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PARETO APPROXIMATE SUBDIFFERENTIAL COMPOSITION RULE FOR CONE-CONVEX SET-VALUED MAPPINGS AND ITS APPLICATIONS

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Abstract. In this paper, we develop a composition rule for the Pareto approximate subdifferential of the convex set-valued mapping $F+G\circ H$, where F, G, and H are convex set-valued mappings with G being nondecreasing. Necessary optimality conditions for constrained convex set-valued optimization problems are considered as an application.
Keywords. Cone-convex set-valued mappings; Optimality conditions; Pareto approximate subdifferential; Set-valued analysis; Set-valued Optimization.
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1. INTRODUCTION

In recent years, optimization problems involving set-valued mappings have garnered intensive interest from researchers due to their broad applications across fields such as optimal control, differential inclusions, and economics (see, e.g., [1, 2, 7, 8]). The concept of subdifferentials plays a crucial role in analyzing optimality conditions for set-valued optimization problems. Various types of subdifferentials have been defined in the literature, and calculus rules have been developed under specific qualification conditions. These rules facilitated the establishment of necessary and sufficient optimality conditions for various types of solutions to set-valued optimization problems (see [1, 2, 4, 7, 10] and the references therein). Recently, the authors [3] explored several properties and calculus rules for weakly and properly Pareto approximate subdifferentials, including an exact sum rule with applications to general constrained convex optimization problems.

This paper aims to introduce a composition rule for approximate Pareto subdifferentials (both weak and proper). Specifically, we focus on the case where a set-valued mapping is combined with another composite mapping. By applying this new composition rule, we establish the existence of approximate Lagrange multipliers for general convex set-valued optimization problems.

The structure of the paper is as follows. Section 2 provides essential definitions, notations, and preliminary concepts related to the set-valued mappings. Section 3 presents the composition rule for approximate Pareto subdifferentials. In Section 4, the last section, this rule is

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applied to derive approximate efficient optimality conditions for constrained convex set-valued optimization problems.

2. PRELIMINARIES

In this paper, let X, Y, and Z denote real separated topological vector spaces, with X^*, Y^*, and Y^* representing their respective topological duals. These duals are paired by the duality pairing $\langle .,. \rangle$. The origins of X, Y, and Z are denoted by 0_X, 0_Y, and 0_Z, respectively. For simplicity, 0 is used to refer to 0_X, 0_Y, and 0_Z when no confusion is possible. We also consider a pointed closed convex cone $Y_+ \subset Y$ (and similarly, $Z_+ \subset Z$) which defines partial preorders on Y (and Z)

$$\begin{array}{ll} y_1 \leq_{Y_+} y_2, \quad if \quad y_2 - y_1 \in Y_+ & (resp., z_1 \leq_{Z_+} z_2, \quad if \quad z_2 - z_1 \in Z_+), \\ y_1 \lneqq_{Y_+} y_2, \quad if \quad y_2 - y_1 \in Y_+ \backslash \{0_Y\} & (resp., z_1 \lneqq_{Z_+} z_2, \quad if \quad z_2 - z_1 \in Z \backslash \{0_Z\}), \\ y_1 <_{Y_+} y_2, \quad if \quad y_2 - y_1 \in \operatorname{int} Y_+ & (resp., z_1 <_{Z_+} z_2, \quad if \quad z_2 - z_1 \in \operatorname{int} Z_+), \end{array}$$

where $y_1, y_2 \in Y$ (resp., $z_1, z_2 \in Z$) and $\operatorname{int} Y_+$ (resp., $\operatorname{int} Z_+$) stands for the topological interior of Y_+ (resp., of Z). Given two nonempty subsets $A, B \subset Y$ and $\alpha \in \mathbb{R}$, we write $A + B := \{a + b : (a,b) \in A \times B\}$ and $\alpha A := \{\alpha a : a \in A\}$. If B is the empty set, then $A + \emptyset = \emptyset + A = \emptyset$ and $\alpha \emptyset = \emptyset$. Let F be a set-valued mapping from X into Y, i.e. $F(x)$ is a subset of Y for each $x \in X$. The domain, graph, and image of F are defined respectively by

$$\begin{array}{lcl} \operatorname{dom} F & := & \{x \in X : F(x) \neq \emptyset\}, \\ \operatorname{gr} F & := & \{(x,y) \in X \times Y : y \in F(x)\}, \\ \operatorname{Im} F & := & \bigcup_{x \in X} F(x). \end{array}$$

If we define the set-valued mapping $F + A$ from X into Y by $(F + A)(x) := F(x) + A$ for any $x \in X$, then the set

$$\operatorname{epi} F := \operatorname{gr}(F + Y_+) = \{(x,y) \in X \times Y : y \in F(x) + Y_+\}$$

is called the epigraph of F.

The positive polar cone and the strict positive polar cone of Y_+ are denoted by Y_+^* and Y_+^{s*}, respectively, i.e.,

$$\begin{array}{lcl} Y_+^* & := & \{y^* \in Y^* : \ \langle y^*, y \rangle \geq 0, \quad \forall y \in Y_+\}, \\ Y_+^{s*} & := & \{y^* \in Y^* : \ \langle y^*, y \rangle > 0, \quad \forall y \in Y_+ \backslash \{0_Y\}\}. \end{array}$$

Let $G : Y \rightrightarrows Z$. The composite set-valued mapping $G \circ F : X \rightrightarrows Z$ is defined by

$$(G \circ F)(x) := \begin{cases} G(F(x)) = \bigcup_{y \in F(x)} G(y), & \text{if } x \in \operatorname{dom} F, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We have $\operatorname{dom}(G \circ F) = F^{-1}(\operatorname{dom} G) \cap \operatorname{dom} F$, where

$$F^{-1}(\operatorname{dom} G) := \{x \in X : F(x) \cap \operatorname{dom} G \neq \emptyset\}.$$

