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INEXACT PROJECTION METHODS FOR VARIATIONAL INQUALITY PROBLEMS AND APPLICATIONS

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Abstract. In this paper, we propose two new inexact projection algorithms, which can be easily implemented, for solving pseudomonotone variational inequality problems based on self-adaptive step sizes, viscosity technique, and inexact projections. We obtain two strongly convergent theorems of solutions in a real Hilbert space. Numerical experiments illustrate and compare the performances of the proposed algorithms with three other known results.
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1. INTRODUCTION

In a real Hilbert space $\mathbb{H}$, let D be a nonempty, convex, and closed subset of $\mathbb{H}$ and denote $\langle\cdot,\cdot\rangle$ by the inner product, $\|\cdot\|$ by the induced norm, $\mathscr{P}_D$ by the metric projection from $\mathbb{H}$ onto D, $\rightharpoonup$ by the weak convergence, and $\to$ by the strong convergence. The well-known variational inequality problem, shortly $VIPs$, is a model of the form:

$$\text{Find } x^* \in D \text{ such that } \langle \mathscr{F}(x^*), x - x^*\rangle \geq 0, \quad \forall x \in D,$$

where $\mathscr{F}: D \to \mathbb{H}$ is usually called *cost mapping*. Let us denote $\mathscr{S}(D,\mathscr{F})$ by the solution set of the problem $VIPs$. The problem was first introduced by Kinderlehrer and Stampacchia in [21]. Note that problem $VIPs$ has been developed rapidly in the last few years and has been successfully used as a tool in medicine, biology, economics, heat conduction modeling, tomography, and many others branches of science and technology [1]. Various solution algorithms were introduced and studied due to active links with applied fields, such as poroelasticity for petroleum engineering [26], financial analysis in economics [23], the reconstruction of images in imaging processing [19], telecommunication networks and noncooperative games [29], and many others [23, 24, 25]. It is remarkably known that if $\mathscr{F} = \nabla f$, where $f: D \to \mathbb{R}$ is a convex and differentiable function with its gradient ∇f, then problem $VIPs$ is equivalent to the problem: $\min\{f(x) : x \in D\}$. To address $VIPs$, one of the earliest solution algorithms employed is

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the projection algorithm, although this approach is challenging to implement numerically. One notable feature for solving problem $VIPs$ involves the metric projection. Let $\mathscr{F}: D \to \mathbb{H}$ be a cost mapping. Then, the following statements are equivalent:

(i) $x^* \in \mathscr{S}(D, \mathscr{F})$;
(ii) x^* is a fixed point of the solution mapping $\mathscr{A}_\lambda : D \to D$ defined in the form:

$$\mathscr{A}_\lambda(x) = \mathscr{P}_D[x - \lambda \mathscr{F}(x)], \quad \forall x \in D,$$

where $\lambda \in (0, \infty)$ and $\mathscr{P}_D$ is the metric projection from $\mathbb{H}$ onto D.

This is an important basis to the gradient projection method, first proposed by Goldstein in [16]:

$$x^0 \in D, x^{k+1} = \mathscr{A}_\lambda(x^k).$$

Under the β-strongly monotone and L-Lipschitz continuous assumptions on mapping $\mathscr{F}$, and $\lambda \in (0, \frac{2\beta}{L})$, the solution mapping $\mathscr{A}_\lambda$ is contractive. By the Banach Contraction Principle, the sequence $\{x^k\}$ defined above converges strongly to the unique solution of problem $VIPs$. There are modified instances of the projection algorithm for solving problem $VIPs$ such as in the papers [2, 3, 4, 9, 10, 14, 17, 18, 19, 27, 28, 30, 31] and some results in books [13, 23]. Most of these algorithms, at each iteration, require to compute the metric projection of iteration points onto constraint set D. However, this may not be easy to calculate unless D has a simple structure. In fact, computing the projection onto D requires solving a convex quadratic optimization problem constrained to D at each iteration, which can significantly raise the cost per iteration if the number of unknowns is large.

Recently, there are some inexact algorithms, which become increasingly accurate as the optimization solution of the auxiliary problem at each iteration under consideration is approached. These approaches have been proposed in an effort to reduce the computational cost required for the metric projections, thus leading to more effective projection algorithms. For using the inexact projection to solve the problem $VIPs$, let us briefly recall some popular algorithms such as Outer inexact Schemes proposed by Burachik and Lopes in [8], Block-Iterative Outer Inexact Methods introduced by Combettes in [11], Outer Inexact Methods of Gibali et al. in [15], Outer Proximal Algorithms introduced by Anh et al. in [2, 5, 6, 7, 20], and some other interesting algorithms (see, e.g., [13, 34] and the references therein).

There exists a natural question that: *Can we propose a vicosity inexact projection-type approach with one evaluation of the cost mapping $\mathscr{F}$, one calculation of the inexact projection without $\mathscr{P}_D$, and inertial steps to solve problem $VIPs$?*

Our contribution in this paper is to answer the above question affirmatively and offers a brief overview of our results and their distinction from prior results. We propose two new inexact projection algorithms to solve the problem $VIPs$ with the following details:

- It differs from existing algorithms, even in special cases that one inexact projection onto the constraint set D at each iteration is instead of the metric projection $\mathscr{P}_D$;
- our iteration algorithms only use two evaluations of the cost mapping $\mathscr{F}$ at each iteration;
- inertial extrapolation step is incorporated to speed up the iterations;
- we use viscosity technique via a contractive mapping to show that the cluster point of iteration sequences is a unique solution of problem $VIPs$;

- finally, as an application of our proposed algorithms, we apply the algorithms to solve image restoration models and compare the performances of the algorithms with some popular results.

The remainder of this paper is structured as follows:

- Section 2 introduces a foundation by basic definitions, comparing the metric projection with the approximation projection, and reviewing essential concepts and relevant prior lemmas. One important concept is the "inexact projection onto constraint set D";
- building upon the inexact projection, Section 3 presents two new viscosity inexact algorithms and shows their strong convergence;
- in Section 4, the numerical experiences are performed to evaluate our proposed algorithms. We apply the algorithms to the adaptive image restoration and compare them with three known algorithms.

2. Preliminaries

We recall several concepts which are needed in this paper. These definitions, lemmas, and properties are known and can be found, e.g., in two popular books [13, 23].

Let D be a nonempty, convex, and closed subset of a real Hilbert space $\mathbb{H}$. The metric projection $a \in \mathbb{H}$ onto D is denoted by $\mathscr{P}_D(a)$. It is the unique solution to the quadratical convex programming:

$$\mathscr{P}_D(a) = \operatorname{argmin}\left\{\|a-y\|^2 : a \in D\right\}. \tag{2.1}$$

It is clear that $\mathscr{P}_D$ is from $\mathbb{H}$ onto D, and $\|\mathscr{P}_D(a) - \mathscr{P}_D(b)\| \le \|a-b\|$ for all $a,b \in \mathbb{H}$. This property is said to be *nonexpansive*. Moreover, $\mathscr{P}_D$ is the 1-strongly quasi-nonexpansive, i.e.,

$$\left\|\mathscr{P}_D(a) - \bar{a}\right\|^2 \le \|a-\bar{a}\|^2 - \left\|a - \mathscr{P}_D(a)\right\|^2, \quad \forall a \in \mathbb{H}, \bar{a} \in D.$$

Let us recall some definitions of Lipschitz continuous and monotone mappings in $\mathbb{H}$, used in problem $VIPs$. Cost mapping $\mathscr{F}$ is said to be *monotone* on D if

$$\langle \mathscr{F}(u) - \mathscr{F}(v), u-v\rangle \ge 0, \quad \forall u,v \in D;$$

pseudomonotone on D if

$$\langle \mathscr{F}(v), u-v\rangle \ge 0 \Rightarrow \langle \mathscr{F}(u), u-v\rangle \ge 0, \quad \forall u,v \in D;$$

Lipschitz continuous with constant $L > 0$ on D if

$$\|\mathscr{F}(u) - \mathscr{F}(v)\| \le L\|u-v\|, \quad \forall u,v \in D;$$

partially pseudomonotone with constant $\eta > 0$ on $C \subset D$ if

$$\langle \mathscr{F}(z), v-z\rangle \ge 0 \Rightarrow \langle \mathscr{F}(v), v-z\rangle \ge \eta\|v-z\|^2, \quad \forall v \in D, z \in C;$$

partially pseudomonotone on $C \subset D$, if we have

$$\langle \mathscr{F}(z), v-z\rangle \ge 0 \Rightarrow \langle \mathscr{F}(v), v-z\rangle \ge 0, \quad \forall v \in D, z \in C.$$

Now, we use an *inexact projection* $u \in \mathbb{H}$ onto D relatives to any point $z \in \mathbb{H}$ with computational error $\tau \ge 0$. Denote this projection by $\mathscr{P}_D^{\tau,z}(u)$. It is defined by

$$\mathscr{P}_D^{\tau,z}(u) = \left\{w \in D : \langle u-w, v-w\rangle \le \tau\|w-z\|^2, \quad \forall v \in D\right\}. \tag{2.2}$$

By the definition (2.1) of the metric projection $\mathscr{P}_D$ of a point $u \in \mathbb{H}$ onto D, we see that $\mathscr{P}_D(u) \in D$ and $\langle u - \mathscr{P}_D(u), y - \mathscr{P}_D(u)\rangle \leq 0$ for all $y \in D$. From this and $\tau \geq 0$, it yields

$$\langle u - \mathscr{P}_D(u), y - \mathscr{P}_D(u)\rangle \leq \tau\|\mathscr{P}_D(u) - z\|^2,$$

and hence $\mathscr{P}_D(u) \in \mathscr{P}_D^{\tau,z}(u)$ for all $u \in \mathbb{H}, z \in \mathbb{H}$ and $\tau \geq 0$. This shows that $\mathscr{P}_D^{\tau,z}$ is a multivalued mapping from $\mathbb{H}$ to D. In the case that $\tau = 0$, we have $\mathscr{P}_D^{0,z} = \{\mathscr{P}_D\}$ for all $z \in \mathbb{H}$. Thus, for each $\tau \geq 0$ and $z \in \mathbb{H}$, the inexact projection $\mathscr{P}_D^{\tau,z}$ is an extended formulation of the metric projection $\mathscr{P}_D$. However, there exists a problem that: *Why do we have to use the inexact projection $\mathscr{P}_D^{\tau,z}$ without the metric projection $\mathscr{P}_D$?* First, computing the projection of a point x onto D is to solve the quadratical convex programming:

$$\min\{\|x - y\|^2 : y \in D\}.$$

Computing an inexact projection $y_x \in \mathscr{P}_D^{\tau,z}(x)$ of a point x onto the constraint D with respect to the iteration point $z \in \mathbb{H}$ and the parameter $\tau \geq 0$ is very simple as follows:

Procedure 2.1. (for finding $y_x \in \mathscr{P}_D^{\tau,z}(x)$)

St. 1: Take $k = 0, y^0 \in D, z \in \mathbb{H}$ and $\tau \in (0,\infty)$.

