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OPTIMALITY CONDITIONS FOR A CLASS OF PROPERLY EFFICIENT SOLUTIONS OF STRONG MULTIOBJECTIVE BILEVEL OPTIMIZATION PROBLEMS

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Abstract. In this paper, we are concerned with a strong vectorial nonlinear bilevel programming problem whose upper and lower levels are vectorial. For such a problem, we give a conjugate duality approach based on Scalarization, regularization, and conjugate duality. We show that any accumulation point of the sequence of scalarized-regularized solutions solves the bilevel programming problem. Via this duality approach, we establish necessary optimality conditions for the scalarized-regularized problem. We also provide necessary and sufficient optimality conditions for a class of properly efficient solutions of the bilevel programming problem.
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1. INTRODUCTION

In this paper, we are concerned with the following vector bilevel minimization problem

$$(S) \qquad v-\min_{\substack{x\in X\\ y\in\mathscr{M}(x)}} F(x,y),$$

where $\mathscr{M}(x)$ is the set of properly efficient solutions of the vector minimization problem

$$\mathscr{P}(x) \qquad v-\min_{y\in Y} f(x,y),$$

where $F:\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^k$, $f:\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^m$, $k\geq 2$, $m\geq 2$, are convex functions, X and Y are two nonempty, compact, and convex subsets of $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, and "$v-\min$" stands for vector minimization.

Problem (S) is called a multiobjective strong bilevel programming problem or multiobjective strong Stackelberg problem. It corresponds to a two-player game in which a leader plays against a follower. The leader, having all information about the follower, announces first a strategy $x\in X$ to minimize his objective vector function F. Then, the follower reacts optimally by selecting a strategy $y(x)\in Y$, to minimize his objective vector function f. It is assumed that the game is cooperative.

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A multiobjective bilevel programming problem is a bilevel problem which one or both levels are vectorial. Bilevel optimization problems with multiple objectives in both levels were investigated seldom in the literature. Let us summarize some interesting results. In [15], Yin emphasized the importance of formulating the transportation decision-making problems as a multiobjective bilevel model and then proposed a solution algorithm by using genetic algorithms. Note that Yin's algorithm was efficient to search simultaneously the pareto optimal solutions. In [9], Eichfelder showed that the set of feasible points of the upper level problem can be expressed completely as the solution set of a multiobjective optimization problem. This problem was solved based on a scalarization approach. Eichfelder presented an algorithm for the first time in the case of bicriteria optimization problems on both levels and for a one dimentional upper level variable.

In [8], Dempe and Frank considered a linear bilevel programming problem with multivalued objective functions in both upper and lower levels. Using vector optimization theory, the multiobjective problem was suitably reformulated into a parametric bilevel programming problem. Then, a respective solution algorithm was presented and illustrated via an example. Even though multiobjective bilevel optimization problems where both levels are vectorial have not yet received a broad attention in the literature, real-world decision-making processes always have several social concern and thus multiple objectives need to be achieved simultaneously. For illustration of such a class of bilevel problems, let us give the following practical example [9]. Consider a city bus transportation system financed by the public authorities. They have two objectives to achieve; The first one is the reduction of the money losses, and the second one is to bring as many people as possible to use the buses instead of their own cars in order to reduce the overall traffic. The public authorities can decide about the bus ticket price but with taking into account the customers in their usage of the buses. The customers may have several objectives like minimizing their transportation time and costs. Therefore, the transportation system can be modeled as a bilevel multiobjective optimization problem where the first level includes the objectives and the constraints of the public authorities and the lower level includes the objectives and the constraints of the public.

The aim of this paper is to provide necessary and sufficient optimality conditions for (S), the multiobjective bilevel problem via the Fenchel-Lagrange duality approach. This duality was first introduced for convex programming problems in [14], and afterwards extended to some generalized convex programming problems (see, e.g., [5, 7]). In [1], the authors presented a Fenchel-Lagrange duality approach using conjugacy for a semivectorial bilevel problem where the upper and lower levels are vectorial and scalar respectively and for a one upper and lower level variable. In [3], a Fenchel-Lagrange duality approach and optimality conditions were given for a class of semivectorial bilevel problem where the upper level is vectorial and the lower level is scalar. In this paper, we extend this duality approach via scalarization to the multiobjective case where the corresponding upper and lower levels are both vectorial. The approach considered is based on the use of four operations: Scalarization, regularization, decomposition, and a conjugate duality. In the first step, we scalarize problem (S) into problem $(S^{\lambda}), \lambda \in \text{int}\mathbb{R}^p$. In order to establish strong duality, we need the so-called Slater condition. Unfortunately, due to the constraint $y \in \mathscr{M}(x)$, problem (S) and its scalarized one in the sense of Geoffrion ([10]) do not satisfy this condition. In order to avoid this situation, we start by regularizing problem (S^{λ}) into $(S^{\lambda}_{\varepsilon})$ and the regularization is based on the use of ε-properly efficient solutions of problem (S^{λ}). As a main result, we show that any accumulation point of a sequence of solutions of the scalarized-regularized problem $(S_{\varepsilon}^{\lambda})$ solves (S). Next, we decompose the scalarized-regularized problem $(S_{\varepsilon}^{\lambda})$ according to the second variable. This decomposition is obtained via the link that exists between the $\varepsilon-$properly efficient solutions of the lower level problem $\widehat{\mathscr{M}^{\varepsilon}}(x)$ and the set of $\mu^{\top}\varepsilon-$solutions of the scalarized problem $\mathscr{P}^{\mu}(x)$. In order to start our procedure of dualization, we consider in a second time a decomposition of problem $(S_{\varepsilon,\mu}^{\lambda})$ into a family of scalar convex minimization problems $(S_{\varepsilon,\mu}^{\lambda,\tilde{x}}), \tilde{x} \in \mathbb{R}^p$. The key of this decomposition is that $(S_{\varepsilon,\mu}^{\lambda})$ can be viewed as a minimization problem of a convex scalar objective function under d.c. constraints. Note that the technique used to obtain such a decomposition is inspired from the work of Martinez-Legaz and Volle ([12]). Then, based on the study given in [6], we give the Fenchel-Lagrange dual to every subproblem $(S_{\varepsilon,\mu}^{\lambda,\tilde{x}})$. Using the decomposition, we define a duality for problem $(S_{\varepsilon}^{\lambda})$ which we call the extended Fenchel-Lagrange duality. Under appropriate assumptions, we show that strong extended Fenchel-Lagrange duality holds for $(S_{\varepsilon}^{\lambda})$. Based on the obtained results, we provide necessary optimality conditions for the scalarized-regularized problem $(S_{\varepsilon}^{\lambda})$. Via this duality and some stability results related to the regularization, we give necessary optimality conditions for the class of properly efficient solutions of (S) which are accumulation points of a sequence of scalarized-regularized solutions. Finally, sufficient optimality conditions are given for problem (S) without resorting to duality. Note that this duality approach extends the one given in [2] from the scalar case to the multiobjective one, where both levels are vectorials.

The paper is organized as follows. We start the second section by some results related to convex analysis. Then, we recall some definitions and results concerning multiobjective optimization. After that we give some preliminary results concerning the scalarized problem associated to the lower level problem. In Section 3, we present the link that exist between the scalarized-regularized problem and the original bilevel problem that are needed in what follows. In Section 4, we present our duality approach and provide necessary and sufficient optimality conditions for the scalarized-regularized problem. In Section 5, we provide necessary and sufficient optimality conditions for the original multiobjective bilevel programming problem (S). Finally, Section 6 ends this paper.

2. PRELIMINARIES

In this section, we first recall some results related to convex analysis. Then, we remind some definitions and results concerning multiobjective optimization. We close this section by providing some preliminary results concerning the scalarized problem associated to the lower level problem.

2.1. **Background of convex analysis.** Let A be a nonempty subset of $\mathbb{R}^n$. We denote by ψ_A the indicator function of set A, i.e., $\psi_A(x) = 0$ if $x \in A$, and $\psi_A(x) = +\infty$ otherwise. In what follows, set $\mathbb{R}^n$ is equipped with the usual topology and the following conventions in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ will be adopted

$$(+\infty) - (+\infty) = (-\infty) - (-\infty) = (+\infty) + (-\infty) = +\infty$$

$$\begin{cases} 0 \times (+\infty) = +\infty \\ 0 \times (-\infty) = 0 \end{cases} \qquad \begin{cases} \alpha(-\infty) = -\infty, & \alpha(+\infty) = +\infty \quad \text{for } \alpha \in \mathbb{R}_+^* \\ \alpha(-\infty) = +\infty, & \alpha(+\infty) = -\infty \quad \text{for } \alpha \in \mathbb{R}_-^*. \end{cases}$$

Definition 2.1. Let $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a function. The conjugate function of h relative to set A is denoted by h_A^* and defined on $\mathbb{R}^n$ by $h_A^*(p) = \sup_{x\in A}\{\langle p,x\rangle - h(x)\}$, where $\langle .,.\rangle$ denotes the inner product for two vectors in $\mathbb{R}^n$. If $A = \mathbb{R}^n$, then we have the usual Legendre-Fenchel conjugate function of h, simply denoted by h^*.