Recall that the set-valued mapping $G : Y \rightrightarrows Z$ is said to be (Y_+, Z_+)-nondecreasing on subset $A \subseteq Y$ if, $G(y_2) \subseteq G(y_1) + Z_+$ for all $(y_1, y_2) \in A \times A$ satisfying $y_1 \leq_{Y_+} y_2$. If G is (Y_+, Z_+)-nondecreasing on $ImF + Y_+$, Z_+-convex and F is Y_+-convex, then $G \circ F$ is Z_+-convex (see [10]). The set-valued indicator mapping $R_S^v : X \rightrightarrows Y$ is defined for the nonempty subset $S \subseteq X$ by

$$R_S^v(x) := \begin{cases} \{0_Y\}, & \text{if } x \in S, \\ \emptyset, & \text{elsewhere.} \end{cases}$$

Definition 2.1. [5] The set-valued mapping F is said to be

(i) Y_+-convex if its epigraph is a convex subset of $X \times Y$.
(ii) Proper if its effective domain $\operatorname{dom} F \neq \emptyset$.
(iii) Connected at $x_0 \in X$ if there exists a continuous mapping $h : X \to Y$ such that $h(v) \in F(v)$ for all v in some neighborhood of x_0.

In what follows, we recall two concepts of lower semicontinuity adapted to set-valued mappings, namely respectively, Y_+-epi-closedness and star Y_+-epi-closedness.

Definition 2.2. [3] Let $F : X \rightrightarrows Y$ be a set-valued mapping.

(i) F is said to be Y_+-epi-closed if its epigraph is closed in the product topology on $X \times Y$.
(ii) F is said to be star Y_+-epi-closed if, for any $y^* \in Y_+^*$, the real set-valued mapping $y^* \circ F$ is $\mathbb{R}_+$-epi-closed.

Now, we consider the following vector set-valued optimization problem

$$(\mathrm{P}_S) \quad \begin{cases} \text{Min F(x)}, \\ x \in S, \end{cases}$$

where S is a nonempty subset of X and $F : X \rightrightarrows Y$ is a given set-valued mapping. There are several types of ε-solutions for (P_S): a pair $(\bar{x}, \bar{y}) \in (S \times Y) \cap \operatorname{gr} F$ is said to be

(a) strongly efficient solution of (P_S) if $F(x) \subseteq \bar{y} - \varepsilon + Y_+, \quad \forall x \in S$,
(b) Pareto or efficient solution of (P_S) if $F(x) \subseteq \bar{y} - \varepsilon + (Y \backslash -Y_+), \quad \forall x \in S$,
(c) weak Pareto or weakly efficient solution of (P_S) if $F(x) \subseteq \bar{y} - \varepsilon + Y \backslash -\operatorname{int} Y_+, \quad \forall x \in S$,
(d) proper Pareto or (Henig) properly efficient solution of (P_S) if there exists $\hat{Y}_+ \in \mathscr{C}(Y_+)$ such that $F(x) \subseteq \bar{y} - \varepsilon + (Y \backslash -\hat{Y}_+) \cup (\hat{Y}_+ \cap -\hat{Y}_+)$, $\forall x \in S$, where

$$\mathscr{C}(Y_+) := \{\hat{Y}_+ \subset Y : \ \hat{Y}_+ \text{ is a proper convex cone such that } Y_+ \backslash \{0_Y\} \subseteq \operatorname{int} \hat{Y}_+\}.$$

The ε-efficient set, strongly, properly, and weakly ε-efficient sets for (P_S) are denoted, respectively, by $K_{\varepsilon,e}(F(S), Y_+)$, $K_{\varepsilon,s}(F(S), Y_+)$, $K_{\varepsilon,p}(F(S), Y_+)$ and $K_{\varepsilon,w}(F(S), Y_+)$. To unify the presentation, we denote by $K_{\varepsilon,\sigma}(F(S), Y_+)$ the set of ε-σ-efficient pairs depending on the choice of $\sigma \in \{s, e, w, p\}$.

Remark 2.1. The vector ε must belong to some set in order to have consistent approximate notions. One can see easily that $K_{\varepsilon,\sigma}(F(S), Y_+) \neq \emptyset \Longrightarrow \varepsilon \in D^\sigma$, where

$$D^\sigma := \begin{cases} Y_+, & \text{if } \sigma = s\ , \\ Y \backslash -\operatorname{int} Y_+, & \text{if } \sigma = w, \\ Y \backslash (-Y_+ \backslash l(Y_+)), & \text{if } \sigma = p. \end{cases}$$

We can easily see that

$$K_{\varepsilon,w}(F(S),Y_+) = K_{\varepsilon,e}(F(S),\operatorname{int}Y_+ \cup \{0\})$$

and

$$K_{\varepsilon,p}(F(S),Y_+) = \bigcup_{\hat{Y}_+ \in \mathscr{C}(Y_+)} K_{\varepsilon,e}(F(S),\hat{Y}_+).$$

Moreover, $K_{\varepsilon,p}(F(S),Y_+) \subseteq K_{\varepsilon,e}(F(S),Y_+) \subseteq K_{\varepsilon,w}(F(S),Y_+)$. For ε_1, $\varepsilon_2 \in Y$, $\sigma \in \{s,p,w\}$, we have $\varepsilon_1 \leq_{Y_+} \varepsilon_2 \Longrightarrow K_{\varepsilon_1,\sigma}(F(S),Y_+) \subseteq K_{\varepsilon_2,\sigma}(F(S),Y_+)$. The concept of the approximate subdifferential in the Pareto sense is crucial for addressing vector set-valued optimization problems. By defining various approximate efficient sets, we can introduce the notion of the ε-σ-subdifferential for set-valued mappings.

