Appl. Set-Valued Anal. Optim. 8 (2026), No. 2, pp. 147-166
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.8.2026.2.02

VARIATIONAL ANALYSIS AND NUMERICAL SIMULATION FOR A DYNAMIC CONTACT PROBLEM WITH COULOMB'S FRICTION IN THERMO-VISCOELASTICITY

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Abstract. The focus of this study is a dynamic frictional contact model involving a viscoelastic body with thermal effects and a conductive foundation. Coulomb's law describes the frictional behavior, while a normal compliance model simulates the contact. We derive a variational formulation for the problem and establish the existence of a unique weak solution by using the Banach fixed point theorem. To solve the problem, we propose a fully discrete scheme that combines the finite element method for spatial approximation with the Euler scheme for the time discretization. Error estimates for the solutions are derived, and linear convergence is achieved under suitable regularity assumptions. Finally, numerical simulations are presented to demonstrate the performance of the proposed method.
Keywords. Error estimates; Frictional contact; Finite element approximation; Heat transfer; Thermo-viscoelastic material.
2020 Mathematics Subject Classification. 47J20, 49J52, 74K15.

1. INTRODUCTION

Contact problems arise in numerous applications across mechanics, physics, and engineering. Examples from the automotive industry include the contact between brake pads and rotors or between pistons and cylinders. Thermal effects in contact processes influence the composition and stiffness of contacting surfaces, while also inducing thermal stresses in the contacting bodies (see [16]). Conversely, temperature can significantly affect the elastic response of materials. The literature contains a variety of works that study and develop thermomechanical frictional problems, such as those in [9, 10, 11, 16, 19] and the references therein. These works rigorously constructed mathematical models of contact with thermal effects and established the unique weak solvability of the models using variational and hemivariational inequality techniques.

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Received 22 December 2024; Accepted 19 July 2025; Published online 17 January 2026.

In [4], Chau, Goeleven, and Oujja analyzed a class of dynamic thermal contact problems for viscoelastic materials involving the normal compliance condition and friction. They proposed a numerical scheme for approximating solution fields and performed corresponding numerical computations. Other works in the literature, such as [2, 12, 13, 15, 18], presented numerical solutions to frictional contact problems that account for thermal effects. Bouallala and Essouf [3] addressed a dynamic contact problem between a thermo-viscoelastic body and a conductive foundation under normal compliance and Coulomb's friction. Building on this work, we study the same problem and prove the existence-uniqueness of a weak solution by employing dynamic nonlinear quasi-variational inequalities, nonlinear parabolic variational equalities, and the fixed point method.

To address the problem numerically, we present a discrete formulation by using the finite element method for spatial discretization and a backward Euler scheme for time discretization. We also demonstrate the convergence of the numerical solution. This study faced significant challenges due to the nonlinearity of the boundary conditions and the dynamic nature of the problem. A key novelty of this work is the inclusion of numerical simulations that analyze various problem parameters. Notably, our results are consistent with those found in [4] for a non-clamped body. However, unlike [4], this work assumes Dirichlet boundary conditions on part of the body's surface. Furthermore, our model includes a heat exchange condition in which the heat transfer coefficient is assumed to be constant along the contact boundary.

The rest of the paper is structured as follows. The model of the dynamic process of the thermo-viscoelastic body is presented in Section 2, together with its variational formulation. In Section 3, we state and prove our main existence and uniqueness result, Theorem 3.1. The main result concerning the error estimate for fully discrete numerical scheme is presented in Section 4. Finally, in Section 5, we present numerical simulations for a two-dimensional test problem to illustrate the theoretical error estimate and the evolution of the displacement and the temperature fields.

2. Problem Statement and Weak Formulation

In this paper, we denote by $\mathbb{S}^d$, $(d=2,3)$, the space of second order symmetric tensor on $\mathbb{R}^d$ and by "$\cdot$" and $\|\cdot\|$ the inner product and the Euclidean norm on the space $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively, that is, for all $u,\ v\in\mathbb{R}^d$ and for all $\sigma,\ \tau\in\mathbb{S}^d$,

$$u\cdot v=u_iv_i,\ \|v\|=\sqrt{v\cdot v},\ \sigma\cdot\tau=\sigma_{ij}\tau_{ij},\ \|\tau\|=\sqrt{\tau\cdot\tau}.$$

Also, we denote by $t\in[0,T]$ and $x\in\Omega$ the time and spatial variables, respectively, where $T>0$.

We consider a body made of a thermo-viscoelastic material that occupies the domain $\Omega\subset\mathbb{R}^d$ with a smooth boundary $\Gamma=\partial\Omega$. The boundary is divided into three disjoint measurable parts: Γ_D, Γ_N, and Γ_C, such that $\text{meas}(\Gamma_D)>0$. Additionally, $\nu=\{\nu_i\}$ represents the unit outward normal vector to the boundary. The body is subjected to body forces of density f_0 and a volumetric heat source of constant intensity q_0 in Ω. It is clamped on Γ_D, where the displacement field vanishes. Surface traction forces of density f_N act on Γ_N. The temperature is assumed to vanish on $\Gamma_D\cup\Gamma_N$). The body may come into frictional contact with a thermally conductive foundation, whose temperature is maintained at θ_F. The normal gap between Γ_C and the foundation is denoted by g.

For the displacement field $u : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$ and the stress tensor $\sigma : \Omega \times (0,T) \longrightarrow \mathbb{S}^d$, the symbols u_ν, σ_ν, u_τ, and σ_τ represent their normal and tangential components on the boundary, respectively, and are defined as follows:

$$u_\nu = u \cdot \nu, \quad u_\tau = u - u_\nu \nu,$$
$$\sigma_\nu = (\sigma \nu) \cdot \nu, \quad \sigma\tau = \sigma\nu - \sigma_\nu \nu.$$

We denote by $q = (q_i) : \Omega \times (0,T) \longrightarrow \mathbb{R}$ the heat flux vector, $\theta : \Omega \times (0,T) \longrightarrow \mathbb{R}$ the temperature and by $\varepsilon(u)$ the linearized strain tensor given by $\varepsilon(u) = \big(\varepsilon_{ij}(u)\big)$, $\varepsilon_{ij}(u) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$, where $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$. Here and below $Div(\sigma) = \sigma_{ij,j}$ and $div(q) = q_{i,i}$ denote the divergence operator for tensor and vector valued function, respectively.

The classical model for a dynamic contact problem with Coulomb's friction in thermo-viscoelasticity is as follows.

• **Problem (P)** : Find a displacement field $u : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$ and a temperature field $\theta : \Omega \times (0,T) \longrightarrow \mathbb{R}$ such that, for all $t \in (0,T)$,

$$\sigma(t) = \mathscr{A}\varepsilon(\dot{u}(t)) + \mathscr{F}\varepsilon(u(t)) - \theta(t)\mathscr{M} \text{ in } \Omega, \tag{2.1}$$
$$q(t) = -\mathscr{K}\nabla\theta(t) \text{ in } \Omega, \tag{2.2}$$

$$\rho\ddot{u}(t) - Div\,\sigma(t) = f_0(t) \text{ in } \Omega, \tag{2.3}$$
$$\dot{\theta}(t) + div\,q(t) - \mathscr{R}\varepsilon(\dot{u}(t)) = q_0(t) \text{ in } \Omega, \tag{2.4}$$

$$u(t) = 0 \text{ on } \Gamma_D, \tag{2.5}$$
$$\sigma(t)\nu = f_N(t) \text{ on } \Gamma_N, \tag{2.6}$$
$$\theta(t) = 0 \text{ on } \Gamma_D \cup \Gamma_N, \tag{2.7}$$

$$-\sigma_\nu(u(t) - g) = p_\nu(u_\nu(t) - g), \text{ on } \Gamma_C, \tag{2.8}$$

$$\left.\begin{aligned} &\|\sigma_\tau(t)\| \le p_\tau(u_\nu(t) - g), \\ &\dot{u}_\tau(t) \neq 0 \longrightarrow \sigma_\tau(t) = -p_\tau(u_\nu(t) - g)\frac{\dot{u}_\tau(t)}{\|\dot{u}_\tau(t)\|} \end{aligned}\right\} \text{ on } \Gamma_C, \tag{2.9}$$

$$q(t) \cdot \nu = k_c(\theta(t) - \theta_F) \text{ on } \Gamma_C, \tag{2.10}$$

$$u(0) = u_0,\ \dot{u}(0) = v_0,\ \theta(0) = \theta_0 \text{ in } \Omega. \tag{2.11}$$

Equations (2.1) and (2.2) represent the thermo-viscoelastic constitutive law, where $\mathscr{F} = (\mathscr{F}_{ijkl})$, $\mathscr{A} = (\mathscr{A}_{ijkl})$, $\mathscr{M} = (\mathscr{M}_{ij})$, and $\mathscr{K} = (\mathscr{K}_{ij})$ are, respectively, the elastic tensor, the viscosity tensor, the thermal expansion tensor, and the thermal conductivity tensor. Equation (2.3) describes the equation of motion with a mass density $\rho = 1$. Equation (2.4) represents Fourier's law of heat conduction, where the function $\mathscr{R} = (\mathscr{R}_{ij})$ captures the influence of the displacement field.

In addition, (2.5)–(2.7) define the displacement and thermal boundary conditions. The normal compliance contact condition is specified in (2.8), where p_ν is a prescribed function. When $p_\nu > 0$, the term $u_\nu - g$ represents the penetration of the surface of the body into the foundation. Relation (2.9) represents Coulomb's law of friction, where p_τ is a prescribed non-negative function, known as the friction bound. Relation (2.10) describes a thermal contact condition, where $k_c > 0$ is the coefficient of heat exchange and θ_F is the temperature of the foundation. Finally, the initial conditions are specified in Equation (2.11).