St. 2: Solve the linear programming with the convex constraint:

$$v^k = \operatorname{argmin}\{\langle y^k - x, y - y^k\rangle : y \in D\}.$$

If $\langle y^k - x, y^k - v^k\rangle \leq \tau\|y^k - z\|^2$, then Stop, i.e., $y_x = y^k$. Otherwise, compute $y^{k+1} = y^k + \delta_k(v^k - y^k)$ with the stepsize $\delta_k = \min\left\{1, \frac{\langle y^k - x, y^k - v^k\rangle}{\|v^k - y^k\|^2}\right\}$.

St. 3: Repeat $k := k+1$ and come back to *St. 2*.

In fact, computing y_x of a point x on computer via the inexact projection $\mathscr{P}_D^{\tau_j,x}$ proves more efficient than obtaining the metric projection $\mathscr{P}_D(x)$ on Mathlab Software.

Remark 2.1. Let $\tau \geq 0$ and $\xi > 0$. A point $x^* \in D$ is a solution to problem $VIPs$ if and only if $x^* \in \mathscr{P}_D^{\tau,x^*}(x^* - \xi\mathscr{F}(x^*))$.

Indeed, by the definition of $\mathscr{P}_D^{\tau,x}$, $x^* \in \mathscr{P}_D^{\tau,x^*}(x^* - \xi\mathscr{F}(x^*))$ is equivalent to

$$\langle x^* - \xi\mathscr{F}(x^*) - x^*, y - x^*\rangle \leq \tau\|x^* - x^*\|^2, \quad \forall y \in D,$$

and hence $\langle\mathscr{F}(x^*), y - x^*\rangle \geq 0$ for all y in D. Thus x^* also is a solution of the problem $VIPs$.

Lemma 2.1. [27, Remark 4.4] *Let $\{a_k\}$ be a positive sequence. For each any positive integer h, there exists a positive integer $p > h$ such that $a_p \leq a_{p+1}$. For each positive integer k_0 such that $a_{k_0} \leq a_{k_0+1}$, set $\xi(k) = \max\{i \in \mathscr{N} : k_0 \leq i \leq k, a_i \leq a_{i+1}\}$. Then, $0 \leq a_k \leq a_{\xi(k)+1}$ for all $k \geq k_0$. Moreover, $\{\xi(k)\}_{k\geq k_0}$ is nondecreasing and $\lim_{k\to\infty}\xi(k) = +\infty$.*

Lemma 2.2. [38, Lemma 2.5] *Assume that $\{a_k\}$ is a positive sequence such that $a_{k+1} \leq (1-\theta_k)a_k + \theta_k\tau_k$ for all $k \geq 1$. Let $\{\theta_k\}$ and $\{\tau_k\}$ be two real sequences satisfying the following conditions:*

(i) $\{\theta_k\} \subset (0,1)$ *and* $\sum_{k=1}^{\infty}\theta_k = \infty$;

(ii) $\limsup_{k\to\infty}\tau_k \leq 0$ *or* $\sum_{k=1}^{\infty}|\theta_k\tau_k| < +\infty$.

Then, $\lim_{k\to\infty} a_k = 0$.

Lemma 2.3. [12, lemma 2.1] *Consider the problem $VIPs$ with D being a nonempty, convex, and closed subset of a real Hilbert space $\mathbb{H}$ and $\mathscr{F} : D \to \mathbb{H}$ being pseudomonotone and continuous. Then, x^* is a solution to $VIPs$ if and only if $\langle \mathscr{F}x, x - x^*\rangle \geq 0$ for all $x \in D$.*

Lemma 2.4. [33, Lemma 3] *Consider the constraint D defined in $VIPs$. Suppose that $\{x^k\} \subset \mathbb{H}$ satisfies the following conditions:*

(i) for all $x \in D$, $\lim_{k\to\infty} \|x^k - x\|$ exists;

(ii) every sequential weak cluster point of $\{x^k\}$ is belong to D.

Then, $\{x^k\}$ converges weakly to a point belonging to D.

3. Algorithms and Their Convergence

In this paper, we use the contraction mapping $f : \mathbb{H} \to \mathbb{H}$ with constant $\delta \in [0,1)$ and the viscosity sequence $\{\alpha_k\}$ is chosen at each iteration. To solve problem $VIPs$, we need the following assumptions on constraint set D and cost mapping $\mathscr{F}$:

- (C_1) constraint set D is a nonempty, closed and convex subset of $\mathbb{H}$;
- (C_2) cost mapping $\mathscr{F} : \mathbb{H} \to \mathbb{H}$ is pseudomonotone, $L-$Lipschitz continuous, and sequentially weakly continuous on D;
- (C_3) solution set $\mathscr{S}(D,\mathscr{F})$ of problem $VIPs$ is nonempty.

Parameters satisfies the following restrictions:

$$\begin{cases} b \in (0,1), 0 < \lambda_k < a \leq \frac{\sqrt{1-b}}{L}, \\ 1 - 2\varepsilon_k - \lambda_k^2 L^2 \geq b, \\ \beta_k \in (0,1), \sum_{k=0}^{\infty} \beta_k = \infty, \lim_{k\to\infty} \beta_k = 0. \end{cases} \tag{3.1}$$

Algorithm 3.1 (VAPA - Viscosity Approximation Projection Algorithm)

Initialization: Choose $x^{-1}, x^0 \in \mathbb{H}$.

Iterative Steps: Calculate x^{k+1} as follows, $k = 0, 1, \cdots,$

Step 1. Evaluate $w^k = x^k + \alpha_k(x^k - x^{k-1})$ *(inertial technique)*. Find a point y^k via inexact projection:

$$y^k \in \mathscr{P}_D^{\varepsilon_k, w^k}\left(w^k - \lambda_k \mathscr{F} w^k\right).$$

If $y^k = w^k$, then Stop. Otherwise, go to *Step 2*.

Step 2. Compute *(viscosity technique)*

$$x^{k+1} = \beta_k f(x^k) + (1-\beta_k)\left[y^k - \lambda_k(\mathscr{F}y^k - \mathscr{F}w^k)\right].$$

Step 3. Increase k by 1 and come back *Step 1*.

Remark 3.1. (i) In the case $y^k = w^k$, from *Step 1*, it follows $w^k \in \mathscr{P}_D^{\varepsilon_k, w^k}[w^k - \lambda_k \mathscr{F} w^k]$. By Remark 2.1, w^k is a solution to problem $BVIs$ under Condition (3.1).

(ii) At each iteration k, we choose the sequence $\{\alpha_k\}$ satisfying the following condition:

$$\lim_{k\to\infty} \alpha_k \frac{\|x^k - x^{k-1}\|}{\beta_k} = 0. \tag{3.2}$$

An example is as

$$\alpha_k = \begin{cases} \frac{\beta_k}{k\|x^k - x^{k-1}\|} & \text{if } \|x^k - x^{k-1}\| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The next lemma shows the relation between the iteration point $y^k - \lambda_k(\mathscr{F}y^k - \mathscr{F}w^k)$ in *Step 2* and any solution of problem $BVIs$.

Lemma 3.1. *Set* $z^k = y^k - \lambda_k(\mathscr{F}y^k - \mathscr{F}w^k)$. *It holds:*

$$\|z^k - p\|^2 \leq \|w^k - p\|^2 - (1 - 2\varepsilon_k - \lambda_k^2 L^2)\|y^k - w^k\|^2, \quad \forall p \in \mathscr{S}(D, \mathscr{F}).$$

Proof. Since $z^k = y^k - \lambda_k(\mathscr{F}y^k - \mathscr{F}w^k)$, we have

$$\begin{aligned} \|z^k - p\|^2 =& \|y^k - p\|^2 + \lambda_k^2\|\mathscr{F}y^k - \mathscr{F}w^k\|^2 - 2\lambda_k\langle y^k - p, \mathscr{F}y^k - \mathscr{F}w^k\rangle \\ =& \|w^k - p\|^2 + \|w^k - y^k\|^2 + 2\langle y^k - w^k, w^k - p\rangle + \lambda_k^2\|\mathscr{F}y^k - \mathscr{F}w^k\|^2 \\ & - 2\lambda_k\langle y^k - p, \mathscr{F}y^k - \mathscr{F}w^k\rangle \\ =& \|w^k - p\|^2 + \|w^k - y^k\|^2 - 2\langle y^k - w^k, y^k - w^k\rangle + 2\langle y^k - w^k, y^k - p\rangle \\ & + \lambda_k^2\|\mathscr{F}y^k - \mathscr{F}w^k\|^2 - 2\lambda_k\langle y^k - p, \mathscr{F}y^k - \mathscr{F}w^k\rangle \\ =& \|w^k - p\|^2 - \|w^k - y^k\|^2 + 2\langle y^k - w^k, y^k - p\rangle + \lambda_k^2\|\mathscr{F}y^k - \mathscr{F}w^k\|^2 \\ & - 2\lambda_k\langle y^k - p, \mathscr{F}y^k - \mathscr{F}w^k\rangle. \end{aligned} \tag{3.3}$$