Definition 2.3. [3] Let $F : X \rightrightarrows Y$ be a Y_+-convex set-valued mapping and $(\bar{x},\bar{y}) \in \operatorname{gr}F$. The ε-σ-subdifferential of F at $(\bar{x},\bar{y})$ with $\sigma \in \{s,p,e,w\}$ is defined as

$$\partial_\varepsilon^\sigma F(\bar{x},\bar{y}) := \{T \in L(X,Y) : \quad (\bar{x},\bar{y} - T(\bar{x})) \in K_{\varepsilon,\sigma}((F-T)(X),Y_+)\},$$

where $L(X,Y)$ is the set of all continuous linear operators from X into Y.

This definition is substantiated by the relevance of the following immediate property.

$$(\bar{x},\bar{y}) \in K_{\varepsilon,\sigma}(F(X),Y_+) \quad \Longleftrightarrow \quad 0 \in \partial_\varepsilon^\sigma F(\bar{x},\bar{y}). \tag{2.1}$$

Remark 2.2. These notions of approximate subdifferentials can be explicitly expressed as follows:

$$\begin{aligned}
\partial_\varepsilon^s F(\bar{x},\bar{y}) &:= \{T \in L(X,Y) : \quad \forall x \in X, \forall y \in F(x),\ T(x-\bar{x}) \leq_{Y_+} y - \bar{y} + \varepsilon\},\\
\partial_\varepsilon^e F(\bar{x},\bar{y}) &:= \{T \in L(X,Y) : \quad \nexists x \in X,\ \exists y \in F(x),\ y - \bar{y} + \varepsilon \lneqq_{Y_+} T(x-\bar{x})\},\\
\partial_\varepsilon^w F(\bar{x},\bar{y}) &:= \{T \in L(X,Y) : \quad \nexists x \in X,\ \exists y \in F(x),\ y - \bar{y} + \varepsilon <_{Y_+} T(x-\bar{x})\},\\
\partial_\varepsilon^p F(\bar{x},\bar{y}) &:= \{T \in L(X,Y) : \quad \exists \hat{Y}_+ \in \mathscr{C}(Y_+) \text{ such that, } \nexists x \in X,\ \exists y \in F(x)\\
&\qquad\qquad y - \bar{y} + \varepsilon \lneqq_{\hat{Y}_+} T(x-\bar{x})\}.
\end{aligned}$$

By convention, we take $\partial_\varepsilon^\sigma F(\bar{x},\bar{y}) = \emptyset$ if $(\bar{x},\bar{y}) \notin \operatorname{gr}F$ and we say that F is ε-σ-subdifferentiable at $(\bar{x},\bar{y})$ with $\sigma \in \{s,e,p,w\}$ and $\varepsilon \in Y$ if $\partial_\varepsilon^\sigma F(\bar{x},\bar{y}) \neq \emptyset$. For the case $Y = \mathbb{R}$, all these approximate subdifferentials coincide with the following approximate subdifferential

$$\partial_\varepsilon F(\bar{x},\bar{y}) := \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq y - \bar{y} + \varepsilon, \quad \forall (x,y) \in \operatorname{gr}F\}.$$

For $\varepsilon = 0_Y$, we have $\partial_{0_Y}^\sigma F(\bar{x},\bar{y}) = \partial^\sigma F(\bar{x},\bar{y})$, where $\partial^\sigma F(\bar{x},\bar{y})$ stands for the exact Pareto σ-subdifferential (see [5, 9]).

The following theorem describes how the $\varepsilon - \sigma$-subdifferential, with $\varepsilon \in D^\sigma$ and $\sigma \in \{s,p,w\}$, can be characterized via a scalarization technique involving scalar set-valued mappings.

Theorem 2.1. [3] *Let $F : X \rightrightarrows Y$ be a set-valued mapping, $(\bar{x},\bar{y}) \in \operatorname{gr}F$ and $\varepsilon \in D^\sigma$. Then, for $\sigma = s$,*

$$\partial_\varepsilon^s F(\bar{x},\bar{y}) \subseteq \bigcap_{y^* \in Y_+^* \setminus \{0\}} \{T \in L(X,Y) : y^* \circ T \in \partial_{\langle y^*,\varepsilon\rangle}(y^* \circ F)(\bar{x},\langle y^*,\bar{y}\rangle)\},$$

with equality if Y is locally convex and Y_+ is closed. For $\sigma \in \{w, p\}$,

$$\partial_\varepsilon^\sigma F(\bar{x},\bar{y}) \supseteq \bigcup_{y^* \in Y_+^{*\sigma}} \{T \in L(X,Y) : y^* \circ T \in \partial_{\langle y^*,\varepsilon\rangle}(y^* \circ F)(\bar{x}, \langle y^*, \bar{y}\rangle)\},$$

with equality if F is Y_+-convex and the cone Y_+ is pointed for $\sigma = p$, where

$$Y_+^{*\sigma} := \begin{cases} Y_+^* \backslash \{0\}, & \text{if } \sigma = w, \\ Y_+^{s*}, & \text{if } \sigma = p. \end{cases}$$

Proposition 2.1. [3] *Let $\theta : X \rightrightarrows Y$ be the set-valued mapping defined by $\theta(x) := \{0_Y\}$ for all $x \in X$ and $\sigma \in \{p, w\}$ with Y_+ being pointed as $\sigma = p$. Then $\partial_\varepsilon^\sigma \theta(x, 0_Y) = \vartheta_\sigma(X,Y)$ for all $\varepsilon \in D^\sigma$, where $\vartheta_\sigma(X,Y) := \{T \in L(X,Y) : \exists y^* \in Y_+^{*\sigma}, y^* \circ T = 0\}$ is the set of σ-zerolike linear continuous operators.*

Definition 2.4. [3] Let $F : X \rightrightarrows Y$ be a set-valued mapping, $(\bar{x},\bar{y}) \in \mathrm{gr}F$ and $\eta \in \mathbb{R}_+$. The map F is said to be