Definition 2.2. The effective domain of h denoted by dom$h$ is the set defined by dom$h = \{x \in \mathbb{R}^n \ / \ h(x) < +\infty\}$. We say that h is proper if $h(x) > -\infty$, for all $x \in \mathbb{R}^n$, and dom$h$ is nonempty.

Definition 2.3. Let $h : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a proper convex function and $\bar{x} \in$ dom$h$. The subdifferential in the sense of convex analysis of h at $\bar{x}$ denoted by $\partial h(\bar{x})$ is the set defined by

$$\partial h(\bar{x}) = \{x^* \in \mathbb{R}^n \ / \ h(x) \geq h(\bar{x}) + \langle x^*, x-\bar{x}\rangle \quad \forall x \in \mathbb{R}^n\}.$$

An element $x^* \in \partial g(\bar{x})$ is called a subgradient of h at $\bar{x}$.

Remark 2.1. *i)* $x^* \in \partial h(\bar{x}) \Longleftrightarrow \langle x^*, \bar{x}\rangle = h(\bar{x}) + h^*(x^*)$.
ii) $h(x) + h^*(x^*) \geq \langle x^*, x\rangle$ for all $x, x^* \in \mathbb{R}^n$, called the Fenchel inequality.

Definition 2.4. let C be a nonempty convex subset of $\mathbb{R}^n$ and $\bar{x} \in C$. The normal cone $\mathscr{N}_C(\bar{x})$ to C at $\bar{x}$ in the sense of convex analysis is defined by $\mathscr{N}_C(\bar{x}) = \{x^* \in \mathbb{R}^n : \ \langle x^*, x-\bar{x}\rangle \leq 0, \ \forall x \in C\}$.

Theorem 2.1. [13] *Let $h : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex, and lower semicontinuous function, and let C be a nonempty and compact subset of int(domh). Then, $\bigcup_{x\in C}\partial h(x)$ is compact.*

Theorem 2.2. [4] *Let $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a proper convex functions. Assume that there exists $x_0 \in domh_1$ such that h_2 is continuous at x_0. Then, for every $x \in \mathbb{R}^n$, $\partial(h_1+h_2)(x) = \partial h_1(x) + \partial h_2(x)$.*

2.2. **Background of multiobjective optimization.** Let us recall some definitions and results concerning multiobjective optimization. Consider the following vector minimization problem

$$(\mathscr{Q}) \qquad v - \min_{x\in\mathscr{A}} g(x),$$

where $g = (g_1,...,g_k)^\top : \mathbb{R}^p \to \mathbb{R}^k$ is a function and $\mathscr{A}$ is a nonempty subset of $\mathbb{R}^p$.

Definition 2.5. [10] An element $\bar{x} \in \mathscr{A}$ is called an efficient solution to problem $(\mathscr{Q})$ if $g(x) \leq g(\bar{x})$, for $x \in \mathscr{A}$, $g(x) = g(\bar{x})$. An efficient solution is also called a pareto-efficient solution.

Throughout the paper, we adopt the following definition of properly efficient solution in the sense of Geoffrion .

Definition 2.6. [10] An element $\bar{x} \in \mathscr{A}$ is called a properly efficient solution to problem $(\mathscr{Q})$ if it is efficient and if there exists a positive real number M such that, for each $i \in \{1,...,k\}$ and $x \in \mathscr{A}$ satisfying $g_i(x) < g_i(\bar{x})$, there exists $j \in \{1,...,k\}$ such that

i) $g_j(\bar{x}) < g_j(x)$

ii) $\dfrac{g_i(\bar{x}) - g_i(x)}{g_j(x) - g_j(\bar{x})} \leq M$.

For $\lambda = (\lambda_1,...,\lambda_k)^\top \in int\mathbb{R}_+^k$, we consider the following scalar minimization problem

$$(\mathscr{Q}_\lambda) \qquad \min_{x\in\mathscr{A}} \sum_{i=1}^k \lambda_i g_i(x)$$

associated to the vector minimization problem $(\mathscr{Q})$.

Theorem 2.3. [10] *Let $\lambda_i > 0\,(i=1,...,k)$ be fixed. If $\bar{x}$ is optimal solution to $(\mathscr{Q}_\lambda)$, then $\bar{x}$ is properly efficient solution to $(\mathscr{Q})$.*

Theorem 2.4. [10] *Let $\mathscr{A}$ be a convex set, and let g_i be convex on $\mathscr{A}$. Then $\bar{x}$ is properly efficient of $(\mathscr{Q})$ if and only if there exists $\lambda = (\lambda_1,...,\lambda_k)^\top \in int\mathbb{R}^k_+$ with $\sum_{i=1}^k \lambda_i = 1$ such that $\bar{x}$ solves the scalarized problem $(\mathscr{Q}_\lambda)$, i.e., $Argmin\mathscr{Q} = \bigcup_{\lambda \in int\mathbb{R}^k_+} Argmin\mathscr{Q}_\lambda$.*

Definition 2.7. ([10, 11]) Let $\varepsilon = (\varepsilon_1,...,\varepsilon_k)^\top \in \mathbb{R}^k_+$. A point $\bar{x} \in \mathscr{A}$ is said to be an ε-efficient (or pareto ε-efficient) solution to problem $(\mathscr{Q})$ if, for $x \in \mathscr{A}$ such that $g(x) \leq g(\bar{x}) - \varepsilon$, $g(x) = g(\bar{x}) - \varepsilon$.

Definition 2.8. [11] Let $\varepsilon = (\varepsilon_1,...,\varepsilon_k)^\top \in \mathbb{R}^k_+$. A point $\bar{x} \in \mathscr{A}$ is said to be an ε-properly efficient solution to problem $(\mathscr{Q})$ if it is ε-efficient and there exists a positive real number M such that, for each $i \in \{1,...,k\}$ and $x \in \mathscr{A}$ satisfying $g_i(x) < g_i(\bar{x}) - \varepsilon_i$, there exists $j \in \{1,...,k\}$ such that

i) $g_j(\bar{x}) - \varepsilon_j < g_j(x)$

ii) $\dfrac{g_i(\bar{x}) - g_i(x) - \varepsilon_i}{g_j(x) - g_j(\bar{x}) + \varepsilon_j} \leq M.$

The following result gives a characterization of ε-proper efficiency via scalarization.

Theorem 2.5. [11] *Let $\varepsilon = (\varepsilon_1,...,\varepsilon_k)^\top \in \mathbb{R}^k_+$. Assume that the set $\mathscr{A}$ and the function g are convex. Let $\bar{x} \in \mathscr{A}$. Then, $\bar{x}$ is an ε-properly efficient solution to problem $(\mathscr{Q})$ if and only if , there exists $\lambda = (\lambda_1,...,\lambda_k)^\top \in int\mathbb{R}^k_+$ with $\sum_{i=1}^k \lambda_i = 1$ such that $\bar{x}$ is a $\lambda^\top\varepsilon$-solution of the scalar minimization problem $(\mathscr{Q}_\lambda)$ i.e., $\sum_{i=1}^k \lambda_i g_i(\bar{x}) \leq \sum_{i=1}^k \lambda_i g_i(x) + \sum_{i=1}^k \lambda_i \varepsilon_i \qquad \forall x \in \mathscr{A}$.*

2.3. **Scalarization of the lower level problem $\mathscr{P}(x)$.** In this section, we present the scalarized problem associated to the lower level problem. Then, we establish some results which are needed for our further investigation.

For a fixed $\mu = (\mu_1,...,\mu_m)^\top \in int\mathbb{R}^m_+$, we consider the following scalarized problem of $\mathscr{P}(x)$

$$\mathscr{P}^\mu(x) \qquad \min_{y \in Y} \sum_{j=1}^m \mu_j f_j(x,y).$$

Define $\hat{f}_\mu(x,y) = \sum_{j=1}^m \mu_j f_j(x,y)$ on $\mathbb{R}^p \times \mathbb{R}^q$. Set $v_\mu(x) = \inf_{y \in Y} \hat{f}_\mu(x,y)$ and

$$\mathscr{M}^\mu(x) = \left\{ y \in Y / \hat{f}_\mu(x,y) \leq v_\mu(x) \right\}$$

the infimal value and the set of solutions of the scalar problem $\mathscr{P}^\mu(x)$ respectively.

Remark 2.2.
i) $\hat{f}_\mu$ is continuous since f_j, $j = 1,...,m$, are continuous as finite convex functions.
ii) Since Y is compact and $\hat{f}_\mu$ is continuous, then $\mathscr{P}^\mu(x)$, the scalarized problem, admits at least one solution. Hence $v_\mu(x) \in \mathbb{R}$ for all $x \in X$.

