \text{ for all } \nu \in \mathbb{N}.$$

Therefore $\Sigma = \bigcap_{\nu\in\mathbb{N}} \Sigma_\nu$, where, for any $\nu \in \mathbb{N}$, $\Sigma_\nu : \Omega \rightsquigarrow H$ is given by

$$\Sigma_\nu(\omega) := \{\hat{y} \in K : \Psi_\omega(z_\nu, \hat{y}) + \varphi(\hat{y}) \leq \varphi(z_\nu)\}\,.$$

Then $\Sigma_\nu(\omega)$ is closed by continuity, and by $\mathbb{A}\otimes\mathbb{B}(H) - \mathbb{B}(\mathbb{R})$ measurability of Carathéodory functions (see, e.g., [21, Lemma 8.2.6]), the graph of the set valued map Σ_ν belongs to $\mathbb{A}\otimes\mathbb{B}(H)$. Since

$$\text{graph}\,\Sigma = \bigcap_{\nu\in\mathbb{N}} \text{graph}\,\Sigma_\nu \,\in \mathbb{A}\otimes\mathbb{B}(H),$$

by the Castaing characterization theorem (see [21, Theorem 8.1.4]), the claimed measurability of Σ follows. ☐

Using (3.4) and (3.8), we can derive from (3.2), (3.5), and (3.6) the following a priori estimate:

$$\begin{aligned}&(c_\Phi - c_J\|\gamma\|^2)\,\|x^*_\omega - z_0\|^2\\ &\leq \varphi(z_0) + c_\varphi(1+\|x^*_\omega\|) + [c_J\|\gamma\|_{H\to X}(1+\|\gamma\|_{H\to X}\|z_0\|) + \|\lambda\|_{L^\infty(\Omega;H)}]\|x^*_\omega - z_0\|\end{aligned}$$

with some arbitrary fixed $z_0 \in K$. Hence, using the definition $\hat{u}(\omega) := x^*_\omega$,

$$\|\hat{u}(\omega)\| \leq \tilde{c}\left(c_\varphi, c_\Phi, c_J, \varphi(z_0), \|z_0\|, \|\gamma\|, \|\lambda\|\right). \tag{3.9}$$

Then we can exploit (3.9) and can conclude that $\hat{u}$ belongs to the Bochner-Lebesgue space $L^\infty(\Omega,\mu,H)$, the Banach space of (classes of) measurable, μ-essentially bounded maps $u : \Omega \to H$.

Theorem 3.2. *Let $(\Omega,\mathbb{A},\mu)$ be a complete σ-finite measure space and H a separable Hilbert space. Then, under assumptions (3.2), (3.3), (3.5), and (3.6), the random variational inequality (3.1) admits a unique solution $\hat{u} : \omega \in \Omega \mapsto \hat{u}(\omega) \in K$. If, in addition, that μ is a finite measure, then we have $\hat{u} \in L^\infty(\Omega,\mu,H)$.*

4. Solvability and Regularity of Time-dependent Hemivariational Inequalities

In this section, we consider the particular case of a time-dependent HVI. We specialize $\Omega := [0,T]$ for given $T > 0$ and μ becomes the Lebesgue measure. As above, $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ denotes a separable Hilbert space with $K \subset H$ closed, convex, and nonempty. Moreover, $\Phi \in C([0,T] \times H; H)$, that is, $(t,x) \in [0,T] \times H \mapsto \Phi(t,x) \in H$ is continuous in both variables t and x, and similarly as above, suppose that $\Phi(t,\cdot)$ is a monotone operator on H. Here let us simply write $\Phi(t) := \Phi(t,\cdot)$. Furthermore, $f \in C([0,T];H)$ is given that gives rise to $\tilde{f} : [0,T] \times H \to \mathbb{R}$ via $\tilde{f}(t,x) := \langle f(t), x\rangle$. Moreover, we have the non-time-dependent data: $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ that is now a convex lower semicontinuous proper function with its nonempty convex effective domain $\operatorname{dom}\varphi$, while again, $\gamma := \gamma_{H\to Z}$ is a a linear continuous operator into a real Banach space Z, and J is a real-valued locally Lipschitz functional defined on Z giving rise to the Clarke generalized directional derivative J^0 as made precise above.