- regular η-subdifferentiable at $(\bar{x},\bar{y})$ if

$$\partial_\eta(y^* \circ F)(\bar{x}, \langle y^*, \bar{y}\rangle) = \bigcup_{\substack{\varepsilon \in \eta Y_+ \\ \langle y^*, \varepsilon\rangle = \eta}} y^* \circ \partial_\varepsilon^s F(\bar{x},\bar{y}), \ \forall y^* \in Y_+^*.$$

- σ-regular η-subdifferentiable at $(\bar{x},\bar{y})$ with $\sigma \in \{w, p\}$ if

$$\partial_\eta(y^* \circ F)(\bar{x}, \langle y^*, \bar{y}\rangle) = \bigcup_{\substack{\varepsilon \in \eta Y_+ \\ \langle y^*, \varepsilon\rangle = \eta}} y^* \circ \partial_\varepsilon^s F(\bar{x},\bar{y}), \ \forall y^* \in Y_+^{*\sigma},$$

  where $y^* \circ \partial_\varepsilon^s F(\bar{x},\bar{y}) := \{y^* \circ T : T \in \partial_\varepsilon^s F(\bar{x},\bar{y})\}$.

Now, we recall the exact formula for the Pareto (properly or weakly) ε-subdifferential of the sum of two cone-convex vector set-valued mappings taking values in a partially preordered topological linear space.

Theorem 2.2. [3] *Let $F, G : X \rightrightarrows Y$ be Y_+-convex set-valued mappings and $\sigma \in \{w, p\}$ with Y_+ being pointed as $\sigma = p$. Then, for any $(\bar{x},\bar{u}) \in \mathrm{gr}\,F$ and $(\bar{x},\bar{v}) \in \mathrm{gr}\,G$,*

$$\partial_\varepsilon^\sigma(F+G)(\bar{x},\bar{u}+\bar{v}) \supseteq \bigcup_{\substack{\varepsilon_1 \in D^\sigma,\ \varepsilon_2 \in Y_+^\varepsilon \\ \varepsilon_1+\varepsilon_2=\varepsilon}} \partial_{\varepsilon_1}^\sigma F(\bar{x},\bar{u}) + \partial_{\varepsilon_2}^s G(\bar{x},\bar{v}), \quad \forall \varepsilon \in D^\sigma.$$

Assume now that G is σ-regular η-subdifferentiable at $(\bar{x},\bar{v})$ for all $\eta \geq 0$, and one of the following qualification condition is satisfied

$$(\mathrm{MR})_1 \quad \begin{cases} X \textit{ is a Hausdorff locally convex space,} \\ F \textit{ or } G \textit{ is connected at some point of } \mathrm{dom}F \cap \mathrm{dom}G. \end{cases}$$

$$(\mathrm{AB})_1 \quad \begin{cases} X \textit{ is a Banach space and } F, G \textit{ are star } Y_+\textit{-epi-closed,} \\ \mathbb{R}_+[\mathrm{dom}F - \mathrm{dom}G] \textit{ is a closed vector subspace of } X. \end{cases}$$

Then,

$$\partial_\varepsilon^\sigma(F+G)(\bar{x},\bar{u}+\bar{v}) = \bigcup_{\substack{\varepsilon_1 \in D^\sigma,\ \varepsilon_2 \in Y_+^\varepsilon \\ \varepsilon_1+\varepsilon_2=\varepsilon}} \partial_{\varepsilon_1}^\sigma F(\bar{x},\bar{u}) + \partial_{\varepsilon_2}^s G(\bar{x},\bar{v}), \quad \forall \varepsilon \in D^\sigma,$$

where $Y_+^{0_Y} = \{0_Y\}$ *and* $Y_+^{\varepsilon} = Y_+$, *if* $\varepsilon \neq 0_Y$.

3. Composition

In this section, we provide some results related to the σ-subdifferential calculus for the composition of two set-valued mappings. Our approach to determining the Pareto subdifferential of the composed set-valued mapping involves transforming it into the Pareto subdifferential of the sum of two set-valued mappings. We work with the following definitions: For all $(x,z) \in X \times Z$ and $(A,B) \in L(X,Y) \times L(Z,Y)$, and $y^* \in Y_+^*$, one has $(A,B)(x,z) = A(x) + B(z)$ and $y^* \circ (A,B) = (y^* \circ A, y^* \circ B)$. Consider the following set valued maps $\tilde{F}, \tilde{H} : X \times Z \rightrightarrows Y$, $\tilde{F}(x,z) = F(x) + R^{v}_{epiG}(x,z)$, $\tilde{H}(x,z) = H(z)$, where $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$, $H : Z \rightrightarrows Y$ are three set valued maps.

Remark 3.1. Note that $dom\tilde{F} = (domF \times Z) \cap epiG$, $dom\tilde{H} = X \times domH$, and $gr\tilde{H} = X \times grH$.

The next lemma is necessary in the sequel.

Lemma 3.1. *Let* $\bar{x} \in domF \cap dom(H \circ G)$, $\bar{z} \in G(\bar{x})$, $\bar{y}_1 \in F(\bar{x})$, *and* $\bar{y}_2 \in H(\bar{z})$ ($\sigma \in \{p,w\}$).

(i) *If* H *is* (Z_+,Y_+)*-nondecreasing on* $ImG + Z_+$, *then*

$$A \in \partial_{\varepsilon}^{\sigma}(F + H \circ G)(\bar{x}, \bar{y}_1 + \bar{y}_2) \ \Leftrightarrow \ (A,0) \in \partial_{\varepsilon}^{\sigma}(\tilde{F} + \tilde{H})((\bar{x},\bar{z}), \bar{y}_1 + \bar{y}_2), \ \forall \varepsilon \in D^{\sigma}.$$

(ii) $\partial_{\varepsilon}^{s}\tilde{H}((\bar{x},\bar{z}),\bar{y}_2) = \{0\} \times \partial_{\varepsilon}^{s} H(\bar{z},\bar{y}_2)$, $\forall \varepsilon \in D^{s}$.