Proposition 2.1. *Let $x \in X$. Then, $\bar{y}$ is a properly efficient solution to $\mathscr{P}(x)$ if and only if there exists $\mu_x = (\mu_{x,1}, \mu_{x,2},...,\mu_{x,m})^\top \in int(\mathbb{R}^m_+)$ such that $\bar{y}$ is a solution to the scalarized minimization problem $\mathscr{P}^{\mu_x}(x)$ and $\mathscr{M}(x) = \bigcup_{\mu_x \in int(\mathbb{R}^m_+)} \mathscr{M}^{\mu_x}(x)$.*

Proof. Following Theorem 2.4, we can conclude the result immediately. □

Proposition 2.2. *Let $\varepsilon \in int(\mathbb{R}^m_+)$ and $x \in X$. Then, $\bar{y}$ is an ε-properly efficient solution to $\mathscr{P}(x)$ if and only if there exists $\mu_x = (\mu_{x,1}, \mu_{x,2}, ..., \mu_{x,m})^\top \in int(\mathbb{R}^m_+)$ such that $\bar{y}$ is a $\mu_x^\top \varepsilon-$ solution to the scalarized minimization problem $\mathscr{P}^{\mu_x}(x)$ and $\widehat{\mathscr{M}^\varepsilon}(x) = \bigcup_{\mu_x \in int(\mathbb{R}^m_+)} \widetilde{\mathscr{M}}^{\mu_x}_{\mu_x^\top \varepsilon}(x)$ with $\widehat{\mathscr{M}^\varepsilon}(x)$ being the set of ε-properly efficient solutions of $\mathscr{P}(x)$ and $\widetilde{\mathscr{M}}^{\mu_x}_{\mu_x^\top \varepsilon}(x)$ being the set of $\mu_x^\top\varepsilon$-solutions of $\mathscr{P}^{\mu_x}(x)$, i.e., $\widetilde{\mathscr{M}}^{\mu_x}_{\mu_x^\top \varepsilon}(x) = \left\{ y \in Y / \hat{f}_{\mu_x}(x,y) \leq v_{\mu_x}(x) + \mu_x^\top \varepsilon \right\}$.*

Proof. It immediately follows from Theorem 2.5. □

3. The Scalarized-Regularized Problem

As mentioned in the introduction, we need the Slater constraint qualification condition for the application of the Fenchel-Lagrange duality in our study. Since (S) does not satisfy this condition, we first proceed to its scalarization and then its regularization. This scalarization-regularization method uses ε-properly efficient solutions of lower level problem $\mathscr{P}(x)$. As a main result, we show that any accumulation point of a sequence of scalarized-regularized solutions of problem $(S^\lambda_{\varepsilon_n})_{\lambda \in \text{int}(\mathbb{R}^k_+), \varepsilon_n \in \text{int}(\mathbb{R}^m_+)}$ is a properly efficient solution of (S).

For a given $\lambda = (\lambda_1, ..., \lambda_k)^\top \in int(\mathbb{R}^k_+)$, we consider the following scalarized problem of (S)

$$(S^\lambda) \qquad \min_{\substack{x \in X \\ y \in \mathscr{M}(x)}} \sum_{i=1}^k \lambda_i F_i(x,y).$$

Let $\lambda \in \text{int}(\mathbb{R}^k_+)$ and $\varepsilon \in int(\mathbb{R}^m_+)$. We consider the following regularized problem of (S^λ)

$$(S^\lambda_\varepsilon) \qquad \min_{\substack{x \in X \\ y \in \widehat{\mathscr{M}^\varepsilon}(x)}} \sum_{i=1}^k \lambda_i F_i(x,y).$$

In what follows, for $\varepsilon_n \searrow 0^+_{\mathbb{R}^m}$, we denote the problem $(S^\lambda_{\varepsilon_n})$ by (S^λ_n). The following theorem establishes that any accumulation point of a sequence of solutions of the scalarized-regularized problem (S^λ_n) solves (S).

Theorem 3.1. *Let $\varepsilon_n \searrow 0^+_{\mathbb{R}^m}, \lambda \in int(\mathbb{R}^k_+)$ and $(\bar{x}_n, \bar{y}_n)_n$ be a sequence of solutions of scalarized-regularized problem (S^λ_n), $n \in \mathbb{N}$. Let $(\bar{x}, \bar{y})$ be an accumulation point of the sequence $(\bar{x}_n, \bar{y}_n)$. Then $(\bar{x}, \bar{y})$ is a properly efficient solution to (S).*

Proof. Feasibility. Obviously, we have $\bar{x} \in X$. Let us show that $\bar{y} \in \mathscr{M}(\bar{x})$, i.e., $\bar{y}$ is a properly efficient solution to $\mathscr{P}(\bar{x})$.

Efficiency: Let $y \in Y$ such that

$$f_i(\bar{x}, y) \leq f_i(\bar{x}, \bar{y}), \quad \forall i \in \{1, ..., m\} = I. \tag{3.1}$$

Let us show that, for all $i \in I$, $f_i(\bar{x}, \bar{y}) = f_i(\bar{x}, y)$. Note that $(\bar{x}_n, \bar{y}_n)$ is a solution to $(S^\lambda_{\varepsilon_n}), n \in \mathbb{N}$. Then, $\bar{y}_n \in \widehat{\mathscr{M}^{\varepsilon_n}}(x_n)$. Then, $\bar{y}_n$ is an ε_n-efficient solution to $\mathscr{P}(\bar{x}_n)$. We distinguish the following cases:

1) Assume that there exists $n_0 \in \mathscr{N}$ such that $f_i(\bar{x}_n, y) \leq f_i(\bar{x}_n, \bar{y}_n) - \varepsilon^i_n$ for all $n \in \mathscr{N}, n \geq n_0, i \in I$. For $n \in \mathscr{N}$, since $\bar{y}_n$ is an ε_n-efficient solution to $\mathscr{P}(\bar{x}_n)$, then $f_i(\bar{x}_n, y) = f_i(\bar{x}_n, \bar{y}_n) - \varepsilon^i_n$, $i \in I$. Passing to the limit as $n \to +\infty, n \in \mathscr{N}$, we obtain $f_i(\bar{x}, y) = f_i(\bar{x}, \bar{y})$.

2) Assume that there exists an infinite subset $\mathcal{N}' \subset \mathcal{N}$ such that $f_i(\bar{x}_n, y) > f_i(\bar{x}_n, \bar{y}_n) - \varepsilon_n^i$ for all $n \in \mathcal{N}'$. Then, passing to the limit as $n \to +\infty$, we obtain $f_i(\bar{x}, y) \geq f_i(\bar{x}, \bar{y})$. Using (3.1), we obtain $f_i(\bar{x}, y) = f_i(\bar{x}, \bar{y})$. By means of the two cases and the fact that i is arbitrary in I, we deduce that $f(\bar{x}, \bar{y}) = f(\bar{x}, y)$.

Proper efficiency: Now, let us show that $\bar{y}$ is a properly efficient solution to $\mathscr{P}(\bar{x})$. Assume the contrary. Let $M > 0$ be arbitrary. Then, there exist $y^* \in Y$ and $i \in I$ such that

$$f_i(\bar{x}, y^*) < f_i(\bar{x}, \bar{y}) \tag{3.2}$$

and

$$\frac{f_i(\bar{x}, \bar{y}) - f_i(\bar{x}, y^*)}{f_j(\bar{x}, y^*) - f_j(\bar{x}, \bar{y})} > M \tag{3.3}$$

for all $j \in I \setminus \{i\}$, verifying $f_j(\bar{x}, \bar{y}) < f_j(\bar{x}, y^*)$. Set $I(\bar{y}) = \{j \in I \setminus \{i\} \,/\, f_j(\bar{x}, \bar{y}) < f_j(\bar{x}, y^*)\}$. Since I is finite, then we easily deduce the following property:

$(\mathscr{L})$ There exists $n_3 \in \mathcal{N}$ such that, for all $n \geq n_3, n \in \mathcal{N}$,

i) $f_i(\bar{x}_n, y^*) < f_i(\bar{x}_n, \bar{y}_n) - \varepsilon_i^n$,

ii) $f_j(\bar{x}_n, \bar{y}_n) - \varepsilon_j^n < f_j(\bar{x}_n, y^*)$, $\forall j \in I(\bar{y})$

iii) $\frac{f_i(\bar{x}_n, \bar{y}_n) - f_i(\bar{x}_n, y^*) - \varepsilon_i^n}{f_j(\bar{x}_n, y^*) - f_j(\bar{x}_n, \bar{y}_n) + \varepsilon_j^n} > M$, $\forall j \in I(\bar{y})$.