Then we consider the following problem: For each $t \in [0,T]$, find $x_t^* \in K \cap \operatorname{dom}\varphi$ such that

$$\langle \Phi(t,x_t^*), x - x_t^*\rangle + J^0(\gamma x_t^*; \gamma(x - x_t^*)) + \varphi(x) - \varphi(x_t^*) \geq \langle f(t), x - x_t^*\rangle, \quad \forall x \in K. \tag{4.1}$$

Similarly, we assume that Φ is uniformly strongly monotone in the sense that there exists a constant $c_\Phi > 0$ such that

$$c_\Phi \|x-y\|^2 \leq \langle \Phi(t,x) - \Phi(t,y), x-y\rangle, \quad \forall t \in [0,T], \forall x,y \in H. \tag{4.2}$$

We again assume that the locally Lipschitz function J satisfies the linear growth condition, $\|\zeta\|_{X^*} \leq d_J(1 + \|z\|_X)$ for all $z \in X$ and $\zeta \in \partial J(z)$ and for some $d_J > 0$, which implies the estimate, $J^0(z;w) \leq d_j(1 + \|z\|_X)\|w\|_X$ for all $z,w \in X$, Further, similar to [12], we suppose the one-sided Lipschitz condition for the generalized directional derivative J^0: There exists $c_J > 0$ such that

$$J^0(y_1; y_2 - y_1) + J^0(y_2; y_1 - y_2) \leq c_J \|y_1 - y_2\|_X^2, \quad \forall y_1, y_2 \in X \tag{4.3}$$

and in addition the smallness condition

$$c_J \|\gamma\|_{H\to X}^2 < c_\Phi. \tag{4.4}$$

Next, we define the bifunction

$$\Psi_t(x,y) := \langle \Phi(t,x), y-x\rangle + J^0(\gamma x; \gamma y - \gamma x) - \langle f(t), y-x\rangle.$$

Then, under assumptions (4.2), (4.3), and (4.4), by [5, Proposition 1], we have that Ψ_t is strongly monotone and the above HVI (4.1) in the pathwise formulation falls into the framework of an *extended real-valued equilibrium problem of monotone type* in the sense of [1]. Moreover, since the convex proper lower semicontinuous function φ is conically minorized, that is, it enjoys the estimate $\varphi(x) \geq -c_\varphi(1 + \|x\|), x \in H$ with some $c_\varphi > 0$, and the asymptotic coercivity condition in [1] holds. Therefore, the solution $x_t^* \in K \cap \operatorname{dom}\varphi$ of (4.1) exists uniquely; see [1, Theorem 6.2]. Thus we can extend [4, Theorem 3.12] to time-dependent HVIs in the following.

Theorem 4.1. *Suppose (4.2), (4.3), and (4.4). Then the unique solution $x_t^* \in K \cap \operatorname{dom}\varphi$ of (4.1) for any fixed $t \in [0,T]$ gives rise to the time-dependent solution $u^* \in C([0,T];H)$, defined by $u^*(t) := x_t^*$ for $t \in [0,T]$. Moreover, if $f \in W^{1,p}(0,T;H)$ for some $p \in [1,\infty]$, then also $u^* \in W^{1,p}(0,T;H)$.*