(iii) *If* H *is* σ*-regular* η*-subdifferentiable at* $(\bar{z},\bar{y}_2)$ *for all* $\eta \geq 0$, *then* $\tilde{H}$ *is* σ*-regular* η*-subdifferentiable at* $((\bar{x},\bar{z}),\bar{y}_2)$ *for all* $\eta \geq 0$.

(iv) *If* $W = \mathbb{R}_+[G(domF \cap domG) + Z_+ - domH]$ *is a closed vector subspace of* Z, *then* $X \times W = \mathbb{R}_+[dom\tilde{F} - dom\tilde{H}]$ *is a closed vector subspace of* $X \times Z$.

Proof. (i) For $\sigma = w$, let $A \in \partial_{\varepsilon}^{w}(F + H \circ G)(\bar{x}, \bar{y}_1 + \bar{y}_2)$. For all x in X, we have

$$F(x) + (H \circ G)(x) - \bar{y}_1 - \bar{y}_2 - A(x - \bar{x}) + \varepsilon \subset Y \setminus -intY_+.$$

For all (x,z) in $X \times Z$, we have

$$F(x) + R_{epiG}(x,z) + (H \circ G)(x) - \bar{y}_1 - \bar{y}_2 - A(x - \bar{x}) + \varepsilon \subset Y \setminus -intY_+,$$

which implies that, for all $(x,z) \in epiG$,

$$\tilde{F}(x,z) + (H \circ G)(x) - \bar{y}_1 - \bar{y}_2 - A(x - \bar{x}) + \varepsilon \subset Y \setminus -intY_+. \tag{3.1}$$

On the other hand, as H is (Z_+,Y_+)-nondecreasing on $ImG + Z_+$, for $(x,z) \in epiG$, we see that $H(z) \subset (H \circ G)(x) + Y_+$. From (3.1), we see that

$$\begin{aligned}
&\tilde{F}(x,z) + H(z) - \bar{y}_1 - \bar{y}_2 - A(x - \bar{x}) + \varepsilon \\
&\subset \tilde{F}(x,z) + (H \circ G)(x) + Y_+ - \bar{y}_1 - \bar{y}_2 - A(x - \bar{x}) + \varepsilon \\
&\subset Y \setminus -intY_+ + Y_+.
\end{aligned}$$

By using the fact that $Y \setminus -intY_+ + Y_+ \subset Y \setminus -intY_+$, we deduce that, for all $(x,z) \in X \times Z$,

$$\tilde{F}(x,z) + \tilde{H}(x,z) - \bar{y}_1 - \bar{y}_2 - A(x - \bar{x}) + \varepsilon \subset Y \setminus -intY_+.$$

Thus $(A,0) \in \partial_{\varepsilon}^{\sigma}(\tilde{F} + \tilde{H})((\bar{x},\bar{z}), \bar{y}_1 + \bar{y}_2)$. Similarly, we prove the reverse implication. The case $\sigma = p$ is obtained similarly.

(ii) Let $(A,B) \in \partial^s_\varepsilon \tilde{H}((\bar{x},\bar{z}),\bar{y}_2)$. Then, for all $((x,z),y) \in gr\tilde{H} = X \times grH$,

$$A(x-\bar{x}) + B(z-\bar{z}) \leq_{Y_+} y - \bar{y}_2 + \varepsilon. \tag{3.2}$$

By taking $z = \bar{z}$ and $y = \bar{y}_2$ in (3.2), it follows that, for all $x \in X$, $n \in \mathbb{N}^*$,

$$-\frac{\varepsilon}{n} \leq_{Y_+} A(x) \leq_{Y_+} \frac{\varepsilon}{n}.$$

As Y_+ is closed and with $n \nearrow +\infty$, we obtain that, for all $x \in X$, $A(x) \in Y_+ \cap -Y_+ = \{0_Y\}$. Consequently $A = 0$, which means that $\partial^s \tilde{H}((\bar{x},\bar{z}),\bar{y}_2) \subseteq \{0\} \times \partial^s H(\bar{z},\bar{y}_2)$.

For the reverse inclusion, let $B \in \partial^s_\varepsilon H(\bar{z},\bar{y}_2)$. Then, for all $(z,y) \in grH$, $B(z-\bar{z}) \leq_{Y_+} y - \bar{y}_2 + \varepsilon$. As $gr\tilde{H} = X \times grH$, we deduce that $\{0\} \times \partial^s_\varepsilon H(\bar{z},\bar{y}_2) \subseteq \partial^s \tilde{H}_\varepsilon((\bar{x},\bar{z}),\bar{y}_2)$.

(iii) Suppose that H is σ-regular η-subdifferentiable at $(\bar{z},\bar{y}_2)$ for all $\eta \geq 0$, and let $y^* \in Y_+^{*\sigma}$. By (ii), we have

$$\partial_\eta (y^* \circ \tilde{H})((\bar{x},\bar{z}),\langle y^*,\bar{y}_2\rangle) = \{0\} \times \partial_\eta (y^* \circ H)(\bar{z},\langle y^*,\bar{y}_2\rangle).$$

Thus

$$\begin{aligned}(0,B) \in \partial_\eta (y^* \circ \tilde{H})((\bar{x},\bar{z}),\langle y^*,\bar{y}_2\rangle) \quad &\Leftrightarrow \quad B \in \partial_\eta (y^* \circ H)(\bar{z},\langle y^*,\bar{y}_2\rangle) \\ &\Leftrightarrow \quad B \in \bigcup_{\substack{\varepsilon \in \eta Y_+ \\ \langle y^*,\varepsilon\rangle = \eta}} y^* \circ \partial^s_\varepsilon H(\bar{z},\bar{y}_2).\end{aligned}$$