Indeed, assume by contradiction that there exists an infinite subset $\mathcal{N}' \subset \mathcal{N}$ such that, $\forall n \in \mathcal{N}'$,

a) $f_i(\bar{x}_n, y^*) \geq f_i(\bar{x}_n, \bar{y}_n) - \varepsilon_i^n$, or,

b) $\exists j \in I(\bar{y})$ such that $f_j(\bar{x}_n, \bar{y}_n) - \varepsilon_j^n \geq f_j(\bar{x}_n, y^*)$, or,

c) $\exists j \in I(\bar{y})$ such that $\frac{f_i(\bar{x}_n, \bar{y}_n) - f_i(\bar{x}_n, y^*) - \varepsilon_i^n}{f_j(\bar{x}_n, y^*) - f_j(\bar{x}_n, \bar{y}_n) + \varepsilon_j^n} \leq M$.

We distinguish the following cases:

1) If *a)* is satisfied, then we obtain $f_i(\bar{x}, y^*) \geq f_i(\bar{x}, \bar{y})$ by passing to the limit as $n \to +\infty$. This inequality contradicts (3.2).

2) If *b)* is satisfied, then we obtain $f_j(\bar{x}, \bar{y}) \geq f_j(\bar{x}, y^*)$ by passing to the limit as $n \to +\infty$. This contradicts the fact that $j \in I(\bar{y})$.

3) If *c)* is satisfied, then we obtain

$$\exists j \in I(\bar{y}) \text{ such that } \frac{f_i(\bar{x}, \bar{y}) - f_i(\bar{x}, y^*)}{f_j(\bar{x}, y^*) - f_j(\bar{x}, \bar{y})} \leq M$$

by passing to the limit as $n \to +\infty$. This inequality contradicts (3.3).

Then, we obtain a contradiction. Set $I_{n_3} = \{j \in I \setminus \{i\} \,/\, f_j(\bar{x}_n, \bar{y}_n) - \varepsilon_j^n < f_j(\bar{x}_n, y^*),\ \forall n \geq n_3,\ n \in \mathcal{N}\}$. Let us show that the third assertion in $(\mathscr{L})$ is also true for all $j \in I_{n_3}$. Let $j \in I_{n_3}$. We distinguish the following cases.

1) If $j \in I(\bar{y})$, then there is nothing to prove. Note that *iii)* is satisfied for all $j \in I(\bar{y})$.

2) If $j \notin I(\bar{y})$, then $f_j(\bar{x}, \bar{y}) \geq f_j(\bar{x}, y^*)$. We distinguish the following subcases:

2.1) Assume that $f_j(\bar{x}, \bar{y}) > f_j(\bar{x}, y^*)$. Hence, there exists $n_4 \in \mathcal{N}$ such that

$$f_j(\bar{x}_n, \bar{y}_n) - \varepsilon_j^n > f_j(\bar{x}_n, y^*),\ \forall n \geq n_4, \quad n \in \mathcal{N}. \tag{3.4}$$

Set $n_5 = \max\{n_4, n_3\}$. Then, for all $n \geq n_5, n \in \mathcal{N}$, we get a contradiction between (3.4) and the fact that $j \in I_{n_3}$.

2.2) Assume that $f_j(\bar{x},\bar{y}) = f_j(\bar{x},y^*)$. Assume that there exists an infinite subset $\mathcal{N}' \subset \{n \in \mathcal{N} \ / \ n \geq n_3\}$ such that

$$\frac{f_i(\bar{x}_n,\bar{y}_n) - f_i(\bar{x}_n,y^*) - \varepsilon_i^n}{f_j(\bar{x}_n,y^*) - f_j(\bar{x}_n,\bar{y}_n) + \varepsilon_j^n} \leq M, \ \forall n \in \mathcal{N}'. \tag{3.5}$$

We have $\lim_{\substack{n\to+\infty \\ n\in\mathcal{N}'}} f_j(\bar{x}_n,y^*) - f_j(\bar{x}_n,\bar{y}_n) + \varepsilon_j^n = f_j(\bar{x},y^*) - f_j(\bar{x},\bar{y}) = 0$. Since $f_i(\bar{x},y^*) < f_i(\bar{x},\bar{y})$, then $\lim_{\substack{n\to+\infty \\ n\in\mathcal{N}'}} \frac{f_i(\bar{x}_n,\bar{y}_n) - f_i(\bar{x}_n,y^*) - \varepsilon_i^n}{f_j(\bar{x}_n,y^*) - f_j(\bar{x}_n,\bar{y}_n) + \varepsilon_j^n} = +\infty$, which leads to a contradiction in (3.5) $(+\infty \leq M)$.

In summary, we have the following property:

$(\mathscr{R})$ For arbitrary $M > 0$, there exists $y^* \in \mathbb{R}^q$ and $i \in I$ such that

i) $y^* \in Y$,
ii) $f_i(\bar{x}_n,y^*) < f_i(\bar{x}_n,\bar{y}_n) - \varepsilon_i^n$,
iii) $\frac{f_i(\bar{x}_n,\bar{y}_n) - f_i(\bar{x}_n,y^*) - \varepsilon_i^n}{f_j(\bar{x}_n,y^*) - f_j(\bar{x}_n,\bar{y}_n) + \varepsilon_j^n} > M, \ \forall j \in I_{n_3}$.

Therefore, the property $(\mathscr{R})$ gives a contradiction with the fact that $\bar{y}_n \in \widehat{\mathscr{M}^{\varepsilon_n}}(\bar{x}_n)$.

Optimality. Let us show that $(\bar{x},\bar{y})$ is a properly efficient solution of (S). For this, let us show that there exists $\tilde{\lambda} \in \text{int}(\mathbb{R}^k_+)$ such that $(\bar{x},\bar{y})$ is a solution to $(S^{\tilde{\lambda}})$ (see Theorem 2.3). We have that $(\bar{x}_n,\bar{y}_n)$ is a solution to scalarized regularized problem (S^{λ}_n), i.e.,

$$\sum_{i=1}^{k} \lambda_i F_i(\bar{x}_n,\bar{y}_n) \leq \sum_{i=1}^{k} \lambda_i F_i(x,y) \quad \forall (x,y) \in X \times \widehat{\mathscr{M}^{\varepsilon_n}}(x). \tag{3.6}$$

On the other hand, we have $\mathscr{M}(x) \subset \widehat{\mathscr{M}^{\varepsilon_n}}(x)$. Hence, from (3.6) we obtain $\sum_{i=1}^{k} \lambda_i F_i(\bar{x}_n,\bar{y}_n) \leq \sum_{i=1}^{k} \lambda_i F_i(x,y)$, $\forall (x,y) \in X \times \mathscr{M}(x)$. Using the continuity of the function $\sum_{i=1}^{k} \lambda_i F_i$ and passing to the limit as $n \to +\infty$, we obtain $\sum_{i=1}^{k} \lambda_i F_i(\bar{x},\bar{y}) \leq \sum_{i=1}^{k} \lambda_i F_i(x,y)$ for all $(x,y) \in X \times \mathscr{M}(x)$. Then, $\exists \tilde{\lambda} = \lambda \in \text{int}(\mathbb{R}^k_+)$ such that $(\bar{x},\bar{y})$ is a solution to $(S^{\tilde{\lambda}})$. Therefore, $(\bar{x},\bar{y})$ is a properly efficient solution to (S). □

4. Optimality Conditions for the Scalarized-regularized Problem

In this section, we give a duality approach and provide optimality conditions for scalarized-regularized problem $(S^{\lambda}_{\varepsilon})$. This duality approach is achieved in three steps. We first decompose problem $(S^{\lambda}_{\varepsilon})$ according to the second variable into a family of subproblems $(S^{\lambda}_{\varepsilon,\mu})_{\mu\in\text{int}(\mathbb{R}^m_+)}$. Then, due to the lack of convexity of $(S^{\lambda}_{\varepsilon,\mu})$, we give a decomposition of them by a family of convex minimization problems $(S^{\lambda,\tilde{x}}_{\varepsilon,\mu}), \tilde{x} \in \mathbb{R}^p$. Next, we define an extended duality for problem $(S^{\lambda}_{\varepsilon})$ via the Fenchel-Lagrange duality applied to every subproblem $(S^{\lambda,\tilde{x}}_{\varepsilon,\mu}), \tilde{x} \in \mathbb{R}^p$.