Proof. To show the claimed continuity of u^* with respect to the time variable, we consider $t_1, t_2 \in [0,T]$, and test (4.1) for $t = t_1, x = u(t_2)$, respectively for $t = t_2, x = u(t_1)$. Thus

$$\begin{aligned}&\langle \Phi(t_1,u^*(t_1),u^*(t_2)-u^*(t_1)\rangle + \langle \Phi(t_2,u^*(t_2),u^*(t_1)-u^*(t_2)\rangle\\&+J^0(\gamma u^*(t_1);\gamma(u^*(t_2)-u^*(t_1)) + J^0(\gamma u^*(t_2);\gamma(u^*(t_1)-u^*(t_2))\\&\geq \langle f(t_1),u^*(t_2)-u^*(t_1)\rangle + \langle f(t_2),u^*(t_1)-u^*(t_2)\rangle\,.\end{aligned}$$

Hence

$$\left(c_\Phi - c_J\|\gamma\|^2\right)\|u^*(t_2)-u^*(t_1)\| \leq \|f(t_2)-f(t_1)\|, \tag{4.5}$$

which shows the continuity of u^*. Now, assume $f \in W^{1,p}(0,T;H)$. Then f is absolutely continuous. (4.5) entails that u^* is absolutely continuous and

$$\left(c_\Phi - c_J\|\gamma\|^2\right)\|\frac{d}{dt}u^*(t)\| \leq \|\dot{f}(t)\| \quad \text{a.a. } t\in(0,T)\,.$$

Thus $\dot{f} \in L^p(0,T;H)$ implies $\dfrac{d}{dt}u^* \in L^p(0,T;X)$, concluding the proof. □

In the following section, we return to random HVI (3.1) and proceed to give a stability result.

5. Random Hemivariational Inequalities - Stability in a Bochner-Lebesgue Space

Again $(H,\langle\cdot,\cdot\rangle,\|\cdot\|)$ is a separable Hilbert space. Here we specialize to a probability space $(\Omega,\mathscr{A},P)$ and consider the above random HVI (3.1) in the Bochner-Lebesgue space $\mathscr{V} := L^\infty(\Omega,P,H)$ of all H - valued P - measurable random variables V such that

$$\|V\|_{\mathscr{V}} = \operatorname*{ess\,sup}_{\omega\in\Omega}\|V(\omega)\| < \infty.$$

Thus, due to Theorem 3.2, the above unique solution $x^*_\omega \in K$, $\omega \in \Omega$ of (3.1) gives $V^* \in \mathscr{V}$ via $V^*(\omega) := x^*_\omega$.

Let us now study the stability with respect to the convex function φ and the right hand side λ. Consider sequences $\{\varphi^{(\nu)}\}_{\nu\in\mathbb{N}}$ and $\{\lambda^{(\nu)}\}_{\nu\in\mathbb{N}}$, where $\varphi^{(\nu)}$ are convex continuous on H and $\lambda^{(\nu)} \in \mathscr{V}$. The Carathéodory monotone operator Φ and the locally Lipschitz function J are given as before.

Thus we are led to the perturbed HVI in pathwise formulation: For each $\omega\in\Omega$, find $x^{(\nu)}_\omega \in K$ such that

$$\begin{aligned}&\langle \Phi(\omega,x^{(\nu)}_\omega),x-x^{(\nu)}_\omega\rangle + J^0(\gamma x^{(\nu)}_\omega;\gamma x-\gamma x^{(\nu)}_\omega) + \varphi^{(\nu)}(x) - \varphi^{(\nu)}(x^{(\nu)}_\omega)\\&\geq \lambda^{(\nu)}(\omega)(x-x^{(\nu)}_\omega), \quad \forall x\in K.\end{aligned} \tag{5.1}$$

Likewise, due to Theorem 3.2, the unique solution $x_\omega \in K$, $\omega\in\Omega$ of (5.1) gives $V^{(\nu)} \in \mathscr{V}$ via $V^{(\nu)}(\omega) := x^{(\nu)}_\omega$. Then we obtain from Lemma 2.1 and Theorem 2.2 the following stability result for the pathwise formulation. To obtain the stability in the Bochner-Lebesgue space $\mathscr{V}$, we introduce the convergence condition

(CC) $|(\varphi^{(\nu)}(x)-\varphi(x)) - (\varphi^{(\nu)}(y)-\varphi(y))| \leq \varepsilon_\nu\|x-y\| \quad \forall x,y\in K\,,$

where $\varepsilon_\nu \to 0$ for $\nu\to\infty$.