Hence, there exist $\varepsilon \in \eta Y_+$ and $T \in \partial^s_\varepsilon H(\bar{z},\bar{y}_2)$ with $\langle y^*,\varepsilon\rangle = \eta$ such that $B = y^* \circ T$. Since $\partial^s_\varepsilon \tilde{H}((\bar{x},\bar{z}),\bar{y}_2) = \{0\} \times \partial^s_\varepsilon H(\bar{z},\bar{y}_2)$, we can write $(0,B) = y^* \circ (0,T) \in y^* \circ \partial^s_\varepsilon \tilde{H}((\bar{x},\bar{z}),\bar{y}_2)$. Therefore, we obtain

$$\partial_\eta (y^* \circ \tilde{H})((\bar{x},\bar{z}),\langle y^*,\bar{y}_2\rangle) = \bigcup_{\substack{\varepsilon \in \eta Y_+ \\ \langle y^*,\varepsilon\rangle = \eta}} y^* \circ \partial^s_\varepsilon \tilde{H}((\bar{x},\bar{z}),\bar{y}_2), \quad \forall y^* \in Y_+^{*\sigma},$$

which means that $\tilde{H}$ is σ-regular η-subdifferentiable at $((\bar{x},\bar{z}),\bar{y}_2)$ for all $\eta \geq 0$.

On other hand, it is easy to conclude the proof of (iv) by following [10]. Then the proof of lemma is complete.

□

Let us consider the following conditions.

$$(\mathrm{MR})_2 \quad \begin{cases} X, Z \text{ are locally convex spaces,} \\ F, H \text{ are } Y_+\text{-convex and } G \text{ is } Z_+\text{-convex,} \\ \exists a \in \mathrm{dom} F \cap \mathrm{dom} G \text{ such that } H \text{ is connected at some point } b \in G(a). \end{cases}$$

$$(\mathrm{AB})_2 \quad \begin{cases} X, Z \text{ are Banach spaces,} \\ F, H \text{ are } Y_+\text{-convex, star } Y_+\text{-epi-closed.} \\ G \text{ is } Z_+\text{-convex and } Z_+\text{-epi-closed,} \\ W = \mathbb{R}_+[G(\mathrm{dom}\, F \cap \mathrm{dom}\, G) - \mathrm{dom}\, H] \text{ is a closed vector subspace of } X. \end{cases}$$

Lemma 3.2. [6]

(i) *If condition* $(\mathrm{MR})_2$ *holds, then* $\tilde{F}_2$ *is connected at* $(a,b) \in \mathrm{dom}\, \tilde{F}_1$.

(ii) *If condition* $(\mathrm{AB})_2$ *holds, then* $X \times W = \mathbb{R}_+[\mathrm{dom}\, \tilde{F}_1 - \mathrm{dom}\, \tilde{F}_2]$ *is a closed vector subspace of* $X \times Z$ *and* $\tilde{F}_1$, $\tilde{F}_2$ *are star* Y_+*-epi-closed.*

Theorem 3.1. *Let $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$, and $H : Z \rightrightarrows Y$ be three set-valued mappings such that F, H are Y_+-convex, star epi-closed, and G is Z_+-convex, epi-closed, $(\bar{x}, \bar{y}_1) \in grF$, $(\bar{x}, \bar{z}) \in grG$, $(\bar{z}, \bar{y}_2) \in H(\bar{z})$. Then, for all $\varepsilon \in D^\sigma$,*

$$\partial^\sigma_\varepsilon (F + H \circ G)(\bar{x}, \bar{y}_1 + \bar{y}_2) \supseteq \bigcup_{\substack{\varepsilon_1 \in D^\sigma, \varepsilon_2 \in \varepsilon Y_+ \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{\partial^\sigma_{\varepsilon_1} (F + A \circ G)(\bar{x}, \bar{y}_1 + A(\bar{z})) \,|\, A \in \partial^s_{\varepsilon_2} H(\bar{z}, \bar{y}_2)\}$$

Assume that H is (Z_+, Y_+)-nondecreasing on $ImG + Z_+$, and σ-regular η-subdifferentiable at $(\bar{z}, \bar{y}_2)$ for all $\eta \geq 0$, Suppose that condition $(\mathrm{MR})_2$ or $(\mathrm{AB})_2$ holds. Then,

$$\partial^\sigma_\varepsilon (F + H \circ G)(\bar{x}, \bar{y}_1 + \bar{y}_2) = \bigcup_{\substack{\varepsilon_1 \in D^\sigma, \varepsilon_2 \in \varepsilon Y_+ \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{\partial^\sigma_{\varepsilon_1} (F + A \circ G)(\bar{x}, \bar{y}_1 + A(\bar{z})) \,|\, A \in \partial^s_{\varepsilon_2} H(\bar{z}, \bar{y}_2)\}.$$

Proof. Let us prove the first inclusion for $\sigma = w$. Let ε, $\varepsilon_1 \in D^\sigma$ and $\varepsilon_2 \in \varepsilon Y_+$ with $\varepsilon_1 + \varepsilon_2 = \varepsilon$ and $B \in \partial^w_{\varepsilon_1}(F + A \circ G)(\bar{x}, \bar{y}_1 + A(\bar{z}))$ for some $A \in \partial^s_{\varepsilon_2} H(\bar{z}, \bar{y}_2)$. We proceed by contradiction: if $B \notin \partial^w_\varepsilon (F + H \circ G)(\bar{x}, \bar{y}_1 + \bar{y}_2)$, then there exist $x_0 \in (domF \cap (dom(H \circ G)))$, $y_0 \in F(x_0)$, and $z_0 \in (H \circ G)(x_0)$ such that

$$y_0 + z_0 - \bar{y}_1 - \bar{y}_2 - B(x_0 - \bar{x}) + \varepsilon \in -intY_+. \tag{3.3}$$

Note that $A \in \partial^s_{\varepsilon_2} H(\bar{z}, \bar{y}_2)$ implies

$$\bar{y}_2 - z_0 + A(z_0) - A(\bar{z}) - \varepsilon_2 \in -Y_+. \tag{3.4}$$

Adding (3.3) and (3.4) and using the fact that $-Y_+ - intY_+ \subseteq -intY_+$, we obtain

$$y_0 - \bar{y}_1 + A(z_0) - A(\bar{z}) - B(x_0 - \bar{x}) + \varepsilon_1 \in -intY_+.$$

