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STEPANOV OSCILLATORY TYPE SOLUTIONS FOR RENEWAL EQUATIONS WITH INFINITE DELAY: APPLICATION TO AN EPIDEMIC MODEL WITH WANING IMMUNITY

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Abstract. We establish a new result on the reduction principle for nonhomogeneous autonomous linear renewal equations with the forcing term bounded by mean values. This result is then used, together with a variation of constants formula, to prove Stepanov's version of Massera and Bohr-Neugebauer-type results. We also investigate uniqueness conditions in both the linear and nonlinear cases. Finally, we apply our theoretical results to an epidemic model with waning immunity.
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1. INTRODUCTION

The main aim of this work is to prove some results concerning the existence and the uniqueness of almost periodic (resp. almost automorphic) solutions in Stepanov sense for the following nonhomogeneous linear renewal equation with infinite delay:

$$x(t) = \int_{-\infty}^{t} K(t-s)x(s)\,ds + f(t) \text{ for } t \in \mathbb{R}, \tag{1.1}$$

where $K : [0,+\infty[\to \mathbb{C}^{m\times m}$ is a measurable function satisfies the following assumption:

$$\|K\|_{1,\rho} := \int_0^{\infty} \|K(\tau)\| e^{\rho\tau}\,d\tau < \infty, \|K\|_{\infty,\rho} := ess\sup\{\, \|K(\tau)\| e^{\rho\tau}, \quad \tau \geq 0\} < \infty, \tag{\textbf{H1}}$$

for some $\rho > 0$ and f is a Stepanov almost periodic (resp. Stepanov almost automorphic) function.

Renewal equations (REs) are a type of mathematical equation that does not involve derivatives and describe the present value of an unknown function in terms of its past values. This is the difference between delay differential equations and (REs), for the former the rule gives the

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derivative of the function at the current time, while for the latter the rule specifies the value of the function itself. (REs) can be expressed as follows

$$x(t)=F(t,x_t) \text{ for } t\in\mathbb{R},$$

where $F:\mathbb{R}\times X\to\mathbb{C}^m$ is a function that satisfies certain conditions, X is a linear normed space of mappings from $]-\infty,0]$ into $\mathbb{C}^m$ and $x_t\in X$ is the history function, given as

$$x_t(\theta)=x(t+\theta) \text{ for } \theta\leq 0.$$

Renewal equations arise in the formulation of various phenomena in population dynamics including birth processes, cannibalism, physiologically structured populations, epidemic models, and so on. In this context, we mention the work of Kermack and McKendrick [23] and Lotka [26] on problems in mathematical biology. For instance, in a closed population, let $S(t)$ be the density of susceptible individuals, $F(t)$ be the force of infection, and $A(a)$ be transmission potential. Then, the force of infection can be described by the following (RE):

$$F(t)=\int_0^{+\infty}S(t-\tau)F(t-\tau)A(\tau)d\tau \text{ for } t\geq 0.$$

The population growth structured by age can be described by the linear renewal equation:

$$B(t)=\int_0^{+\infty}B(t-a)\beta(a)F(\tau)da \text{ for } t\geq 0,$$

where $B(t)$ is the birth rate, $F(a)$ is the age dependent survival probability, and $\beta(a)$ is the age dependent fertility. In [9], Barril et al. developed a size-structured model to describe the dynamics of trees growth in the forest, associated with the following renewal equation

$$b(t)=\int_0^{\infty}\beta(x_m+\int_0^a g(e^{-\mu(\tau-a)}\int_a^{\infty}e^{-\mu s}b(t-s)ds)d\tau)e^{-\mu a}b(t-a)da,$$

where $b(t)$ is the tree's population birth rate at time t, $g(x)$ denotes the growth rate of an individual of height x, the positive parameter $\mu>0$ represents the per capita death rate and $\beta(x)$ is the per capita reproduction rate (depending only on height x). It was also assumed in [9] that all newborn individuals have the minimal height $x_m\geq 0$ and that Mappings $\beta:[x_m,\infty)\longrightarrow[0,\infty)$, $g:[0,\infty)\longrightarrow(0,\infty)$ are continuous, β is increasing and g is decreasing. For more examples, we refer to [4, 9, 15, 16, 23, 26, 30, 31, 32, 36].

A fundamental progress has been achieved in the treatment of (REs) by using a functional analytic methods involving the semigroup theory. Diekmann and Gyllenberg [16] utilized perturbation theory for adjoint semigroups to demonstrate the principle of linearized stability for nonlinear renewal equations with infinite delay. In [29], the authors established a new variation of constants formula for equation (1.1) in the phase space, along with a decomposition of the phase space corresponding to the solution semigroup associated with equation (1.1) when $f=0$. Building on this, they developed a fundamental reduction principle which plays a crucial role in the study of bounded solutions of equation (1.1). This approach allows for the study of asymptotic behaviors of solutions from a dynamical systems viewpoint (see, e.g., [28, 29]).

The concept of almost periodicity, originally introduced by Bohr [13], provides a mathematical framework for a class of continuous functions endowed with certain structural properties. This concept serves as a generalization of pure periodicity and affords a more comprehensive understanding of oscillatory behavior. In the last decades, the theory of almost periodic functions has undergone considerable development. In particular, many studies directed to generalizations of Bohr's theory from different points of view. The first generalization was an essential one given by Stepanov [35]. He succeeded in removing the continuity and boundedness restrictions and characterized the generalized almost periodicity, known as Stepanov almost periodic, not by values of the functions at each point, but by mean values over intervals of fixed length. In [11] Bochner extended Bohr's theory to general Banach spaces and gave a very practical characterization of almost periodicity in the sense of Bohr using sequences. In his quest to gain a deeper understanding of Bohr's almost periodicity, Bochner [11, 12] introduced a new generalization of almost periodicity, called almost automorphic functions. He observed that almost automorphic functions can sometimes be used in obtaining simpler proofs of certain results concerning almost periodic functions by first proving these results for almost automorphic functions. In [34], the authors introduced a new concept of almost automorphy defined just on measurable functions that are locally integrable, namely, Stepanov-like almost automorphy. For further contributions, we refer to [6, 22, 25, 33, 34].

In the context of dynamical systems, almost periodicity and almost automorphy are used to understand the long term behavior of solutions. The question that are of interest to many researchers is when an evolution equation has an almost periodic or almost automorphic solution. In this direction, there are two (best) classical results that are fundamental in characterizing the existence of such solutions, and they serve as the basis for much of the research in this area. The first one is known as known as the Bohr–Neugebauer Theorem ([14]), which claims, for an autonomous linear ordinary differential equations with almost periodic non homogeneous, that every bounded solution on the real line is almost periodic. The second theorem, which is called Massera Theorem (see [22]), asserts that the existence of a bounded solution on $\mathbb{R}^+$, implies the existence of an almost periodic solution on $\mathbb{R}$ (not necessarily the same). Liu et al. [25] (see also [33]) extended the previous results for a class of evolution equations with almost automorphic nonhomogeneous. Ezzinbi et al. (see [1, 3, 10, 17, 20, 21]) used the reduction principle to extend the Bohr-Neugebauer Theorem and Massera-type result to linear partial functional differential equations with almost periodic (resp. almost automorphic) forcing term, both in the finite and infinite delay cases.

Recently, Afoukal et al. [2] studied, by employing a reduction principle and a variation of constants formula established in [29], the existence of almost periodic and almost automorphic solutions for equation (1.1). Results of the Massera and Bohr-Neugebauer type results were provided with uniqueness conditions. They also observed, in contrast to ordinary differential equations, where solutions become strong (Bohr-almost periodic or compact almost automorphic) sense under forcing terms in Stepanov sense (my be not bounded or continuous) (see [3, 6, 10, 17, 19, 20]), and this scenarios are not possible for renewal equations. In other words, equation (1.1) has an almost periodic (resp. compact almost automorphic, almost automorphic) solution except that f almost periodic (resp. compact almost automorphic, almost automorphic). Motivated by this fact, the aim of the present work is to examine the case when f is continuous purely almost periodic (resp. almost automorphic) in Stepanov sense. To that purpose, we use the variation of constants formula with spectral decomposition of the phase space obtained in [29], to obtain Stepanov version of Massera and Bohr-Neugebauer type results for equation (1.1). Some findings on uniqueness for both equation (1.1) and the nonlinear case, when the semigroup generated by the linear part is hyperbolic, are also achieved.

This work is organized as follows. In Section 2, we provide a brief overview of the concepts and main properties related to almost periodicity and almost automorphy, along with the well-posedness of equation (1.1). Section 3 is devoted to a new result on the reduction principle. In Section 4, we study the behavior of solutions of equation (1.1) with bounded mean values. Section 5 is about the existence and uniqueness for a fully nonlinear equation under Lipshizian condition. In Section 6, we present an application that illustrates the abstract results obtained.

2. Preliminary Results

This section presents some preliminaries needed to elaborate our main results in this work, namely, the concepts of oscillatory functions in Stepanov sense and basic facts on the well-posedness of equation (1.1) with a variation of constants formula in the phase space.

2.1. **Stepanov type oscillatory functions.** Let $(\mathscr{Y}, \|.\|)$ be a Banach space and $BC(\mathbb{R}, \mathscr{Y})$ be the space of bounded continuous functions from $\mathbb{R}$ to $\mathscr{Y}$, provided with the uniform norm topology.

Definition 2.1. [22] A bounded continuous function $f : \mathbb{R} \to \mathscr{Y}$ is said to be almost periodic if, for every $\varepsilon > 0$, there exists a positive number ℓ such that every interval of length ℓ contains a number τ such that $\|f(t+\tau) - f(t)\| < \varepsilon$ for $t \in \mathbb{R}$. The space of all such functions is denoted by $AP(\mathbb{R}, \mathscr{Y})$.

The following results describe some of the properties of almost periodic functions.

Theorem 2.1. [11] *Each almost periodic function is uniformly continuous and with relatively compact range.*

Theorem 2.2. [22] *Let $(f_n)_{n\geq 0} \subset AP(\mathbb{R}, \mathscr{Y})$. If $(f_n)_{n\geq 0}$ converges uniformly to a function f, then $f \in AP(\mathbb{R}, \mathscr{Y})$. Hence, $AP(\mathbb{R}, \mathscr{Y})$ is a closed subspace of $BC(\mathbb{R}, \mathscr{Y})$.*

A useful characterization of Bochner's almost periodic functions is given by the following result.

Theorem 2.3. [22] *A function $f : \mathbb{R} \longrightarrow \mathscr{Y}$ is almost periodic if and only if, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exist a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s'_n)_{n\in\mathbb{N}}$ and a continuous function g such that $\lim_{n\to+\infty} \sup_{t\in\mathbb{R}} \|f(t+s_n) - g(t)\| = 0$.*

Let $(\mathscr{Z}, \|.\|)$ be another Banach space.

Definition 2.2. [6] A continuous function $f : \mathbb{R} \times \mathscr{Y} \longrightarrow \mathscr{Z}$ is said to be almost periodic in t uniformly with respect to x in $\mathscr{Y}$ if, for each compact set K in $\mathscr{Y}$ and for all $\varepsilon > 0$, there exists $l_{\varepsilon,K} > 0$, such that, for every $a \in \mathbb{R}$, there exists $\tau \in [a, a + l_{\varepsilon,K}]$ satisfying

$$\sup_{x\in K} \|f(t+\tau, x) - f(t,x)\| < \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

The space of all such functions is denoted by $APU(\mathbb{R} \times \mathscr{Y}, \mathscr{Z})$.

Let us now turn to the concept of almost periodicity in the sense of Stepanov. Fixed $p \geq 1$.