For $\lambda \in \text{int}(\mathbb{R}^k_+), (\varepsilon,\mu) \in (\text{int}(\mathbb{R}^m_+))^2$, we consider the following problem

$$(S^{\lambda}_{\varepsilon,\mu}) \qquad \min_{\substack{x\in X \\ y\in\widehat{\mathscr{M}}^{\mu}_{\mu^\top\varepsilon}(x)}} \sum_{i=1}^{k} \lambda_i F_i(x,y).$$

Theorem 4.1. *Let $\lambda \in int(\mathbb{R}^k_+)$ and $(\varepsilon,\mu) \in (int(\mathbb{R}^m_+))^2$. Then, $(S^{\lambda}_{\varepsilon,\mu})$ has at least one solution.*

Proof. The result follows from the continuity of F_i $i=1,...,k$ and the compactness of X and $\widetilde{\mathscr{M}}^{\mu}_{\mu^\top\varepsilon}(x)\subset Y$. □

From Proposition 2.2, the problem (S^λ_ε) can be written in the following form

$$(S^\lambda_\varepsilon) \qquad \min_{\substack{x\in X\\ y\in \bigcup\limits_{\mu\in int\mathbb{R}^m_+}\widetilde{\mathscr{M}}^{\mu}_{\mu^\top\varepsilon}(x)}} \sum_{i=1}^k \lambda_i F_i(x,y).$$

Then,

$$\inf S^\lambda_\varepsilon = \inf\left\{\sum_{i=1}^k \lambda_i F_i(x,y):\ x\in X,\ y\in \bigcup_{\mu\in \mathrm{int}(\mathbb{R}^m_+)} \mathscr{M}^{\mu}_{\mu^\top\varepsilon}(x)\right\} = \inf_{\mu\in \mathrm{int}(\mathbb{R}^m_+)} \inf S^\lambda_{\varepsilon,\mu}.$$

Hence, we obtain a decomposition of (S^λ_ε) according to the second variable into a family of scalar subproblems $(S^\lambda_{\varepsilon,\mu}), \lambda\in \mathrm{int}(\mathbb{R}^k_+), (\varepsilon,\mu)\in \mathrm{int}(\mathbb{R}^m_+)^2$.

4.1. **A formulation of problem $(S^\lambda_{\varepsilon,\mu})$ by conjugacy.** In this subsection, based on the study given by Martinez-Legaz and Volle [12], we give a formulation of problem $(S^\lambda_{\varepsilon,\mu})$ that uses the conjugate of the functions involved. For $\varepsilon\in \mathrm{int}(\mathbb{R}^k_+)$ and $\mu\in \mathrm{int}(\mathbb{R}^m_+)$, we define on $\mathbb{R}^p\times\mathbb{R}^q$ $h^\mu_{1,\varepsilon}(x,y)=0$ and $h^\mu_{2,\varepsilon}(x,y)=v_\mu(x)+\mu^\top\varepsilon$. We have

$$(S^\lambda_{\varepsilon,\mu}) \qquad \min_{\substack{(x,y)\in X\times Y\\ \hat{f}_\mu(x,y)\le v_\mu(x)+\mu^\top\varepsilon}} \sum_{i=1}^k \lambda_i F_i(x,y).$$

Then, this problem can be written in the following form

$$(S^\lambda_{\varepsilon,\mu}) \qquad \min_{\substack{(x,y)\in \mathbb{R}^p\times\mathbb{R}^q\\ \psi_{X\times Y}(x,y)-h^\mu_{1,\varepsilon}(x,y)\le 0\\ \hat{f}_\mu(x,y)-h^\mu_{2,\varepsilon}(x,y)\le 0}} \sum_{i=1}^k \lambda_i F_i(x,y),$$

which under the data is a minimization problem of a convex function under d.c. constraints. For $\lambda\in \mathrm{int}(\mathbb{R}^k_+), (\varepsilon,\mu)\in \mathrm{int}(\mathbb{R}^m_+)^2$, let $\mathscr{B}^\lambda_{\varepsilon,\mu}$ denote the feasible set of problem $(S^\lambda_{\varepsilon,\mu})$, i.e.,

$$\mathscr{B}^\lambda_{\varepsilon,\mu} = \left\{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q / \psi_{X\times Y}(x,y)-h^\mu_{1,\varepsilon}(x,y)\le 0, \hat{f}_\mu(x,y)-h^\mu_{2,\varepsilon}(x,y)\le 0\right\}.$$

Then, from [12, Lemma 2.1], we obtain

$$\mathscr{B}^\lambda_{\varepsilon,\mu} = \bigcup_{\substack{(x^*,y^*)\in\mathbb{R}^p\times\mathbb{R}^q\\ (t^*,z^*)\in\mathbb{R}^p\times\mathbb{R}^q\\ (u^*,v^*)\in\mathbb{R}^p\times\mathbb{R}^q\\ h^*_{1,\varepsilon}(x^*,y^*)-\psi^*_{X\times Y}(x^*,y^*)\le 0\\ h^*_{2,\varepsilon}(t^*,z^*)-f^*(t^*,z^*)\le 0}} \Big\{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q \ /\ h^*_{1,\varepsilon}(x^*,y^*)+\psi_{X\times Y}(x,y)$$

$$-\langle x^*,x\rangle-\langle y^*,y\rangle\le 0, h^*_{2,\varepsilon}(t^*,z^*)+\hat{f}(x,y)-\langle t^*,x\rangle-\langle z^*,y\rangle\le 0\Big\}.$$

For $\widetilde{x} \in \mathbb{R}^p, \lambda \in \mathrm{int}(\mathbb{R}^k_+), (\varepsilon, \mu) \in (\mathrm{int}(\mathbb{R}^m_+))^2$, set

$$\mathscr{B}^{\lambda,\widetilde{x}}_{\varepsilon,\mu} = \left\{(x,y) \in X \times Y / \hat{f}(x,y) + \hat{f}^*_Y(\widetilde{x}, 0_{\mathbb{R}^q}) - \langle \widetilde{x}, x\rangle \leq \mu^\top \varepsilon \right\}.$$

Proposition 4.1. *Let* $(\varepsilon,\mu) \in (int(\mathbb{R}^m_+))^2, \lambda \in int(\mathbb{R}^k_+)$. *Then* $\mathscr{B}^{\lambda}_{\varepsilon,\mu} = \bigcup\limits_{\widetilde{x}\in\mathbb{R}^p} \mathscr{B}^{\lambda,\widetilde{x}}_{\varepsilon,\mu}$.

Proof. The proof is obvious, and it is omitted here. □

4.2. **Duality for the decomposed problem** $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$**.** For $\widetilde{x} \in \mathbb{R}^p$, $(\varepsilon,\mu) \in (\mathrm{int}\mathbb{R}^m_+)^2$, and $\lambda \in \mathrm{int}\mathbb{R}^k_+$, consider the following problem

$$(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu}) \qquad \min_{(x,y)\in\beta^{\lambda,\widetilde{x}}_{\varepsilon,\mu}} \sum_{i=1}^k \lambda_i F_i(x,y),$$

and the following constraint qualification
$(CQ)^{\lambda,\widetilde{x}}_{\varepsilon,\mu}$ There exists $(x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}, y^{\lambda,\widetilde{x}}_{\varepsilon,\mu}) \in X \times Y$ such that $\hat{f}(x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}, y^{\lambda,\widetilde{x}}_{\varepsilon,\mu}) + \hat{f}^*_Y(\widetilde{x}, 0_{\mathbb{R}^q}) - \langle \widetilde{x}, x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}\rangle < \mu^\top\varepsilon$.

Remark 4.1.
1) For $\widetilde{x} \in \mathbb{R}^p$, $(\varepsilon,\mu) \in (\mathrm{int}(\mathbb{R}^m_+))^2$, and $\lambda \in \mathrm{int}(\mathbb{R}^k_+)$, $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$ is a convex minimization problem.
2) The qualification condition $(CQ)^{\lambda,\widetilde{x}}_{\varepsilon,\mu}$ says that the Slater condition is satisfied by the problem $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$. Thanks to our regularization, we can assume the possible satisfaction of the qualification constraint.

From Proposition 4.1, problem $(S^{\lambda}_{\varepsilon,\mu})$ can be written in the following form

$$(S^{\lambda}_{\varepsilon,\mu}) \qquad \min_{(x,y)\in \bigcup\limits_{\widetilde{x}\in\mathbb{R}^p} \mathscr{B}^{\lambda,\widetilde{x}}_{\varepsilon,\mu}} \sum_{i=1}^k \lambda_i F_i(x,y). \tag{4.1}$$

Then,

$$\begin{aligned}\inf S^{\lambda}_{\varepsilon,\mu} &= \inf\left\{\sum_{i=1}^k \lambda_i F_i(x,y) : (x,y) \in \bigcup_{\widetilde{x}\in\mathbb{R}^p} \mathscr{B}^{\lambda,\widetilde{x}}_{\varepsilon,\mu}\right\} \\ &= \inf_{\widetilde{x}\in\mathbb{R}^p} \inf\left\{\sum_{i=1}^k \lambda_i F_i(x,y) : (x,y) \in \mathscr{B}^{\lambda,\widetilde{x}}_{\varepsilon,\mu}\right\} = \inf_{\widetilde{x}\in\mathbb{R}^p} \inf S^{\lambda,\widetilde{x}}_{\varepsilon,\mu}.\end{aligned}$$

The formulation of $(S^{\lambda}_{\varepsilon,\mu})$ in (4.1) gives a decomposition of problem $(S^{\lambda}_{\varepsilon,\mu})$ to a family of convex minimization subproblems $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu}), \widetilde{x} \in \mathbb{R}^p$. We define the following function

$$g_{0,\varepsilon}(x,y) = \hat{f}(x,y) + \hat{f}^*_Y(\widetilde{x}, 0_{\mathbb{R}^q}) - \langle \widetilde{x}, x\rangle - \mu^\top\varepsilon.$$