This leads to a contradiction with $B \in \partial^w_{\varepsilon_1}(F + A \circ G)(\bar{x}, \bar{y}_1 + A(\bar{z}))$. The case $\sigma = p$ is similarly obtained. For the reverse inclusion, let $B \in \partial^\sigma_\varepsilon (F + H \circ G)(\bar{x}, \bar{y}_1 + \bar{y}_2)$ and $\tilde{F}$, $\tilde{H}$ be the set valued mappings defined above. By Lemma 3.1 (i), we have that $(B, 0) \in \partial^\sigma_\varepsilon (\tilde{F} + \tilde{H})((\bar{x}, \bar{z}), \bar{y}_1 + \bar{y}_2)$. We can easily see that $\tilde{F}$ and $\tilde{H}$ are Y_+-convex and star epi-closed. Thus, by Theorem 2.2 and Lemma 3.1 (iii), there exist $(T, A) \in L(X, Y) \times L(Z, Y)$, $\varepsilon_1 \in D^w$, $\varepsilon_2 \in \varepsilon Y_+$, and $\varepsilon_1 + \varepsilon_2 = \varepsilon$ such that

$$(B - T, -A) \in \partial^\sigma_{\varepsilon_1} \tilde{F}((\bar{x}, \bar{z}), \bar{y}_1) \quad \textit{and} \quad (T, A) \in \partial^s_{\varepsilon_2} \tilde{H}((\bar{x}, \bar{z}), \bar{y}_2).$$

On other hand, by Lemma 3.1 ((ii) and (iv)), we obtain $T = 0$, $A \in \partial^s_{\varepsilon_2} H(\bar{z}, \bar{y}_2)$, and $(B, -A) \in \partial^\sigma_{\varepsilon_1} \tilde{F}((\bar{x}, \bar{z}), \bar{y}_1)$. Note that, for $\sigma = w$, $(B, -A) \in \partial^w_{\varepsilon_1} \tilde{F}((\bar{x}, \bar{z}), \bar{y}_1)$ is equivalent to

$$F(x) - \bar{y}_1 - B(x - \bar{x}) + A(z - \bar{z}) + \varepsilon_1 \subset Y \setminus -intY_+, \quad (x, z) \in X \times Z \cap \mathrm{epi}G.$$

Hence, for all $x \in X$, we have

$$(F + A \circ G)(x) - (\bar{y}_1 + A(\bar{z})) - B(x - \bar{x}) + \varepsilon_1 \subset Y \setminus -intY_+.$$

Therefore, $B \in \partial^w_{\varepsilon_1}(F + A \circ G)(\bar{x}, \bar{y}_1 + A(\bar{z}))$. Case $\sigma = p$ can be obtained similarly. □

By taking $F(x) = \{0_Y\}$ for any $x \in X$, we obtain the following corollary.

Corollary 3.1. *Let $G : X \rightrightarrows Z$ and $H : Z \rightrightarrows Y$ be two set-valued mappings, and let $\bar{z} \in G(\bar{x})$ and $\bar{y} \in H(\bar{z})$. Assume that H is (Z_+, Y_+)-nondecreasing on $ImG + Z_+$ and σ-regular η-subdifferentiable at $(\bar{z}, \bar{y})$ for all $\eta \leq 0$. Suppose that one of the following conditions holds*

$$(\mathrm{MR})_3 \quad \begin{cases} X, Z \text{ are locally convex spaces,} \\ G \text{ is } Z_+\text{-convex and } H \text{ is } Y_+\text{-convex,} \\ H \text{ is connected at some point of } \mathrm{Im}\, G. \end{cases}$$

$$(\mathrm{AB})_3 \quad \begin{cases} X, Z \text{ are Banach spaces,} \\ H \text{ is } Y_+\text{-convex, star } Y_+\text{-epi-closed,} \\ G \text{ is } Z_+\text{-convex and } Z_+\text{-epi-closed,} \\ \mathbb{R}_+[G(\mathrm{dom}\, G) - \mathrm{dom}\, H] \text{ is a closed vector subspace of } X. \end{cases}$$

Then, for all $\varepsilon \in D^\sigma$,

$$\partial_\varepsilon^\sigma (H \circ G)(\bar{x}, \bar{y}) = \bigcup_{\substack{\varepsilon_1 \in D^\sigma, \varepsilon_2 \in \varepsilon Y_+ \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} \{\partial_{\varepsilon_1}^\sigma (A \circ G)(\bar{x}, A(\bar{z})) | A \in \partial_{\varepsilon_2}^s H(\bar{z}, \bar{y})\}$$

4. Applications to Constrained SVOPs

In this section, we provide the approximate σ-efficient optimality conditions in terms of the Lagrange–Kuhn–Tucker (operator) multiplier for the following general convex set valued mathematical programming problem

$$(\mathrm{P}_S) \quad \begin{cases} \text{Minimize F(x),} \\ G(x) \cap -Z_+ \neq \emptyset, \\ x \in C, \end{cases}$$

where $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ are two set-valued mappings, Z is a real locally convex topological vector space, Z_+ is a closed convex pointed cone with nonempty topological interior and C be a nonempty closed convex set of X. For establishing the σ-efficient optimality conditions of this problem, we need the following lemmas.