The variational formulation of **Problem (P)** requires some additional notations and preliminaries. First, we define the following spaces:

$$H = \left\{u = (u_i) : u_i \in L^2(\Omega)\right\}, \; \mathscr{H} = \left\{\sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},$$
$$H_1 = \left\{u \in L^2\left(\Omega; \mathbb{R}^d\right) : \varepsilon(u) \in \mathscr{H}\right\}.$$

These are real Hilbert spaces endowed with the following inner products:

$$(u,v)_H = \int_\Omega u_i v_i \, dx, \; \forall u, v \in H, \quad (\sigma, \tau)_{\mathscr{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx, \quad \forall \sigma, \tau \in \mathscr{H},$$
$$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathscr{H}},$$

and the associated norms: $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, and $\|\cdot\|_{\mathscr{H}}$.

For the mechanical and thermal unknowns, we introduce the following spaces:

$$V = \{v \in H : v = 0 \text{ on } \Gamma_D\}, \; Q = \{\eta \in H_1 : \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\},$$

endowed with the inner products and norms given by:

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \; \|v\|_V = \sqrt{(v,v)_V}, \; (\theta, \eta)_Q = (\nabla\theta, \nabla\eta)_H, \; \|\eta\|_Q = \sqrt{(\eta,\eta)_Q}.$$

The following Korn and Friedrichs-Poincaré inequalities hold

$$\|\varepsilon(v)\|_{\mathscr{H}} \geq c_k \|v\|_{H_1}, \text{ for all } v \in V, \quad \|\nabla\eta\|_{L^2(\Omega)} \geq c_p \|\eta\|_Q, \text{ for all } \eta \in Q,$$

where c_k and c_p are two positive constants depend on Ω and Γ_D. By the Sobolev trace theorem,

$$\|v\|_{\left[L^2(\Gamma_C)\right]^d} \leq c_1 \|v\|_V, \text{ for all } v \in V, \quad \|\eta\|_{L^2(\Gamma_C)} \leq c_2 \|\eta\|_Q, \text{ for all } \eta \in Q,$$

where c_1 and c_2 are two positive constants depend on Ω, Γ_D and Γ_C. We denote by V^* the dual space of V and by identifying H with its own dual, and we have $V \subset H = H^* \subset V^*$. We denote $(\cdot,\cdot)_{V^* \times V}$ the duality pairing between V^* and V. Next, we consider the following mappings

$$(f,v)_{V^*\times V} := \int_\Omega f_0.v dx + \int_{\Gamma_N} f_N.v da,$$
$$(q_c, \eta)_{Q^*\times Q} := \int_\Omega q_0 \eta dx,$$
$$j_d(u,v) := \int_{\Gamma_C} p_\nu(u_\nu - g) v_\nu da + \int_{\Gamma_C} p_\tau(u_\tau - g)\|v_\tau\| da,$$
$$j_c(\theta, \eta) := \int_{\Gamma_C} k_c(\theta - \theta_F)\eta da.$$

We now introduce assumptions regarding the data in the study of **Problem (P)**:

(H1) The viscosity operator $\mathscr{A} = \mathscr{A}_{ijkl} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

i) There exists $m_b > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{S}^d$ and a.e. $x \in \Omega$

$$\left(\mathscr{A}(x,\xi_1) - \mathscr{A}(x,\xi_2)\right).(\xi_1 - \xi_2) \geq m_b \|\xi_1 - \xi_2\|^2.$$

ii) There exists $M_b > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{S}^d$ and a.e. $x \in \Omega$

$$\left\|\mathscr{A}(x,\xi_1) - \mathscr{A}(x,\xi_2)\right\| \leq M_b \|\xi_1 - \xi_2\|.$$

iii) The mapping $x \mapsto \mathscr{A}(x,\xi)$ is measurable on Ω, for all $\xi \in \mathbb{S}^d$.
iv) $\mathscr{A}(x,0) = 0$ a.e. $x \in \Omega$.

(H2) The elasticity operator $\mathscr{F} = \mathscr{F}_{ijkl} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies
i) There exists $m_a > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{S}^d$ and a.e. $x \in \Omega$
$$\big(\mathscr{F}(x,\xi_1) - \mathscr{F}(x,\xi_2)\big).\big(\xi_1 - \xi_2\big) \geq m_a \|\xi_1 - \xi_2\|^2.$$
ii) There exists $M_a > 0$ such that for all $\xi_1, \xi_2 \in \mathbb{S}^d$ and a.e. $x \in \Omega$
$$\big\|\mathscr{F}(x,\xi_1) - \mathscr{F}(x,\xi_2)\big\| \leq M_a \|\xi_1 - \xi_2\|.$$
iii) The mapping $x \mapsto \mathscr{F}(x,\xi)$ is measurable on Ω, for all $\xi \in \mathbb{S}^d$.
iv) $\mathscr{F}(x,0) = 0$ a.e. $x \in \Omega$.

(H3) The thermal conductivity tensor $\mathscr{K} = (\mathscr{K}_{ij}) : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfies
i) $\mathscr{K}_{ij} = \mathscr{K}_{ji} \in L^\infty(\Omega)$.
ii) $\mathscr{K}_{ij}(x)\xi_i\xi_j \geq m_d \|\xi\|^2$, with $m_d > 0$, for all $\xi \in \mathbb{R}^d$, $x \in \Omega$.
iii) $\|(\mathscr{K}\nabla\theta, \nabla\eta)\|_H \leq M_d \|\theta\|_Q \|\eta\|_Q$, with $M_d > 0$, for all $\theta, \eta \in Q$.

(H4) The thermal expansion tensor $\mathscr{M} = (\mathscr{M}_{ij}) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies
i) $\mathscr{M}_{ij} = \mathscr{M}_{ji} \in L^\infty(\Omega)$.
ii) $\|(\mathscr{M}\theta, \varepsilon(v))\|_{\mathscr{H}} \leq M_m \|\theta\|_Q \|v\|_V$, with $M_m > 0$, for all $\theta \in Q, v \in V$.

(H5) The influence of the displacement field tensor $\mathscr{R} = (\mathscr{R}_{ij}) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies
i) $\mathscr{R}_{ij} = \mathscr{R}_{ji} \in L^\infty(\Omega)$.
ii) $\|(\mathscr{R}\varepsilon(v), \eta)\|_H \leq M_e \|v\|_V \|\eta\|_Q\|$, with $M_e > 0$, for all $\eta \in Q, v \in V$.

(H6) i) The forces, the traction and the thermal flux satisfy
$$f_0 \in L^2(0,T;L^2(\Omega)^d), \;\; f_N \in L^2(0,T;L^2(\Gamma_N)^d) \;\text{ and }\; q_0 \in L^2(0,T;L^2(\Omega));$$
ii) the gap function, the thermal potential, and the initial data satisfy
$$g \geq 0,\; g \in L^\infty(\Gamma_C), \text{ and } \theta_F \in L^2(0,T;L^2(\Gamma_C)),\; u_0, v_0 \in V, \theta_0 \in L^2(\Omega);$$
iii) the functional j_d is proper, convex, and lower semi-continuous on V.

(H7) The normal compliance function p_ν and the friction bound p_τ satisfy the following hypothesis for $r = \nu, \tau$
i) $p_r : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}_+$;
ii) $x \longrightarrow p_r(x,u)$ is measurable on Γ_C, for all $u \in \mathbb{R}$;
iii) $x \longrightarrow p_r(x,u) = 0$ for $u \leq 0$, a.e. $x \in \Gamma_C$;
iv) there exists $L_r > 0$ such that $|p_r(x,u) - p_r(x,v)| \leq L_r|u - v|$, for all $u, v \in \mathbb{R}_+$, a.e. $x \in \Gamma_C$.

(H8)
$$m_a < (L_\nu + L_\tau)\, c_1^2, \text{ and } m_d > M_{k_c} c_2^2. \tag{2.12}$$

For the sake of simplification, let us assume that
$$\begin{aligned}
a : V \times V \longrightarrow \mathbb{R}, &\quad a(u,v) &&:= (\mathscr{F}\varepsilon(u), \varepsilon(v))_{\mathscr{H}},\\
b : V \times V \longrightarrow \mathbb{R}, &\quad b(u,v) &&:= (\mathscr{A}\varepsilon(u), \varepsilon(v))_{\mathscr{H}},\\
d : Q \times Q \longrightarrow \mathbb{R}, &\quad d(\theta,\eta) &&:= (\mathscr{K}\nabla\theta, \nabla\eta)_H,\\
m : Q \times V \longrightarrow \mathbb{R}, &\quad m(\theta,v) &&:= (\mathscr{M}\theta, \varepsilon(v))_{\mathscr{H}},\\
e : V \times Q \longrightarrow \mathbb{R}, &\quad e(u,\eta) &&:= (\mathscr{R}\varepsilon(v), \eta)_{L^2(\Omega)}.
\end{aligned}$$

According to this notation and through a standard derivation, we have the following variational formulation in terms of displacement field and temperature.