Corollary 5.1. *Let $K, \Phi, J, \varphi, \varphi^{(\nu)}, f, f^{(\nu)}$ be given as above. Suppose that $\varphi^{(\nu)} \xrightarrow{\mathrm{M}} \varphi$ and $\lambda^{(\nu)} \to \lambda$ in $\mathscr{V}$ for $\nu \to \infty$. Then, for each $\omega \in \Omega$, $x_\omega^{(\nu)} \to x^*$ in H for $\nu \to \infty$. If, moreover,* (CC) *holds, then* $\lim_{\nu\to\infty} \|V^{(\nu)} - V^*\|_{\mathscr{V}} = 0$ *in* $\mathscr{V} = L^\infty(\Omega, P, H)$.

Proof. Fix $\omega \in \Omega$. By Lemma 2.1, one has $\varphi^{(\nu)} + \lambda^{(\nu)}(\omega) \xrightarrow{\mathrm{M}} \varphi + \lambda(\omega)$ in H. Thanks to Theorem 2.2, one sees that $\lim_{\nu\to\infty} \|x_\omega^{(\nu)} - x_\omega^*\| = 0$. Now, suppose (CC). Test (3.1) with $x = x_\omega^{(\nu)}$ and test (5.1) with $x = x_\omega^*$, add, and obtain

$$\begin{aligned}&\langle \Phi(\omega, x_\omega^{(\nu)}) - \Phi(\omega, x_\omega^*), x_\omega^{(\nu)} - x_\omega^*\rangle + \varphi^{(\nu)}(x_\omega^{(\nu)}) - \varphi^{(\nu)}(x_\omega^*) - (\varphi(x_\omega^{(\nu)}) - \varphi(x_\omega^*))\\ &\le J^0(\gamma x_\omega^{(\nu)}; \gamma x_\omega^* - \gamma x_\omega^{(\nu)}) + J^0(\gamma x_\omega^*; \gamma x_\omega^{(\nu)} - \gamma x_\omega^*) + \langle \lambda^{(\nu)}(\omega) - \lambda(\omega), x_\omega^{(\nu)} - x_\omega^*\rangle.\end{aligned}$$

By (3.2), (3.5), and (3.6), one obtains

$$(c_\Phi - c_J\|\gamma\|^2)\,\|x_\omega^* - x_\omega^{(\nu)}\|^2 \le (\varepsilon_\nu + \|\lambda^{(\nu)} - \lambda\|_{\mathscr{V}})\,\|x_\omega^* - x_\omega^{(\nu)}\|,$$

which proves the claim. □

6. A Random Nonsmooth Boundary Value Problem under Uncertainty

As an application, in this section, we consider a random nonsmooth boundary value problem, which augments the random boundary value problem in [14] with nonmonotone boundary conditions .

Let R, S, and T be real-valued random variables on a probability space $(\Omega, \mathscr{A}, P)$. Further let $D \subset \mathbb{R}^d$ $(d \ge 2)$ be a bounded Lipschitz domain with outer unit normal ν. Then, for the pathwise strong formulation, the nonsmooth boundary value problem under study reads as follows: For each $\omega \in \Omega$, find $u_\omega = u(\omega, x)$ such that - taking the divergence and the gradient ∇ with respect to the spatial x variable -

$$-\operatorname{div}\Big(S(\omega)\, p(|\nabla u_\omega|)\, \nabla u_\omega\Big) = R(\omega)\, g \quad \text{a.e. in } D, \tag{6.1}$$