Definition 2.3. [6] Let $f \in L^p_{loc}(\mathbb{R},\mathscr{Y})$. f is said to be S^p-almost periodic or almost periodic in Stepanov sense if, for all $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that, for all $a \in \mathbb{R}$, there exists $\tau_\varepsilon \in [a, a+l(\varepsilon)]$ such that

$$\left(\int_t^{t+1} \|f(s+\tau_\varepsilon) - f(s)\|^p ds\right)^{\frac{1}{p}} < \varepsilon \text{ for all } t \in \mathbb{R}.$$

Definition 2.4. [6] Let $f \in L^p_{loc}(\mathbb{R},\mathscr{Y})$. The Bochner transformation of f is the function $f^b : \mathbb{R} \to L^p([0,1],\mathscr{Y})$ defined, for $t \in \mathbb{R}$, by $f^b(t)(s) = f(t+s)$ for all $s \in [0,1]$.

Proposition 2.1. [6] *The following statements are true:*

(i) A function $f \in L^p_{loc}(\mathbb{R},\mathscr{Y})$ is S^p-almost periodic if and only if f^b is almost periodic.
(ii) If $1 \le q \le p$ and f is S^p-almost periodic, then f is S^q-almost periodic.
(iii) Every almost periodic function is S^p-almost periodic for all $p \ge 1$.
(iv) An uniformly continuous Stepanov almost periodic function is almost periodic.

Proposition 2.2. [3] *A function $f \in L^p_{loc}(\mathbb{R},\mathscr{Y})$ is S^p-almost periodic if and only if, for each $\{\eta'_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, there exist a subsequence $\{\eta_n\}_{n\in\mathbb{N}}$ of $\{\eta'_n\}_{n\in\mathbb{N}}$ and $\tilde{f} \in L^p_{loc}(\mathbb{R};\mathscr{Y})$ such that $\tilde{f}^b$ is bounded and*

$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(s+\eta_n) - \tilde{f}(s)\|^p\right)^{\frac{1}{p}} = 0.$$

Example 2.1. [6]

- Every continuous periodic function is almost periodic. The following function
$$f_1(t) = \sum_{k=0}^{+\infty} \frac{1}{k^2}\sin(\frac{\pi t}{k^3}),$$
is almost periodic but not periodic.
- Let $H \in C_0^\infty(\mathbb{R},\mathbb{R})$ with support in $(-\frac{1}{2},\frac{1}{2})$ such that $H \ge 0$, $H(0) = 1$ and $\int_{-\frac{1}{2}}^{\frac{1}{2}} H(t)\mathrm{d}t = 1$. Let β_n be defined on $\mathbb{R}$ by: $\beta_n(t) = \sum\limits_{k\in P_n} H(n^2(t-k))$ with $P_n = 3^n(2\mathbb{Z}+1)$. Let f be given by $\mathrm{f_2(t) = \sum_{n\ge1}\beta_n(t)}$ for $t \in \mathbb{R}$. Then, f is S^1-almost periodic which is not bounded. Hence, it is not almost periodic (see [3]).
- The function
$$\mathrm{f_3(t) = \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right)}$$
is S^p-almost periodic for all $p \ge 1$, but not almost periodic.

Definition 2.5. [7] Let $1 \le p < \infty$. A function $f : \mathbb{R}\times\mathscr{Y} \longrightarrow \mathscr{X}$ such that $f(\cdot,x) \in L^p_{loc}(\mathbb{R},\mathscr{X})$ for each $x \in \mathscr{Y}$ is said to be S^p-almost periodic in t uniformly with respect to x in $\mathscr{Y}$ if for each compact set K in $\mathscr{Y}$, for every $\varepsilon > 0$ there exists $l_{\varepsilon,K} > 0$, such that for all $a \in \mathbb{R}$ there exits $\tau \in [a, a+l_{\varepsilon,K}]$ such that

$$\sup_{x\in K}\left(\int_t^{t+1} \|f(s+\tau,x) - f(s,x)\|^p ds\right)^{\frac{1}{p}} < \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

The space of all such functions is denoted by $APS^pU(\mathbb{R}\times\mathscr{Y},\mathscr{Z})$.

Theorem 2.4. [24, Theorem 3.2] *Let $f\in APS^pU(\mathbb{R}\times\mathscr{Y},\mathscr{Z})$ and $\varsigma\in AP(\mathbb{R},\mathscr{Y})$. Then, the function $t\mapsto f(t,\varsigma(t))$ is S^p-almost periodic.*

In the rest of this part, we will have a look at the basic properties of the concept of almost automorphy.

Definition 2.6. [12] A continuous function $f:\mathbb{R}\to\mathscr{Y}$ is said to be almost automorphic if, for any sequence of real numbers $(t_n')_{n\in\mathbb{N}}$, there exists a subsequence $(t_n)_{n\in\mathbb{N}}$ of $(t_n')_{n\in\mathbb{N}}$ such that

$$\lim_{m\to+\infty}\lim_{n\to+\infty} f(t+s_n-s_m)=f(t)$$

for any $t\in\mathbb{R}$. This is equivalent to the following:

$$g(t)=\lim_{n\to+\infty} f(t+t_n), \tag{2.1}$$

is well defined for any $t\in\mathbb{R}$ and

$$\lim_{n\to+\infty} g(t-t_n)=f(t)\quad \text{for all } t\in\mathbb{R}. \tag{2.2}$$

Moreover, if the limits in (2.1) and (2.2) are uniform on any compact subset $K\subset\mathbb{R}$, we say that f is compact almost automorphic.

Denote by $AA(\mathbb{R},\mathscr{Y})$ (resp. $AA_c(\mathbb{R},\mathscr{Y})$) the space of all (resp. compact) almost automorphic $\mathscr{Y}$-valued functions.

Theorem 2.5. [33] *The following assertions are hold:*

(i) Every almost automorphic function has a relatively compact range.
(ii) The spaces $AA(\mathbb{R},\mathscr{Y})$ and $AA_c(\mathbb{R},\mathscr{Y})$ are closed subspace of $BC(\mathbb{R},\mathscr{Y})$.
(iii) Let $f\in AA(\mathbb{R},\mathscr{Y})$. Then, $f\in AA_c(\mathbb{R},\mathscr{Y})$ if and only if f is uniformly continuous.

Definition 2.7. [33] A continuous function $f:\mathbb{R}\times\mathscr{Y}\longrightarrow\mathscr{Z}$ is said to be almost automorphic in t uniformly with respect to x in $\mathscr{Y}$ if, for each compact set K in $\mathscr{Y}$ and for every sequence $(s_n')_{n\geq 0}$ of real numbers, there exist a subsequence $(s_n)_{n\geq 0}\subset(s_n')_{n\geq 0}$ and a measurable function $g:\mathbb{R}\times\mathscr{Y}\longrightarrow\mathscr{Z}$, such that

$$\lim_{n\to+\infty}\sup_{x\in K}\|f(t+s_n,x)-g(t,x)\|=0 \quad\text{and}\quad \lim_{n\to+\infty}\sup_{x\in K}\|g(t-s_n,x)-f(t,x)\|=0$$

are well-defined for each $t\in\mathbb{R}$. The space of all such functions is denoted by $AAU(\mathbb{R}\times\mathscr{Y},\mathscr{Z})$.

Now, let us shift our focus to the concept of almost automorphy in the Stepanov sense.

Definition 2.8. [34] A function $f\in L^p_{loc}(\mathbb{R},\mathscr{Y})$ is said to be S^p-almost automorphic for some $p\geq 1$ if the function $f^b:\mathbb{R}\mapsto L^p_{loc}([0,1],\mathscr{Y})$ is almost automorphic.

The following characterization of almost automorphy in the sense of Stepanov is essential for the remainder of this work.

Proposition 2.3. [34]*A function $f\in L^p_{loc}(\mathbb{R},\mathscr{Y})$ is S^p-almost automorphic if and only if, for every sequence of real numbers $(s_n')_n$, there exist a subsequence $(s_n)_n$ of $(s_n')_n$ and a function $g\in L^p_{loc}(\mathbb{R},\mathscr{Y})$ such that g^b is bounded and, for each $t\in\mathbb{R}$,*

$$\lim_{n\to+\infty}\left(\int_t^{t+1}\|f(s+s_n)-g(s)\|^p ds\right)^{\frac{1}{p}}=0$$

and

$$\lim_{n\to+\infty}\left(\int_t^{t+1}\|g(s-s_n)-f(s)\|^p ds\right)^{\frac{1}{p}}=0.$$

Let $SAA^p(\mathbb{R},\mathscr{Y})$ denote the space of S^p-almost automorphic $\mathscr{Y}$-valued functions on $\mathbb{R}$. Then, for all $p\geq 1$, $AA(\mathbb{R},\mathscr{Y})\subset SAA^p(\mathbb{R},\mathscr{Y})$. Moreover, if $p\geq q$, then $SAA^p(\mathbb{R},\mathscr{Y})\subset SAA^q(\mathbb{R},\mathscr{Y})$.

Example 2.2. [33]

(1) The function f_3 defined in Example 2.1 is an almost automorphic but not almost periodic, since it is not uniformly continuous.

(2) Let $(a_n)_{n\in\mathbb{Z}}$ be an almost automorphic sequence and $\varepsilon_0\in]0,\dfrac{1}{2}[$. Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by:

$$f(t)=\begin{cases} a_n & \text{if } \ t\in]n-\varepsilon_0,n+\varepsilon_0[,\\ \\ 0 & \text{otherwise.}\end{cases}$$

Then, f is S^p-almost automorphic for all $p\geq 1$ but not almost automorphic.

Definition 2.9. [34] Let $1\leq p<\infty$. A function $f:\mathbb{R}\times\mathscr{Y}\longrightarrow\mathscr{Z}$ such that $f(\cdot,x)\in L^p_{loc}(\mathbb{R},\mathscr{Z})$ for each $x\in\mathscr{Y}$ is said to be S^p-almost automorphic in t uniformly with respect to x in $\mathscr{Y}$, if for each compact set K in $\mathscr{Y}$ for every sequence $(s')_{n\geq 0}$ of real numbers, there exists a subsequence $(s_n)_{n\geq 0}\subset(s'_n)_{n\geq 0}$ and a function $g(\cdot,x)\in L^p_{loc}(\mathbb{R},\mathscr{Y})$ for each $x\in\mathscr{Y}$, such that

$$\lim_{n\to+\infty}\sup_{x\in K}\left(\int_t^{t+1}\|f(s+s_n,x)-g(s,x)\|^p ds\right)^{\frac{1}{p}}=0$$

and

$$\lim_{n\to+\infty}\sup_{x\in K}\left(\int_t^{t+1}\|g(s-s_n,x)-f(s,x)\|^p ds\right)^{\frac{1}{p}}=0.$$

The space of all such functions is denoted by $AAS^pU(\mathbb{R}\times\mathscr{Y},\mathscr{Z})$.

Theorem 2.6. [8, Theorem 3.1] *Let $f\in AAS^pU(\mathbb{R}\times\mathscr{Y},\mathscr{Z})$ and $\varsigma\in AA(\mathbb{R},\mathscr{Y})$. Then, the function $t\mapsto f(t,\varsigma(t))$ is S^p-almost automorphic.*

2.2. **Variation of constants formula.** Throughout the rest of this work, we consider $X=L^1_\rho(\mathbb{R}^-,\mathbb{C}^m)$, the phase space of equation (1.1), consists of equivalence classes of measurable function $\varphi:\mathbb{R}^-\to\mathbb{C}^m$ such that

$$\int_{-\infty}^0|\varphi(\theta)|e^{\rho\theta}d\theta<+\infty.$$

X is a Banach space equipped with the following norm

$$\|\varphi\|=\int_{-\infty}^0|\varphi(\theta)|\,e^{\rho\theta}\,d\theta\quad\text{for}\quad\varphi\in X,$$

where $\rho > 0$ is the constant mentioned in assumption (**H1**). To equation (1.1), we associate the following initial value problem:

$$\begin{cases} x(t) = \int_{-\infty}^{0} K(-\theta)x_t(\theta)d\theta + f(t) \text{ for } t \geq \sigma, \\ \\ x_\sigma = \varphi, \end{cases} \tag{2.3}$$

where $\sigma \in \mathbb{R}$, $K : \mathbb{R}^+ \to \mathbb{C}^{m\times m}$ is a measurable function verifies condition (**H1**), and $f : \mathbb{R} \to \mathbb{C}^m$ is a continuous function. For $t \geq \sigma$, $x_t : \mathbb{R}^- \to \mathbb{C}^m$ is the history function defined, for $\theta \leq 0$, by $x_t(\theta) = x(t+\theta)$.