Problem (PV) : Find a displacement field $u : \Omega \times (0,T) \longrightarrow \mathbb{R}^d$ and a temperature field $\theta : \Omega \times (0,T) \longrightarrow \mathbb{R}$ such that for all $v \in V$, $\eta \in Q$ and a.e. $t \in (0,T)$

$$\begin{aligned} &(\ddot{u}(t), v - \dot{u}(t))_H + b(\dot{u}(t), v - \dot{u}(t)) + a(u(t), v - \dot{u}(t)) - m(\theta(t), v - \dot{u}(t)) \\ &\qquad + j_d(u(t), v) - j_d(u(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_{V^* \times V}, \end{aligned} \tag{2.13}$$

$$(\dot{\theta}(t), \eta)_{L^2(\Omega)} + d(\theta(t), \eta) - e(\dot{u}(t), \eta) + j_c(\theta(t), \eta) = (q_c(t), \eta)_{Q^* \times Q}, \tag{2.14}$$

$$u(0) = u_0,\ \dot{u}(0) = v_0,\ \theta(0) = \theta_0.$$

3. An Existence and Uniqueness Result

In this section, we present, and we demonstrate an existence and uniqueness result.

Theorem 3.1. *Assume that (H1)-(H8) and condition (2.12) hold. Then, there exists a unique solution (u, θ) to **Problem (PV)** which satisfies the following regularity conditions*

$$u \in L^2(0,T;V),\ \dot{u} \in L^2(0,T;V^*),\ \ \theta \in L^2(0,T;Q).$$

The proof of this result is carried out in several steps and based on Banach fixed point theorem. First, let $\zeta \in L^2(0,T;V^*)$ and $\xi \in L^2(0,T;Q^*)$ given by

$$(\zeta(t), v)_{V^* \times V} = m\left(\theta_\xi(t), v\right),\ \forall v \in V, \quad (\xi(t), \eta)_{Q^* \times Q} = -e\left(\dot{u}_\zeta(t), \eta\right),\ \forall \eta \in Q.$$

Applying Riesz' representation theorem, we define the elements

$$\begin{aligned} \left(f_\zeta(t), v\right)_{V^* \times V} &= (f(t), v)_{V^* \times V} - (\zeta(t), v)_{V^* \times V}, \\ \left(q_\xi(t), \eta\right)_{Q^* \times Q} &= (q_c(t), \eta)_{Q^* \times Q} - (\xi(t), \eta)_{Q^* \times Q}, \end{aligned}$$

for all $v \in V$, $\eta \in Q$ and a.e. $t \in (0,T)$.

Next, we consider the following intermediates problems.

Problem (PD): Find a displacement field $u_\zeta : (0,T) \longrightarrow V$ such that for all $v \in V$ and a.e. $t \in (0,T)$

$$\begin{aligned} &\left(\ddot{u}_\zeta(t), v - \dot{u}_\zeta(t)\right)_H + b\left(\dot{u}_\zeta(t), v - \dot{u}_\zeta(t)\right) + a\left(u_\zeta(t), v - \dot{u}_\zeta(t)\right) \\ &\qquad j_d\left(u_\zeta(t), v\right) - j_d\left(u_\zeta(t), \dot{u}_\zeta(t)\right) \geq \left(f_\zeta(t), v - \dot{u}_\zeta(t)\right)_{V^* \times V}, \\ &u_\zeta(0) = u_0,\ \dot{u}_\zeta(0) = v_0. \end{aligned} \tag{3.1}$$

Problem (PT): Find a temperature field $\theta_\xi : (0,T) \longrightarrow Q$ such that for all $\eta \in Q$ and a.e. $t \in (0,T)$

$$\begin{aligned} &\left(\dot{\theta}_\xi(t), \eta\right)_{L^2(\Omega)} + d\left(\theta_\xi(t), \eta\right) + j_c\left(\theta_\xi(t), \eta\right) = \left(q_\xi(t), \eta\right)_{Q^* \times Q}, \\ &\theta_\xi(0) = \theta_0. \end{aligned}$$

In the second step, we present the existence and uniqueness result of the intermediates problems.

Lemma 3.1. *For all $v \in V$, **Problem (PD)** has a unique solution u_ζ which satisfies $u_\zeta \in L^2(0,T;V)$ and $\dot{u}_\zeta \in L^2(0,T;V^*)$.*

The proof is based on similar arguments to those used in [8, Theorem 5.15].

Lemma 3.2. *For all $\eta \in Q$, **Problem (PT)** has a unique solution θ_ξ satisfies $\theta_\xi \in L^2(0,T;Q)$.*

The proof of this result is presented in [5, Lemma 3.3] using the Galerkin method.
In the last step, we define the operator

$$\Lambda(\zeta,\xi)(t) := \big(\Lambda_1(\zeta,\xi)(t), \Lambda_2(\zeta,\xi)(t)\big) \in V^* \times Q^*,$$

given by

$$\begin{aligned}
\big(\Lambda_1(\zeta,\xi)(t), v\big)_{V^*\times V} &= m\left(\theta_\xi(t), v\right),\\
\big(\Lambda_2(\zeta,\xi)(t), \eta\big)_{Q^*\times Q} &= -e\left(\dot{u}_\zeta(t), \eta\right).
\end{aligned}$$

We have the following lemma.

Lemma 3.3. *The operator Λ is continuous and has a unique fixed point $(\zeta^*,\xi^*) \in L^2(0,T;V^* \times Q^*)$.*

Proof. Let $(\zeta,\xi) \in L^2(0,T;V \times L^2(\Omega))$ and $t_1,t_2 \in [0,T]$. By assumption (H2) and (H5), we have

$$\big\|\Lambda_1(\zeta,\xi)(t_1) - \Lambda_2(\zeta,\xi)(t_2)\big\|_{V^*\times Q^*} \le M_m \left\|\theta_\xi(t_1) - \theta_\xi(t_2)\right\|_Q, \tag{3.2}$$

$$\big\|\Lambda_2(\zeta,\xi)(t_1) - \Lambda_2(\zeta,\xi)(t_2)\big\|_{V^*\times Q^*} \le M_e \left\|\dot{u}_\zeta(t_1) - \dot{u}_\zeta(t_2)\right\|_V. \tag{3.3}$$

Taking account the regularities of θ_ξ and $\dot{u}_\xi$, we deduce that Λ is continuous.

Now, let $(\zeta_1,\xi_1),(\zeta_2,\xi_2) \in L^2(0,T;V^* \times Q^*)$. For $t \in [0,T]$, similar to (3.2)-(3.3), we obtain

$$\big\|\Lambda(\zeta_1,\xi_1)(t) - \Lambda(\zeta_2,\xi_2)(t)\big\|_{V^*\times Q^*} \le c\left(\left\|\dot{u}_{\zeta_1}(t) - \dot{u}_{\zeta_2}(t)\right\|_V + \left\|\theta_{\xi_1}(t) - \theta_{\xi_2}(t)\right\|_Q\right). \tag{3.4}$$

Using (3.1), we obtain that for a.e. $t \in (0,T)$

$$\begin{aligned}
&\left(\ddot{u}_{\zeta_1}(t) - \ddot{u}_{\zeta_2}(t), \dot{u}_{\zeta_1}(t) - \dot{u}_{\zeta_2}(t)\right)_H + b\left(\dot{u}_{\zeta_1}(t) - \dot{u}_{\zeta_2}(t), \dot{u}_{\zeta_1}(t) - \dot{u}_{\zeta_2}(t)\right)\\
&+ a\left(u_{\zeta_1}(t) - u_{\zeta_2}(t), \dot{u}_{\zeta_1}(t) - \dot{u}_{\zeta_2}(t)\right) + \left(\zeta_1(t) - \zeta_2(t), \dot{u}_{\zeta_1}(t) - \dot{u}_{\zeta_2}(t)\right)\\
&+ j_d\left(u_{\zeta_1}(t), \dot{u}_{\zeta_1}(t)\right) - j_d\left(u_{\zeta_1}(t), \dot{u}_{\zeta_2}(t)\right) - j_d\left(u_{\zeta_2}(t), \dot{u}_{\zeta_1}(t)\right) + j_d\left(u_{\zeta_2}(t), \dot{u}_{\zeta_2}(t)\right) \le 0.
\end{aligned} \tag{3.5}$$

From the hypothesis on the operator j_d, we have

$$\begin{aligned}
&\left| j_d\left(u_{\zeta_1}(t), \dot{u}_{\zeta_1}(t)\right) - j_d\left(u_{\zeta_1}(t), \dot{u}_{\zeta_2}(t)\right) - j_d\left(u_{\zeta_2}(t), \dot{u}_{\zeta_1}(t)\right) + j_d\left(u_{\zeta_2}(t), \dot{u}_{\zeta_2}(t)\right)\right|\\
&\le c_1^2(L_\nu + L_\tau)\left\|u_{\zeta_1}(t) - u_{\zeta_2}(t)\right\|_V \left\|\dot{u}_{\zeta_1}(t) - \dot{u}_{\zeta_2}(t)\right\|_V,
\end{aligned} \tag{3.6}$$

for a.e. $t \in (0,T)$. Integrating inequality (3.5) over $[0,T]$ and coupling (3.6), (H1), it follows that for a.e. $t \in (0,T)$

$$\begin{aligned}
&\frac{1}{2}\left\|\dot{u}_{\zeta_1}(t)-\dot{u}_{\zeta_2}(t)\right\|_V^2+m_b\int_0^t\left\|\dot{u}_{\zeta_1}(s)-\dot{u}_{\zeta_2}(s)\right\|_V^2 ds\\
&\leq -\int_0^t\left(\zeta_1(s)-\zeta_2(s),\dot{u}_{\zeta_1}(s)-\dot{u}_{\zeta_2}(s)\right)_{V^*\times V} ds\\
&+(M_a+c_1^2(L_\nu+L_\tau))\int_0^t\left\|u_{\zeta_1}(s)-u_{\zeta_2}(s)\right\|_V\left\|\dot{u}_{\zeta_1}(s)-\dot{u}_{\zeta_2}(s)\right\|_V ds.
\end{aligned}$$