where $g \in L^2(D)$ and $p : [0,\infty) \to [0,\infty)$ is a continuous function with $t \cdot p(t)$ being monotonously increasing with t. The PDE (6.1) has to be complemented by boundary conditions. To this end, decompose the boundary $\Gamma := \partial D$ in mutually disjoint open parts, namely the Dirichlet part Γ_D, the Neumann part Γ_N, the Signorini part Γ_S, the Tresca part Γ_T, and the Clarke part Γ_C, such that $\partial D = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_S \cup \overline{\Gamma}_T \cup \overline{\Gamma}_C$ with meas $(\Gamma_D) > 0$. Further, let $h \in L^2(\Gamma_N) \cup L^2(\Gamma_S)$ and $k \in L^2(\Gamma_T)$ with $k > 0$ a.e. Then, we demand that $U : \omega \mapsto u_\omega$ satisfies P-almost surely (a.s.)

$$\left.\begin{aligned}
&Q_\nu := S\, p(|\nabla U|)\, \nabla U \cdot \nu && \text{on } \partial D\\
&U = 0 && \text{on } \Gamma_D\,,\\
&Q_\nu = T\, h && \text{on } \Gamma_N\,,\\
&U \le 0,\, Q_\nu - T\, h \le 0,\, U(Q_\nu - Th) = 0 && \text{on } \Gamma_S\,,\\
&|Q_\nu| \le k,\, U\, Q_\nu + k\,|U| = 0 && \text{on } \Gamma_T\,,\\
&p(|\nabla U|)\, \tfrac{\partial U}{\partial \nu}\Big|_{\Gamma_C} \in \partial j(\cdot, U\big|_{\Gamma_C}) && \text{on } \Gamma_C\,.
\end{aligned}\right\} \tag{6.2}$$

In accordance with the standard case of uniformly strongly elliptic operators, we assume that P-almost surely, S is contained in a compact interval that is included in $(0, +\infty)$. Hence

S belongs to $L^\infty(\Omega)$. Similarly, we assume that the random variables R and T are in $L^\infty(\Omega)$. The function $j : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ is such that $j(\cdot,\xi) : \Gamma_C \to \mathbb{R}$ is measurable on Γ_C for all $\xi \in \mathbb{R}$ and $j(s,\cdot) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz for almost all (a.a.) $s \in \Gamma_C$ with $\partial j(s,\xi) := \partial j(s,\cdot)(\xi)$, the generalized gradient of $j(s,\cdot)$ at ξ. Further, require the following growth condition on the so-called superpotential j: There exist positive constants $c_{j,1}$ and $c_{j,2}$ such that, for a.a. $s \in \Gamma_C$, all $\xi \in \mathbb{R}$ and for all $\eta \in \partial j(s,\xi)$, the following inequalities hold

$$(i) \quad \eta\, \xi \geq -c_{j,2}|\xi|\,, (ii) \quad |\eta| \leq c_{j,1}(1+|\xi|)\,. \tag{6.3}$$

Note that it follows from (6.3) (i) and (6.3) (ii), respectively, that for a.a. $s \in \Gamma_C$

$$j^0(s,\xi;-\xi) \leq c_{j,2}|\xi|, \left| j^0(s,\xi;\varsigma) \right| \leq c_{j,1}(1+|\xi|)|\varsigma|, \quad \forall \xi,\varsigma \in \mathbb{R}.$$

Altogether the boundary value problem consists in finding a random variable U distributed in D that satisfies (6.1) and (6.2) in a weak formulation.

Let us remark in passing that the scalar unilateral Signorini boundary conditions on Γ_S, scalar Tresca-like boundary conditions on Γ_T, and the scalar nonmonotone boundary conditions on Γ_C, which are based on Clarke's generalized differential calculus, model unilateral contact conditions with Tresca friction and with nonmonotone friction in continuum mechanics, respectively, as already discussed in [15]. Further, such scalar nonlinear boundary conditions result from frictional contact problems under antiplane shear; see [4, Chapter 8] for the treatment of more special mixed variational inequalities and also [4, 22] for more complicated evolutionary antiplane frictional contact problems.