Definition 2.10. [29] A function $x : \mathbb{R} \to \mathbb{C}^m$ is said to be a solution of the initial value problem (2.3) on $\mathbb{R}$ if x satisfies the following conditions:

(i) $x_\sigma = \varphi$ on $\mathbb{R}^-$,
(ii) $x \in L^1_{loc}([\sigma,+\infty[;\mathbb{C}^m)$,
(iii) $x(t)$ satisfies equation (2.3) for $t \geq \sigma$.

The following result ensures the existence and uniqueness of the solution of equation (2.3).

Proposition 2.4. [29] *Under the above conditions, equation* (2.3) *has a unique solution defined for $t \geq \sigma$.*

For $\sigma \in \mathbb{R}$ and $\varphi \in X$, we denote by $x(.,\sigma,\varphi,f)$ the unique solution of equation (2.3) on $[\sigma,\infty[$. Observe that $x(.,\sigma,\varphi,f)$ is continuous on $]\sigma,+\infty[$ and $\lim_{t\to\sigma^+} x(t,\sigma,\varphi,f)$ exists. Consequently, the function $t \mapsto x_t(.,\sigma,\varphi,f)$ is also continuous on $]\sigma,+\infty[$, with values in X. In addition, if $\varphi(\theta) = \psi(\theta)$ a.e $\theta \in \mathbb{R}^-$, for some $\varphi,\psi \in X$, then $x(.,\sigma,\varphi,f) = x(.,\sigma,\psi,f)$. We refer to [29] for the basic properties of equation (2.3).

For $t \geq 0$, we define operator $T(t) : X \to X$ by

$$T(t)\varphi = x_t(.,0,\varphi,0),$$

where $x_t(.,0,\varphi,0)$ is the unique solution of equation (2.3) with $\sigma = 0$ and $f = 0$.

Theorem 2.7. [29] *The family $(T(t))_{t\geq 0}$ is a linear strongly continuous semigroup on X. Moreover, the operator $(\mathscr{A},D(\mathscr{A}))$ defined by*

$$\begin{cases} D(\mathscr{A}) = \{\varphi \in X \mid \varphi(\theta) = \tilde{\varphi}(\theta) \text{ a.e. } \theta \in \mathbb{R}^- \text{ for some } \tilde{\varphi} \in \tilde{X}\} \\ \\ \mathscr{A}\varphi = \dfrac{d\tilde{\varphi}}{d\theta}, \quad \varphi \in D(\mathscr{A}), \end{cases}$$

where

$$\tilde{X} = \left\{ \tilde{\varphi} \in X \mid \tilde{\varphi} \text{ is locally absolutely continuous on } \mathbb{R}^-, \frac{d\tilde{\varphi}}{d\theta} \in X \text{ and } \tilde{\varphi}(0) = \int_{-\infty}^{0} K(-\theta)\tilde{\varphi}(\theta)d\theta \right\},$$

is its infinitesimal generator.

For $n \geq 0$, we define $\Gamma^n : \mathbb{R}^- \longrightarrow \mathbb{R}^+$ by

$$\Gamma^n(\theta) = \begin{cases} 2n(n\theta+1), & \frac{-1}{n} \leq \theta \leq 0 \\ 0, & \theta < \frac{-1}{n}. \end{cases}$$

For each $n \in \mathbb{N}^*$, $\int_{-\infty}^{0} \Gamma^n(\theta)\, d\theta = 1$. Moreover, for all $v \in \mathbb{C}^m$, we have $\Gamma^n v \in X$ and $\|\Gamma^n v\| \leq |v|$.

The following result provides a representation of the solutions of equation (2.3) in the phase space X in terms of $(T(t))_{t\geq 0}$ and f. This outcome allows us to study the asymptotic behavior of the solutions from the perspective of an infinite-dimensional dynamical system, which plays a crucial role in this study.

Theorem 2.8. [29] *The unique solution $x(.,\sigma,\varphi,f)$ of equation (2.3) on $\mathbb{R}$ satisfies the following variation of constants formula in X:*

$$x_t(.,\sigma,\varphi,f) = T(t-\sigma)\varphi + \lim_{n\longrightarrow\infty} \int_\sigma^t T(t-s)(\Gamma^n f(s))ds \quad \text{for} \quad t \geq \sigma. \tag{VCF}$$

In addition, the above limit exists uniformly on $[\sigma, T]$ for any $T > \sigma$.

In the following result, we give the relation between the solutions of equation (2.3) and a continuous function with values in X which satisfies (VCF). For that purpose, let us define

$$\overline{X} := \left\{ \phi \in X : \phi \text{ is countinous on } [-\varepsilon_\phi, 0] \text{for some } \varepsilon_\phi > 0 \right\}$$

and

$$X_0 = \left\{ \varphi \in X : \varphi = \phi \quad \text{a.e. on } \mathbb{R}^- \quad \text{for some } \phi \in \overline{X} \right\}.$$

The space X_0 is a normed space equipped with the following norm:

$$\|\varphi\|_{X_0} = \|\varphi\| + |\varphi[0]| \quad \text{for} \quad \varphi \in X_0,$$

where $\varphi[0] = \phi(0)$ represents the value of φ at zero.

Theorem 2.9. [29] *If $x(.,\sigma,\varphi,f)$ is a solution to equation (2.3) on $\mathbb{R}$, then $\xi(t) = x_t(.,\sigma,\varphi,f)$ is continuous on $\mathbb{R}$ with values in X_0 and satisfies in X the following formula:*

$$\xi(t) = T(t-\sigma)\xi(\sigma) + \lim_{n\longrightarrow\infty} \int_\sigma^t T(t-s)(\Gamma^n f(s))ds \quad \textit{for} \quad t \geq \sigma. \tag{2.4}$$

Conversely, if a continuous function $\xi : \mathbb{R} \longrightarrow X$ satisfies formula (2.4), then $\xi \in C(\mathbb{R}, X_0)$ and the function $x(t,\sigma,\xi(\sigma),f) = \xi(t)[0]$ is a solution to equation (2.3) on $\mathbb{R}$.

3. Asymbtoic Behaviour of $(T(t))_{t\geq 0}$ and Reduction Principal

In this section, we develop a new result on the reduction principle by employing the variation of constants formula (VCF) and a spectral decomposition of the phase space. This allows us to analyze the behavior of mean values bounded solutions. To this end, let us begin with some basic definitions and proprieties in spectral theory. Recall that $\mathscr{Y}$ is always a Banach space.

Definition 3.1. [36] The Kuratowski measure of noncompactness of a bounded set D on $\mathscr{Y}$ is defined by:

$\chi(D) = \inf\{\varepsilon > 0; D$ admits a cover by a finite number of sets of diameter less than strictly $\varepsilon\}$.

For a bounded linear operator B on $\mathscr{Y}$, the Kuratowski measure of noncompactness of B is defined by

$$|B|_{ess} := \inf\{k > 0 : \chi(B(D)) \leq k\chi(D) \text{ for any bounded set } D \text{ of } \mathscr{Y}\}$$

and for a C_0-semigroup $(V(t))_{t\geq 0}$, we define the essential growth bound $\omega_{ess}(V)$ by:

$$\begin{aligned} \omega_{ess}(V) &:= \lim_{t\to+\infty} \frac{\log\left(|V(t)|_{ess}\right)}{t} \\ &= \inf_{t>0} \frac{\log\left(|V(t)|_{ess}\right)}{t}. \end{aligned}$$

Definition 3.2. [36] Let $(B, D(B))$ be a closed operator on $\mathscr{Y}$. The essential spectrum, $\sigma_{ess}(B)$, of B is the set of all $\lambda \in \sigma(B)$, when at least one of the following conditions is satisfied:

(1) $R(\lambda I_{\mathscr{Y}} - B)$, the range of $\lambda I_{\mathscr{Y}} - B$, is not closed;
(2) $\bigcup\limits_{k\geq 0} \ker((\lambda I_{\mathscr{Y}} - B)^k)$ is infinity dimensional;
(3) $\lambda \in \overline{\sigma(B)\setminus\{\lambda\}}$.

Remark 3.1. [36] If B is a bounded linear operator on $\mathscr{Y}$, then

$$r_{ess}(B) := \sup\{|\lambda| : \lambda \in \sigma_{ess}(B)\} = \lim_{n\to+\infty} |B^n|_{ess}^{\frac{1}{n}} < \infty.$$

$r_{ess}(B)$ is said to be the essential spectral radius of B.

Next, we summarize the spectral properties of $\mathscr{A}$. Before that, we denote by $\sigma(\mathscr{A})$ (respectively, $\sigma_p(\mathscr{A})$) the spectrum (respectively, point spectrum) of $\mathscr{A}$.

Theorem 3.1. [29, Proposition 5 and Theorem 1] *The following statements are true:*

(1) $\Big\{\lambda \in \sigma(A) : Re(\lambda) > -\rho\Big\} \subset \sigma_p(\mathscr{A})$.
(2) Let $\lambda \in \mathbb{C}$ *such that* $Re(\lambda) > -\rho$. *Then,* $\lambda \in \sigma(A)$ *if and only if* $\det \Delta(\lambda) = 0$, *where* $\Delta(\lambda) = I_m - \int_{-\infty}^{0} K(-\theta)e^{\lambda\theta}d\theta$ *and* I_m *is the identity* $m\times m$ *matrix.*
(3) $r_{ess}(T(t)) \leq e^{-\rho t}$ *for all* $t > 0$.

By [36, Proposition 4.13] and the statement (3) in the above Theorem, we see that $\omega_{ess}(T) \leq -\rho < 0$. As a result of [18, Corollary 2.11, Chapter IV and Theorem 3.1, Chapter V], we have the following spectral decomposition result.

Theorem 3.2. [29] *The space* X *is decomposed as* $X = S \oplus U$ *such that*

(i) S *and* U *are closed subspace of* X *which are* $T(.)$*-invariant,*
(ii) dim $U < \infty$, *and the restriction of* $(T(t))_{t\geq 0}$ *to* U, *denoted by* $(T^U(t))_{t\in\mathbb{R}}$, *can be extended to a* C_0*-group,*
(iii) $\sigma(\mathscr{A}|_U) = \Big\{\lambda \in \sigma(\mathscr{A}) : Re(\lambda) \geq 0\Big\}$ *is finite and* $\sigma(\mathscr{A}|_{S\cap D(\mathscr{A})}) = \sigma(\mathscr{A}) \setminus \sigma(\mathscr{A}|_U)$,

(iv) there exist $C \geq 1$ and $\alpha > 0$ such that

$$\|T^S(t)\varphi\| \leq Ce^{-\alpha t}\|\varphi\| \quad \text{for} \quad \varphi \in S \quad \text{and} \quad t \geq 0,$$

where $(T^S(t))_{t\geq 0}$ is the restriction of $(T(t))_{t\geq 0}$ to S.

In addition, if $(T(t))_{t\geq 0}$ is hyperbolic; that is,

$$\sigma(\mathscr{A}) \cap i\mathbb{R} = \emptyset, \tag{H.P}$$

then

$$\|T^U(t)\varphi\| \leq Ce^{\alpha t}\|\varphi\| \quad \text{for} \quad \varphi \in U \quad \text{and} \quad t \leq 0.$$

Let $\Phi = \{\phi_1, \phi_2 \cdots, \phi_d\}$ be the vector basis of U with $d := \dim U$. Then, there exist d-elements $(\psi_1, \psi_2, \cdots, \psi_d)$ in X^* (the dual space of X) such that

$$\begin{cases} \langle \psi_i, \phi_j \rangle = \delta_{ij}, \\ \langle \psi_i, \phi \rangle = 0 \text{ for all } \phi \in \mathscr{S} \text{ and } i \in \{1, \cdots, d\}, \end{cases} \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^* and

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Let $\Psi = col\{\psi_1, \psi_2, \cdots, \psi_d\}$, $\langle \Psi, \Phi \rangle$ be a $(d \times d)$-matrix, where the (i, j)-component is $\langle \psi_i, \phi_j \rangle$. Denote by Π^U (resp. Π^S) the projection operator from X into U (resp. S). For each $\phi \in X$, we have

$$\Pi^U \phi \quad = \quad \Phi\langle \Psi, \phi \rangle,$$

where $\langle \Psi, \phi \rangle := (\langle \psi_1, \phi \rangle, \cdots, \langle \psi_d, \phi \rangle)$. Since $T^U(t)$ is a C_0-group on U, then there exists a $(d \times d)$-matrix G such that $T^U(t)\Phi = \Phi e^{Gt}$, $t \geq 0$, where the spectrum of the matrix G is identical with the set $\{\lambda \in \sigma(\mathscr{A}) : Re(\lambda) \geq 0\}$.