We recall that, for $i=1,2$, we have $u_{\zeta_i}(t)=\int_0^t \dot{u}_{\zeta_i}(s)ds+u_0$. Thus

$$\left\|u_{\zeta_1}(t)-u_{\zeta_2}(t)\right\|_V\leq c\int_0^t\left\|\dot{u}_{\zeta_1}(s)-\dot{u}_{\zeta_2}(s)\right\|_V ds.$$

Combining the two last relations with the young's inequality, $ab\leq \varepsilon a^2+\frac{1}{4\varepsilon}b^2$, $\varepsilon>0$, and Gornwall's inequality, we deduce that

$$\left\|\dot{u}_{\zeta_1}(t)-\dot{u}_{\zeta_2}(t)\right\|_V\leq c\left\|\zeta_1(t)-\zeta_2(t)\right\|_{V^*}. \tag{3.7}$$

By using the same idea, we get that for a.e. $t\in(0,T)$

$$\begin{aligned}
&\left(\dot{\theta}_{\xi_1}(t)-\dot{\theta}_{\xi_2}(t),\theta_{\xi_1}(t)-\theta_{\xi_2}(t)\right)_{L^2(\Omega)}+d\left(\theta_{\xi_1}(t)-\theta_{\xi_2}(t),\theta_{\xi_1}(t)-\theta_{\xi_2}(t)\right)\\
&+\left(\xi_1(t)-\xi_2(t),\theta_{\xi_1}(t)-\theta_{\xi_2}(t)\right)_{Q^*\times Q}\\
&+j_c\left(\theta_{\xi_1}(t),\theta_{\xi_1}(t)-\theta_{\xi_2}(t)\right)-j_c\left(\theta_{\xi_2}(t),\theta_{\xi_1}(t)-\theta_{\xi_2}(t)\right)=0.
\end{aligned}$$

By (2.12) and (H7), we conclude

$$\left|j_c\left(\theta_{\xi_1}(t),\theta_{\xi_1}(t)-\theta_{\xi_2}(t)\right)-j_c\left(\theta_{\xi_2}(t),\theta_{\xi_1}(t)-\theta_{\xi_2}(t)\right)\right|\leq k_c c_2^2\left\|\theta_{\xi_1}(t)-\theta_{\xi_2}(t)\right\|_Q^2.$$

Using condition (2.12), we find that there exists a positive constant c such that

$$\left\|\theta_{\xi_1}(t)-\theta_{\xi_2}(t)\right\|_Q\leq c\left\|\xi_1(t)-\xi_2(t)\right\|_{Q^*}. \tag{3.8}$$

Now, combining (3.4), (3.7), and (3.8), we find

$$\left\|\Lambda(\zeta_1,\xi_1)-\Lambda(\zeta_2,\xi_2)\right\|_{L^2(0,T;V^*\times Q^*)}\leq c\left\|(\zeta_1,\xi_1)-(\zeta_2,\xi_2)\right\|_{L^2(0,T;V^*\times Q^*)}.$$

Reiterating this inequality n times leads to

$$\left\|\Lambda^n(\zeta_1,\xi_1)-\Lambda^n(\zeta_2,\xi_2)\right\|_{L^2(0,T;V^*\times Q^*)}\leq\frac{(cT)^n}{n!}\left\|(\zeta_1,\xi_1)-(\zeta_2,\xi_2)\right\|_{L^2(0,T;V^*\times Q^*)},$$

which implies that, for n sufficiently large, operator Λ is a contraction in $L^2(0,T;V^*\times Q^*)$. Therefore, there exists a unique fixed point (ζ^*,ξ^*) of Λ. □

Now, we have all ingredients to prove **Theorem 3.1**

Proof of Theorem 3.1.
Existence: Let $\left(\zeta^*,\xi^*\right) \in L^2\left(0,T;V^*\times Q^*\right)$ be the fixed point of the operator Λ and let $u^*_{\zeta^*}$, $\theta^*_{\xi^*}$ be the solutions of **Problem (PD)** and **Problem (PT)**, respectively. For $(\zeta,\xi)=(\zeta^*,\xi^*)$, by the definition of Λ, we find that $\left(u_{\zeta^*},\theta_{\xi^*}\right)$ is a solution to **Problem (PV)**.
Uniqueness: The uniqueness results from the uniqueness of the fixed point of Λ. □

4. Fully Discrete Scheme and Error Estimate

In this section, we present a fully discrete scheme for the variational formulated in **Problem (PV)**, and we establish a result on error estimate. Let $\left\{\mathscr{T}^h\right\}$ be a regular family of triangular finite element partition of $\overline{\Omega}$ which are compatible with the boundary decomposition $\overline{\Gamma}=\overline{\Gamma}_D\cup\overline{\Gamma}_N\cup\overline{\Gamma}_C$, where $h>0$ denotes a spatial discretization parameter. Let V^h and Q^h be a finite dimensional subspace of V and Q respectively given by

$$V^h=\left\{v^h\in\left[C(\overline{\Omega})\right]^d;\ v^h_{|Tr}\in\left[\mathbb{P}_1(Tr)\right]^d\ \forall Tr\in\mathscr{T}^h;\ v^h=0 \text{ on } \overline{\Gamma}_D\right\}\subset V,$$
$$Q^h=\left\{\eta^h\in C(\overline{\Omega});\ \eta^h_{|Tr}\in\mathbb{P}_1(Tr)\ \forall Tr\in\mathscr{T}^h;\ \eta^h=0 \text{ on } \overline{\Gamma}_D\cup\overline{\Gamma}_N\right\}\subset Q.$$

For a positive integer N, we define a uniform partition of $[0,T]$ given by $0=t_0<t_1<...<t_N=T$, and the time step size $k_n=t_n-t_{n-1}$, and let $k=\max_n\{k_n\}$ is the maximal step size. For a time continuous function $u=u(t)$, we write $u_n=u(t_n)$ and $\delta u_n=\dfrac{u_n-u_{n-1}}{k}$ for $n=1,...,N$. Also, we introduce the velocity field $\left\{w_n^{hk}\right\}_{n=0}^N$ which is related with the displacement field with the following relations

$$w_n^{hk}=\delta u_n^{hk} \quad \text{and} \quad u_n^{hk}=u_0^h+\sum_{j=1}^n k_j w_j^{hk}.$$

Using the backward Euler scheme, the fully discrete approximation of **Problem (PV)** is the following.

Problem (PF) : Find a displacement field $\left\{u_n^{hk}\right\}_{n=0}^N\subset V^h$ and a temperature field $\left\{\theta_n^{hk}\right\}_{n=0}^N\subset Q^h$ for all $v^h\in V^h$ and $\eta^h\in Q^h$ such that

$$\begin{aligned}&\left(\delta w_n^{hk},v^h-w_n^{hk}\right)_H+b\left(w_n^{hk},v^h-w_n^{hk}\right)+a\left(u_n^{hk},v^h-w_n^{hk}\right)\\&\quad-m\left(\theta_{n-1}^{hk},v^h-w_n^{hk}\right)+j_d\left(u_n^{hk},v^h\right)-j_d\left(u_n^{hk},w_n^{hk}\right)\geq\left(f_n,v^h-w_n^{hk}\right)_{V^*\times V},\end{aligned}\tag{4.1}$$

$$\left(\delta\theta_n^{hk},\eta^h\right)_{L^2(\Omega)}+d\left(\theta_n^{hk},\eta^h\right)-e\left(w_{n-1}^{hk},\eta^h\right)+j_c\left(\theta_n^{hk},\eta^h\right)=\left(q_{c_n},\eta^h\right)_{Q^*\times Q},\tag{4.2}$$

and

$$u_0^{hk}=u_0^h,\ w_0^{hk}=w_0^h,\ \theta_0^{hk}=\theta_0^h,$$

where $u_0^h\in V^h$, $w_0^h\in V^h$, and $\theta_0^h\in Q^h$ are respectively approximates of u_0, v_0, and θ_0.

Under the assumptions of **Theorem 3.1**, and following the same arguments, used in previous section, there exists a unique solution of **Problem (PF)**. Next, we recall the following discrete Gornwall's inequality [17, Lemma 4.1].

Lemma 4.1. *Let $T > 0$ be given. For a positive integer N, define $k = T/N$. Assume that $\{g_n\}_{n=1}^N$ and $\{e_n\}_{n=1}^N$ are two sequences of nonnegative numbers satisfying, for all $n = 1,...,N$, $e_n \leq cg_n + c\sum_{j=1}^n ke_j$ for a positive constant c independent of N or k. Then, there exists a positive constant c, independent of N or k, such that*

$$\max_{1\leq n\leq N} e_n \leq c \max_{1\leq n\leq N} g_n.$$

Now we state a result on error estimation.