By using the definition of the inexact projection $y^k \in \mathscr{P}_D^{\varepsilon_k, w^k}(w^k - \lambda_k\mathscr{F}w^k)$, one has

$$\langle y^k - w^k + \lambda_k\mathscr{F}w^k, y^k - p\rangle \leq \varepsilon_k\|y^k - w^k\|^2,$$

which is equivalent to

$$\langle y^k - w^k, y^k - p\rangle \leq -\lambda_k\langle\mathscr{F}w^k, y^k - p\rangle + \varepsilon_k\|y^k - w^k\|^2. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\begin{aligned} \|z^k - p\|^2 =& \|w^k - p\|^2 - \|w^k - y^k\|^2 + 2\langle y^k - w^k, y^k - p\rangle + \lambda_k^2\|\mathscr{F}y^k - \mathscr{F}w^k\|^2 \\ & - 2\lambda_k\langle y^k - p, \mathscr{F}y^k - \mathscr{F}w^k\rangle \\ \leq& \|w^k - p\|^2 - \|w^k - y^k\|^2 - 2\lambda_k\langle\mathscr{F}w^k, y^k - p\rangle + 2\varepsilon_k\|y^k - w^k\|^2 \\ & + \lambda_k^2\|\mathscr{F}y^k - \mathscr{F}w^k\|^2 - 2\lambda_k\langle y^k - p, \mathscr{F}y^k - \mathscr{F}w^k\rangle. \\ =& \|w^k - p\|^2 - (1 - 2\varepsilon_k)\|w^k - y^k\|^2 + \lambda_k^2\|\mathscr{F}y^k - \mathscr{F}w^k\|^2 - 2\lambda_k\langle y^k - p, \mathscr{F}y^k\rangle \\ \leq& \|w^k - p\|^2 - (1 - 2\varepsilon_k - \lambda_k^2 L^2)\|y^k - w^k\|^2, \end{aligned}$$

where the last inequality is deduced from the assumptions $p \in \mathscr{S}(D, \mathscr{F})$, $y^k \in D$ of the inexact projection and pseudomonotonicity of $\mathscr{F}$ in (C_2), i.e.,

$$\langle\mathscr{F}p, y^k - p\rangle \geq 0 \Rightarrow \langle\mathscr{F}y^k, y^k - p\rangle \geq 0.$$

This completes the proof. □

Lemma 3.2. *The sequences* $\{x^k\},\{w^k\},\{z^k\},\{f(x^k)\}$, *and* $\{y^k\}$ *are bounded.*

Proof. By Lemma 3.1 and the condition $1-2\varepsilon_k-\lambda_k^2L^2>0$ of (3.1), one has

$$\|z^k-p\|\le\|w^k-p\|,\quad \forall k\ge 0. \tag{3.5}$$

According to w^k in *Step 1*, one sees

$$\|w^k-p\|\le\|x^k-p\|+\alpha_k\|x^k-x^{k-1}\|=\|x^k-p\|+\beta_k\alpha_k\frac{\|x^k-x^{k-1}\|}{\beta_k}. \tag{3.6}$$

Using Remark (ii) that $\lim_{k\to\infty}\alpha_k\frac{\|x^k-x^{k-1}\|}{\beta_k}=0$, one sees that there exists a constant $M_1>0$ such that $\alpha_k\frac{\|x^k-x^{k-1}\|}{\beta_k}\le M_1$ for all $k\ge1$. Combining (3.5) and (3.6), one deduces $\|z^k-p\|\le\|x^k-p\|+\beta_kM_1$ for all $k\ge1$. Since f is $\delta-$contractive and $\beta_k\in(0,1)$, for each $k\ge1$, we have

$$\begin{aligned}\|x^{k+1}-p\|&\le\beta_k\|f(x^k)-p\|+(1-\beta_k)\|z^k-p\|\\&\le\beta_k\delta\|x^k-p\|+\beta_k\|f(p)-p\|+(1-\beta_k)\|x^k-p\|+\beta_kM_1\\&=[1-(1-\delta)\beta_k]\|x^k-p\|+(1-\delta)\beta_k\frac{M_1+\|f(p)-p\|}{1-\delta}\\&\le\max\left\{\|x^k-p\|,\frac{M_1+\|f(p)-p\|}{1-\delta}\right\},\end{aligned}$$

where z^k is defined in Lemma 3.1. By induction, we have

$$\|x^k-p\|\le\max\left\{\|x^0-p\|,\frac{M_1+\|f(p)-p\|}{1-\delta}\right\},$$

which implies that $\{x^k\}$ is bounded. By (3.5) and (3.6), both $\{z^k\}$ and $\{w^k\}$ are also bounded. Using Lemma 3.1 and the condition $1-2\varepsilon_k-\lambda_k^2L^2\ge b>0$ of (3.1), it follows

$$b\|y^k-w^k\|^2\le\|w^k-p\|^2-\|z^k-p\|^2,\quad\forall k\ge1.$$

Consequently, $\{y^k\}$ is bounded. This finishes the proof. □

Lemma 3.3. *Let the subsequence* $\{w^{k_j}\}$ *of* $\{w^k\}$ *satisfy the conditions that* $\{w^{k_j}\}$ *converges weakly to* $z\in\mathbb{H}$ *and* $\lim_{j\to\infty}\|w^{k_j}-y^{k_j}\|=0$. *Then,* $z\in\mathscr{S}(D,\mathscr{F})$.

Proof. From the assumptions that $w^{k_j}\rightharpoonup z$ and $\lim_{j\to\infty}\|w^{k_j}-y^{k_j}\|=0$, it follows $y^{k_j}\rightharpoonup z$. Note that $\{y^{k_j}\}\subset D$ and D is convex and closed. It yields $z\in D$. From *Step 2* and the definition of the inexact projection $y^{k_j}\in\mathscr{P}_D^{\varepsilon_{k_j},w^{k_j}}\left(w^{k_j}-\lambda_{k_j}\mathscr{F}w^{k_j}\right)$, we have

$$\langle w^{k_j}-\lambda_{k_j}\mathscr{F}w^{k_j}-y^{k_j},x-y^{k_j}\rangle\le\varepsilon_{k_j}\|y^{k_j}-w^{k_j}\|^2,\quad\forall x\in D.$$

Using $\lambda_k>0$, we obtain

$$\frac{1}{\lambda_{k_j}}\langle w^{k_j}-y^{k_j},x-y^{k_j}\rangle\le\langle\mathscr{F}w^{k_j},x-y^{k_j}\rangle+\frac{\varepsilon_{k_j}}{\lambda_{k_j}}\|y^{k_j}-w^{k_j}\|^2,\quad\forall x\in D,$$

which is equivalent to

$$\frac{1}{\lambda_{k_j}}\langle w^{k_j}-y^{k_j},x-y^{k_j}\rangle+\langle\mathscr{F}w^{k_j},y^{k_j}-w^{k_j}\rangle-\frac{\varepsilon_{k_j}}{\lambda_{k_j}}\|y^{k_j}-w^{k_j}\|^2\le\langle\mathscr{F}w^{k_j},x-w^{k_j}\rangle.$$

Using the boundedness of the sequences $\{y^k\}$ and $\{w^k\}$ in Lemma 3.2, $0 < \lambda_k \le a < \infty$ in Condition (3.1) and passing the liminf as $j \to \infty$, we obtain

$$\liminf_{j\to\infty} \langle \mathscr{F} w^{k_j}, x - w^{k_j} \rangle \ge 0, \quad \forall x \in D. \tag{3.7}$$

Otherwise,

$$\langle \mathscr{F} y^{k_j}, x - y^{k_j} \rangle = \langle \mathscr{F} y^{k_j} - \mathscr{F} w^{k_j}, x - w^{k_j} \rangle + \langle \mathscr{F} w^{k_j}, x - w^{k_j} \rangle + \langle \mathscr{F} y^{k_j}, w^{k_j} - y^{k_j} \rangle. \tag{3.8}$$

Since $\mathscr{F}$ is $L-$Lipschitz continuous and the assumption $\lim_{j\to\infty} \|w^{k_j} - y^{k_j}\| = 0$, we deduce

$$0 \le \lim_{j\to\infty} \|\mathscr{F} w^{k_j} - \mathscr{F} y^{k_j}\| \le \lim_{j\to\infty} L\|w^{k_j} - y^{k_j}\| = 0.$$

Combining this, (3.7) and (3.8) yields $\liminf_{j\to\infty} \langle \mathscr{F} y^{k_j}, x - y^{k_j} \rangle \ge 0$. We can take a sequence $\{\xi_j\} \subset (0,1)$ satisfying $\lim_{j\to\infty} \xi_j = 0$ so that, for all $j \ge 1$, there exists the smallest positive integer $m_j \ge k_j$ such that

$$\langle \mathscr{F} y^i, x - y^i \rangle + \xi_j \ge 0, \quad \forall i \ge m_j. \tag{3.9}$$

Note that $\|\mathscr{F} y^k\| \ne 0$. So we can set $g^{m_j} = \frac{\mathscr{F} y^{m_j}}{\|\mathscr{F} y^{m_j}\|^2}$. Then, $\langle \mathscr{F} y^{m_j}, g^{m_j} \rangle = 1$, $\forall j \ge 1$. From (3.9), it follows $\langle \mathscr{F} y^{m_j}, x + \xi_j g^{m_j} - y^{m_j} \rangle \ge 0$. Combining this and the pseudomonotonicity of $\mathscr{F}$ yields $\langle \mathscr{F}(x + \xi_j g^{m_j}), x + \xi_j g^{m_j} - y^{m_j} \rangle \ge 0$, and hence

$$\langle \mathscr{F} x, x - y^{m_j} \rangle \ge \langle \mathscr{F} x - \mathscr{F}(x + \xi_j g^{m_j}), x + \xi_j g^{m_j} - y^{m_j} \rangle - \xi_j \langle \mathscr{F} x, g^{m_j} \rangle, \quad \forall j \ge 1. \tag{3.10}$$

Next, we prove that $\lim_{j\to\infty} \xi_j \|g^{m_j}\| = 0$. As the above proof, $y^{k_j} \rightharpoonup z$ and $z \in D$. Since $\mathscr{F}$ is sequentially weakly continuous on D, then $\{\mathscr{F} y^{k_j}\}$ converges weakly to $\mathscr{F} z$. The proof is complete with $\mathscr{F} z = 0$, i.e., $z \in \mathscr{S}(D, \mathscr{F})$.