Lemma 3.1. [29, Lemma 2] *Let $\{e_1, \cdots, e_m\}$ be the canonical basis of $\mathbb{C}^m$. Then, there exits a $d \times m$-matrix H such that, for any $i \in \{1, \cdots, m\}$, $\lim\limits_{n \longrightarrow \infty} \langle \Psi, \Gamma^n e_i \rangle = He_i$.*

Let consider the following reduced ordinary differential equation:

$$z'(t) = Gz(t) + Hf(t) \text{ for } t \in \mathbb{R}. \tag{3.2}$$

The following Theorem shows that the dynamics of bounded solutions of equation (1.1) can be described by the ordinary differential equation (3.2) posed in the finite-dimensional space U. This result pertains to the reduction principle.

Theorem 3.3. [29, Proposition 6 and Theorem 8] *Assume that $f : \mathbb{R} \to \mathbb{C}^m$ is continuous and bounded. Then, the following statements are true:*

(1) For any $t \in \mathbb{R}$, the limit $Y(t) := \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S(\Gamma^n f(s))ds$ exits in X. Moreover, if $\eta : \mathbb{R} \to S$ is a continuous functions satisfies the following formula:

$$\eta(t) = T^S(t-\sigma)\eta(\sigma) + \lim_{n\to+\infty} \int_{\sigma}^{t} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \text{ for } t \geq \sigma,$$

then $\eta(t) = Y(t)$ for all $t \in \mathbb{R}$.

(2) If x is a (resp. bounded) solution of equation (1.1) on $\mathbb{R}$, then the function $z(t) := \langle \Psi, x_t \rangle$ is a (resp. bounded) solution of equation (3.2) on $\mathbb{R}$.

(3) *If z is a (resp. bounded) solution of equation (3.2) on $\mathbb{R}$, then the function $\xi:\mathbb{R}\to X$ defined by*

$$\xi(t)=\Phi z(t)+\lim_{n\longrightarrow\infty}\int_{-\infty}^{t}T^S(t-s)\Pi^S(\Gamma^n f(s))ds,\ \text{ for } t\in\mathbb{R},$$

is continuous from $\mathbb{R}$ to X_0 and the function $t\mapsto\xi(t)[0]$ is a (resp. bounded) solution of equation (1.1) on $\mathbb{R}$.

We obtain the same result in the above Theorem by weakening the boundedness assumption of f, the forcing term. Namely, f is only bounded by mean values.

Theorem 3.4. *Assume that $f:\mathbb{R}\to\mathbb{C}^m$ is continuous and $\sup\limits_{t\in\mathbb{R}}\int_t^{t+1}|f(s)|ds<+\infty$. Then, the following statements are true:*

(1) *For any $t\in\mathbb{R}$, the limit $Y(t):=\lim\limits_{n\to+\infty}\int_{-\infty}^{t}T^S(t-s)\Pi^S(\Gamma^n f(s))ds$ exists in X. Moreover, if $\eta:\mathbb{R}\to S$ is a continuous functions satisfies the following formula:*

$$\eta(t)=T^S(t-\sigma)\eta(\sigma)+\lim_{n\to+\infty}\int_{\sigma}^{t}T^S(t-s)\Pi^S(\Gamma^n f(s))ds \text{ for } t\geq\sigma \tag{SVCF}$$

and $\sup\limits_{t\in\mathbb{R}}\|\eta(t)\|<+\infty$, then $\eta(t)=Y(t)$ for all $t\in\mathbb{R}$.

(2) *If x is a solution of equation (1.1) on $\mathbb{R}$, then the function $z(t):=\langle\Psi,x_t\rangle$ is a solution of equation (3.2) on $\mathbb{R}$.*

(3) *If z is a solution of equation (3.2) on $\mathbb{R}$, then the function $\xi:\mathbb{R}\to X$ defined by*

$$\xi(t)=\Phi z(t)+\lim_{n\longrightarrow\infty}\int_{-\infty}^{t}T^S(t-s)\Pi^S(\Gamma^n f(s))ds,\quad \text{for}\quad t\in\mathbb{R},$$

is continuous from $\mathbb{R}$ to X_0 and the function $t\mapsto\xi(t)[0]$ is a solution of equation (1.1) on $\mathbb{R}$.

Proof. (1) Let $t\in\mathbb{R}$ and n sufficiently large. Then,

$$\begin{aligned}\left\|\int_{-\infty}^{t}T^S(t-s)\Pi^S(\Gamma^n f(s))\,ds\right\| &\leq C\|\Pi^S\|\int_{\infty}^{t}e^{-\alpha(t-s)}\|\Pi^S\||f(s)|ds\\ &\leq C\|\Pi^S\|\sum_{k=1}^{\infty}\int_{t-k}^{t-k+1}e^{-\alpha(t-s)}|f(s)|ds\\ &\leq C\|\Pi^S\|\sum_{k=1}^{\infty}e^{-\alpha(k-1)}\int_{t-k}^{t-k+1}|f(s)|ds\\ &\leq K,\end{aligned}$$

where

$$K=C\|\Pi^s\|\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}|f(s)|\,ds\right)\frac{1}{1-e^{-\alpha}}.$$

Let

$$H(n,s,t)=T^S(t-s)\Pi^S(\Gamma^n f(s))\quad \text{for } s\leq t\quad\text{and}\quad n\in\mathbb{N}.$$

Then, for n and m large enough and $\sigma \leq t$, we have

$$\begin{aligned}\left\| \int_{-\infty}^{t} H(n,s,t)ds - \int_{-\infty}^{t} H(m,s,t)ds \right\| &\leq \left\| \int_{-\infty}^{\sigma} H(n,s,t)ds \right\| + \left\| \int_{-\infty}^{\sigma} H(m,s,t)ds \right\| \\ &+ \left\| \int_{\sigma}^{t} H(n,s,t)ds - \int_{\sigma}^{t} H(m,s,t)ds \right\| \\ &\leq 2Ke^{-\alpha(t-\sigma)} + \left\| \int_{\sigma}^{t} H(n,s,t)ds - \int_{\sigma}^{t} H(m,s,t)ds \right\|.\end{aligned}$$

Since $\lim_{n\longrightarrow\infty} \int_{\sigma}^{t} H(n,s,t)ds$ exists in X, then

$$\limsup_{n,m\longrightarrow\infty} \left\| \int_{-\infty}^{t} H(n,s,t)ds - \int_{-\infty}^{t} H(m,s,t)ds \right\| \leq 2Ke^{-\alpha(t-\sigma)}.$$

By letting $\sigma \longrightarrow -\infty$, we obtain

$$\limsup_{n,m\longrightarrow\infty} \left\| \int_{-\infty}^{t} H(n,s,t)ds - \int_{-\infty}^{t} H(m,s,t)ds \right\| = 0.$$

Thus, by the completeness of X, we deduce that the limit $\lim_{n\longrightarrow\infty} \int_{-\infty}^{t} H(n,s,t)ds$ exists in X.

Now, let $\eta : \mathbb{R} \to S$ be a bounded function satisfies the formula (SVCF). For $t \in \mathbb{R}$, we observe that

$$\begin{aligned}\|\eta(t) - Y(t)\| &= \left\| T^S(t-\sigma)\eta(\sigma) + \lim_{n\to+\infty} \int_{\sigma}^{t} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \right. \\ &- \left. \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \right\| \\ &\leq Ce^{-\alpha(t-\sigma)} \sup_{t\in\mathbb{R}} \|\eta(t)\| + \left\| \lim_{n\to+\infty} \int_{-\infty}^{\sigma} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \right\| \\ &\leq Ce^{-\alpha(t-\sigma)} \sup_{t\in\mathbb{R}} \|\eta(t)\| + \left\| Y(0) - \lim_{n\to+\infty} \int_{\sigma}^{0} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \right\|.\end{aligned}$$

By the first part of the proof, we assert, for any $t \in \mathbb{R}$, that

$$\lim_{\sigma\to-\infty} \lim_{n\to+\infty} \int_{\sigma}^{t} T^S(t-s)\Pi^S(\Gamma^n f(s))ds = Y(t).$$

Then,

$$\lim_{\sigma\to-\infty} \left\| Y(0) - \lim_{n\to+\infty} \int_{\sigma}^{0} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \right\| = 0.$$

Consequently,

$$\|\eta(t) - Y(t)\| \leq \lim_{\sigma\to-\infty} \left(Ce^{-\alpha(t-\sigma)} \sup_{t\in\mathbb{R}} \|\eta(t)\| + \left\| Y(0) - \lim_{n\to+\infty} \int_{\sigma}^{0} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \right\| \right) = 0.$$

Hence, for all $t \in \mathbb{R}$, $\eta(t) = Y(t)$.

(2) and (3) can be deduced from [29, Theorem 7] easily.

□

4. Stepanov Version of Massera and Bohr–Neugebauer Type Results

In this section, we use the reduction principal to investigate the nature of mean values bounded solutions of equation(1.1).

4.1. Almost periodicity.

We start by the following Lemma.

Lemma 4.1. *Let f be a continuous S^p-almost periodic for some $p \geq 1$. Then, the function $Y : \mathbb{R} \to X$ defined, for $t \in \mathbb{R}$, by $Y(t) = \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)(\Pi^S \Gamma^n f(s))ds$, is almost periodic.*

Proof. For $k \geq 1$, we define the following function

$$Y_k(t) = \lim_{n\to+\infty} \int_{t-k}^{t-k+1} T^S(t-s)\Pi^S(\Gamma^n f(s))ds.$$

By the strongly continuity of $(T(t))_{t\geq 0}$, we can assert that, for all $k \geq 1$, Y_k is continuous on $\mathbb{R}$. Let $\varepsilon > 0$. Since, f is S^p-almost periodic, then it is S^1-almost periodic. Thus, there exists $l(\varepsilon) > 0$ such that for all $a \in \mathbb{R}$, there exists $\tau_\varepsilon \in [a, a + l(\varepsilon)]$ such that

$$\sup_{t\in\mathbb{R}} \int_{t}^{t+1} |f(s+\tau_\varepsilon) - f(s)|ds < \frac{\varepsilon}{C\|\Pi^S\|}.$$

Therefore, for $t \in \mathbb{R}$, we have

$$\begin{aligned}
\|Y_k(t+\tau_\varepsilon) - Y_k(t)\| &\leq \Bigg\| \lim_{n\to+\infty} \int_{t+\tau_\varepsilon-k}^{t+\tau_\varepsilon-k+1} T^S(t+\tau_\varepsilon - s)\Pi^S(\Gamma^n f(s))ds \\
&- \lim_{n\to+\infty} \int_{t-k}^{t-k+1} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \Bigg\| \\
&\leq \left\| \lim_{n\to+\infty} \int_{t-k}^{t-k+1} T^S(t-s)\Pi^S\Big(\Gamma^n (f(s+\tau_\varepsilon) - f(s)\Big)ds \right\| \\
&\leq \int_{t-k}^{t-k+1} Ce^{-\alpha(t-s)}\|\Pi^S\||f(s+\tau_\varepsilon) - f(s)|ds \\
&\leq C\|\Pi^S\| \sup_{t\in\mathbb{R}} \int^{t+1} |f(s+\tau_\varepsilon) - f(s)|ds \\
&\leq C\|\Pi^S\| \times \frac{\varepsilon}{C\|\Pi^S\|} \\
&\leq \varepsilon.
\end{aligned}$$

Then, we deduce that Y_k is almost periodic for all $k \geq 1$.