Lemma 4.2. *Let (u,θ) and $\left(u_n^{hk},\theta_n^{hk}\right)$ be solutions to **Problem (PV)** and **Problem (PF)**, respectively. Assume (H1)-(H8) and (2.12). Then the following bound holds for all $\left\{v_j^n\right\}_{j=1}^N \subset V^h$ and $\left\{\eta_j^n\right\}_{j=1}^N \subset Q^h$*

$$\begin{aligned}
&\left\|w_n - w_n^{hk}\right\|_H^2 + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + ck\sum_{j=1}^n \left(\left\|w_j - w_j^{hk}\right\|_V^2 + \left\|\theta_j - \theta_j^{hk}\right\|_Q^2\right) \\
&\leq c\left\{\left\|w_0 - w_0^{hk}\right\|_H^2 + \left\|u_0 - u_0^{hk}\right\|_V^2 + \left\|\theta_0 - \theta_0^{hk}\right\|_{L^2(\Omega)}^2 + \left\|\theta_0 - \theta_0^{hk}\right\|_Q^2\right\} \\
&+ c\left\{\left\|w_1 - v_1^h\right\|_H^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2\right\} + c\left\{\left\|w_n - v_n^h\right\|_H^2 + \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2\right\} \\
&+ ck^2\left\|\theta\right\|_{H^1(0,T;Q)}^2 + ck^2\left\|u\right\|_{H^2(0,T;V)}^2 + ckR\left(w_j, v_j^h\right) \\
&+ ck\sum_{j=1}^n\left(\left\|\dot{w}_j - \delta w_j\right\|_H^2 + \left\|w_j - v_j^h\right\|_V^2 + \left\|\dot{\theta}_j - \delta\theta_j\right\|_{L^2(\Omega)}^2 + \left\|\theta_j - \eta_j^h\right\|_Q^2\right) \\
&+ \frac{1}{k}\sum_{j=1}^{n-1}\left(\left\|\left(w_j - v_j^n\right) - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2 + \left\|\left(\theta_j - \eta_j^n\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2\right),
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
R\left(w_n, v^h\right) &= \left(\delta w_n, v^h - w_n\right)_H + b\left(w_n, v^h - w_n\right) + a\left(u_n, v^h - w_n\right) \\
&\quad - m\left(\theta_n, v^h - w_n\right) + j_d\left(u_n, v^h\right) - j_d\left(u_n, w_n\right) - \left(f_n, v^h - w_n\right)_{V^*\times V},
\end{aligned}$$

Proof. Using inequality (4.1), we have

$$\begin{aligned}
&-\left(\delta w_n^{hk}, w_n - w_n^{hk}\right)_H + b\left(-w_n^{hk}, w_n - w_n^{hk}\right) \\
&\leq a\left(u_n^{hk}, w_n - w_n^{hk}\right) - m\left(\theta_{n-1}^{hk}, w_n - w_n^{hk}\right) - \left(\delta w_n^{hk}, w_n - v^h\right)_H - b\left(w_n^{hk}, w_n - v^h\right) \\
&\quad - a\left(u_n^{hk}, w_n - v^h\right) - m\left(\theta_{n-1}^{hk}, v^h - w_n\right) + j_d\left(u_n^{hk}, v^h\right) - j_d\left(u_n^{hk}, w_n^{hk}\right) - \left(f_n, v^h - w_n^{hk}\right)_{V^*\times V}.
\end{aligned}$$

Taking $v = w_n^{hk}$ in (2.13) at time $t = t_n$, we obtain

$$\begin{aligned}
&\left(\dot{w}_n, w_n - w_n^{hk}\right)_H + b\left(w_n, w_n - w_n^{hk}\right) \\
&\leq a\left(u_n, w_n^{hk} - w_n\right) - m\left(\theta_n, w_n^{hk} - w_n\right) + j_d\left(u_n, w_n^{hk}\right) - j_d\left(u_n, w_n\right) - \left(f_n, w_n^{hk} - w_n\right)_{V^*\times V}.
\end{aligned}$$

Adding the inequality above yields

$$\begin{aligned}
&\left(\dot{w}_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H + b\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) \\
&\leq a\left(u_n - u_n^{hk}, w_n^{hk} - w_n\right) - m\left(\theta_n - \theta_{n-1}^{hk}, w_n^{hk} - w_n\right) - \left(\delta w_n^{hk}, w_n - v^n\right)_H \\
&\quad - b\left(w_n^{hk}, w_n - v^n\right) - a\left(u_n^{hk}, w_n - v^n\right) + m\left(\theta_{n-1}^{hk}, w_n - v^n\right) + j_d\left(u_n^{hk}, v^h\right) - j_d\left(u_n^{hk}, w_n^{hk}\right) \\
&\quad - \left(f_n, v^h - w_n\right)_{V^*\times V} + j_d\left(u_n, w_n^{hk}\right) - j_d\left(u_n, w_n\right).
\end{aligned} \tag{4.4}$$

Using the relation

$$\begin{aligned}
\left(\dot{w}_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H = &\left(\delta w_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H + \left(\dot{w}_n - \delta w_n, w_n - v^h\right)_H \\
&+ \left(\dot{w}_n - \delta w_n^{hk}, v^h - w_n^{hk}\right)_H,
\end{aligned} \tag{4.5}$$

we deduce that

$$\begin{aligned}
&\left(\delta w_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H + b\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) \\
&\leq \left(\delta w_n - \delta w_n^{hk}, w_n - v^h\right)_H + \left(\delta w_n - \dot{w}_n, v^h - w_n^{hk}\right)_H + a\left(u_n - u_n^{hk}, w_n^{hk} - w_n\right) \\
&\quad - m\left(\theta_n - \theta_{n-1}^{hk}, w_n^{hk} - w_n\right) + b\left(w_n - w_n^{hk}, w_n - v^h\right) + a\left(u_n - u_n^{hk}, w_n - v^h\right) \\
&\quad - m\left(\theta_n - \theta_{n-1}^{hk}, w_n - v^h\right) + R\left(w_n, v^h\right) + R_1,
\end{aligned} \tag{4.6}$$

where

$$R_1 = j_d\left(u_n, w_n^{hk}\right) - j_d\left(u_n, v^h\right) + j_d\left(u_n^{hk}, v^h\right) - j_d\left(u_n^{hk}, w_n^{hk}\right). \tag{4.7}$$

Using the formula $2(x-y,x)_H = \|x-y\|_H^2 + \|x\|_H^2 - \|y\|_H^2$ for $x = w_n - w_n^{hk}$ and $b = w_{n-1} - w_{n-1}^{hk}$, we have

$$\left(\delta w_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H \geq \frac{1}{2k}\left(\left\|w_n - w_n^{hk}\right\|_H^2 - \left\|w_{n-1} - w_{n-1}^{hk}\right\|_H^2\right). \tag{4.8}$$

By the lipschitz continuity of b and the continuity of a, b and m, we have

$$\begin{aligned}
&b\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) \geq m_b \left\|w_n - w_n^{hk}\right\|_V^2, \\
&b\left(w_n - w_n^{hk}, w_n - v^h\right) \leq M_b \left\|w_n - w_n^{hk}\right\|_V \left\|w_n - v^h\right\|_V, \\
&a\left(u_n - u_n^{hk}, w_n^{hk} - v^h\right) \leq M_a \left\|u_n - u_n^{hk}\right\|_V \left\|w_n^{hk} - v^h\right\|_V, \\
&a\left(u_n - u_n^{hk}, w_n^{hk} - w_n\right) \leq M_a \left\|u_n - u_n^{hk}\right\|_V \left\|w_n - w_n^{hk}\right\|_V, \\
&m\left(\theta_n - \theta_{n-1}^{hk}, w_n - v^h\right) \leq M_m \left\|\theta_n - \theta_{n-1}^{hk}\right\|_Q \left\|w_n - v^h\right\|_V, \\
&m\left(\theta_n - \theta_{n-1}^{hk}, w_n^{hk} - w_n\right) \leq M_m \left\|\theta_n - \theta_{n-1}^{hk}\right\|_Q \left\|w_n^{hk} - w_n\right\|_V.
\end{aligned} \tag{4.9}$$

Taking into account (H8), we find that

$$|R_1| \leq \sqrt{meas(\Gamma_C)}\, c_1^2 (L_\nu + L_\tau) \left\|u_n - u_n^{hk}\right\|_V \left\|w_n^{hk} - v^h\right\|_V. \tag{4.10}$$

Using $\left\|w_n^{hk}-v^h\right\|_V \le \left\|w_n^{hk}-w_n\right\|_V + \left\|w_n-v^h\right\|_V$, (4.4)-(4.10) yield

$$\begin{aligned}
&\left\|w_n-w_n^{hk}\right\|_H^2-\left\|w_{n-1}-w_{n-1}^{hk}\right\|_H^2+ck\left\|w_n-w_n^{hk}\right\|_V^2\\
&\le ck\left\{\left\|u_n-u_n^{hk}\right\|_V^2+\left\|\theta_n-\theta_{n-1}^{hk}\right\|_Q^2+\left\|w_n-v^h\right\|_V\right\}\\
&+ck\left\{\|\dot{w}_n-\delta w_n\|_H^2+\left\|w_n-v^h\right\|_H^2+R\left(w_n,v^h\right)\right\}+2k\left(\delta w_n-\delta w_n^{hk},w_n-v^h\right)_H.
\end{aligned}$$