To proceed further in the functional analytical setting, we introduce the separable Hilbert space

$$H := H_0^1(D) := \{v \in H^1(D) : v|_{\Gamma_D} = 0\},$$

which in virtue of the Poincaré inequality can be equipped with the norm $\|v\| = \|\nabla v\|_{L^2(D)}$, associated to the scalar product $\langle v,w\rangle = \langle \nabla v, \nabla w\rangle_{L^2(D)\times L^2(D)}$.

Next, from the Signorini boundary condition, we set $C := \{v \in H : v|_{\Gamma_S} \leq 0\}$, a closed convex subset. In fact, it is a cone in H. Moreover, let $Z := L^2(\Gamma)$, introduce the linear continuous embedding operator, $\gamma := \gamma_{H\to Z}$, and introduce the nonsmooth convex function $K(z) = \int_{\Gamma_T} k(s)\,|z(s)|\; ds$, $z \in Z$. Further, we introduce the real-valued locally Lipschitz functional $J(y) := \int_{\Gamma_C} j(s,y(s))\; ds$, $y \in Z$. Then by Lebesgue's theorem of majorized convergence, we have

$$J^0(y;z) = \int_{\Gamma_C} j^0(s,y(s);z(s))\; ds\,, \qquad (y,z) \in Z\times Z\,,$$

where $j^0(s,\cdot\,;\,\cdot)$ denotes the generalized directional derivative of $j(s,\cdot)$.

Next, we define the real-valued functional $G(u) := \int_\Omega g(|\nabla u|)dx$, $u \in H^1(D)$ where the function g is given by p by

$$g : [0,\infty) \to [0,\infty), t \mapsto g(t) = \int_0^t s\cdot p(s)\, ds,$$

where we also assume that p is C^1 and $0 \leq p(t) \leq p_0 < \infty$. Then, $0 \leq g(t) \leq \frac{1}{2}p_0 \cdot t^2$ and the functional G is strictly convex. The Gateaux derivative of G,

$$DG(u,v) = \int_\Omega p(|\nabla u|)(\nabla u)^T \cdot \nabla v\, dx \qquad u,v \in H^1(D)$$

is Lipschitz continuous and strongly monotone in H, that is, there exists a constant $c_G > 0$ such that

$$c_G \|u-v\|^2 \le DG(u,u-v) - DG(v,u-v), \quad \forall u,v \in H. \tag{6.4}$$

Now, we switch to the probabilistic setting. Introduce the Bochner-Sobolev space $\mathscr{V} := L^\infty(\Omega;H) = L^\infty(\Omega;H_0^1(D))$. Define the Carathéodory operator $\Phi : \Omega \times H \to H$ by

$$\langle \Phi(\omega,u),v\rangle := S(\omega)\, DG(u,v) = S(\omega) \int_\Omega p(|\nabla u|)(\nabla u)^T \cdot \nabla v\, dx, \quad \omega \in \Omega; u,v \in H^1(D)$$

and the Carathéodory function $\lambda : \Omega \times H \to \mathbb{R}$ by

$$\lambda(\omega,u) := R(\omega) \int_\Omega g\, u\, dx + T(\omega) \int_{\Gamma_N \cup \Gamma_S} h\, \gamma u\, ds, \quad \omega \in \Omega; u \in H^1(D).$$

Then it can be proved (see, e.g., [23, Theorem 1] for a similar result) that the boundary value problem (6.1) - (6.2) in the pathwise formulation is equivalent in the sense of distributions to the following HVI problem: For each $\omega \in \Omega$, find $\hat{u}_\omega \in C$ such that, for all $u \in C$, there holds, for $\delta u_\omega := u - \hat{u}_\omega$,

$$\langle \Phi(\omega,\hat{u}_\omega), \delta u_\omega\rangle + J^0(\gamma \hat{u}_\omega; \gamma \delta u_\omega) + K(\gamma u) - K(\gamma \hat{u}_\omega) \ge \lambda(\omega, \delta u_\omega). \tag{6.5}$$

Again, we suppose that the generalized directional derivative J^0 satisfies the one-sided Lipschitz condition (3.5) and the smallness condition (3.6), where $c_J \|\gamma\|^2_{H\to Z} < c_G$ with c_G from (6.4). Due to Theorem 3.2, the unique solution $\hat{u}_\omega \in C$, $\omega \in \Omega$ of (6.5) gives $\hat{U} \in \mathscr{V}$ via $\hat{U}(\omega) := \hat{u}_\omega$.