On the other hand, we observe that

$$\begin{aligned}
\|Y_k(t)\| &\leq \left\| \lim_{n\to+\infty} \int_{t-k}^{t-k+1} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \right\| \\
&\leq C\|\Pi^S\| \int_{t-k}^{t-k+1} e^{-\alpha(t-s)}|f(s)|ds \\
&\leq C\|\Pi^S\| e^{-\alpha(k-1)} \int_{t-k}^{t-k+1} |f(s)|ds \\
&\leq C\|\Pi^S\| e^{\alpha} e^{-\alpha k} \sup_{t\in\mathbb{R}} \int_{t}^{t+1} |f(s)|ds.
\end{aligned}$$

Then, by Weierstrass Theorem's, we deduce that the series $\sum_{k=1}^{\infty} Y_k(t)$ is uniformly convergent on $\mathbb{R}$. For $t \in \mathbb{R}$ and $p \geq 1$, we have

$$\begin{aligned}
\left\| \sum_{k=1}^{p} Y_k(t) - Y(t) \right\| &\leq \left\| \lim_{n\to+\infty} \int_{t-p}^{t} T^S(t-s)\Pi^S(\Gamma^n f(s))ds - \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \right\| \\
&\leq \left\| \lim_{n\to+\infty} \int_{-\infty}^{t-p} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \right\| \\
&\leq C\|\Pi^S\| \int_{-\infty}^{t-p} e^{-\alpha(t-s)}|f(s)|ds \\
&\leq C\|\Pi^S\| \sum_{k=p+1}^{\infty} \int_{t-k}^{t-k+1} e^{-\alpha(t-s)}|f(s)|ds \\
&\leq C\|\Pi^S\| e^{\alpha} \sup_{t\in\mathbb{R}} \left(\int_{t}^{t+1} |f(s)|ds \right) \sum_{k=p+1}^{\infty} e^{-\alpha k}.
\end{aligned}$$

Consequently, for any $t \in \mathbb{R}$, $\lim_{p\to\infty} \left\| \sum_{k=1}^{p} Y_k(t) - Y(t) \right\| = 0$, which means that $\sum_{k=1}^{\infty} Y_k(t)$ converges uniformly to $Y(t)$. Hence, Y is almost periodic. □

Now, we are able to establish the main result ensuring the existence of Stepanov almost periodic solutions of (1.1).

Theorem 4.1. *Let f be a continuous S^p-almost periodic for some $p \geq 1$. Then,*

(1) If equation (1.1) has a solution $x(\cdot)$ on $\mathbb{R}^+$ such that

$$\sup_{t\in\mathbb{R}^+} \int_{t}^{t+1} |x(s)|ds < \infty,$$

then it has an S^p-almost periodic solution on $\mathbb{R}$.

(2) If $x(\cdot)$ is a solution of equation (1.1) on $\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \int_{t}^{t+1} |x(s)|ds < \infty,$$

then the function $t \mapsto x_t$ is almost periodic and $x(\cdot)$ is S^p-almost periodic.

(3) If equation (1.1) has an almost periodic in Bohr sense, then f is Bohr almost periodic.

Proof. First of all, we notice that n $t \mapsto Hf(t)$ is also S^p-almost periodic.

(1) Let x be a solution of equation (1.1) on $\mathbb{R}^+$ with

$$\sup_{t\in\mathbb{R}^+} \int_{t}^{t+1} |x(s)|ds < \infty.$$

Then, for $t \geq 0$, we have

$$
\begin{aligned}
\|x_t\| &= \int_{-\infty}^{0} |x(t+\theta)| e^{\rho\theta} d\theta \\
&= \int_{-\infty}^{-t} |x(t+\theta)| e^{\rho\theta} d\theta + \int_{-t}^{0} |x(t+\theta)| e^{\rho\theta} d\theta \\
&\leq \int_{-\infty}^{0} |x_0(\theta)| e^{-t\rho} e^{\rho\theta} d\theta + \int_{0}^{t} e^{-\rho(t-s)} |x(s)| ds \\
&\leq \|x_0\| + e^{-\rho[t]} \int_{0}^{[t]+1} e^{\rho s} |x(s)| ds \\
&\leq \|x_0\| + e^{-\rho[t]} \sum_{k=0}^{[t]} \int_{k}^{k+1} e^{\rho s} |x(s)| ds \\
&\leq \|x_0\| + \left(\sup_{t\in\mathbb{R}^+} \int_{t}^{t+1} |x(s)| ds \right) e^{-\rho[t]} \sum_{k=0}^{[t]} e^{\rho(k+1)} \\
&\leq \|x_0\| + \left(\sup_{t\in\mathbb{R}^+} \int_{t}^{t+1} |x(s)| ds \right) e^{\rho} e^{-\rho[t]} \times \left(\frac{1-e^{\rho([t]+1)}}{1-e^{\rho}} \right) \\
&\leq \|x_0\| + \left(\sup_{t\in\mathbb{R}^+} \int_{t}^{t+1} |x(s)| ds \right) e^{\rho} \times \left(\frac{e^{-\rho[t]}-e^{\rho}}{1-e^{\rho}} \right).
\end{aligned}
$$

Consequently, the function $t \mapsto x_t$ is bounded on $\mathbb{R}^+$ with values in X. Therefore, $t \mapsto \langle \Psi, x_t \rangle$ is a bounded solution of equation (3.2) on $\mathbb{R}^+$ (see Theorem 3.3). Thanks to [3, Corollary 6.4], we obtain that equation (3.2) has an almost periodic solution z. Using Lemma 4.1, we conclude that the function $\xi : \mathbb{R} \to X$ defined by

$$\xi(t) = \Phi z(t) + \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S \Gamma^n f(s) ds,$$

for $t \in \mathbb{R}$ is almost periodic. By Theorem 3.3, we see that $\tilde{x}(t) = \xi(t)[0]$ is a solution of equation (1.1) on $\mathbb{R}$ with $\tilde{x}_t = \xi(t)$ for all $t \in \mathbb{R}$. Then,

$$\tilde{x}(t) = \int_{-\infty}^{0} K(-\theta)\xi(t)(\theta) d\theta + f(t) \text{ for } t \in \mathbb{R}.$$

Hence, $\tilde{x}$ is an S^p-almost periodic solution of equation (1.1).

(2) Let x be a solution of equation (1.1) on $\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \int_{t}^{t+1} |x(s)| ds < \infty.$$

Then, the function $t \mapsto x_t$ is bounded, since

$$\begin{aligned}
\|x_t\| &= \int_{-\infty}^{0} |x(t+\theta)|e^{\rho\theta} d\theta \\
&= \int_{-\infty}^{t} e^{-\rho(t-\theta)}|x(\theta)|d\theta \\
&\le \sum_{k=1}^{\infty} \int_{t-k}^{t-k+1} e^{-\rho(t-\theta)}|x(\theta)|d\theta \\
&\le \sum_{k=1}^{\infty} e^{-\rho(k-1)} \int_{t-k}^{t-k+1} |x(\theta)|d\theta \\
&\le \frac{e^{\rho}}{e^{\rho}-1} \sup_{t\in\mathbb{R}} \int_{t}^{t+1} |x(s)|ds.
\end{aligned}$$

Therefore, using Theorem 3.3, the function $z(t) := \langle \Psi, x_t \rangle$ is a bounded solution of equation (3.2) on $\mathbb{R}$. By [3, Theorem 6.2], we deduce that $z(t)$ is almost periodic. Since $t \mapsto \Pi^S x_t$ is also bounded, then by using the (1) in Theorem 3.3, for all $t \in \mathbb{R}$,

$$\Pi^S x_t = \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S\Gamma^n f(s)ds.$$

Thus, $t \mapsto \Pi^S x_t$ is almost periodic (see Lemma 4.1). Using the (3) in Theorem 3.3, we have that

$$x_t = \Phi\langle \Psi, x_t \rangle + \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S\Gamma^n f(s)ds.$$

It follows that $t \mapsto x_t$ is almost periodic. As for any $t \in \mathbb{R}$ $x(t) = \int_{-\infty}^{0} K(\theta)x_t(\theta)d\theta + f(t)$, one has $x(\cdot)$ is S^p-almost periodic.

(3) Assume that equation (1.1) has an almost periodic solution $x(\cdot)$ in Bohr sense. Then, by (2), the function $t \mapsto x_t$ is also almost periodic. Consequently, the function $t \mapsto \int_{-\infty}^{0} K(\theta)x_t(\theta)d\theta$ is almost periodic. Since, for $t \in \mathbb{R}$, $f(t) = x(t) - \int_{-\infty}^{0} K(\theta)x_t(\theta)d\theta$, then f is Bohr almost periodic, which ends the proof. $\square$

Now, we give a sufficient condition for the existence and uniqueness of a Stepanov almost periodic solution.

Theorem 4.2. *Let f be a continuous S^p-almost periodic for some $p \ge 1$ and assume that $(T(t))_{t\ge 0}$ is hyperbolic. Then, equation (1.1) has a unique S^p-almost periodic solution $x(\cdot)$. Moreover, for any $t \in \mathbb{R}$,*

$$x_t = \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S\Gamma^n f(s)ds - \lim_{n\to+\infty} \int_{t}^{+\infty} T^U(t-s)\Pi^U\Gamma^n f(s)ds. \tag{4.1}$$

In addition, if $\{\lambda \in \sigma(\mathscr{A}):Re(\lambda) \ge 0\} = \emptyset$, then the unique S^p-almost periodic $x(\cdot)$ is globally attractive. Namely, for any solution $\tilde{x}(\cdot)$ of equation (1.1) on $\mathbb{R}^+$, $\lim_{t\to+\infty} |\tilde{x}(t) - x(t)| = 0$.

Proof. Under condition (H.P), we can affirm that

$$\sigma(G) = \{\lambda \in \sigma(\mathscr{A}) : Re(\lambda) > 0\}.$$

Then, equation (3.2) has a unique bounded solution on $\mathbb{R}$ given by

$$z(t) = -\int_{t}^{+\infty} e^{(t-s)G}Hf(s)ds \text{ for } t \in \mathbb{R},$$

which is almost periodic (see [27, Lemma 3.3]). Consequently, in the light of Theorem 3.4 and Lemma 4.1, the function $\xi : \mathbb{R} \to X$ defined by

$$\xi(t) = \Phi z(t) + \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S\Gamma^n f(s)ds \text{ for } t \in \mathbb{R},$$

is almost periodic and $x(t) := \xi(t)[0]$ is an S^p-almost periodic solution of equation (1.1).