Lemma 4.1. [10] *The indicator set-valued mapping $R^v_{-Z_+}$ is (Z_+, Y_+)-nondecreasing on Z.*

Lemma 4.2. [3]

(i) *For a convex and closed subset S, R^v_S is proper, Y_+-convex and epi-closed. Furthermore, for all $x \in S$, $\varepsilon \geq 0$, $\partial^s_\varepsilon R^v_S(\bar{x}, 0) = N^v_\varepsilon(\bar{x}, S)$, where $N^v_\varepsilon(\bar{x}, S) = \{T \in L(X,Y) : \forall x \in S, T(x - \bar{x}) \leq_{Y_+} \varepsilon\}$ is the ε-normal vectors at $\bar{x} \in S$. In particular,*

$$\partial^s_\varepsilon R^v_{-Z_+}(\bar{z}, 0) = \{A \in L_+(Z,Y) : \ -\varepsilon \leq_{Y_+} A(\bar{z}) \leq_{Y_+} 0\}.$$

(ii) *If $\mathrm{int}(S) \neq \emptyset$, then R^v_S is connected on $\mathrm{int}(S)$.*

(iii) *R^v_S is σ-regular η-subdifferentiable on $S \times \{0_Y\}$ for all $\eta \geq 0$, $(\sigma \in \{p, w\})$.*

Theorem 4.1. *Let $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ be two set-valued mappings and $(\bar{x}, \bar{y}) \in \mathrm{gr}\, F$ with $\bar{x} \in C$ and $G(\bar{x}) \cap (-Z_+) \neq \emptyset$, and let $\varepsilon \in D^\sigma$ and $\sigma \in \{w, p\}$. If one of the following conditions* *holds*

$$(\mathrm{MR})_4 \quad \begin{cases} X,\ Z \text{ are locally convex vector spaces}, \\ F \text{ is } Y_+\text{-convex and } G \text{ is } Z_+\text{-convex}, \\ \operatorname{int}(-Z_+)\cap G(C\cap \operatorname{dom} F\cap \operatorname{dom} G)\neq \emptyset. \end{cases}$$

$$(\mathrm{AB})_4 \quad \begin{cases} X,\ Z \text{ are Banach spaces}, \\ F \text{ is } Y_+\text{-convex and star } Y_+\text{-epi-closed}, \\ G \text{ is } Z_+\text{-convex and epi-closed }, \\ \mathbb{R}_+[G(\operatorname{dom} F\cap C\cap \operatorname{dom} G)+Z_+] \text{ is a closed vector subspace of } Z, \end{cases}$$

then $(\bar{x},\bar{y})$ *is an* ε-σ-*efficient solution to problem* (P_S) *if and only if, for any* $\bar{z}\in G(\bar{x})\cap -Z_+$, *there exist* $\varepsilon_1\in D^\sigma$, $\varepsilon_2\in\varepsilon Y_+$ *and* $A\in\{A\in L_+(Z,Y):\ \ \varepsilon_2\leq_{Y_+} A(\bar{z})\leq_{Y_+} 0_Y\}$ *such that* $0\in \partial^\sigma_{\varepsilon_1}(F+R^v_C+A\circ G)(\bar{x},\bar{y}+A(\bar{z}))$.

Proof. The feasible set associated to problem (P_S) is given by $S=\{x\in X:\ G(x)\cap -Z_+\neq\emptyset\}\cap C$, and it is easy to check that $R^v_S=R^v_C+R^v_{-Z_+}\circ G$. Hence problem (P_S) becomes equivalent to the unconstrained set-valued minimization problem

$$\begin{cases} \text{Minimize } (\mathrm{F}+R^v_C+R^v_{-Z_+}\circ G)(x), \\ x\in X. \end{cases}$$

We can see easily that $K_{\varepsilon,\sigma}(F(S),Y_+)=K_\sigma((F+R^v_C+R^v_{-Z_+}\circ G)(X),Y_+)$. From relation (2.1), we can write

$$(\bar{x},\bar{y})\in K_{\varepsilon,\sigma}(F(S),Y_+)\Longleftrightarrow 0\in\partial^\sigma_\varepsilon(F+R^v_C+R^v_{-Z_+}\circ G)(\bar{x},\bar{y}).$$

Observe that $\operatorname{epi}(F+R^v_C)=\operatorname{epi} F\cap(C\times Y)$, which asserts that the convexity of the set-valued mapping $F+R^v_C$ follows from the convexity of the epigraph of F and the convexity of C. Also, we note that, for any $y^*\in Y^*_+$,

$$\operatorname{epi}(y^*\circ(F+R^v_C))=\operatorname{epi}(y^*\circ F+y^*\circ R^v_C)=\operatorname{epi}(y^*\circ F)\cap(C\times\mathbb{R}).$$

Thus the star Y_+-epi-closedness of mapping $F+R^v_C$ comes from the star Y_+-epi-closedness of F and the closedness of subset C.

Note that the conditions $(\bar{x},\bar{y})\in\operatorname{gr} F$ with $\bar{x}\in C$ and $G(\bar{x})\cap(-Z_+)\neq\emptyset$ could be written equivalently as $(\bar{x},\bar{y})\in\operatorname{gr}(F+R^v_C)$, $(\bar{x},\bar{z})\in\operatorname{gr} G$ and $(\bar{z},0_Y)\in\operatorname{gr} R^v_{-Z_+}$ for any $\bar{z}\in G(\bar{x})\cap(-Z_+)$.

According to Lemma 4.2, set-valued mappings $F+R^v_C$, G and $R^v_{-Z_+}$ satisfy all the assumptions of Theorem 3.1 and thus we obtain $(\bar{x},\bar{y})\in K_{\varepsilon,\sigma}(F(S),Y_+)$, if and only if there exist $\varepsilon_1\in D^\sigma$, $\varepsilon_2\in\varepsilon Y_+$, and $A\in\partial^s_{\varepsilon_2}R^v_{-Z_+}(\bar{z},0_Y)=\{A\in L_+(Z,Y):\ \ \varepsilon_2\leq_{Y_+}A(\bar{z})\leq_{Y_+}0_Y\}$ such that $0\in\partial^\sigma_{\varepsilon_1}(F+R^v_C+A\circ G)(\bar{x},\bar{y}+A(\bar{z}))$. The proof of theorem is complete. □

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