Now we consider the case $\mathscr{F} z \ne 0$. By using the sequentially weak lower semicontinuity of the norm $\|\cdot\|$, we have $0 < \|\mathscr{F} z\| \le \liminf_{j\to\infty} \|\mathscr{F} y^{k_j}\|$. From $\{y^{m_j}\} \subset \{y^{k_j}\}, \{\xi_j\} \subset (0, \infty)$, and $\lim_{j\to\infty} \xi_j = 0$, it follows

$$0 \le \limsup_{j\to\infty} \|\xi_j g^{m_j}\| = \limsup_{j\to\infty} \left(\frac{\xi_j}{\|\mathscr{F} y^{k_j}\|} \right) \le \frac{\limsup_{j\to\infty} \xi_j}{\liminf_{j\to\infty} \|\mathscr{F} y^{k_j}\|} \le \frac{\lim_{j\to\infty} \xi_j}{\|\mathscr{F} z\|} = 0,$$

which means that $\lim_{j\to\infty} \xi_j \|g^{m_j}\| = 0$. Taking the limit as $j \to \infty$, the right hand side of (3.10) tends to zero under the fact that $\mathscr{F}$ is Lipschitz continuous, $\{x^{m_j}\}$ and $\{g^{m_j}\}$ are bounded, and $\lim_{j\to\infty} \xi_j g^{m_j} = 0$. From (3.10) it follows $\liminf_{j\to\infty} \langle \mathscr{F} x, x - y^{m_j} \rangle \ge 0$. Since $\{y^{m_j}\}$ converges weakly to $z \in D$, we deduce

$$\langle \mathscr{F} x, x - z \rangle = \lim_{j\to\infty} \langle \mathscr{F} x, x - y^{m_j} \rangle = \liminf_{j\to\infty} \langle \mathscr{F} x, x - y^{m_j} \rangle \ge 0, \quad \forall x \in D.$$

By Lemma 2.3, we have $z \in \mathscr{S}(D, \mathscr{F})$. The proof is completed. □

Theorem 3.1. *Let the cost mapping $\mathscr{F}$ satisfy Assumptions $(C_1) - (C_3)$. Then, under Conditions (3.1) and (3.2), the sequence $\{x^k\}$ generated by Algorithm 3.1 converges strongly to an element $p \in \mathscr{S}(D, \mathscr{F})$. Moreover, the point p is a unique solution to the following variational inequality problem:*

$$\textit{Find } x^* \in \mathscr{S}(D, \mathscr{F}) \textit{ such that } \langle (f - I)(x^*), y - x^* \rangle \le 0, \quad \forall y \in \mathscr{S}(D, \mathscr{F}). \tag{3.11}$$

Proof. Note that, under Assumptions $(C_1)-(C_3)$, set $\mathscr{S}(D,\mathscr{F})$ is nonempty, closed, and convex. Since f is $\delta-$contractive, then the existence of solutions for problem (3.11) is guaranteed. Firstly, we show that there exists a positive constant M_4 satisfying

$$(1-2\varepsilon_k-\lambda_k^2L^2)\|w^k-y^k\|^2 \le \|x^k-p\|^2-\|x^{k+1}-p\|^2+\beta_kM_4.$$

Indeed, using the viscosity technique in *Step 2* and the inequality

$$\|a+b\|^2 \le \|a\|^2+2\langle b,a+b\rangle, \quad \forall a,b\in\mathbb{H}, \tag{3.12}$$

we have

$$\begin{aligned}
\|x^{k+1}-p\|^2 &= \|z^k-p+\beta_k(f(x^k)-z^k)\|^2\\
&\le \|z^k-p\|^2+2\beta_k\langle f(x^k)-z^k,x^{k+1}-p\rangle\\
&\le \|z^k-p\|^2+2\beta_k\|f(x^k)-z^k\|\|x^{k+1}-p\|\\
&\le \|z^k-p\|^2+\beta_kM_2,
\end{aligned}\tag{3.13}$$

where $M_2=\sup\{2\|f(x^k)-z^k\|\|x^{k+1}-p\| : k\ge 1\}$. By using lemma 3.2, the sequences $\{x^k\}$ and $\{z^k\}$ are bounded and hence $M_2<\infty$. Lemma 3.1 shows that

$$\|z^k-p\|^2 \le \|w^k-p\|^2-(1-2\varepsilon_k-\lambda_k^2L^2)\|w^k-y^k\|^2, \quad \forall k\ge 1. \tag{3.14}$$

By (3.6), it yields

$$\begin{aligned}
\|w^k-p\|^2 &\le (\|x^k-p\|+\beta_kM_1)^2\\
&= \|x^k-p\|^2+\beta_k(2M_1\|x^k-p\|+\beta_kM_1^2)\\
&\le \|x^k-p\|^2+\beta_kM_3,
\end{aligned}\tag{3.15}$$

where $M_3=\sup\{2M_1\|x^k-p\|+\beta_kM_1^2 : k\ge 1\}<\infty$. Combining (3.15) and (3.14), we have

$$\|z^k-p\|^2 \le \|x^k-p\|^2-(1-2\varepsilon_k-\lambda_k^2L^2)\|w^k-y^k\|^2+\beta_kM_3. \tag{3.16}$$

By using (3.13) and (3.16), it follows that

$$\|x^{k+1}-p\|^2 \le \|x^k-p\|^2-(1-2\varepsilon_k-\lambda_k^2L^2)\|w^k-y^k\|^2+\beta_kM_3+\beta_kM_2,$$

which means that

$$(1-2\varepsilon_k-\lambda_k^2L^2)\|w^k-y^k\|^2 \le \|x^k-p\|^2-\|x^{k+1}-p\|^2+\beta_kM_4, \tag{3.17}$$

where $M_4:=M_2+M_3$.

Now we show the following relation:

$$\begin{aligned}
\|x^{k+1}-p\|^2 \le& [1-\beta_k(1-\delta^2)]\|x^k-p\|^2+\beta_k(1-\delta^2)\Bigg[\frac{M_5(1-\beta_k)}{(1-\delta^2)}\alpha_k\frac{\|x^k-x^{k-1}\|}{\beta_k}\\
&+\frac{2\langle f(p)-p,x^{k+1}-p\rangle}{1-\delta^2}\Bigg],
\end{aligned}\tag{3.18}$$

where $M_5 = \sup\{2\|w^k - p\| : k \geq 1\} < \infty$. Indeed, from $w^k = x^k + \alpha_k(x^k - x^{k-1})$ in *Step 1* and the formula (3.12), we have

$$\begin{aligned}\|w^k - p\|^2 &= \|x^k - p + \alpha_k(x^k - x^{k-1})\|^2 \\ &\leq \|x^k - p\|^2 + 2\alpha_k \left\langle x^k - x^{k-1}, w^k - p \right\rangle \\ &\leq \|x^k - p\|^2 + 2\alpha_k \|x^k - x^{k-1}\| \|w^k - p\|, \\ &\leq \|x^k - p\|^2 + \alpha_k \|x^k - x^{k-1}\| M_5. \end{aligned} \tag{3.19}$$

It follows from (3.5) and (3.19) that

$$\|z^k - p\|^2 \leq \|w^k - p\|^2 \leq \|x^k - p\|^2 + \alpha_k \|x^k - x^{k-1}\| M_5, \quad \forall k \geq 1. \tag{3.20}$$

Combining (3.20) and the contractiveness of f, we deduce that

$$\begin{aligned}\|x^{k+1} - p\|^2 =& \|\beta_k(f(x^k) - f(p)) + (1-\beta_k)(z^k - p) + \beta_k(f(p) - p)\|^2 \\ \leq& \|\beta_k(f(x^k) - f(p)) + (1-\beta_k)(z^k - p)\|^2 + 2\beta_k\langle f(p) - p, x^{k+1} - p\rangle \\ \leq& \beta_k\|f(x^k) - f(p)\|^2 + (1-\beta_k)\|z^k - p\|^2 + 2\beta_k\langle f(p) - p, x^{k+1} - p\rangle \\ \leq& \beta_k\delta^2\|x^k - p\|^2 + (1-\beta_k)\|z^k - p\|^2 + 2\beta_k\langle f(p) - p, x^{k+1} - p\rangle \\ \leq& \beta_k\delta^2\|x^k - p\|^2 + (1-\beta_k)\|x^k - p\|^2 + \alpha_k(1-\beta_k)\|x^k - x^{k-1}\|M_5 \\ &+ 2\beta_k\langle f(p) - p, x^{k+1} - p\rangle \\ =& [1 - \beta_k(1-\delta^2)]\|x^k - p\|^2 + \beta_k(1-\delta^2)\Bigg[\frac{M_5(1-\beta_k)}{(1-\delta^2)}\alpha_k\frac{\|x^k - x^{k-1}\|}{\beta_k} \\ &+ \frac{2\langle f(p) - p, x^{k+1} - p\rangle}{1-\delta^2}\Bigg].\end{aligned}$$

This implies (3.18).

Let us consider two cases as follows.