On the other hand, for any $t \in \mathbb{R}$, we have

$$\begin{aligned}
\Phi z(t) &= -\Phi \int_{t}^{+\infty} e^{(t-s)G} H f(s)ds \\
&= -\lim_{n\to+\infty} \int_{t}^{+\infty} \Phi e^{(t-s)G} \langle \Psi, \Gamma^n f(s)\rangle ds \\
&= -\lim_{n\to+\infty} \int_{t}^{+\infty} T^U(t-s)\Phi\langle \Psi, \Gamma^n f(s)\rangle ds \\
&= -\lim_{n\to+\infty} \int_{t}^{+\infty} T^U(t-s)\Pi^U\Gamma^n f(s)ds.
\end{aligned}$$

Therefore, for $t \in \mathbb{R}$,

$$x_t = \xi(t) = \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S\Gamma^n f(s)ds - \lim_{n\to+\infty} \int_{t}^{+\infty} T^U(t-s)\Pi^U\Gamma^n f(s)ds.$$

Let $\tilde{x}$ be another S^p-almost periodic solution of equation (1.1). Then,

$$\sup_{t\in\mathbb{R}} \int_{t}^{t+1} |\tilde{x}(s)|ds < \infty.$$

Therefor, by Theorem 4.1, $t \mapsto \tilde{x}_t$ is almost periodic and hence $\tilde{z}(t) := \langle \Psi, \tilde{x}_t\rangle$ is an almost periodic solution of (3.2) on $\mathbb{R}$. By the uniqueness of z, we deduce that $\tilde{z} = z$. Thus $\tilde{x}_t = \xi(t)$ and then we conclude the uniqueness of x. Let us assume that $\{\lambda \in \sigma(\mathscr{A}) : Re(\lambda) \geq 0\} = \emptyset$. Consequently, we have $U = \{0_X\}$ and $X = S$. As per the initial part of this theorem, it follows that equation (1.1) has a unique S^p-almost periodic solution $x(\cdot)$. Let $\tilde{x}(\cdot)$ be an arbitrary solution of equation (1.1) on $\mathbb{R}^+$. According to Theorem 2.8, for any $t \geq 0$, we have

$$\|x_t - \tilde{x}_t\| = \|T(t)(x_0 - \tilde{x}_0)\| \leq Ce^{-\alpha t}\|x_0 - \tilde{x}_0\|.$$

Therefore, $\lim_{t\to+\infty} \|x_t - \tilde{x}_t\| = 0$. Moreover, for $t \geq 0$,

$$|x(t) - \tilde{x}(t)| = \left| \int_{-\infty}^{0} K(-\theta)(x(t+\theta) - \tilde{x}(t+\theta))d\theta \right| \leq \|K\|_{\infty,\rho}\|x_t - \tilde{x}_t\|.$$

Consequently, $\lim_{t\to+\infty} |x(t) - \tilde{x}(t)| = 0$, which makes the proof complete. □

Remark 4.1. Let f be purely Stepanov almost periodic. According to the claim presented in part (v) of the above Theorem, equation (1.1) never has a Bohr almost periodic solution.

4.2. **Almost automorphy.** Now, we focus our attention on the almost automorphic case. We begin with the following Lemma.

Lemma 4.2. *Assume that f is a continuous S^p-almost automorphic for some $p \geq 1$. Then, the function $Y : \mathbb{R} \to X$ given by*

$$Y(t) = \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)(\Pi^S\Gamma^n f(s))ds \text{ for } t \in \mathbb{R},$$

is compact almost automorphic.

Proof. For $k \geq 1$, we define the following function

$$\begin{aligned} Y_k(t) &:= \lim_{n\to\infty} \int_{t-k}^{t-k+1} T^S(t-s)\Pi^S(\Gamma^n f(s))ds \\ &= \lim_{n\to\infty} \int_0^1 T^S(k-s)\Pi^S(\Gamma^n f(t-k+s))ds \quad \text{for } t \in \mathbb{R}. \end{aligned}$$

Obviously, $Y_k(\cdot)$ is continuous. Let $(s'_j)_{j\in\mathbb{N}}$ be any sequence in $\mathbb{R}$. Since f is S^p-almost automorphic, then it is S^1-almost automorphic. Therefore, there exists a subsequence $(s_j)_{j\in\mathbb{N}}$ of $(s'_j)_{j\in\mathbb{N}}$ such that, for any $t \in \mathbb{R}$,

$$\lim_{m\to+\infty} \lim_{j\to+\infty} \int_0^1 |f(t+s_j-s_m+s)-f(t+s)|ds = 0.$$

Then, for $t \in \mathbb{R}$, on has

$$\begin{aligned} \|Y_k(t+s_j-s_m)-Y_k(t)\| &\leq \left\| \lim_{n\to\infty} \int_0^1 T^S(k-s)\Pi^S(\Gamma^n f(t+s_j-s_m-k+s)-f(t-k+s))ds \right\| \\ &\leq C\|\Pi^S\| \int_0^1 |f(t+s_j-s_m-k+s)-f(t-k+s)|ds. \end{aligned}$$

Thus,

$$\lim_{m\to+\infty} \lim_{j\to+\infty} \|Y_k(t+s_j-s_m)-Y_k(t)\| = 0.$$

On the other hand, by using similar arguments in the proof of Lemma 4.1, we can assert that the series $\sum_{k=1}^{\infty} Y_k(\cdot)$ converges uniformly to $Y(\cdot)$. Hence, $Y(\cdot)$ is almost automorphic.

Now, we claim that Y is uniformly continuous. Let $(t'_j)_{j\in\mathbb{N}}$ and $(s'_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ be any real sequences such that

$$\lim_{j\to+\infty} |t'_j - s'_j| = 0. \tag{4.2}$$

Then, if $t'_j \geq s'_j$ (see Theorem 3.4), we have

$$\begin{aligned} \|Y(t'_j)-Y(s'_j)\| &= \left\| T^S(t'_j-s'_j)Y(s'_j)-Y(s'_j)+\lim_{n\to+\infty}\int_{s'_j}^{t'_j} T^S(t'_j-\sigma)\Pi^S(\Gamma^n f(\sigma))d\sigma \right\| \\ &\leq \sup_{\phi\in\overline{\mathscr{Q}}} \|T^S(t'_j-s'_j)\phi-\phi\| + \int_0^{t'_j-s'_j} Ce^{-\alpha(t'_j-s'_j-\sigma)}\|\Pi^S\||f(s'_j+\sigma)|d\sigma \\ &\leq \sup_{\phi\in\overline{\mathscr{Q}}} \|T^S(t'_j-s'_j)\phi-\phi\| + C\|\Pi^S\| \int_0^{t'_j-s'_j} |f(s'_j+\sigma)|d\sigma, \end{aligned}$$

where $\mathscr{Q} := \{Y(t) : t \in \mathbb{R}\}$. Since f is S^1-almost automorphic, there exists a subsequence $(s_j)_{j\in\mathbb{N}}$ of $(s'_j)_{j\in\mathbb{N}}$ and a function $\tilde{f} \in L^1_{\text{loc}}(\mathbb{R};\mathbb{C}^m)$ such that, for any $t \in \mathbb{R}$,

$$\lim_{j\to+\infty} \int_0^1 |f(t+s_j+\sigma)-\tilde{f}(t+\sigma)|d\sigma = 0,$$

and

$$\lim_{j\to+\infty} \int_0^1 |\tilde{f}(t-s_j+\sigma)-f(t+\sigma)|d\sigma = 0.$$

In particular,

$$\lim_{j\to+\infty}\int_0^1 |f(s_j+\sigma)-\tilde{f}(\sigma)|d\sigma = 0. \tag{4.3}$$

Therefore,

$$\|Y(t_j)-Y(s_j)\| \le \sup_{\phi\in\overline{\mathscr{Q}}}\|T^S(t_j-s_j)\phi-\phi\| + C\|\Pi^S\|\int_0^{t_j-s_j}|f(s_j+\sigma)-\tilde{f}(\sigma)|d\sigma$$
$$+C\|\Pi^S\|\int_0^{t_j-s_j}|\tilde{f}(\sigma)|d\sigma.$$

Consequently, for all $n\in\mathbb{N}$ large enough such that $|t_n-s_n|\le 1$, on has

$$\|Y(t_j)-Y(s_j)\| \le \sup_{\phi\in\overline{\mathscr{Q}}}\|T^S(|t_n-s_n|)\phi-\phi\| + C\|\Pi^S\|\int_0^{1}|f(s_j+\sigma)-\tilde{f}(\sigma)|d\sigma$$
$$+C\|\Pi^S\|\int_0^{|t_n-s_n|}|\tilde{f}(\sigma)|d\sigma.$$

Since Y is almost automorphic, $\overline{\mathscr{Q}}$ is compact in S. Therefore,

$$\lim_{n\to+\infty}\sup_{\phi\in\overline{\mathscr{Q}}}\|T^S(|t_n-s_n|)\phi-\phi\| = 0.$$

By using (4.2) and (4.3), we deduce $\lim_{n\to+\infty}\|Y(t)-Y(s)\|=0$. Hence, Y is uniformly continuous, which concludes the proof. □

We have the following extension of Massera and Bohr–Neugebauer type results.

Theorem 4.3. *Assume that f is a continuous S^p-almost automorphic for some $p\ge 1$. Then, the following assertions are hold:*

(i) The existence of a solution $x(\cdot)$ on $\mathbb{R}^+$ of equation (1.1) such that

$$\sup_{t\in\mathbb{R}^+}\int_t^{t+1}|x(s)|ds<\infty,$$

implies the existence of an S^p-almost automorphic solution.

(ii) Let x is a solution on $\mathbb{R}$ of equation (1.1) such that

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}|x(s)|ds<\infty.$$

Then, the function $t\mapsto x_t$ is compact almost automorphic and x is S^p-almost automorphic.

(iii) If equation (1.1) has an almost automorphic (resp. compact automorphic) solution, then f must be almost automorphic (resp. compact almost automorphic).

Proof. Obviously, we have that the function $t\mapsto Hf(t)$ is also S^p-almost automorphic.

(i) Consider x as a solution of equation (1.1) on $\mathbb{R}^+$ such that $\sup_{t\in\mathbb{R}^+}\int_t^{t+1}|x(s)|ds<\infty$. By employing a similar argument to the one used in establishing the (1) in Theorem 4.1, we can affirm that $t\mapsto x_t$ is bounded on $\mathbb{R}^+$. Then, $z_0(t):=\langle\Phi,x_t\rangle$ is a bounded solution of equation

(3.2) on $\mathbb{R}^+$. According to [19, Corollary 35], equation (3.2) has a compact almost automorphic solution z. By Lemma 4.2, the function $\xi : \mathbb{R} \to X$ given, for $t \in \mathbb{R}$, by

$$\xi(t) = \Phi z(t) + \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S\Gamma^n f(s)ds,$$

is compact almost automorphic. Consequently, $\tilde{x}(t) := \xi(t)[0]$ is an S^p-almost automorphic solution of equation (1.1).

(ii) Let x be a solution of equation (1.1) on $\mathbb{R}$ with $\sup_{t\in\mathbb{R}} \int_t^{t+1} |x(s)|ds < \infty$. As in the proof of the (2) in Theorem 4.1, we can show that the function $t \mapsto x_t$ is bounded. Obviously, $z(t) := \langle \Psi, x_t \rangle$ is a bounded solution of equation (3.2) on $\mathbb{R}$, which is almost automorphic (see [10, Theorem 4.14]). Due to [19, Proposition 34], z is uniformly continuous and, consequently, compact almost automorphic. On the other hand, for any $t \in \mathbb{R}$, we have

$$x_t = \Phi z(t) + \lim_{n\to\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S\Gamma^n f(s)ds.$$

Thus the map $t \mapsto x_t$ is also compact almost automorphic. Hence, x is S^p-almost automorphic.

(iii) Assume that equation (1.1) has an almost automorphic (resp. compact almost automorphic) solution $x(\cdot)$. Then, by (ii), the function $t \mapsto x_t$ is compact almost automorphic. Consequently, the function $t \mapsto \displaystyle\int_{-\infty}^{0} K(\theta)x_t(\theta)d\theta$ is compact almost automorphic. Since, for $t \in \mathbb{R}$,

$$f(t) = x(t) - \int_{-\infty}^{0} K(\theta)x_t(\theta)d\theta,$$

then f is almost automorphic (resp. compact almost automorphic). The proof is complete. □

Now we study the uniqueness almost automorphic solution.

Theorem 4.4. *Let f be a continuous S^p-almost automorphic for some $p \geq 1$ and assume that $(T(t))_{t\geq 0}$ is hyperbolic. Then, equation (1.1) has a unique S^p-almost automorphic solution. Moreover, for any $t \in \mathbb{R}$*

$$x_t = \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Gamma^n f(s)ds - \lim_{n\to+\infty} \int_{t}^{+\infty} T^U(t-s)\Gamma^n f(s)ds. \tag{4.4}$$

In addition, if $\{\lambda \in \sigma(\mathscr{A}) : Re(\lambda) \geq 0\} = \emptyset$, then the unique S^p-almost automorphic $x(\cdot)$ is globally attractive.