Next, we intend to give a novel stability result for the nonsmooth boundary value problem described above. We consider the dependence of the solution $\hat{u}_\omega \in C$, $\omega \in \Omega$ of (6.5) and of the random distributed solution $\hat{U}$ with respect to the right hand side λ and to the convex function K. By the existence and uniqueness discussed above, we have the solution map $(\lambda;K) \mapsto S_\omega(\lambda;K) := \hat{u}_\omega \in C\, (\forall \omega \in \Omega)$, the solution of (6.5).

To describe explicitly the dependence on the right hand side, we consider $R_n \in L^\infty(\Omega), T_n \in L^\infty(\Omega)$ and $g_n \in L^2(D), h_n \in L^2(\Gamma_N) \cup L^2(\Gamma_S)$ for $n \in \mathbb{N}$. Then $R_n \to R$ in $L^\infty(\Omega)$, $T_n \to T$ in $L^\infty(\Omega)$, $g_n \to g$ in $L^2(D)$, $h_n \to h$ in $L^2(\Gamma_N) \cup L^2(\Gamma_S)$ for $n \to \infty$ implies $\lambda_n \to \lambda$ in $\mathscr{V}$, where

$$\lambda_n(\omega,u) := R_n(\omega) \int_\Omega g_n\, u\, dx + T_n(\omega) \int_{\Gamma_N \cup \Gamma_S} h_n\, \gamma u\, ds, \qquad \omega \in \Omega; u \in H^1(D).$$

To describe explicitly the dependence on e convex function K, we introduce the linear continuous trace map $\gamma_T : H := H_0^1(D) \to L^2(\Gamma_T)$ and let $K_n(w) := \int_{\Gamma_T} k_n |\gamma_T w|$ with $k_n \in L^\infty(\Gamma_T)$ and $k_n > 0$ a.e. such that $k_n \to k$ in $L^\infty(\Gamma_T)$ for $n \to \infty$. Then $K_n \xrightarrow{\mathrm{M}} K$ on H. Indeed, let $w_n \rightharpoonup w$ in H. By the Trace Theorem, see, e.g., [19, Theorem 5.6.1], $\gamma_T w_n \rightharpoonup \gamma_T w$ in $L^2(\Gamma_T)$. This means

$$\langle \ell, \gamma_T w_n - \gamma_T w\rangle_{L^2(\Gamma_T)\times L^2(\Gamma_T)} \to 0, \quad \forall \ell \in L^2(\Gamma_T).$$

We claim that $k_n \gamma_T w_n \rightharpoonup k\gamma_T w$ in $L^2(\Gamma_T)$. By the Uniform Boundedness Principle, $\gamma_T w_n$ is bounded in $L^2(\Gamma_T)$, say by $C > 0$. Further, for all $\ell \in L^2(\Gamma_T)$,

$$|\langle \ell, k_n \gamma_T w_n\rangle - \langle \ell, k\gamma_T w\rangle| \le |\langle k\ell, (\gamma_T w_n - \gamma_T w)\rangle| + |\langle \ell, (k_n - k)\gamma_T w_n\rangle|.$$

Both summands above converge to zero for $n \to \infty$; the first is due to the weak convergence $\gamma_T w_n \rightharpoonup \gamma_T w$, the second is due o the convergence $k_n \to k$ in $L^\infty(\Gamma_T)$ and the estimate

$$|\langle \ell, (k_n - k)\gamma_T w_n\rangle| \le C\, \|k_n - k\|_{L^\infty(\Gamma_T)} \|\ell\|_{L^2(\Gamma_T)}.$$

Thus the claim is proven.