Let $g_\varepsilon = (\psi_{X\times Y}, g_{0,\varepsilon})^\top$. Then, the first step to define a new duality for problem $(S^{\lambda}_{\varepsilon})$ is to consider the following dual for every problem $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$, called Fenchel-Lagrange dual ([14])

$$(\widehat{\mathscr{D}}^{\lambda,\widetilde{x}}_{\varepsilon,\mu}) \sup_{\substack{(p_1,p_2)\in\mathbb{R}^p\times\mathbb{R}^q \\ \beta=(\beta_0,\beta_1)\in\mathbb{R}^2_+}} \left\{-\left(\sum_{i=1}^k \lambda_i F_i\right)^*(p_1,p_2) - (\beta^\top g_\varepsilon)^*(-p_1,-p_2)\right\}.$$

Let us give an explicit expression of the objective function of problem $(\widehat{\mathscr{D}}^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$. We have

$$
\begin{aligned}
(\beta^\top g_\varepsilon)^*(-p_1,-p_2) &= \sup_{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q}\left\{\left\langle\begin{pmatrix}-p_1\\-p_2\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right\rangle-(\beta^\top g_\varepsilon)(x,y)\right\}\\
&= \sup_{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q}\left\{\left\langle\begin{pmatrix}-p_1\\-p_2\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right\rangle-(\beta_0\psi_{X\times Y}+\beta_1 g_{0,\varepsilon})(x,y)\right\}.
\end{aligned}
$$

For $(x,y)\in\mathbb{R}^p\times\mathbb{R}^q$, we have $\beta_0\psi_{X\times Y}(x,y)=\psi_{X\times Y}(x,y)$. Then

$$
\begin{aligned}
(\beta^\top g_\varepsilon)^*(-p_1,-p_2) &= \sup_{(x,y)\in\mathbb{R}^p\times\mathbb{R}^q}\left\{\left\langle\begin{pmatrix}-p_1\\-p_2\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right\rangle-(\psi_{X\times Y}+\beta_1 g_{0,\varepsilon})(x,y)\right\}\\
&= (\beta g_{0,\varepsilon})^*_{X\times Y}(-p_1,-p_2).
\end{aligned}
$$

Then, problem $(\widehat{\mathscr{D}}^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$ and the following problem

$$
(\mathscr{D}^{\lambda,\widetilde{x}}_{\varepsilon,\mu})\quad \sup_{\substack{(p_1,p_2)\in\mathbb{R}^p\times\mathbb{R}^q\\ \beta\in\mathbb{R}_+}}\left\{-\left(\sum_{i=1}^k\lambda_i F_i\right)^*(p_1,p_2)-(\beta g_{0,\varepsilon})^*_{X\times Y}(-p_1,-p_2)\right\},
$$

have the same supremum. Next, in our investigation, we use the above problem which can be developed by simple calculus based on conjugate functions to the following problem

$$
\begin{aligned}
(\mathscr{D}^{\lambda,\widetilde{x}}_{\varepsilon,\mu})\qquad &\sup_{\substack{(p_1,p_2)\in\mathbb{R}^p\times\mathbb{R}^q\\ \beta\in\mathbb{R}_+}}\Bigg\{-\left(\sum_{i=1}^k\lambda_i F_i\right)^*(p_1,p_2)+\\
&\qquad\inf_{(x,y)\in X\times Y}\left\langle\begin{pmatrix}p_1-\beta\widetilde{x}\\p_2\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right\rangle+\beta(\hat{f}^*_Y(\widetilde{x},0_{\mathbb{R}^q})-\mu^\top\varepsilon+\hat{f}(x,y))\Bigg\}.
\end{aligned}
$$

Because of $\bigcap_{i=1}^k ri(domF_i)\neq\emptyset$, we have from [6] that

$$
\left(\sum_{i=1}^k\lambda_i F_i\right)^*(\tilde{p}_1,\tilde{p}_2)=\inf\left\{\sum_{i=1}^k(\lambda_i F_i)^*(\tilde{p}_{1i},\tilde{p}_{2i}):\sum_{i=1}^k\tilde{p}_{1i}=\tilde{p}_1,\sum_{i=1}^k\tilde{p}_{2i}=\tilde{p}_2\right\}
$$

and the dual is

$$
\begin{aligned}
(\mathscr{D}^{\lambda,\widetilde{x}}_{\varepsilon,\mu})\qquad &\sup_{\substack{(\tilde{p}_1,\tilde{p}_2)\in\mathbb{R}^p\times\mathbb{R}^q\\ \beta\in\mathbb{R}_+\\ (\tilde{p}_{1i},\tilde{p}_{2i})\in\mathbb{R}^p\times\mathbb{R}^q\\ \Sigma_{i=1}^k\tilde{p}_{1i}=\tilde{p}_1\\ \Sigma_{i=1}^k\tilde{p}_{2i}=\tilde{p}_2}}\Bigg\{-\sum_{i=1}^k(\lambda_i F_i)^*(\tilde{p}_{1i},\tilde{p}_{2i})+\\
&\qquad\inf_{(x,y)\in X\times Y}\left\langle\begin{pmatrix}\tilde{p}_1-\beta\widetilde{x}\\\tilde{p}_2\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right\rangle+\beta(\hat{f}^*_Y(\widetilde{x},0_{\mathbb{R}^q})-\mu^\top\varepsilon+\hat{f}(x,y))\Bigg\}.
\end{aligned}
$$

Because of $(\lambda_i F_i)^*(\tilde{p}_{1i},\tilde{p}_{2i}) = \lambda_i F_i^*(\frac{\tilde{p}_{1i}}{\lambda_i},\frac{\tilde{p}_{2i}}{\lambda_i}),\ i=1,\dots,k$, we can make the substitution $\frac{\tilde{p}_{1i}}{\lambda_i} = p_{1i}, \frac{\tilde{p}_{2i}}{\lambda_i} = p_{2i},\ i=1,\dots,k$ and $\tilde{p}_1 = \sum_{i=1}^{k}\lambda_i p_{1i},\ \tilde{p}_2 = \sum_{i=1}^{k}\lambda_i p_{2i}$. Then, omitting $\tilde{p}_1$ and $\tilde{p}_2$, we obtain

$$(\mathscr{D}^{\lambda,\widetilde{x}}_{\varepsilon,\mu}) \qquad \sup_{\substack{(p_{1i},p_{2i})\in\mathbb{R}^p\times\mathbb{R}^q\\ i=1,\dots,k\\ \beta\in\mathbb{R}_+}} \Bigg\{ -\sum_{i=1}^{k}\lambda_i F_i^*(p_{1i},p_{2i}) + \inf_{(x,y)\in X\times Y}\Bigg\{ \Bigg\langle \begin{pmatrix}\sum_{i=1}^{k}\lambda_i p_{1i}-\beta\widetilde{x}\\ \sum_{i=1}^{k}\lambda_i p_{2i}\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\Bigg\rangle + \beta(\hat{f}_Y^*(\widetilde{x},0_{\mathbb{R}^q}) - \mu^\top\varepsilon) + \beta\hat{f}(x,y)\Bigg\}\Bigg\}.$$

Then, we have the following result concerning weak duality between $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$ and $(\mathscr{D}^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$.

Proposition 4.2. *Let $\widetilde{x}\in\mathbb{R}^p$, $(\varepsilon,\mu)\in(int(\mathbb{R}^m_+))^2$, and $\lambda\in int(\mathbb{R}^k_+)$. Then,* $\sup(\mathscr{D}^{\lambda,\widetilde{x}}_{\varepsilon,\mu}) \le \inf(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$.

Proof. The result uses the fact that $\sup(\widehat{\mathscr{D}}^{\lambda,\widetilde{x}}_{\varepsilon,\mu}) = \sup(\mathscr{D}^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$ and the known result of weak Fenchel-Lagrange duality between $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$, and $(\widehat{\mathscr{D}}^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$ ([14]). □

The following theorem establishes strong duality between $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$ and $(\mathscr{D}^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$ for a given $\widetilde{x}\in\mathbb{R}^p$, $\varepsilon\in\text{int}\mathbb{R}^m_+$, $\mu\in\text{int}\mathbb{R}^m_+$ and $\lambda\in\text{int}\mathbb{R}^k_+$.

Theorem 4.2. *Let $\widetilde{x}\in\mathbb{R}^p$, $\varepsilon\in int\mathbb{R}^m_+$, $\mu\in int\mathbb{R}^m_+$, and $\lambda\in int\mathbb{R}^k_+$. Assume that the constraint qualification $(CQ)^{\lambda,\widetilde{x}}_{\varepsilon,\mu}$ is satisfied. Then $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$ and $(\mathscr{D}^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$ are in strong Fenchel-Lagrange duality.*

Proof. The result follows from [6, Theorem 3.3] immediately. □

We have the following necessary optimality conditions for problem $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$, $\widetilde{x}\in\mathbb{R}^p$, $\varepsilon\in\text{int}\mathbb{R}^m_+$, $\mu\in\text{int}\mathbb{R}^m_+$, and $\lambda\in\text{int}(\mathbb{R}^k_+)$.