Proof. Given condition (H.P), we can conclude from [27, Lemma 3.3] that equation (3.2) has a unique bounded solution on $\mathbb{R}$ given by

$$z(t) = -\int_{t}^{+\infty} e^{(t-s)G} H f(s)ds \text{ for } t \in \mathbb{R}.$$

From [10, Theorem 4.14], we have that z is almost automorphic. Hence, the function $x(t) := \xi(t)[0]$ is an S^p-almost automorphic solution of equation (1.1), where

$$\xi(t) = \Phi z(t) + \lim_{n\to+\infty} \int_{-\infty}^{t} T^S(t-s)\Pi^S\Gamma^n f(s)ds \text{ for } t \in \mathbb{R}.$$

The rest of the proof can be demonstrated using a similar approach to the one employed in proving Theorem 4.1. □

Remark 4.2. As a result of the (iii) in the above Theorem, if f is purely Stepanov almost automorphic, then equation (1.1) does not ever have an almost automorphic solution.

5. Stepanov Oscillatory Dynamics for Fully Nonlinear Renewal Equations

This part is devoted to the study of the existence and uniqueness of an oscillatory solution for the following nonlinear equation:

$$x(t)=\int_{-\infty}^{t}K(t-s)x(s)\,ds+g(t,x_t) \text{ for } t\in\mathbb{R}, \tag{5.1}$$

where $g:\mathbb{R}\times X\to\mathbb{C}^m$ is a continuous function and Lipschitzian with respect to the second argument. That is, there exists $L_g>0$ such that

$$|g(t,\varphi)-g(t,\psi)|\leq L_g\|\varphi-\psi\| \text{ for all } t\in\mathbb{R} \text{ and } \varphi,\psi\in X. \tag{Lip}$$

Lemma 5.1. *Let $x(\cdot)$ be a solution of equation (5.1) on $\mathbb{R}$. Then, the statements below are hold:*

(i) *If $x(\cdot)$ is S^p-almost periodic for some $p\geq 1$, then the function $t\mapsto x_t$ is almost periodic.*
(ii) *If $x(\cdot)$ is S^p-almost automorphic for some $p\geq 1$, then the function $t\mapsto x_t$ is almost automorphic.*

Proof. (i) Let $\varepsilon>0$. Since, $x(\cdot)$ is S^p-almost periodic, then it is S^1-almost periodic. Consequently, there exists $l(\varepsilon)>0$ such that, for all $a\in\mathbb{R}$, there exists $\tau_\varepsilon\in[a,a+l(\varepsilon)]$ such that

$$\sup_{t\in\mathbb{R}}\int_t^{t+1}|x(s+\tau_\varepsilon)-x(s)|ds<\frac{\varepsilon\times(e^\rho-1)}{e^\rho}.$$

Therefore, for $t\in\mathbb{R}$, we have

$$\begin{aligned}\|x_{t+\tau_\varepsilon}-x_t\| &= \int_{-\infty}^{0}e^{\rho\theta}|x(t+\tau_\varepsilon+\theta)-x(t+\theta)|d\theta\\ &= \int_{-\infty}^{t}e^{-\rho(t-\theta)}|x(\tau_\varepsilon+\theta)-x(\theta)|d\theta\\ &\leq \sum_{k=1}^{\infty}\int_{t-k}^{t-k+1}e^{-\rho(t-\theta)}|x(\tau_\varepsilon+\theta)-x(\theta)|d\theta\\ &\leq \sum_{k=1}^{\infty}e^{-\rho(k-1)}\int_{t-k}^{t-k+1}|x(\tau_\varepsilon+\theta)-x(\theta)|d\theta\\ &\leq \frac{e^\rho}{e^\rho-1}\sup_{t\in\mathbb{R}}\int_t^{t+1}|x(\tau_\varepsilon+\theta)-x(\theta)|ds\\ &\leq \frac{e^\rho}{e^\rho-1}\times\frac{\varepsilon\times(e^\rho-1)}{e^\rho}\\ &\leq \varepsilon.\end{aligned}$$

Hence, $t\mapsto x_t$ is almost periodic.

(ii) Let $(s'_n)_{j\in\mathbb{N}}$ be any real sequence. Since x is S^p-almost automorphic, then there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ of $(s'_n)_{n\in\mathbb{N}}$ such that, for any $t\in\mathbb{R}$,

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_0^1|x(t+s_n-s_m+s)-x(t+s)|ds=0. \tag{5.2}$$

Consequently, for each $t \in \mathbb{R}$, we have

$$\begin{aligned} \|x_{t+s_n-s_m} - x_t\| &= \int_{-\infty}^{0} e^{\rho\theta}|x(t+s_n-s_m+\theta)-x(t+\theta)|d\theta \\ &= \int_{-\infty}^{t} e^{-\rho(t-\theta)}|x(s_n-s_m+\theta)-x(\theta)|d\theta \\ &\leq \sum_{k=1}^{\infty}\int_{t-k}^{t-k+1} e^{-\rho(t-\theta)}|x(s_n-s_m+\theta)-x(\theta)|d\theta \\ &\leq \sum_{k=1}^{\infty} e^{-\rho(k-1)}\int_{t-k}^{t-k+1}|x(s_n-s_m+\theta)-x(\theta)|d\theta \\ &\leq \sum_{k=1}^{\infty} e^{-\rho(k-1)}\int_{0}^{1}|x(t-k+s_n-s_m+\theta)-x(t-k+\theta)|d\theta. \end{aligned}$$

From (5.2), we have, for all $k \geq 1$, that

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_0^1 |x(t-k+s_n-s_m+s)-x(t-k+s)|ds = 0.$$

In addition, for all $k \geq 1$, on has

$$e^{-\rho(k-1)}\int_0^1 |x(t-k+s_n-s_m+\theta)-x(t-k+\theta)|d\theta \leq 2e^{-\rho(k-1)}\sup_{t\in\mathbb{R}}\int_0^1 |x(t+s)|ds$$

for any $n,m \in \mathbb{N}$. Therefore,

$$\begin{aligned} &\lim_{m\to+\infty}\lim_{n\to+\infty}\|x_{t+s_n-s_m}-x_t\| \\ &\leq \lim_{m\to+\infty}\lim_{n\to+\infty}\sum_{k=1}^{\infty} e^{-\rho(k-1)}\int_0^1 |x(t-k+s_n-s_m+\theta)-x(t-k+\theta)|d\theta \\ &\leq \sum_{k=1}^{\infty} e^{-\rho(k-1)}\lim_{m\to+\infty}\lim_{n\to+\infty}\int_0^1 |x(t-k+s_n-s_m+\theta)-x(t-k+\theta)|d\theta \\ &= 0. \end{aligned}$$

Hence, $t \mapsto x_t$ is almost automorphic.

□

Proposition 5.1. *Assume that $(T(t))_{t\geq 0}$ is hyperbolic, and for each $\varphi \in X$, $g(\cdot,\varphi) \in APS^p(\mathbb{R},\mathbb{C}^m)$ (resp. $g(\cdot,\varphi) \in AAS^p(\mathbb{R},\mathbb{C}^m)$) for some $p \geq 1$. If $x(\cdot)$ is S^p-almost periodic (resp. S^p-almost automorphic) solution of (5.1), then the mapping $t \mapsto x_t$ satisfies, for $t \in \mathbb{R}$,*

$$x_t = \lim_{n\to+\infty}\int_{-\infty}^{t} T^S(t-s)\Pi^S\Gamma^n g(s,x_s)ds - \lim_{n\to+\infty}\int_t^{+\infty} T^U(t-s)\Pi^U\Gamma^n g(s,x_s)ds. \tag{5.3}$$

Proof. Given that $x(\cdot)$ is S^p-almost periodic (resp. S^p-almost automorphic), and by Lemma 5.1, the function $t \mapsto x_t$ is almost periodic (resp. almost automorphic). Since for each $\varphi \in X$, $g(\cdot,\varphi) \in APS^p(\mathbb{R},\mathbb{C}^m)$ (or $g(\cdot,\varphi) \in AAS^p(\mathbb{R},\mathbb{C}^m)$), and g is Lipschitz with respect to the second argument, it follows from [7, Theorem 3.1] (resp. [8, Lemma 3.1 and Theorem 3.1]) that the function $h(t) := g(t,x_t)$ is S^p-almost periodic (or S^p-almost automorphic). In view of $\sigma(\mathscr{A})\cap i\mathbb{R}=\emptyset$ and g Theorem 4.2 (resp. Theorem 4.4), we deduce that the following nonhomogeneous linear equation

$$\tilde{x}(t)=\int_{-\infty}^{t}K(t-s)\tilde{x}(s)\,ds+h(t) \text{ for } t\in\mathbb{R},$$

has a unique S^p-almost periodic (resp. S^p-almost automorphic). Furthermore, for $t\in\mathbb{R}$,

$$\tilde{x}_t=\lim_{n\to+\infty}\int_{-\infty}^{t}T^S(t-s)\Pi^S\Gamma^n h(s)ds-\lim_{n\to+\infty}\int_{t}^{+\infty}T^U(t-s)\Pi^U\Gamma^n h(s)ds.$$

By the uniqueness of $\tilde{x}(\cdot)$, we have $x=\tilde{x}$. Consequently,

$$x_t=\tilde{x}_t=\lim_{n\to+\infty}\int_{-\infty}^{t}T^S(t-s)\Pi^S\Gamma^n g(s,x_s)ds-\lim_{n\to+\infty}\int_{t}^{+\infty}T^U(t-s)\Pi^U\Gamma^n g(s,x_s)ds.$$

This concludes the proof. □

The main result of this section is the following theorem.

Theorem 5.1. *Assume that $(T(t))_{t\geq 0}$ is hyperbolic, and for each $\varphi\in X$, $g(\cdot,\varphi)\in APS^p(\mathbb{R},\mathbb{C}^m)$ (resp. $g(\cdot,\varphi)\in AAS^p(\mathbb{R},\mathbb{C}^m)$) for some $p\geq 1$. Then, there exists $\varepsilon_0>0$ such that if $L_g<\varepsilon_0$ then equation (5.1) has a unique S^p-almost periodic (resp. S^p-almost automorphic).*

Proof. Consider the following operator $\Upsilon: AP(\mathbb{R},X)(\text{ resp. } AA(\mathbb{R},X))\to AP(\mathbb{R},X)(\text{ resp.}AA(\mathbb{R},X))$ defined, for $\upsilon\in AP(\mathbb{R},X)(\text{ resp.}AA(\mathbb{R},X))$, by

$$\begin{aligned}\Upsilon\upsilon(t) &= x_t^{\upsilon}\\ &= \lim_{n\to+\infty}\int_{-\infty}^{t}T^S(t-s)\Pi^S\Gamma^n g(s,\upsilon(s))ds-\lim_{n\to+\infty}\int_{t}^{+\infty}T^U(t-s)\Pi^U\Gamma^n g(s,\upsilon(s))ds\end{aligned}$$

for $t\in\mathbb{R}$, where x^{υ} is the unique S^p-almost periodic (resp. S^p-almost automorphic) of the following nonhomogeneous linear equation:

$$x^{\upsilon}(t)=\int_{-\infty}^{t}K(t-s)x^{\upsilon}(s)\,ds+g(t,\upsilon(t)) \text{ for } t\in\mathbb{R}. \tag{5.4}$$

By the uniqueness of x^{υ} and the fact that $t\mapsto x_t^{\upsilon}$ is almost periodic (resp. almost automorphic), we deduce that Υ is well-defined. We can see, for any $t\in\mathbb{R}$, that

$$\begin{aligned}\|\Upsilon\upsilon(t)-\Upsilon\tilde{\upsilon}(t)\| &\leq \left\|\lim_{n\to+\infty}\int_{-\infty}^{t}T^S(t-s)\Pi^S\Gamma^n\Big(g(s,\upsilon(s))-g(s,\tilde{\upsilon}(s))\Big)ds\right\|\\ &+ \left\|\lim_{n\to+\infty}\int_{t}^{+\infty}T^U(t-s)\Pi^U\Gamma^n\Big(g(s,\upsilon(s))-g(s,\tilde{\upsilon}(s))\Big)ds\right\|\\ &\leq \int_{-\infty}^{t}Ce^{-\alpha(t-s)}\|\Pi^S\|L_g\|\upsilon(s)-\tilde{\upsilon}(s)\|ds\\ &+ \int_{-\infty}^{t}Ce^{\alpha(t-s)}\|\Pi^U\|L_g\|\upsilon(s)-\tilde{\upsilon}(s)\|ds\\ &\leq \frac{2L_gC\max\{\|\Pi^U\|,\|\Pi^S\|\}}{\alpha}\sup_{t\in\mathbb{R}}\|\upsilon(t)-\tilde{\upsilon}(t)\|.\end{aligned}$$