Now, we replace n by j and sum over j from 1 to n to obtain

$$\begin{aligned}
&\left\|w_n-w_n^{hk}\right\|_H^2+ck\sum_{j=1}^{n}\left\|w_j-w_j^{hk}\right\|_V^2\\
&\le ck\sum_{j=1}^{n}\left\{\left\|u_j-u_j^{hk}\right\|_V^2+\left\|w_j-v_j^h\right\|_V^2+\|\dot{w}_j-\delta w_j\|_H^2+\left\|w_j-v_j^h\right\|_H^2\right.\\
&\left.+\left\|\theta_j-\theta_{j-1}^{hk}\right\|_Q^2+R\left(w_j,v_j^h\right)\right\}+ck\sum_{j=1}^{n}\left(\delta w_j-\delta w_j^{hk},w_j-v_j^h\right)_H+\left\|w_0-w_0^{hk}\right\|_H^2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&k\sum_{j=1}^{n}\left(\delta w_j-\delta w_j^{hk},w_j-v_j^h\right)_H=\sum_{j=1}^{n}\left(\left(w_j-w_j^{hk}\right)-\left(w_{j-1}-w_{j-1}^{hk}\right),w_j-v_j^h\right)_H\\
&\quad+\left(w_n-w_n^{hk},w_n-v_j^h\right)_H-\left(w_0-w_0^{hk},w_1-v_1^h\right)_H\\
&\quad+\sum_{j=1}^{n-1}\left(w_j-w_j^{hk},w_j-v_j^h-\left(w_{j+1}-v_{j+1}^h\right)\right)_H\\
&\le c\left\{\left\|w_n-w_n^{hk}\right\|_H^2+\left\|w_n-v_n^h\right\|_H^2+\left\|w_0-w_0^h\right\|_H^2+\left\|w_1-v_1^h\right\|_H^2\right\}\\
&\quad+4\sum_{j=1}^{n-1}k\left\|w_j-w_j^{hk}\right\|_H^2+\sum_{j=1}^{n-1}\frac{1}{k}\left\|w_j-v_j^h-\left(w_{j+1}-v_{j+1}^h\right)\right\|_H^2.
\end{aligned}$$

Recall the following classical inequality

$$\left\|u_j-u_j^{hk}\right\|_V\le\left\|u_0-u_0^h\right\|_V+\sum_{l=1}^{j}k\left\|w_l-w_l^{hk}\right\|_V+I_1,$$

where $I_1=\left\|\int_0^{t_j}w(s)ds-\sum_{l=1}^{j}kw_l\right\|_V\le k\|u\|_{H^2(0,T;V)}$. Then

$$\left\|u_j-u_j^{hk}\right\|_V^2\le c\left\{\left\|u_0-u_0^h\right\|_V^2+j\sum_{l=1}^{j}k^2\left\|w_l-w_l^{hk}\right\|_V^2+k^2\|u\|_{H^2(0,T;V)}\right\}.$$

In view of $j\le n\le N$ and $Nk=T$, we deduce that

$$\sum_{j=1}^{n}k\left\|u_j-u_j^{hk}\right\|_V^2\le cT\left(\left\|u_0-u_0^h\right\|_V^2+k^2\|u\|_{H^2(0,T;V)}\right)+T\sum_{j=1}^{n}k\sum_{l=1}^{j}\left\|w_l-w_l^{hk}\right\|_V^2. \quad (4.11)$$

It follows that

$$\begin{aligned}&\left\|w_n - w_n^{hk}\right\|_H^2 + ck\sum_{j=1}^{n}\left\|w_j - w_j^{hk}\right\|_V^2 \le ck\left\|\theta_n - \theta_{n-1}^{hk}\right\|_Q^2 + ck^2\left\|u\right\|_{H^2(0,T;V)}^2\\&+c\left\{\left\|w_0 - w_0^{hk}\right\|_H^2 + \left\|u_0 - u_0^{hk}\right\|_V^2 + \left\|w_1 - v_1^h\right\|_H^2 + \left\|w_n - v_n^h\right\|_H^2\right\}\\&+ck\sum_{j=1}^{n}\left(\left\|\dot{w}_j - \delta w_j\right\|_V^2 + \left\|w_j - v_j^h\right\|_V^2 + \left\|\theta_j - \theta_{j-1}^{hk}\right\|_Q^2 + R\left(w_j, v_j^h\right)\right)\\&+\frac{1}{k}\sum_{j=1}^{n-1}\left\|w_j - v_j^h - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2.\end{aligned} \tag{4.12}$$

Taking $\eta = \eta^h \in Q^h$ at time $t = t_n$ in (2.14), we arrive at

$$\left(\dot{\theta}_n, \eta^h\right)_{L^2(\Omega)} + d\left(\theta_n, \eta^h\right) - e\left(w_n, \eta^h\right) + j_c\left(\theta_n, \eta^h\right) = \left(q_{c_n}, \eta^h\right).$$

Combing (2.14) with (4.2), we have

$$\begin{aligned}&\left(\dot{\theta}_n - \delta\theta_n^{hk}, \eta^h\right)_{L^2(\Omega)} + d\left(\theta_n - \theta_n^{hk}, \eta^h\right) - e\left(w_n - w_{n-1}^{hk}, \eta^h\right) + j_c\left(\theta_n, \eta^h\right)\\&-j_c\left(\theta_n^{hk}, \eta^h\right) = 0.\end{aligned} \tag{4.13}$$

Substituting η^h by $\eta^h - \theta_n^{hk}$ into (4.13), we obtain

$$\begin{aligned}&\left(\dot{\theta}_n - \delta\theta_n^{hk}, \eta^h - \theta_n^{hk}\right)_{L^2(\Omega)} + d\left(\theta_n - \theta_n^{hk}, \eta^h - \theta_n^{hk}\right) - e\left(w_n - w_{n-1}^{hk}, \eta^h - \theta_n^{hk}\right)\\&+j_c\left(\theta_n, \eta^h - \theta_n^{hk}\right) - j_c\left(\theta_n^{hk}, \eta^h - \theta_n^{hk}\right) = 0.\end{aligned}$$

Using

$$\begin{aligned}\left(\dot{\theta}_n - \delta\theta_n^{hk}, \eta^h - \theta_n^{hk}\right)_{L^2(\Omega)} &= \left(\delta\theta_n - \delta\theta_n^{hk}, \eta^h - \theta_n\right)_{L^2(\Omega)} + \left(\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\right)_{L^2(\Omega)}\\&\quad+\left(\dot{\theta} - \delta\theta_n, \eta^h - \theta_n^{hk}\right)_{L^2(\Omega)},\end{aligned}$$

and

$$d\left(\theta_n - \theta_n^{hk}, \eta^h - \theta_n^{hk}\right) = d\left(\theta_n - \theta_n^{hk}, \eta^h - \theta_n\right) + d\left(\theta_n - \theta_n^{hk}, \theta^h - \theta_n^{hk}\right),$$

we deduce that

$$\begin{aligned}&\left(\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\right)_{L^2(\Omega)} + d\left(\theta_n - \theta_n^{hk}, \theta^h - \theta_n^{hk}\right)\\&= \left(\dot{\theta} - \delta\theta_n, \theta_n^{hk} - \eta^h\right)_{L^2(\Omega)} + \left(\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \eta^h\right)_{L^2(\Omega)}\\&\quad-d\left(\theta_n - \theta_n^{hk}, \eta^h - \theta_n\right) + e\left(w_n - w_{n-1}^{hk}, \eta^h - \theta_n\right) + R_c.\end{aligned} \tag{4.14}$$

where

$$R_c = j_c\left(\theta_n^{hk}, \eta^h - \theta_n^{hk}\right) - j_c\left(\theta_n, \eta^h - \theta_n^{hk}\right).$$

Using (H7), we have

$$|R_c| \le k_c c_2^2\left\|\theta_n - \theta_n^{hk}\right\|_Q\left\|\eta^h - \theta_n^{hk}\right\|_Q.$$

From the following inequalities

$$\left(\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\right)_{L^2(\Omega)} \geq \frac{1}{2k}\left(\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 - \left\|\theta_{n-1} - \theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2\right)$$

and

$$\left\|\eta^h - \theta_n^{hk}\right\|_Q \leq \left\|\eta^h - \theta_n\right\|_Q + \left\|\theta_n - \theta_n^{hk}\right\|_Q,$$

and (H1), (H3), and (H5), we find that there exist a positive constant c such that

$$\begin{aligned}
&\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 - \left\|\theta_{n-1} - \theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2 + ck\left\|\theta_n - \theta_n^{hk}\right\|_Q^2 \\
&\leq ck\left\{\left\|\dot{\theta}_n - \delta\theta_n\right\|_{L^2(\Omega)}^2 + \left\|\theta_n - \eta^h\right\|_Q^2 + \left\|w_n - w_{n-1}^{hk}\right\|_V^2\right\} + ck\left(\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \eta^h\right)_{L^2(\Omega)}.
\end{aligned}$$

Replacing n by j and summing this inequality over j from 1 to n, we obtain

$$\begin{aligned}
&\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + ck\sum_{j=1}^n \left\|\theta_j - \theta_j^{hk}\right\|_Q^2 \\
&\leq \left\|\theta_0 - \theta_0^{hk}\right\|_{L^2(\Omega)}^2 + ck\sum_{j=1}^n\left(\left\|\dot{\theta}_j - \delta\theta_j\right\|_{L^2(\Omega)}^2 + \left\|\theta_j - \eta_j^h\right\|_Q^2\right) \\
&\quad + ck\sum_{j=1}^n \left\|w_j - w_{j-1}^{hk}\right\|_V^2 + ck\sum_{j=1}^n\left(\delta\theta_j - \delta\theta_j^{hk}, \theta_j - \eta_j^h\right)_{L^2(\Omega)}.
\end{aligned} \tag{4.15}$$

Moreover, we have

$$\begin{aligned}
&k\sum_{j=1}^n\left(\delta\theta_j - \delta\theta_j^{hk}, \theta_j - \eta_j^h\right)_{L^2(\Omega)} \\
&= \sum_{j=1}^n\left(\left(\theta_j - \theta_{j-1}\right) - \left(\theta_j^{hk} - \theta_{j-1}^{hk}\right), \theta_j - \eta_j^h\right)_{L^2(\Omega)} \\
&= \sum_{j=1}^n\left(\theta_j - \theta_j^{hk}, \theta_j - \eta_j^h\right)_{L^2(\Omega)} + \sum_{j=1}^n\left(\theta_{j-1} - \theta_{j-1}^{hk}, \theta_j - \eta_j^h\right)_{L^2(\Omega)} \\
&= \sum_{j=1}^n\left(\theta_j - \theta_j^{hk}, \left(\theta_j - \eta_j^h\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right)_{L^2(\Omega)} \\
&\quad + \left(\theta_n - \theta_n^{hk}, \theta_n - \eta_n^h\right)_{L^2(\Omega)} - \left(\theta_0 - \theta_0^{hk}, \theta_1 - \eta_1^h\right)_{L^2(\Omega)}.
\end{aligned} \tag{4.16}$$