Case 1. There exists a positive integer k_0 such that $\|x^{k+1} - p\|^2 \leq \|x^k - p\|^2$ for all $k \geq k_0$. Then, we have $\lim_{k\to\infty} \|x^k - p\|^2 = A \in [0, \infty)$. Taking the limit into (3.17) as $k \to \infty$ and using the condition $1 - 2\varepsilon_k - \lambda_k^2 L^2 \geq b > 0$ in (3.1), one sees that

$$\lim_{k\to\infty} \|w^k - y^k\| = 0. \tag{3.21}$$

Using *Step 1* and Condition (3.2), one obtains

$$\lim_{k\to\infty} \|x^k - w^k\| = \lim_{k\to\infty} \alpha_k\|x^k - x^{k-1}\| = \lim_{k\to\infty} \beta_k\alpha_k\frac{\|x^k - x^{k-1}\|}{\beta_k} = 0. \tag{3.22}$$

Since $\mathscr{F}$ is $L-$Lipschitz continuous and (3.21), we have

$$0 \leq \lim_{k\to\infty} \|z^k - y^k\| = \lim_{k\to\infty} \lambda_k\|\mathscr{F}y^k - \mathscr{F}w^k\| \leq \lim_{k\to\infty} \lambda_k L\|y^k - w^k\| = 0. \tag{3.23}$$

Combining (3.21), (3.22), and (3.23), we obtain

$$0 \leq \lim_{k\to\infty} \|x^k - z^k\| \leq \lim_{k\to\infty} (\|x^k - w^k\| + \|w^k - y^k\| + \|y^k - z^k\|) = 0. \tag{3.24}$$

From Lemma 3.2 and $\lim_{k\to\infty}\beta_k = 0$, it follows

$$\begin{aligned}0 &\le \lim_{k\to\infty}\|x^{k+1}-x^k\|\\ &\le \lim_{k\to\infty}(\|x^{k+1}-z^k\|+\|z^k-x^k\|)\\ &= \lim_{k\to\infty}(\beta_k\|f(x^k)-z^k\|+\|z^k-x^k\|)=0.\end{aligned}$$

Consequently

$$\lim_{k\to\infty}\|x^{k+1}-x^k\| = 0. \tag{3.25}$$

By Lemma 3.2 that $\{x^k\}$ is a bounded sequence, one sees there exists $\{x^{k_j}\}\subset\{x^k\}$ such that $\{x^{k_j}\}$ converges weakly to $z\in\mathbb{H}$ and

$$\limsup_{k\to\infty}\langle f(p)-p, x^k-p\rangle = \lim_{j\to\infty}\langle f(p)-p, x^{k_j}-p\rangle = \langle f(p)-p, z-p\rangle. \tag{3.26}$$

Since $x^{k_j}\rightharpoonup z$, (3.21) and Lemma 3.1, we have $z\in\mathscr{S}(D,\mathscr{F})$. It is clear that $P_{\mathscr{S}(D,\mathscr{F})}f:\mathbb{H}\to\mathscr{S}(D,\mathscr{F})$ is contractive with constant $\delta\in(0,1)$. Therefore, it has a unique fixed point. We assume $p=\mathscr{P}_{\mathscr{S}(D,\mathscr{F})}f(p)$. From the definition of the metric projection $\mathscr{P}_D$ and $z\in D$, it implies $\langle f(p)-p, z-p\rangle\le 0$. Using this and (3.26), we have

$$\limsup_{k\to\infty}\langle f(p)-p, x^k-p\rangle = \langle f(p)-p, z-p\rangle \le 0. \tag{3.27}$$

From (3.25) and (3.27), it follows that

$$\begin{aligned}\limsup_{k\to\infty}\langle f(p)-p, x^{k+1}-p\rangle &\le \limsup_{k\to\infty}\langle f(p)-p, x^{k+1}-x^k\rangle + \limsup_{k\to\infty}\langle f(p)-p, x^k-p\rangle\\ &\le \limsup_{k\to\infty}(\|f(p)-p\|\|x^{k+1}-x^k\|) + \limsup_{k\to\infty}\langle f(p)-p, x^k-p\rangle\\ &\le 0.\end{aligned} \tag{3.28}$$

Applying Lemma 2.2 for (3.18) with

$$a_k := \|x^k-p\|^2, \theta_k := \beta_k(1-\delta^2), \tau_k := \frac{M_5(1-\beta_k)}{(1-\delta^2)}\alpha_k\frac{\|x^k-x^{k-1}\|}{\beta_k} + \frac{2\langle f(p)-p, x^{k+1}-p\rangle}{1-\delta^2},$$

we have the limit $\lim_{k\to\infty}a_k = 0$. Note that $\lim_{k\to\infty}\tau_k\le 0$ by (3.28). Thus, $\{x^k\},\{y^k\}$, and $\{w^k\}$ converge strongly to the unique solution p of problem (3.11).

Case 2. It does not exist a positive integer k_0 such that $\|x^{k+1}-p\|^2\le\|x^k-p\|^2$ for all $k\ge k_0$. Therefore, there exists $\{\|x^{k_j}-p\|^2\}\subset\{\|x^k-p\|^2\}$ satisfying $\|x^{k_j}-p\|^2\le\|x^{k_j+1}-p\|^2$ for all $j\ge 1$. By the interesting results proposed by Maingé in Lemma 2.1, there exists a nondecreasing sequence $\{n_j\}\subset\{1,2,\dots\}$ such that $\lim_{j\to\infty}n_j=\infty$ and the following relations hold:

$$\|x^{n_j}-p\|^2\le\|x^{n_j+1}-p\|^2, \|x^j-p\|^2\le\|x^{n_j+1}-p\|^2, \quad \forall j\ge 1. \tag{3.29}$$

By (3.17), we have

$$b\|w^{n_j}-y^{n_j}\|^2 \le (1-2\varepsilon_k-\lambda_k^2L^2)\|w^{n_j}-y^{n_j}\|^2 \le \|x^{n_j}-p\|^2-\|x^{n_j+1}-p\|^2+\beta_{n_j}M_4 \le \beta_{n_j}M_4.$$

Taking the limit as $j\to\infty$ and using $\liminf_{k\to\infty}\beta_k=0$, we see that $\lim_{j\to\infty}\|y^{n_j}-w^{n_j}\|=0$. As **Case 1**, we also have $\lim_{j\to\infty}\|x^{n_j}-z^{n_j}\|=0$, $\lim_{j\to\infty}\|x^{n_j+1}-x^{n_j}\|=0$, and $\limsup_{j\to\infty}\langle f(p)-$

$p, x^{n_j+1} - p\rangle \leq 0$. Substituting $k := n_j$ into (3.18), it follows that

$$\begin{aligned}\|x^{n_j+1} - p\|^2 &\leq [1 - \beta_{n_j}(1-\delta^2)]\|x^{n_j} - p\|^2 \\ &\quad + \beta_{n_j}(1-\delta^2)\left[\frac{M_5}{(1-\delta^2)}\alpha_{n_j}\frac{\|x^{n_j} - x^{n_j-1}\|}{\beta_{n_j}} + \frac{2\left\langle f(p) - p, x^{n_j+1} - p\right\rangle}{1-\delta^2}\right] \\ &\leq [1 - \beta_{n_j}(1-\delta^2)]\|x^{n_j+1} - p\|^2 \\ &\quad + \beta_{n_j}(1-\delta^2)\left[\frac{M_5}{(1-\delta^2)}\alpha_{n_j}\frac{\|x^{n_j} - x^{n_j-1}\|}{\beta_{n_j}} + \frac{2\left\langle f(p) - p, x^{n_j+1} - p\right\rangle}{1-\delta^2}\right],\end{aligned}$$

which implies that

$$\|x^{n_j+1} - p\|^2 \leq \frac{M_5}{(1-\delta^2)}\alpha_{n_j}\frac{\|x^{n_j} - x^{n_j-1}\|}{\beta_{n_j}} + \frac{2\left\langle f(p) - p, x^{n_j+1} - p\right\rangle}{1-\delta^2}.$$

By (3.29), we obtain

$$\|x^j - p\|^2 \leq \|x^{n_j+1} - p\|^2 \leq \frac{M_5}{(1-\delta^2)}\alpha_{n_j}\frac{\|x^{n_j} - x^{n_j-1}\|}{\beta_{n_j}} + \frac{2\left\langle f(p) - p, x^{n_j+1} - p\right\rangle}{1-\delta^2}.$$

This implies that $\limsup_{j\to\infty}\|x^j - p\|^2 = 0$. Thus s $\{x^k\}, \{y^k\}$, and $\{w^k\}$ converge strongly to the unique solution p of problem (3.11). This finishes the proof. □

Note that algorithm 3.1 requires the knowledge of Lipschitz constant $L > 0$ of cost mapping F. In fact, L is usually difficult to evaluate. In order to overcome these drawbacks, we use Tseng's linesearch technique to present a new modified algorithm of the above algorithm with a pseudomonotone cost mapping without the knowledge of Lipschitz constant L.

Algorithm 3.2 (MVAPA - Modified Viscosity Approximation Projection Algorithm)

Initialization: Choose starting points $x^{-1}, x^0 \in \mathbb{H}, \mu > 0, \gamma \in (0,1), l \in (0,1)$.
Iterative Steps: Calculate x^{k+1} as follows, $k = 0, 1, \cdots,$
Step 1_b**.** Evaluate $w^k = x^k + \alpha_k(x^k - x^{k-1})$. Find an inexact projection point:

$$y^k \in \mathscr{P}_D^{\varepsilon_k, w^k}(w^k - \lambda_k \mathscr{F} w^k),$$

where $\lambda_k := \gamma l^{m_k}$ and m_k is the smallest nonnegative integer m such that

$$\gamma l^m \|\mathscr{F} y^k - \mathscr{F} w^k\| \leq \mu \|y^k - w^k\|. \tag{3.30}$$

If $y^k = w^k$, then Stop. Otherwise, go to *Step* 2_b.
Step 2_b**.**Compute

$$x^{k+1} = \beta_k f(x^k) + (1-\beta_k)\left[y^k - \lambda_k(\mathscr{F} y^k - \mathscr{F} w^k\right].$$

Step 3_b**.** Let $k := k+1$ and go back *Step* 1_b.