Since $L^2(\Gamma_T)$ embeds continuously in $L^1(\Gamma_T)$, $k_n \gamma_T w_n \rightharpoonup k \gamma_T w$ in $L^1(\Gamma_T)$, too. Further, the L^1 norm is convex and continuous, hence weakly lower semicontinuous. Thus $K(w) \leq \liminf_{n\to\infty} K_n(w_n)$ and (M1) is shown. (M2) is obvious from choosing $w_n := w$, the stationary sequence. Thus $K_n \xrightarrow{\text{M}} K$ on H.

Finally we arrive at the following stability result.

Theorem 6.1. *Suppose that the generalized directional derivative J^0 satisfies one-sided Lipschitz condition (3.5) and growth condition (3.3). Moreover, suppose the smallness condition (3.6) with the monotonicity constant c_G from* (6.4). *Let $R_n \to R$ in $L^\infty(\Omega)$, $T_n \to T$ in $L^\infty(\Omega)$, $g_n \to g$ in $L^2(D)$, $h_n \to h$ in $L^2(\Gamma_N) \cup L^2(\Gamma_S)$, and $k_n \to k$ in $L^\infty(\Gamma_T)$ for $n \to \infty$. Then there holds $S_\omega(\lambda_n; K_n) \to S_\omega(\lambda; K)$, $\forall \omega \in \Omega$; and for the associated random distributed solutions $U_n, \hat{U}$, there holds $U_n \to \hat{U}$ in $\mathscr{V}$.*

Proof. Since

$$\begin{aligned}
&\Big|\Big(\int_{\Gamma_T} k_n|y| - \int_{\Gamma_T} k|y|\Big) - \Big(\int_{\Gamma_T} k_n|z| - \int_{\Gamma_T} k|z|\Big)\Big| \\
&\leq \int_{\Gamma_T} |k_n - k||y - z| \leq \|k_n - k\|_{L^2(\Gamma_T)} \|y - z\|_{L^2(\Gamma_T)}, \quad \forall y, z \in L^2(\Gamma_T),
\end{aligned}$$

we see that (CC) is satisfied and the conclusion follows from Corollary 5.1 immediately. □

7. Some Concluding Remarks And An Outlook

Interestingly, our approach to the measurability of solutions of random HVIs via the Castaing theory demands the continuity of the convex function and thus restricts our study to a class of *mixed* HVIs, whereas in the time-dependent case, we can directly show the continuity of the solution of more general *extended real-valued* HVIs. Since an extended real-valued proper lower semicontinuous convex function is conically minorized, the regularity of the solution to a mixed random HVI, respectively to an extended real-valued time-dependent HVI is determined by the regularity of the right hand side; see Theorem 3.2, respectively Theorem 4.1.

Here we focused on the randomness in the uniformly strongly monotone operator and on the randomness in the right hand side, so we only partly extended the results of [14]. It would be interesting to consider randomness in the convex function and in the locally Lipschitz function, too. Here we did not consider perturbations in the bifunction of the variational formulation or stability with respect to the coefficients of the elliptic operator; see [24, 25, 26] for results for deterministic (hemi)variational inequalities. Another open research direction is the stability with respect to convex functions in three-field augmented Lagrangian formulations of nonlinear nonsmooth boundary value problems in continuum mechanics; see, e.g., [27].

Finally, let us point out that stability results lay down the foundation for existence results in optimal control and inverse problems; see, e.g., [5] for various kinds of optimal control problems governed by a more involved interface problem on an unbounded domain, as well as optimal control of the obstacle driven by a related bilateral obstacle interface problem in the deterministic regime, respectively; see [28, 29] for the study of nonlinear inverse problems of estimating stochastic parameters in PDEs with random data, where for inverse problems because of their ill-posed nature, regularization methods come into play in addition that need further analysis.

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