Then,

$$\begin{aligned}
&k\sum_{j=1}^{n}\left(\delta\theta_j-\delta\theta_j^{hk},\theta_j-\eta_j^h\right)_{L^2(\Omega)}\\
&\le c\left\{\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2+\left\|\theta_n-\eta_n^h\right\|_{L^2(\Omega)}^2+\left\|\theta_0-\theta_0^h\right\|_{L^2(\Omega)}^2+\left\|\theta_1-\eta_1^h\right\|_{L^2(\Omega)}^2\right\}\\
&\quad+\sum_{j=1}^{n-1}\left\|\theta_j-\theta_j^{hk}\right\|_{L^2(\Omega)}\left\|\left(\theta_j-\eta_j^h\right)-\left(\theta_{j+1}-\eta_{j+1}^h\right)\right\|_{L^2(\Omega)}\\
&\le c\left\{\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2+\left\|\theta_n-\eta_n^h\right\|_{L^2(\Omega)}^2+\left\|\theta_0-\theta_0^h\right\|_{L^2(\Omega)}^2+\left\|\theta_1-\eta_1^h\right\|_{L^2(\Omega)}^2\right\}\\
&\quad+k\sum_{j=1}^{n-1}\left\|\theta_j-\theta_j^{hk}\right\|_{L^2(\Omega)}^2+\frac{1}{k}\sum_{j=1}^{n-1}\left\|\left(\theta_j-\eta_j^h\right)-\left(\theta_{j+1}-\eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2,
\end{aligned}$$

which together with (4.15) concludes

$$\begin{aligned}
&\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2+ck\sum_{j=1}^{n}\left\|\theta_j-\theta_j^{hk}\right\|_Q^2\\
&\le c\left\{\left\|\theta_0-\theta_0^{hk}\right\|_{L^2(\Omega)}^2+\left\|\theta_1-\eta_1^h\right\|_{L^2(\Omega)}^2+\left\|\theta_n-\eta_n^h\right\|_{L^2(\Omega)}^2\right\}\\
&\quad+ck\sum_{j=1}^{n}\left(\left\|w_j-w_{j-1}^{hk}\right\|_V^2+\left\|\dot\theta_j-\delta\theta_j\right\|_{L^2(\Omega)}^2+\left\|\theta_j-\eta_j^h\right\|_Q^2\right)\\
&\quad+k\sum_{j=1}^{n-1}\left\|\theta_j-\theta_j^{hk}\right\|_{L^2(\Omega)}^2+\frac{1}{k}\sum_{j=1}^{n}\left\|\left(\theta_j-\eta_j^h\right)-\left(\theta_{j+1}-\eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.17}$$

Similarly, one has

$$\sum_{j=1}^{n}k\left\|\theta_j-\theta_{j-1}^{hk}\right\|_Q^2\le cT\left(\left\|\theta_0-\theta_0^h\right\|_Q^2+k^2\|\theta\|_{H^1(0,T;Q)}\right)+T\sum_{j=1}^{n-1}k\sum_{l=1}^{j}\left\|\delta\theta_l-\delta\theta_l^{hk}\right\|_Q^2. \tag{4.18}$$

Finally, combining (4.12), (4.17), and (4.18), we conclude (4.3) immediately. □

The main result of this section is given in the following theorem.

Theorem 4.1. *Under the assumptions stated in* ***Theorem 3.1*** *and the regularity conditions*

$$u\in C^1\left(0,T;H^2\left(\Omega;\mathbb{R}^d\right)\right)\cap H^3\left(0,T;H\right),\ \dot u_{|\Gamma_C}\in C\left(0,T;H^2\left(\Gamma_C,\mathbb{R}^d\right)\right)$$

and

$$\theta\in C\left(0,T;H^2(\Omega)\right)\cap H^2\left(0,T;L^2(\Omega)\right),\ \dot\theta\in L^2\left(0,T;H^1(\Omega)\right),$$

the following order error estimate holds

$$\max_{1\le n\le N}\left\{\left\|w_n-w_n^{hk}\right\|_H+\left\|u_n-u_n^{hk}\right\|_V+\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}\right\}\le c(h+k). \tag{4.19}$$

Proof. Applying the discrete Gronwall inequality 4.1, we conclude that

$$\begin{aligned}\max_{1\leq n}\{e_n\} &\leq c\left\{\left\|w_0-w_0^{hk}\right\|_H^2+\left\|u_0-u_0^{hk}\right\|_V^2+\left\|\theta_0-\theta_0^{hk}\right\|_{L^2(\Omega)}^2\right\}\\ &+ck^2\|\theta\|_{H^1(0,T;Q)}^2+ck^2\|u\|_{H^2(0,T;V)}^2+c\max_{1\leq n}\{g_n\},\end{aligned} \tag{4.20}$$

where

$$e_n=\left\|w_n-w_n^{hk}\right\|_H^2+\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2+ck\sum_{j=1}^{n}\left(\left\|w_j-w_j^{hk}\right\|_V^2+\left\|\theta_j-\theta_j^{hk}\right\|_Q^2\right) \tag{4.21}$$

and

$$\begin{aligned}g_n=\inf_{\substack{v_j^h\in V^h\\ \eta_j^h\in Q^h}}&\left\{k\sum_{j=1}^{n}\left(\left\|\dot{w}_j-\delta w_j\right\|_H^2+\left\|w_j-v_j^h\right\|_V^2\right)\right.\\ &+\sum_{j=1}^{n}\left(\left\|\dot{\theta}_j-\delta\theta_j\right\|_{L^2(\Omega)}^2+\left\|\theta_j-\eta_j^h\right\|_Q^2+R\left(w_j,v_j^h\right)\right)\\ &+\frac{1}{k}\sum_{j=1}^{n-1}\left(\left\|\left(w_j-v_j^n\right)-\left(w_{j+1}-v_{j+1}^h\right)\right\|_H^2+\left\|\left(\theta_j-\eta_j^n\right)-\left(\theta_{j+1}-\eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2\right)\\ &\left.+\left\|w_1-v_1^h\right\|_H^2+\left\|\theta_1-\eta_1^h\right\|_{L^2(\Omega)}^2+\left\|w_n-v_n^h\right\|_H^2+\left\|\theta_n-\eta_n^h\right\|_{L^2(\Omega)}^2\right\}.\end{aligned} \tag{4.22}$$

Let $v_j^h\in V^h$ and $\eta_j^h\in Q^h$ be the finite element interpolate of u_j and θ_j, respectively. Note that [6, 7]

$$\max_{1\leq n\leq N}\left\|w_n-v_n^h\right\|_V\leq ch\|w\|_{C\left(0,T;H^2(\Omega)^d\right)},\quad \max_{1\leq n\leq N}\left\|\theta_n-\eta_n^h\right\|_Q\leq ch\|\theta\|_{C\left(0,T;H^2(\Omega)\right)}, \tag{4.23}$$

$$\left\|w_0-w_0^h\right\|_V\leq ch\|w_0\|_{H^2\left(\Omega,\mathbb{R}^d\right)},\ \left\|u_0-u_0^h\right\|_H\leq ch\|u_0\|_{H^1\left(\Omega,\mathbb{R}^d\right)},\ \left\|\theta_0-\theta_0^h\right\|_{L^2(\Omega)}\leq ch\|\theta_0\|_{L^2(\Omega)}, \tag{4.24}$$

and

$$\begin{aligned}&k\sum_{j=1}^{n}\left(\left\|\dot{w}_j-\delta w_j\right\|_H+\left\|\dot{\theta}_j-\delta\theta_j\right\|_{L^2(\Omega)}\right)\leq ck^2\|u\|_{H^2\left(0,T;L^2(\Omega)\right)}+ck^2\|\theta\|_{H^2(0,T;L^2(\Omega)}^2,\\ &\frac{1}{k}\sum_{j=1}^{n-1}\left(\left\|\left(w_j-v_j^n\right)-\left(w_{j+1}-v_{j+1}^h\right)\right\|_H^2+\left\|\left(\theta_j-\eta_j^n\right)-\left(\theta_{j+1}-\eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2\right)\\ &\leq ch^2\|u\|_{H^2(0,T;V)}^2+ch^2\|\theta\|_{H^2(0,T;Q)}.\end{aligned} \tag{4.25}$$

Following the proof line in [7, 19], we obtain

$$\left|R\left(w_j,v_j^h\right)\right|\leq c\left\|w_n-v_n^h\right\|_{L^2(\Gamma_C)^d}\leq ch^2\|w_n\|_{C\left(0,T;H^2(\Omega)^d\right)}. \tag{4.26}$$

Finally, combing the previous estimates (4.21) and (4.23)-(4.26), we deduce (4.19) immediately. □

5. Numerical Simulations

This section provides computer simulation results on the contact **Problem (PF)**, including numerical evidence of the theoretical error estimates obtained in the previous section for the discrete approximation of the variational problem. The solution of **Problem (PF)** is based on numerical methods described in [1, 14].