Lemma 3.4. *Under Assumptions* $(C_1)-(C_3)$ *of the cost mapping* $\mathscr{F}$*, the rule* (3.30) *is well defined. Moreover, the following relation holds*

$$\min\left\{\gamma, \frac{\mu l}{L}\right\} \leq \lambda_k \leq \gamma, \quad \forall k \geq 0.$$

Proof. Since $\mathscr{F}$ is $L-$Lipschitz continuous on $\mathbb{H}$ in Assumption (C_2), then $\|\mathscr{F}w^k - \mathscr{F}y^k\| \leq L\|w^k - y^k\|$. Consequently

$$\frac{\mu}{L}\|\mathscr{F}w^k - \mathscr{F}y^k\| \leq \mu\|w^k - y^k\|.$$

Thus rule (3.30) is satisfied in the case that $\gamma l^{m_k} \leq \frac{\mu}{L}$. Hence, λ_k is well defined. Since $l \in (0,1)$ and m_k is a nonnegative integer, it follows $\lambda_k = \gamma l^{m_k} \leq \gamma$. If $\lambda_k = \gamma$, then the lemma is proved. Otherwise, we consider the case $\lambda_k < \gamma$ and hence $m_k \geq 1$. Since m_k is the smallest nonnegative integer satisfying the rule (3.30), we deduce that $\gamma l^{m_k-1}\|\mathscr{F}y^k - \mathscr{F}w^k\| > \mu\|y^k - w^k\|$. Consequently,

$$\left\|\mathscr{F}w^k - \mathscr{F}y^k\right\| > \frac{\mu}{\frac{\lambda_k}{l}}\left\|w^k - y^k\right\|.$$

By using $y^k \neq w^k$ and the $L-$Lipschitz continuity of the cost mapping $\mathscr{F}$, one sees

$$\lambda_k > \frac{\mu l}{L} \Rightarrow \lambda_k \geq \min\left\{\gamma, \frac{\mu l}{L}\right\}.$$

The proof is completed. □

Lemma 3.5. *Let the cost mapping* $\mathscr{F}$ *satisfy Conditions* $(C_1)-(C_3)$*, and let* $\{w^k\}$ *and* $\{y^k\}$ *be generated by Algorithm* 3.2*. If there exists a subsequence* $\{w^{k_j}\} \subset \{w^k\}$ *such that* $\{w^{k_j}\}$ *converges weakly to* $z \in H$ *and* $\lim_{j\to\infty}\|w^{k_j} - y^{k_j}\| = 0$*, then* $z \in \mathscr{S}(D,\mathscr{F})$.

Proof. It is the same as in the proof of Lemma 3.3. □

By a similar way as in Lemma 3.1, we also obtain the following result.

Lemma 3.6. *Let* $z^k = y^k - \lambda_k(\mathscr{F}y^k - \mathscr{F}w^k)$ *and* $p \in \mathscr{S}(D,\mathscr{F})$*. Then,*

$$\|z^k - p\|^2 \leq \|w^k - p\|^2 - (1 - 2\varepsilon_k - \mu^2)\|y^k - w^k\|^2. \tag{3.31}$$

Theorem 3.2. *Assume that the cost mapping* $\mathscr{F}$ *satisfy* $(C_1)-(C_3)$*. Then, under Conditions* (3.1) *and* (3.2)*, three sequences* $\{x^k\},\{y^k\}$*, and* $\{w^k\}$ *generated by Algorithm* 3.2 *converges strongly to a common element* $p \in \mathscr{S}(D,\mathscr{F})$*. Moreover, the solution point* p *is the unique solution to problem* (3.11).

Proof. By using Lemma 3.5 and Lemma 3.6 and following the proof of Theorem 3.2, we can conclude the desired conclusion immediately. □

4. Numerical Experiments

In what follows, we provide an instance to show the practicability and feasibility of the proposed viscosity inexact projection algorithms $(VAPA)$ and $(MVAPA)$. At each iteration, the main computational iteration step of the two algorithms is to compute an inexact projection point via w^k as follows: $y^k \in \mathscr{P}_D^{\varepsilon_k,w^k}(w^k - \lambda_k\mathscr{F}w^k)$. By using (2.2), one has

$$\langle u^k - y^k, y - y^k\rangle \leq \varepsilon_k\|y^k - w^k\|^2, \quad \forall y \in D,$$

where $u^k := w^k - \xi_k \mathscr{F}(w^k)$, which is equivalent to

$$\max\left\{\langle u^k - y^k, y - y^k\rangle : y \in D\right\} = -\min\left\{\langle y^k - u^k, y - y^k\rangle : y \in D\right\} \leq \varepsilon_k \|y^k - w^k\|^2. \quad (4.1)$$

Then, by using the celebrated Gradient Descent Method, it is formally defined to compute an inexact projection point y^k of a point u^k onto constraint D by Procedure 2.1. Moreover, solving the problem (4.1) with error $\varepsilon_k \|y^k - w^k\|^2$ is simple and more effective on software Matlab 2023 than computing exact the projection $\mathscr{P}_D(x^k - \lambda_k \mathscr{F} x^k)$ via an auxiliary function such as Quadratic Program.

Example 4.1. Consider adaptive image restoration in $\mathbb{H} := \mathbb{R}^n$, the Euclidean space. In fact, there always exists any noisy observed signal/image $y \in \mathbb{R}^n$. Our main goal is to recover an original signal/image $x \in \mathbb{R}^n$ from a noisy vector $y \in \mathbb{R}^n$. It means that y is some observed data, which are obtained from a noise-free image x. Let $\mathbb{B}$ be a $m \times n$ matrix which is called the *linear blurring operator*. Denote a sample of zero-mean white Gaussian noise by an additive noise vector ε with variance σ^2, where ε is usually called a realization of a Gaussian random variable with zero mean. This means that $p(s) = \mathscr{N}(s|0, \sigma^2 Id)$, where $\mathscr{N}(s|\mu, \Sigma)$ denotes a multivariate Gaussian density with mean μ and covariance Σ, evaluated at s. The image restoration problem is formulated as:

$$y = \mathbb{B}x + \varepsilon. \quad (4.2)$$

Examples of observation mechanisms which are adequately approximated by (4.2) include optical or motion blur, tomographic projections, electronic noise, photoelectric noise, and more. One classical approach for handling (4.1) is the following *Least Absolute Shrinkage and Selection Operator Model* in compressing sensing, mainly known as LASSO proposed by Osher et al. in [32], that aims at minimizing:

$$\min\left\{\frac{1}{2}\|y - \mathbb{B}x\|^2 + \lambda \|x\|_1 : x \in \mathbb{R}^n\right\}, \quad (4.3)$$

where $\|x\|_1 = \sum_{k=1}^n |x_k|$.

Note that (4.3) is an unconstrained convex problem, however the objective function including $\|\cdot\|_1$ which is not easy to solve when n is enough large.

Our main task is to restore the original image x given the data of the blurred image y. Problem (4.3) is convex, subdifferentiable and the objective function bounded by 0. Then, its solution set is nonempty. By using the necessary and sufficient conditions in optimization, a point x^* is a solution to the problem if and only if $0 \in \partial_x(\|y - \mathbb{B}x\|^2 + \lambda \|x\|_1)$. This is equivalent to $0 = \mathbb{B}^\top(\mathbb{B}x - y) + \lambda\ \text{sign}(x)$. The least square problem (4.3) can be expressed as a variational inequality problem by setting $\mathscr{F}(x) := \mathbb{B}^\top(\mathbb{B}x - y) + \lambda\ \text{sign}(x)$. It is not difficult to show that the function $\text{sign}(\cdot)$ is monotone on $\mathbb{R}^n$. So, the cost mapping $\mathscr{F}$ in this case is monotone (hence it is pseudomonotone) and Lipschitz continuous with constant $L = \|\mathbb{B}^\top \mathbb{B}\| + 2\lambda\sqrt{n}$.

This section reports some numerical results to illustrate the effectiveness of the proposed algorithms $(VAPA)$ and $(MVAPA)$ in comparisons with three popular algorithms using the exact metric projections: Subgradient Extragradient Algorithm (SEA) proposed by Censor, Gibali and Reich in [10, Algorithm 4.1], Halpern Subgradient Extragradient Algorithm $(HSEA)$ introduced by Kraikaew and Saejung in [22, Scheme (4)] and Forward - Backward Splitting Algorithm $(FBSA)$ of Tseng in [36].

We consider the grey scale image of m pixels wide and n pixel height, each value is known to be in the closed interval $[0,256]$. The quality of the restored image is measured by the Peak Signal-to-Noise Ratio (*PSNR*) in decibel (dB) in [35] which is defined by:

$$PSNR = 20\lg\left(\frac{\|x\|}{\|x-y\|}\right).$$

Note that the larger the value of *PSNR* is the better the quality of the restored image. The Structural Similarity Index Metric (*SSIM*), in [37], is to explore the structural information between original signal x and noisy image y. The (*SSIM*) is defined in the following form:

$$\mathrm{SSIM}(x,y) = \frac{(2\mu_x\mu_y + C_1)(2\sigma_{xy} + C_2)}{(\mu_x^2 + \mu_y^2 + C_1)(\sigma_x^2 + \sigma_y^2 + C_2)},$$

where

- the parameter μ_x is the mean of x:
$$\mu_x = \frac{1}{n}\sum_{i=1}^{n} x_i;$$
- the parameter μ_y is the mean of y:
$$\mu_y = \frac{1}{n}\sum_{i=1}^{n} y_i;$$
- the parameter σ_x^2 is the variance of x:
$$\sigma_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)^2;$$
- the parameter σ_y^2 is the variance of $\mathbf{y}$:
$$\sigma_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \mu_y)^2;$$
- the parameter σ_{xy} is the covariance between vector x and y:
$$\sigma_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)(y_i - \mu_y);$$
- the parameters $C_1 = (K_1L)^2$ and $C_2 = (K_2L)^2$ are small constants to stabilize the division with
  * the number L is the dynamic range of the pixel values (e.g., $L = 255$ for 8-bit images);
  * the numbers K_1 and K_2 are small constants (e.g., $K_1 = 0.01$, $K_2 = 0.03$).