Theorem 4.3. *Let $\widetilde{x}\in\mathbb{R}^p$, $\varepsilon\in int\mathbb{R}^m_+$, $\mu\in int\mathbb{R}^m_+$, and $\lambda\in int(\mathbb{R}^k_+)$. Assume that the constraint qualification $(CQ)^{\lambda,\widetilde{x}}_{\varepsilon,\mu}$ is satisfied. Let $(x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}, y^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$ be a solution of problem $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$. Then, there exists a solution $(p_{1,\varepsilon},p_{2,\varepsilon},\beta_\varepsilon)$ of the dual $(\mathscr{D}^{\lambda,\widetilde{x}}_{\varepsilon,\mu})$ with $\beta_\varepsilon\in\mathbb{R}^+$, $p_{1,\varepsilon}=(p^\varepsilon_{11},\dots,p^\varepsilon_{1k})\in\mathbb{R}^p\times\mathbb{R}^p\times\dots\times\mathbb{R}^p$, $p_{2,\varepsilon}=(p^\varepsilon_{21},\dots,p^\varepsilon_{2k})\in\mathbb{R}^q\times\mathbb{R}^q\times\dots\times\mathbb{R}^q$, such that the following optimality conditions are satisfied.*

i) $F_i^*(p^\varepsilon_{1i},p^\varepsilon_{2i}) + F_i(x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}, y^{\lambda,\widetilde{x}}_{\varepsilon,\mu}) = \langle p^\varepsilon_{1i}, x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}\rangle + \langle p^\varepsilon_{2i}, y^{\lambda,\widetilde{x}}_{\varepsilon,\mu}\rangle \quad i=1,\dots,k,$

ii) $\beta_\varepsilon(\hat{f}(x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}, y^{\lambda,\widetilde{x}}_{\varepsilon,\mu}) + \hat{f}_Y^*(\widetilde{x},0_{\mathbb{R}^q}) - \langle\widetilde{x}, x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}\rangle - \mu^\top\varepsilon) = 0,$

iii) $$\inf_{(x,y)\in X\times Y}\Bigg\{ \Bigg\langle \begin{pmatrix}\sum_{i=1}^{k}\lambda_i p^\varepsilon_{1i}-\beta_\varepsilon\widetilde{x}\\ \sum_{i=1}^{k}\lambda_i p^\varepsilon_{2i}\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\Bigg\rangle + \beta_\varepsilon\hat{f}(x,y)\Bigg\} = \Bigg\langle \begin{pmatrix}\sum_{i=1}^{k}\lambda_i p^\varepsilon_{1i}-\beta_\varepsilon\widetilde{x}\\ \sum_{i=1}^{k}\lambda_i p^\varepsilon_{2i}\end{pmatrix}, \begin{pmatrix}x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}\\ y^{\lambda,\widetilde{x}}_{\varepsilon,\mu}\end{pmatrix}\Bigg\rangle + \beta_\varepsilon\hat{f}(x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}, y^{\lambda,\widetilde{x}}_{\varepsilon,\mu}).$$

Proof. The desired result direclty follows from [6, Theorem 3.4]. □

Remark 4.2. In term of subdifferential and normal cones, properties i) and iii) in Theorem 4.3 are respectively equivalent to

1) $\begin{pmatrix} p^{\varepsilon}_{1i} \\ p^{\varepsilon}_{2i} \end{pmatrix} \in \partial F_i(x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}, y^{\lambda,\widetilde{x}}_{\varepsilon,\mu}) \quad \forall i \in \{1,...,k\},$

2) $\begin{pmatrix} \beta_\varepsilon \widetilde{x} - \sum_{i=1}^{k} \lambda_i p^{\varepsilon}_{1i} \\ -\sum_{i=1}^{k} \lambda_i p^{\varepsilon}_{2i} \end{pmatrix} \in \partial(\beta_\varepsilon \hat{f})(x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}, y^{\lambda,\widetilde{x}}_{\varepsilon,\mu}) + \mathscr{N}_{X\times Y}(x^{\lambda,\widetilde{x}}_{\varepsilon,\mu}, y^{\lambda,\widetilde{x}}_{\varepsilon,\mu}).$

For $\varepsilon \in \text{int}\mathbb{R}^m_+$ and $\lambda \in \text{int}\mathbb{R}^k_+$, we use $\mathscr{I}^{\lambda}_{\varepsilon} = \left\{(\mu,\widetilde{x}) \in \text{int}\mathbb{R}^m_+ \times \mathbb{R}^p / \inf S^{\lambda}_{\varepsilon} = \inf S^{\lambda,\widetilde{x}}_{\varepsilon,\mu}\right\}$.

Remark 4.3. For every $\varepsilon \in \text{int}\mathbb{R}^m_+$ and $\lambda \in \text{int}\mathbb{R}^k_+$, $\mathscr{I}^{\lambda}_{\varepsilon}$ is nonempty. In fact, let $(x_\varepsilon, y_\varepsilon)$ be a solution to problem $(S^{\lambda}_{\varepsilon})$. Since, $y_\varepsilon \in \widehat{\mathscr{M}^{\varepsilon}}(x_\varepsilon)$ and $\widehat{\mathscr{M}^{\varepsilon}}(x_\varepsilon) = \bigcup_{\mu\in\text{int}\mathbb{R}^m_+} \widetilde{\mathscr{M}}^{\mu}_{\mu^\top\varepsilon}(x_\varepsilon)$, then there exists $\mu_{\varepsilon,x_\varepsilon} \in \text{int}\mathbb{R}^m_+$ such that $y_\varepsilon \in \widetilde{\mathscr{M}}^{\mu_{\varepsilon,x_\varepsilon}}_{\mu^\top_{\varepsilon,x_\varepsilon}\varepsilon}(x_\varepsilon)$. Hence, $(x_\varepsilon, y_\varepsilon)$ is a feasible point of $(S^{\lambda}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}})$, i.e., $(x_\varepsilon, y_\varepsilon) \in \mathscr{B}^{\lambda}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}}$.

On the other hand, $\mathscr{B}^{\lambda}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}} = \bigcup_{\widetilde{x}\in\mathbb{R}^p} \mathscr{B}^{\lambda,\widetilde{x}}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}}$. Hence, there exists $\widetilde{x}^{\lambda}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}} \in \mathbb{R}^p$ such that $(x_\varepsilon, y_\varepsilon) \in \mathscr{B}^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}}}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}}$. Therefore $(x_\varepsilon, y_\varepsilon)$ solves the problem

$$(S^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}}}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}}) \qquad \min_{(x,y)\in\mathscr{B}^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}}}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}}} \sum_{i=1}^{k} \lambda_i F_i(x,y).$$

It follows that $\sum_{i=1}^{k} \lambda_i F_i(x_\varepsilon, y_\varepsilon) = \inf S^{\lambda}_{\varepsilon} = \inf S^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}}}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}}$. Hence, $(\mu_{\varepsilon,x_\varepsilon}, \widetilde{x}^{\lambda}_{\varepsilon,\mu_{\varepsilon,x_\varepsilon}}) \in \mathscr{I}^{\lambda}_{\varepsilon}$.

4.3. **Optimality conditions for the scalarized-regularized problem.** The following theorem gives necessary optimality conditions for problem $(S^{\lambda}_{\varepsilon})$.

Theorem 4.4. *Let $\varepsilon \in int\mathbb{R}^m_+$. Assume that the following constraint qualification is satisfied*

$(CQ)_\varepsilon$ $\forall \widetilde{x} \in \mathbb{R}^p, \mu \in int\mathbb{R}^m_+, \exists (x^{\widetilde{x}}_{\varepsilon,\mu}, y^{\widetilde{x}}_{\varepsilon,\mu}) \in X \times Y,$ *s.t.* $\hat{f}(x^{\widetilde{x}}_{\varepsilon,\mu}, y^{\widetilde{x}}_{\varepsilon,\mu}) + \hat{f}^*_Y(\widetilde{x}, 0_{\mathbb{R}^q}) - \langle \widetilde{x}, x^{\widetilde{x}}_{\varepsilon,\mu}\rangle < \mu^\top\varepsilon.$

Let $\lambda \in int\mathbb{R}^k_+$ and $(x_\varepsilon, y_\varepsilon)$ be a solution to problem $(S^{\lambda}_{\varepsilon})$. Then, there exists $\widetilde{x}^{\lambda}_{\varepsilon} \in \mathbb{R}^p$ and $((p^{\lambda}_{1\varepsilon}, p^{\lambda}_{2\varepsilon}), \beta^{\lambda}_{\varepsilon})$ solves $(\mathscr{D}^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu_\varepsilon})$ with $\beta^{\lambda}_{\varepsilon} \in \mathbb{R}_+$ such that the optimality conditions $i) - iii)$ of Theorem 4.3 are satisfied.