Thus,

$$\sup_{t\in\mathbb{R}}\|\Upsilon\upsilon(t)-\Upsilon\tilde{\upsilon}(t)\|\leq\frac{2L_gC\max\{\|\Pi^U\|,\|\Pi^S\|\}}{\alpha}\sup_{t\in\mathbb{R}}\|\upsilon(t)-\tilde{\upsilon}(t)\|.$$

Observe that

$$\lim_{L_g\to 0^+}\frac{2L_gC\max\{\|\Pi^U\|,\|\Pi^S\|\}}{\alpha}=0.$$

Then, there exists $\varepsilon_0>0$, such that if $L_g<\varepsilon_0$, then

$$\frac{2L_gC\max\{\|\Pi^U\|,\|\Pi^S\|\}}{\alpha}<1.$$

In this case, the operator Υ is a strict contraction, which means it has a unique fixed point, denoted by υ^0. For $\sigma\in\mathbb{R}$ and $t\geq\sigma$, we have

$$\begin{aligned}
\upsilon^0(t) &= \lim_{n\to+\infty}\int_{-\infty}^{t}T^S(t-s)\Pi^S\Gamma^n g(s,\upsilon^0(s))ds-\lim_{n\to+\infty}\int_{t}^{+\infty}T^U(t-s)\Pi^U\Gamma^n g(s,\upsilon^0(s))ds\\
&= \lim_{n\to+\infty}\int_{-\infty}^{\sigma}T^S(t-s)\Pi^S\Gamma^n g(s,\upsilon^0(s))ds+\lim_{n\to+\infty}\int_{\sigma}^{t}T^S(t-s)\Pi^S\Gamma^n g(s,\upsilon^0(s))ds\\
&- \lim_{n\to+\infty}\int_{\sigma}^{+\infty}T^U(t-s)\Pi^U\Gamma^n g(s,\upsilon^0(s))ds+\lim_{n\to+\infty}\int_{\sigma}^{t}T^U(t-s)\Pi^U\Gamma^n g(s,\upsilon^0(s))ds\\
&= T(t-\sigma)\Bigg(\lim_{n\to+\infty}\int_{-\infty}^{\sigma}T^S(\sigma-s)\Pi^S\Gamma^n g(s,\upsilon^0(s))ds\\
&- \lim_{n\to+\infty}\int_{\sigma}^{+\infty}T^U(\sigma-s)\Pi^U\Gamma^n g(s,\upsilon^0(s))ds\Bigg)\\
&+ \lim_{n\to+\infty}\int_{\sigma}^{t}T(t-s)(\Pi^S+\Pi^U)\Gamma^n g(s,\upsilon^0(s))ds\\
&= T(t-\sigma)\upsilon^0(\sigma)+\lim_{n\to+\infty}\int_{\sigma}^{t}T(t-s)\Gamma^n g(s,\upsilon^0(s))ds.
\end{aligned}$$

Then, according to Theorem 2.9, the function $x(t)=\upsilon^0(t)[0]$ is a solution of (5.1) on $\mathbb{R}$, and hence it is S^p-almost periodic (resp. S^p-almost automorphic). Now, let $\tilde{x}$ be another S^p-almost periodic (resp. S^p-almost automorphic) solution of (5.1). Then, by Lemma 5.1 the function $t\mapsto\tilde{x}_t$ is almost periodic (resp. almost automorphic) and from Proposition 5.1, we have, for $t\in\mathbb{R}$, that

$$\begin{aligned}
\tilde{x}_t &= \lim_{n\to+\infty}\int_{-\infty}^{t}T^S(t-s)\Pi^S\Gamma^n g(s,\tilde{x}_s)ds-\lim_{n\to+\infty}\int_{t}^{+\infty}T^U(t-s)\Pi^U\Gamma^n g(s,\tilde{x}_s)ds\\
&= \Upsilon\tilde{x}_\cdot(t).
\end{aligned}$$

Hence, $\tilde{x}=x$. □

Corollary 5.1. *Assume that $(T(t))_{t\geq0}$ is hyperbolic, and for each $\varphi\in X$, $g(\cdot,\varphi)\in AP(\mathbb{R},\mathbb{C}^m)$ (resp. $g(\cdot,\varphi)\in AA(\mathbb{R},\mathbb{C}^m)$). Then, there exists $\varepsilon_0>0$ such that if $L_g<\varepsilon_0$ then equation (5.1) has a unique almost periodic (resp. almost automorphic).*

Proof. By Theorem (5.1), there exists $\varepsilon_0>0$ such that if $L_g<\varepsilon_0$, equation (5.1) has a unique S^p-almost periodic (resp. S^p-almost automorphic), denoted by $x(\cdot)$. From Lemma 5.1, we have the function $t\mapsto x_t$ is almost periodic (resp. almost automorphic). Therefore, the function $t\mapsto g(t,x_t)$ is almost periodic (resp. almost automorphic). This gives the desired result. □

6. Application to An Epidemic Model with Waning Immunity

In order to illustrate our theoretical results, we consider the following general renewal equation arise from analyzing structured epidemic model with waning immunity, formulated by only two variables: the number of susceptible population at time t and the force of infection at time t. This epidemiological model is fundamental to study the role of temporal immunity of recovered individuals in disease transmission dynamics; see for [4, 30, 31, 32] for more details

$$y(t)=\int_0^{+\infty}k(a)y(t-a)da+g(t,y_t) \text{ for } t\in\mathbb{R}, \tag{6.1}$$

where

- $k(a):=\dfrac{\beta(a)F(a)}{R_0}-\dfrac{(R_0-1)}{\int_0^{+\infty}\mathscr{L}(a)da}\mathscr{L}(a)$ for $a\geq 0$.
- $\beta:\mathbb{R}^+\to\mathbb{R}^+$ is the age-specific transmission coefficient of infected individuals whose infection-age is a.
- $F:\mathbb{R}^+\to\mathbb{R}^+$ is a probability function, for an infected individual, to be infectious until his or her infection-age becomes a.
- $R_0:=\displaystyle\int_0^{+\infty}\beta(a)F(a)da$ is the basic reproduction number.
- $\mathscr{L}(a)$ is the probability for individuals who was infected not to obtain susceptibility since the last infection such that
$$\mathscr{L}(a)=1-\int_0^a G(s)ds$$
where $G:\mathbb{R}^+\to\mathbb{R}^+$ is a probability per unit of time to obtain susceptibility after infection satisfying that:
$$\int_0^{+\infty}G(a)da=1.$$
- $g\in APS^pU(\mathbb{R}\times L^1_\rho(\mathbb{R}^-,\mathbb{C})$ (resp. $g\in AAS^pU(\mathbb{R}\times L^1_\rho(\mathbb{R}^-,\mathbb{C}))$.

As we know, the spread of diseases and immunity can be influenced by a fluctuating environment. For example, physical environmental conditions such as temperature and humidity and the availability of food, water, and other resources usually vary over time with seasonal or daily variations, which is represented in the above model by the term $g(t,y_t)$. Therefore, more realistic models for epidemic dynamics should be nonautonomous systems (see [5]). While the periodic case has often been considered, environmental factors, human migration rates, birth rates of vectors, and other parameters may exhibit more complex oscillations, such as almost periodic, almost automorphic, S^p-almost periodic, S^p-almost automorphic, and so on. These facts are the motivation for the conditions imposed on the function g.

The question now is: What is the nature of the behavior of solutions to (6.1)? As mentioned in this work, if g is purely S^p-almost periodic or S^p-almost automorphic, we cannot hope to have an almost periodic or almost automorphic solution. Instead, the solutions may exhibit more chaotic behavior. Thus, S^p-almost periodic or S^p-almost automorphic functions serve as mathematical tools to understand this complex phenomenon. Proving this kind of behavior helps in understanding and predicting the long-term dynamics of the epidemic model, particularly in the presence of environmental and demographic fluctuations.

In the case, when $g(t,y_t)=f(t)$, where is S^p-almost periodic or almost S^p-automorphic function, then the existence of S^p-almost periodic or almost S^p-automorphic solutions for (6.1) can be investigated using the Massera and Bohr-Neugebauer criteria. Which means that, it suffices to analyze the boundedness and the boundedness of the mean values of the solutions.

On the other hand, let $\lambda \in \Big\{\mu \in \sigma(\mathscr{A}) : Re(\mu) \geq 0\Big\}$, then, by Theorem 3.1, λ verifies the following characteristic equation

$$1=\int_0^{+\infty}\frac{\beta(a)F(a)e^{-\lambda a}}{R_0}da-\int_0^{+\infty}\frac{(R_0-1)}{\int_0^{+\infty}\mathscr{L}(\sigma)d\sigma}\mathscr{L}(a)e^{-\lambda a}da.$$

Hence, by [30, Proposition 3.1], there is $\varepsilon_1>0$ such that if

$$1<R_0<1+\varepsilon_1,$$

we have $\Big\{\mu \in \sigma(\mathscr{A}) : Re(\mu) \geq 0\Big\}=\emptyset$. Consequently, thanks to Theorem 4.2 (resp. Theorem 4.4), equation (6.1) has a unique S^p-almost periodic (resp. S^p-almost automorphic) solution which is globally attractive.

To complete the illustration, next we l consider the fully nonlinear equation, as an application for the main result obtained in Section 5. In this case, we assume that g is Lipschitzian with respect to the second argument with L_0 be its Lipschitz constant. As an example of such function: $g(t,\varphi)=g_0(\varphi)+f_0(t)$, where $g_0 : L^1_\rho(\mathbb{R}^-,\mathbb{C}) \to \mathbb{C}$ is L_0-Lipschitz function and $f_0 : \mathbb{R} \to \mathbb{C}$ is an oscillatory function.

By using Theorem 5.1 and Corollary 5.1, if $1<R_0<1+\varepsilon_1$, then, there exists $\varepsilon_0>0$ such that if $L_0<\varepsilon_0$, then equation (6.1) has a unique S^p-almost periodic (resp. S^p-almost automorphic, almost periodic, almost automorphic).

7. Conclusion

In this work, we considered the existence of almost periodic and almost automorphic behaviors for a class of infinite delay renewal equations. In [2], we observed that, unlike ordinary differential equations, where solutions become strong (Bohr-almost periodic or compact almost automorphic) under forcing terms in weak (Stepanov) sense, this is not possible for renewal equations. Starting from this point, in this present work, we investigated the case when the forcing term is only in Stepanov sense. To achieve this goal, we established a new result on the reduction principle for nonhomogeneous autonomous linear renewal equations with the forcing term bounded by mean values. As a consequence, we provided Stepanov versions of the Massera and Bohr-Neugebauer type results. That is, to study the existence of an S^p-almost periodic or S^p-almost automorphic solution, it is sufficient to examine the mean boundedness.

Under the condition that the semigroup generated by the linear part has an exponential dichotomy, we have established the existence of a unique S^p-almost periodic or S^p-almost automorphic solution. Based on this and the Banach fixed point theorem, we have investigated the existence and uniqueness for the nonlinear case with Lipschitz condition. Moreover, we have noted that if $(T(t))_{t\geq 0}$ is exponentially stable, then the unique S^p-almost periodic or S^p-almost automorphic is a global attractor of equation (1.1). More precisely, every solution on $\mathbb{R}^+$ is attracted by the unique S^p-almost periodic or S^p-almost automorphic, resulting in the same long-term behavior.

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