For each $\varepsilon \geq 0$, the stopping condition of all the algorithms is given as follows:

- Algorithms $(VAPA)$ and $(MVAPA)$:
$$\Gamma_k = \|\mathscr{P}_D(x^k - \lambda_k \mathscr{F} x^k) - x^k\| \leq \varepsilon; \tag{4.4}$$
- Algorithms $(VAPA)$ and $(MVAPA)$:
$$\Gamma_k = \|\mathscr{P}_D^{\varepsilon_k, w^k}(w^k - \lambda_k \mathscr{F} w^k) - w^k\| \leq \varepsilon. \tag{4.5}$$

All programming is implemented in Matlab R2023a running on a PC with Intel(R) Core(TM) $i9-9900KS\ CPU$ @ $4.00GHz$ 32.0 $GB\ Ram$. The inexact projection $\mathscr{P}_D$ is computed via the Matlab optimization toolbox by fmincon or quadratic functions.

Test 1. In this test, we apply two our proposed algorithms $(VAPA)$ and $(MVAPA)$ for the image restoration model with $n = 2500$ which is generated by the uniform distribution on the closed interval $[-3,3]$ with 170 non-zero elements. The matrix $\mathbb{B}$ is generated by the normal distribution with mean zero and variance one while the observation y with $m = 550$ is generated by Gaussian noise with $PSNR = 37$. The initial points x^{-1}, x^0 are taken randomly. The quality of restoration is measured by the Mean of Squared Error (MSE) to the original signal x, that is $MSE = \frac{1}{n}\|x-y\|^2$. We compare the performance of each algorithm with respect to the number of iterations, CPU time and MSE values, see Figure 1 and Figure 2.

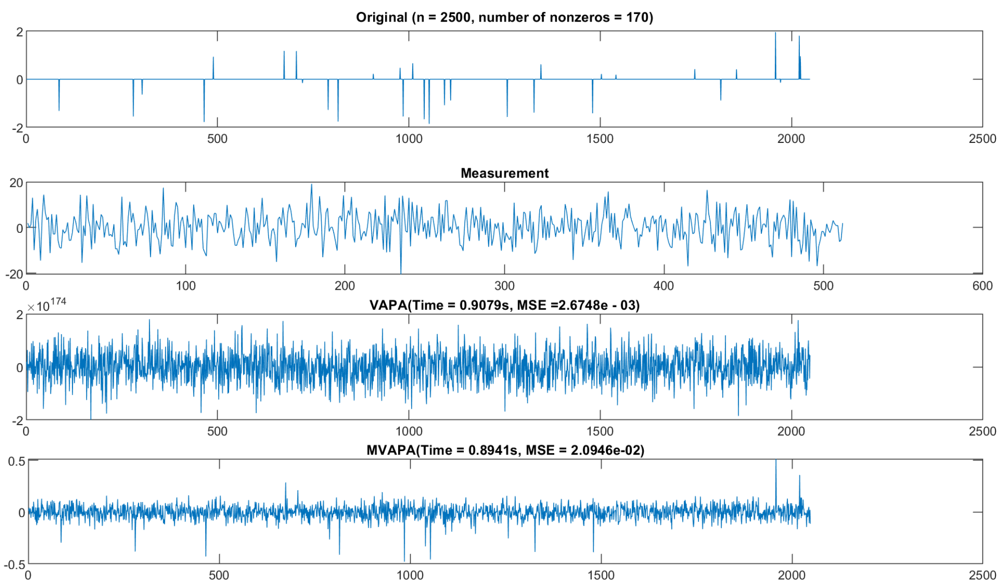

FIGURE 1. Sparse signal recovery includes the original signal, the measurement, the reconstructed signal using Algorithms $(SAPA)$ and $(MSAPA)$, where $\lambda := 0.2683$.

The parameters in all above algorithms are chosen as follows:

- Our algorithms $(VAPA)$ and $(MVAPA)$:

$$a = 10, b = 0,1205, \lambda_k = \frac{\sqrt{1-b}}{L(2k+1)}, \varepsilon_k = \frac{1-L^2\lambda_k^2-b}{4}, \frac{1}{4k+50}, \quad \forall k \geq 1;$$

- Algorithm $(SEA)-$ Censor, Gibali and Reich, and Algorithm $(HSEA)-$ Kraikaew and Saejung:

$$\tau = \frac{1}{2L};$$

- Algorithm $(FBSA)-$ Tseng:

$$\lambda = \frac{1}{2L}.$$

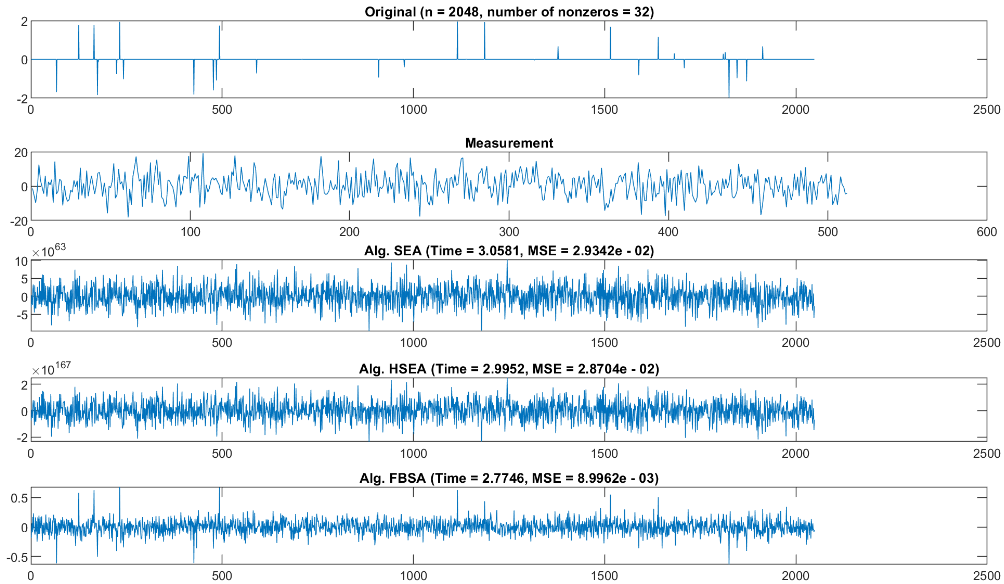

FIGURE 2. Sparse signal recovery includes the original signal, the measurement, the reconstructed signal using Algorithms $(SEA),(HSEA)$ and $(FBSA)$, where $\lambda := 0.2683$.

Test 2. We compare the convergence of the $(VAPA)$, the $(MVAPA)$, the $(SEA),(HSEA)$ and $(FBSA)$ for the image restoration problem by means of $(PSNR)$ and $(SSIM)$. The blurring operator is chosen as

$$\mathbb{B} := fspecial('gaussian', [256\ 256], 4).$$

The results are showed in Figure 3.

Example 4.2. Consider in the real Hilbert space $\mathscr{H} := l_2$, which is given as follows:

$$l_2 = \left\{ (x_k) \subset \mathscr{R} : \sum_{k=1}^{\infty} x_k^2 < +\infty \right\}.$$

The inner product is $\langle x,y\rangle = \sum_{k=1}^{\infty} x_k y_k$, for all $x,y \in l_2$, and its deduced norm is $\|x\| = \sqrt{\sum_{k=0}^{\infty} x_k^2}$, for all $x \in l_2$. For each $\zeta \in (0,1)$, let the viscosity mapping $f : \mathbb{H} \to \mathbb{R}$ be defined by $f(x) = \zeta \cos(\|x\|)$ that is $\zeta-$contractive. The cost mapping $F : \mathbb{H} \to \mathbb{H}$ and the constraint D are given in the forms:

$$D = \{x \in \mathscr{H} : \|x\|^2 \le R^2, \langle r,x\rangle \le l\},\ F(x) = \left[g\sin(p\|x\| + q) + h\cos(e\|x\| + f_0) + m\right] z,$$

where $R,p,q,e,f_0 \in \mathbb{R}, l > 0, g > 0, h > 0, m \in (g+h,\infty), (z,r) \in \mathbb{H}\times\mathbb{H}$.

It is clear that the D is nonempty, closed, and convex. By [4, Proposition 5.1], mapping F is pseudomonotone and $L-$Lipschitz continuous, where $L = (g|p| + h|e|)\|z\|$.