The physical setting used for **Problem (PF)** is depicted in Figure 1. In this case, the body $\Omega = (0,1)\times(0,1)\subset\mathbb{R}^2$ is clamped on $\Gamma_D = [0,1]\times\{1\}$. Traction f_N^1 and f_N^2 are prescribed on the lateral parts Γ_N^1, Γ_N^2 respectively (i.e., $\Gamma_N := \Gamma_N^1 \cup \Gamma_N^2$). The body is in contact with a thermally conductive foundation on its lower boundary $\Gamma_C = [0,1]\times\{0\}$.

The material response is governed by a linear viscoelastic constitutive law in which the elasticity tensor $\mathscr{F}$ and the viscosity tensor $\mathscr{A}$ are given by

$$(\mathscr{F}\tau)_{ij} = \frac{E\chi}{1-\chi^2}(\tau_{11}+\tau_{22})\delta_{ij} + \frac{E}{1+\chi}\tau_{ij}, \qquad 1\le i,j\le 2, \quad \tau\in\mathbb{S}^2,$$

$$(\mathscr{A}\tau)_{ij} = \mu_1(\tau_{11}+\tau_{22})\delta_{ij} + \mu_2\tau_{ij}, \qquad 1\le i,j\le 2, \quad \tau\in\mathbb{S}^2,$$

where E is the Young's modulus, χ is the Poisson's ratio of the material, δ_{ij} denotes the Kronecker symbol ($\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$), and μ_1 and μ_2 are viscosity constants.

The functions p_ν and p_τ in frictional contact conditions (2.8) and (2.9) are given by $p_\nu(r) = c_\nu r_+$ and $p_\tau = \mu_\tau p_\nu$, where c_ν represents large positive constant and μ_τ represents the friction coefficient.

For computation, we use the following data (IS unity):

$$E=2,\ \chi=0.1,\ \mu_1=10,\ \mu_2=10,\ \mathscr{M}_{ij}=\mathscr{K}_{ij}=\mathscr{R}_{ij}=1, \quad 1\le i,j\le 2,$$

$$f_0=(0,-1),\ q_0=1,\ f_N^1=(1.4,0.4),\ f_N^2=(-0.8,0.4),\ c_\nu=10^4,\ \mu_\tau=0.2,\ g=0,$$

$$k_c=1,\ T=1,\ u_0=0, v_0=0,\ \theta_0=0.$$

Our interest in this example is to study the influence of the thermal conductivity of the foun-

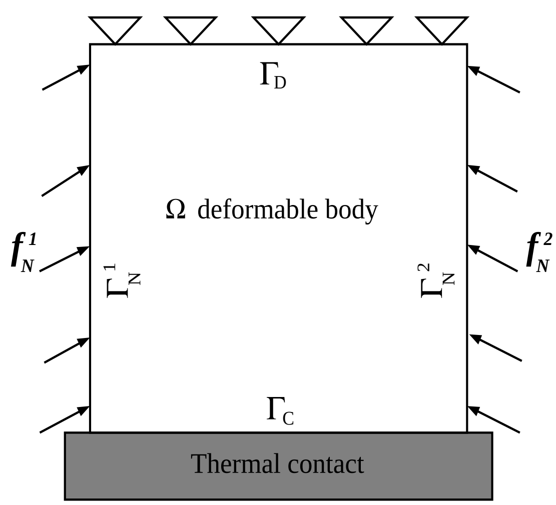

FIGURE 1. Physical setting.

dation on the contact process. Thus, in Figure 2, we show the deformed configurations at final time, and in Figures 4 and 3, the corresponding norm of the temperature and stress fields,

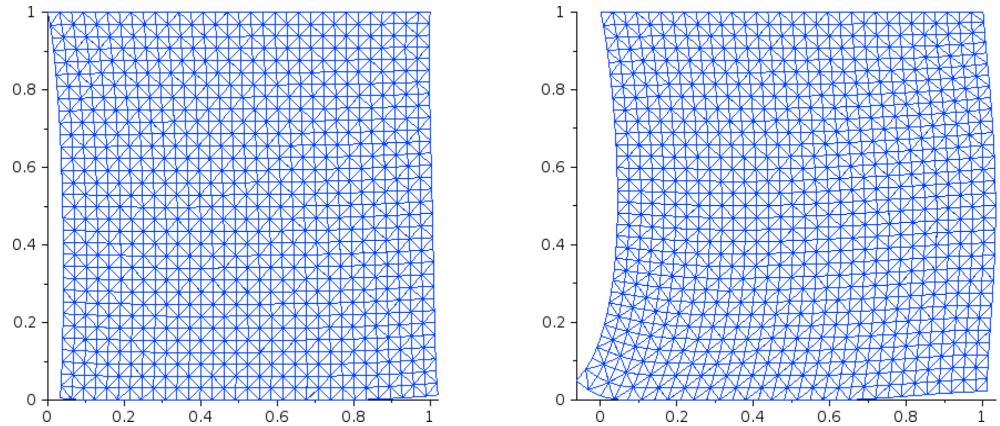

FIGURE 2. Deformed configuration for $\theta_F = 0$ (left) and $\theta_F = 10$ (right).

through the body for two different values of the temperatures of the foundation. These Figures show that, when the temperature of the foundation is more important then the deformations, the norm of the stress and the temperature are larger. To see the convergence behaviour of

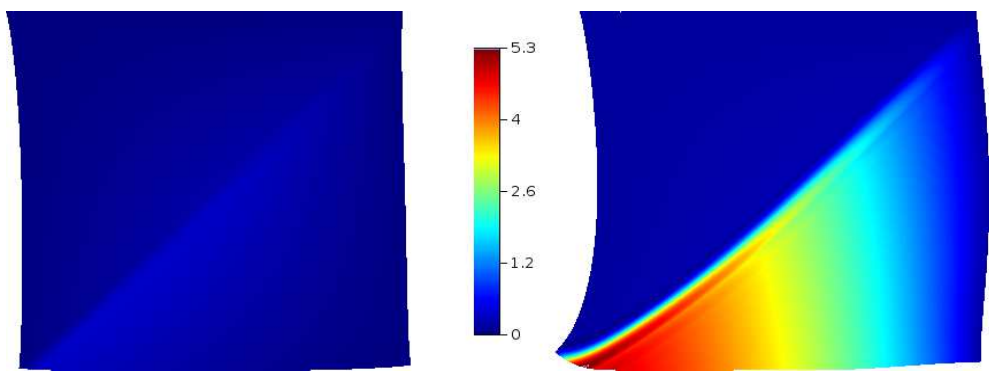

FIGURE 3. Temperature field for $\theta_F = 0$ (left) and $\theta_F = 10$ (right).

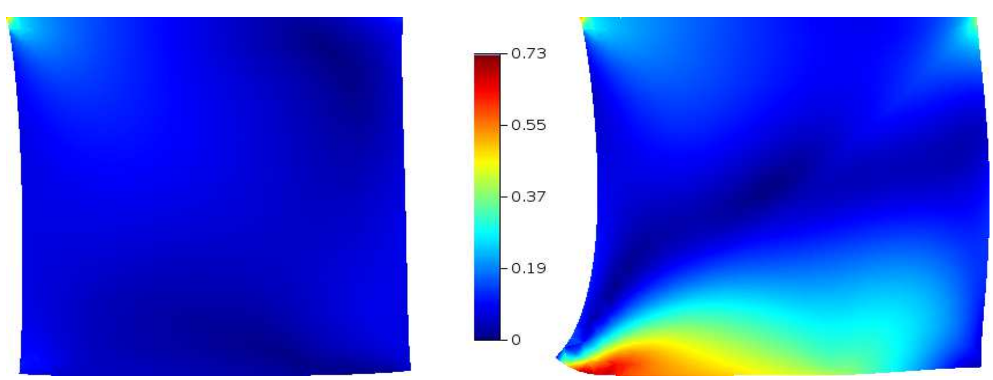

FIGURE 4. Von Mises stress norm for $\theta_F = 0$ (left) and $\theta_F = 10$ (right).

the fully discrete scheme, we compute a sequence of numerical solutions based on uniform partitions of the time interval $[0,T]$, and uniform triangulations of the body. Then, we provide the estimated error values for several discretization parameters h and k. Here, the sides of the square are divided into $1/h$ equal parts and the time interval $[0,T]$ is divided into $1/k$ time steps. We start with $h = 1/16$ and $k = 1/16$ which are successively halved. The numerical solution corresponding to $h = 1/256$ and $k = 1/256$ has been considered as the "exact" solution in order to compute the numerical errors given by

$$E^{hk} = \max_{1 \le n \le N} \left\{ \left\| w_n - w_n^{hk} \right\|_H + \left\| u_n - u_n^{hk} \right\|_V + \left\| \theta_n - \theta_n^{hk} \right\|_{L^2(\Omega)} \right\}.$$

The linear asymptotic convergence behaviour obtained in (4.19) is almost observed (see Figure 5).

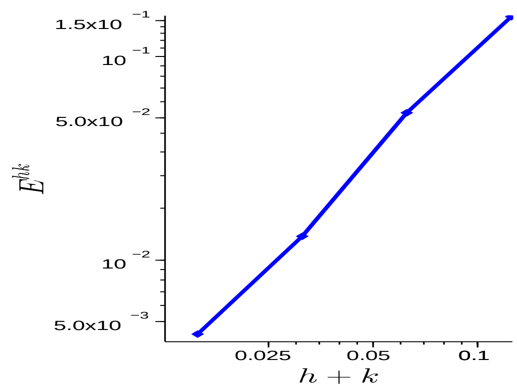

FIGURE 5. Estimated errors.

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