Test 3. We compare our algorithms $(VAPA)$ and $(MVAPA)$ with three the above algorithms with different starting points x^0. The constraint C and the cost mapping $\mathscr{F}$ are defined in

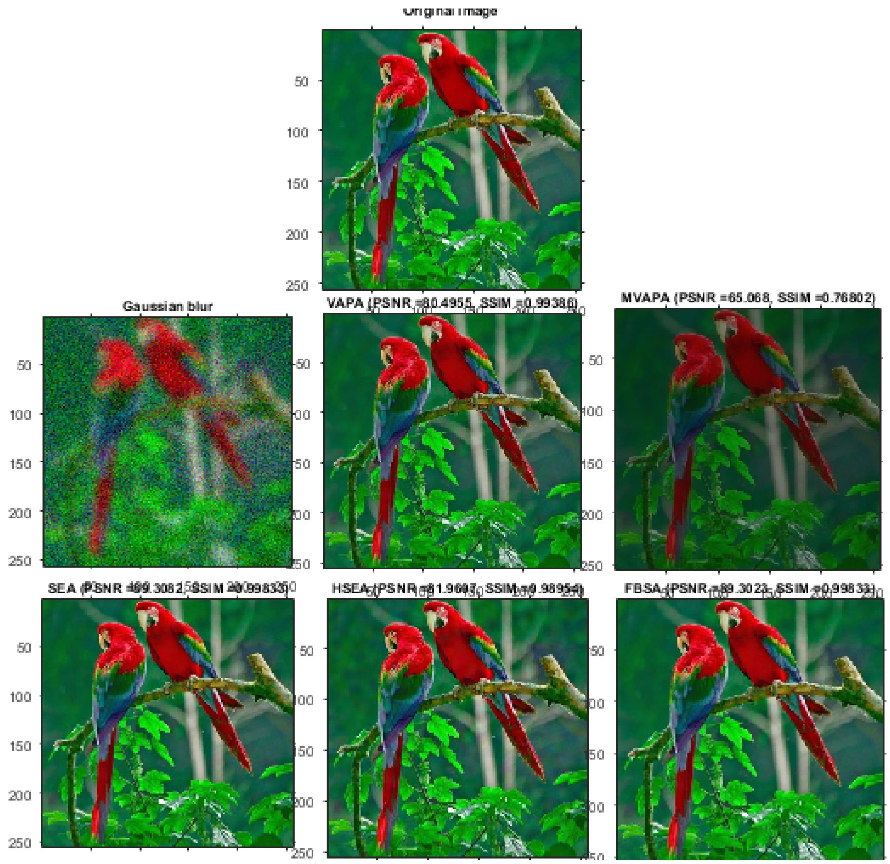

FIGURE 3. Restoration results using five the above algorithms, where $\varepsilon = 10^{-3}$.

Example 4.2. Parameters R, l, g, p, q, h, e, f and m are randomly taken as follows:

$$l = 4, \quad g = 6, \quad p_1 = -2, \quad q_1 = 7, \quad h = 3, \quad e = 10, \quad f_0 = 5, \quad m = 15.$$

The vectors are chosen in the form:

$$r = \left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{k}, \cdots\right)^\top, \quad z = \left(\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \cdots, \frac{1}{2k+1}, \cdots\right)^\top.$$

The numerical results are showed in Table 1.

Algorithm	$x^0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots)^\top$		$x^0 = (\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \cdots)^\top$		$x^0 = (\frac{1}{1^2}, \frac{1}{5^2}, \frac{1}{9^2}, \cdots)^\top$	
	Γ_k	Times	Γ_k	Times	Γ_k	Times
$(VAPA)$	0.2945e-5	37.0942	1.0046e-5	40.97738	2.8055e-5	58.5072
$(MVAPA)$	0.8296e-5	39.1158	1.9905e-5	41.3085	4.0041e-5	63.0230
(SEA)	0.2295e-4	47.0719	0.3061e-4	47.0992	0.5964e-4	59.0782
$(HSEA)$	2.0083e-4	87.0496	3.0965e-4	101.9152	12.5596e-4	162.0458
$(FBSA)$	6.8904e-4	74.8841	6.9930e-4	84.4216	14.6037e-4	92.0052

TABLE 1. Comparison results with different starting points in l_2, where Γ_k is defined in (4.4) and (4.5).

Since the preliminary numerical results reported in Figures $1-3$ and Table 1 of five the above algorithms, based on $(PSNR)$ and $(SSIM)$, we can see that the convergence speed of all the algorithms is very sensitive to the different starting points x^0, the CPU time (second) and the number of iterations of two our proposed algorithms are quite effective and less than three $(SEA),(HSEA)$, and $(FBSA)$ when solving problem $VIPs$.

5. Conclusion

In this paper, we proposed two new inexact projection algorithms for solving pseudomonotone variational inequality problems based on self-adaptive step sizes, viscosity technique, and inexact projections. Then, we proposed two new algorithms and proved strong convergence of their iteration sequences. Primary numerical experiments illustrate and compare the performances of these proposed algorithms with some other known results via image restoration models.

References

[1] F. Alvarez, H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, Set Valued Anal. 9 (2001), 3-11.

[2] P.N. Anh, New outer proximal methods for solving variational inequality problems, J. Optim. Theory Appl. 198 (2023), 479-501.

[3] P.N. Anh, Relaxed projection methods for solving variational inequality problems, J. Glob. Optim. 90 (2024), 909-930.

[4] P.N. Anh, P.Q. Khanh, N.D. Truong, A relaxed projection method for solving bilevel variational inequality problems, Optimization, doi: 10.1080/02331934.2024.2354456.

[5] P.N. Anh, J.K. Kim, Outer approximation algorithms for pseudomonotone equilibrium problems, Comput. Math. Appl. 61 (2011), 2588-2595.

[6] P.N. Anh, L.D. Muu, Coupling the Banach contraction mapping principle and the proximal point algorithm for solving monotone variational inequalites, Acta Math. Vietnamica 129 (2004), 119-133.

[7] P.N. Anh, D.X. Son, A New Method for a Finite Family of Pseudocontractions and Equilibrium Problems, J. Appl. Math. Info. 29 (2011), 1179-1191.

[8] R.S. Burachik, J.O. Lopes, Outer approximation schemes for generalized semi-infinite variational inequality problems, Optimization 59 (2010), 601-617.

[9] Y. Censor, A. Gibali, S. Reich, Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space, Optim. Meth. Softw. 26 (2011), 827-845.

[10] Y. Censor, A. Gibali, S. Reich, The subgradient extragradient method for solving variational inequalities in Hilbert space, J. Optim. Theory Appl. 148 (2011), 318-335.

[11] P.L Combettes, Strong convergence of block-iterative outer approximation methods for convex optimization, SIAM J. Control. Optim. 38 (2000), 538-565.

[12] R.W. Cottle, J.C. Yao, Pseudo-monotone complementarity problems in Hilbert space, J. Optim. Theory Appl. 75 (1992), 281-295.

[13] F. Facchinei, J.-S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. I. Springer, New York, 2003.

[14] J. Fan, L. Liu, X. Qin, A subgradient extragradient algorithm with inertial effects for solving strongly pseudomonotone variational inequalities, Optimization, 69 (2020), 2199-2215.

[15] A. Gibali, S. Reich, R. Zalas, Outer approximation methods for solving variational inequalities in Hilbert space, Optimization, 66 (2017), 417-437.

[16] A.A. Goldstein, Convex programming in Hilbert space, Bull. Amer. Math. Soc. 70 (1963), 709-710.

[17] A.N. Iusem, M. Nasri, Korpelevich's method for variational inequality problems in Banach spaces, J. Global Optim. 50 (2021), 59-76.

[18] A.N. Iusem, B.F. Svaiter, A variant of Korpelevich's method for variational inequalities with a new search strategy, Optimization 42 (1997), 309-321.

[19] C. Izuchukwu, S. Reich, I. Shehu, Convergence of an inertial reflected–forward–backward splitting algorithm for solving monotone inclusion problems with application to image recovery, J. Comput. Appl. Math. 460 (2025), 116405.

[20] J.K. Kim, P.N. Anh, H.G. Hyun, A proximal point-type algorithm for pseudomonotone equilibrium problems, Bull. Korean Math. Soc. 49 (2012), 749-759.

[21] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.

[22] R. Kraikaew, S. Saejung, Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces, J. Optim. Theory Appl. 163 (2014), 399-412.

[23] I.V. Konov, Combined Relaxation Methods for Variational Inequalities, Springer-Verlag, Berlin, 2000.

[24] L. Liu, S.Y. Cho, A Bregman projection algorithm with self adaptive step sizes for split variational inequality problems involving non-Lipschitz operators, J. Nonlinear Var. Anal. 8 (2024), 396-417.

[25] L. Liu, X. Qin, Strong convergence theorems for solving pseudo-monotone variational inequality problems and applications, Optimization, 71 (2022) 3603-3626.

[26] G. Long, S. Liu, G. Xu, S.W. Wong, H. Chen, B. Xiao, A perforation-erosion model for hydraulic-fracturing applications, SPE Prod. Oper. 33 (2018), 770-783.

[27] P.E. Maingé, The viscosity approximation process for quasi-nonexpansive mappings in Hilbert spaces, Comput. Math. Appl. 59 (2010), 74-79.

[28] P.E. Maingé, Numerical approach to monotone variational inequalities by a one-step projected reflected gradient method with line-search procedure, Comput. Math. Appl. 72 (2016), 720-728.

[29] A. Nagurney, Network Economics: A Variational Inequality Approach (Vol. 412). Kluwer Academic Publishers, 1993.

[30] O.I. Oyewole, S. Reich, A survey of iterative methods for approximating solutions to variational inequality problems, Commun. Optim. Theory 2025 (2025), 16.

[31] A.O.E. Owolabi, Strong convergence analysis for solving quasi-monotone variational inequalities and fixed point problems in reflexive Banach spaces, J. Nonlinear Funct. Anal. 2023 (2023), 30.

[32] S. Osher, A. Sole, L. Vese, Image decomposition and restoration using total variation minimization and the H^{-1} norm, Multiscale Model. Simul. 1 (2003), 349-370.

[33] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 591-597.

[34] P. Santos, S. Scheimberg, An inexact subgradient algorithm for equilibrium problems, Comput. Appl. Math. 30 (2011), 91-107.

[35] K. Thung, P. Raveendran, A survey of image quality measures, In: Proceedings of the International Conference for Technical Postgraduates (TECHPOS), Kuala Lumpur, 2009.

[36] P. Tseng, A modified forward-backward splitting method for maximal monotone mappings, SIAM J. Control Optim. 38 (2000), 431–446

[37] Z. Wang, A.k. Bovik, H.R. Sheikh, E.P. Simoncelli, Image quality assessment: From error visibility to structural similarity, IEEE Trans. Image Proc. 13 (2004), 600-612.

[38] H.K. Xu, Iterative algorithms for nonlinear operators, J. Lond. Math. Soc. 66 (2002), 240-256.