Proof. Let $\lambda \in \text{int}\mathbb{R}^k_+$ and $\varepsilon \in \text{int}(\mathbb{R}^m_+)$. We have $(x_\varepsilon, y_\varepsilon)$, a solution to $(S^{\lambda}_{\varepsilon})$. From Remark 4.3, there exist $\widetilde{x}^{\lambda}_{\varepsilon} \in \mathbb{R}^p$ and $\mu^{\lambda}_{\varepsilon} \in \text{int}\mathbb{R}^m_+$ such that $(x_\varepsilon, y_\varepsilon)$ solves problem

$$(S^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}) \qquad \min_{(x,y)\in\mathscr{B}^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}} \sum_{i=1}^{k} \lambda_i F_i(x,y).$$

By using the qualification constraint $(CQ)_\varepsilon$ for $\widetilde{x}^{\lambda}_{\varepsilon}$ and $\mu^{\lambda}_{\varepsilon}$, we deduce that there exists $(x^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}, y^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}) \in X \times Y$ such that $\hat{f}(x^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu_\varepsilon}, y^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu_\varepsilon}) + \hat{f}^*_Y(\widetilde{x}^{\lambda}_{\varepsilon}, 0_{\mathbb{R}^q}) - \langle \widetilde{x}^{\lambda}_{\varepsilon}, x^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu}\rangle < \mu^{\lambda\,\top}_{\varepsilon}\varepsilon$. Then the constraint qualification $(CQ)^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu_\varepsilon}$ of Theorem 4.3 is satisfied by $(S^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}})$. Via this theorem, there exists $((p^{\lambda}_{1\varepsilon}, p^{\lambda}_{2\varepsilon}), \beta^{\lambda}_{\varepsilon}) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}_+$ solution of the dual $(\mathscr{D}^{\lambda,\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}})$ such that the following optimality conditions are satisfied

1) $F^*_i(p^{\lambda,\varepsilon}_{1i}, p^{\lambda,\varepsilon}_{2i}) + F_i(x^{\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}, y^{\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}) = \langle p^{\lambda,\varepsilon}_{1i}, x^{\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}\rangle + \langle p^{\lambda,\varepsilon}_{2i}, y^{\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}\rangle \quad i = 1,\ldots,k,$

2) $\beta^{\lambda}_{\varepsilon}\left(\hat{f}(x^{\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}, y^{\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}) + \hat{f}^*_Y(\widetilde{x}^{\lambda}_{\varepsilon}, 0_{\mathbb{R}^q}) - \langle \widetilde{x}^{\lambda}_{\varepsilon}, x^{\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}\rangle - {\mu^{\lambda}_{\varepsilon}}^{\top}\varepsilon\right) = 0,$

3) $$\inf_{(x,y)\in X\times Y}\left\{\left\langle \begin{pmatrix} \sum_{i=1}^k \lambda_i p^{\lambda,\varepsilon}_{1i} - \beta^{\lambda}_{\varepsilon}\widetilde{x}^{\lambda}_{\varepsilon} \\ \sum_{i=1}^k \lambda_i p^{\lambda,\varepsilon}_{2i}\end{pmatrix}, \begin{pmatrix} x \\ y\end{pmatrix}\right\rangle + \beta^{\lambda}_{\varepsilon}\hat{f}(x,y)\right\} =$$
$$\left\langle \begin{pmatrix} \sum_{i=1}^k \lambda_i p^{\lambda,\varepsilon}_{1i} - \beta^{\lambda}_{\varepsilon}\widetilde{x}^{\lambda}_{\varepsilon} \\ \sum_{i=1}^k \lambda_i p^{\lambda,\varepsilon}_{2i}\end{pmatrix}, \begin{pmatrix} x^{\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}} \\ y^{\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}\end{pmatrix}\right\rangle + \beta^{\lambda}_{\varepsilon}\hat{f}(x^{\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}, y^{\widetilde{x}^{\lambda}_{\varepsilon}}_{\varepsilon,\mu^{\lambda}_{\varepsilon}}).$$

□

Because of the lack of convexity of problem $(S^{\lambda}_{\varepsilon})$, we cannot apply the Fenchel-Lagrange duality to it. However, an extended Fenchel-Lagrange duality for $(S^{\lambda}_{\varepsilon})$ can be defined in the following sense.

Definition 4.1. Let $\varepsilon \in \text{int}(\mathbb{R}^m_+)$. Define the extented Fenchel-Lagrange duality for $(S^{\lambda}_{\varepsilon})$ relative to the redecomposition by the family of subproblems $\left\{(S^{\lambda,\widetilde{x}_{\varepsilon}}_{\varepsilon,\mu_{\varepsilon}}), \widetilde{x}_{\varepsilon} \in \mathbb{R}^p, \mu_{\varepsilon} \in \text{int}\mathbb{R}^m_+\right\}$ in the following sense:

1) We say that weak extended Fenchel-Lagrange duality holds for $(S^{\lambda}_{\varepsilon})$ if there exists $\widetilde{x}_{\varepsilon} \in \mathbb{R}^p$ and $\mu_{\varepsilon} \in int\mathbb{R}^m_+$ s.t. $\inf(S^{\lambda}_{\varepsilon}) \geq \sup(\mathscr{D}^{\lambda,\widetilde{x}_{\varepsilon}}_{\varepsilon,\mu_{\varepsilon}})$, i.e., there exists weak Fenchel-Lagrange duality between $(S^{\lambda}_{\varepsilon})$ and $(\mathscr{D}^{\lambda,\widetilde{x}_{\varepsilon}}_{\varepsilon,\mu_{\varepsilon}})$ for some $\widetilde{x}_{\varepsilon} \in \mathbb{R}^p, \mu_{\varepsilon} \in \text{int}\mathbb{R}^m_+$.
2) We say that strong extended Fenchel-Lagrange duality holds for $(S^{\lambda}_{\varepsilon})$ if there exists $\widetilde{x}_{\varepsilon} \in \mathbb{R}^p, \mu_{\varepsilon} \in \text{int}\mathbb{R}^m_+$ s.t. $\inf(S^{\lambda}_{\varepsilon}) = \sup(\mathscr{D}^{\lambda,\widetilde{x}_{\varepsilon}}_{\varepsilon,\mu_{\varepsilon}})$, i.e., there exists strong Fenchel-Lagrange duality between $(S^{\lambda}_{\varepsilon})$ and $(\mathscr{D}^{\lambda,\widetilde{x}_{\varepsilon}}_{\varepsilon,\mu_{\varepsilon}})$ for some $\widetilde{x}_{\varepsilon} \in \mathbb{R}^p, \mu_{\varepsilon} \in int\mathbb{R}^m_+$.

Remark 4.4. The extended Fenchel-Lagrange duality was first defined by Aboussoror, Adly and Saissi in [2] in order to lead to strong duality between the regularized problem and its decomposed one.

Let $\varepsilon \in \text{int}(\mathbb{R}^m_+)$. The following theorem gives sufficient optimality conditions for the scalarized-regularized problem and show that strong extended Fenchel-Lagrange duality holds for $(S^{\lambda}_{\varepsilon}), \lambda \in \text{int}(\mathbb{R}^k_+)$.

Theorem 4.5. *Let $\varepsilon \in int\mathbb{R}^m_+, \lambda \in int(\mathbb{R}^k_+)$ and $(x_{\varepsilon}, y_{\varepsilon})$ be a feasible point of problem $(S^{\lambda}_{\varepsilon})$. Assume that there exists $(\mu^{\lambda}_{\varepsilon}, \widetilde{x}^{\lambda}_{\varepsilon}) \in \mathscr{I}^{\lambda}_{\varepsilon}$ and $(p^{\lambda}_{1\varepsilon}, p^{\lambda}_{2\varepsilon}, \beta^{\lambda}_{\varepsilon})$ a feasible point of the dual $(\mathscr{D}^{\lambda,\widetilde{x}_{\varepsilon}}_{\varepsilon,\mu_{\varepsilon}})$ with $\beta^{\lambda}_{\varepsilon} \in \mathbb{R}_+$ that satisfies together with $(x_{\varepsilon}, y_{\varepsilon})$ the conditions i)-iii) in Theorem 4.3. Then, $(x_{\varepsilon}, y_{\varepsilon})$ and $((p^{\lambda}_{1\varepsilon}, p^{\lambda}_{2\varepsilon}), \beta^{\lambda}_{\varepsilon})$ solve $(S^{\lambda}_{\varepsilon})$ and $(\mathscr{D}^{\lambda,\widetilde{x}_{\varepsilon}}_{\varepsilon,\mu_{\varepsilon}})$ respectively. Moreover, strong extended Fenchel-Lagrange duality holds for $(S^{\lambda}_{\varepsilon})$.*

Proof. Properties $i) - ii)$ in Theorem 4.3 corresponding to our case are written as follows

1) $F^*_i(p^{\lambda,\varepsilon}_{1i}, p^{\lambda,\varepsilon}_{2i}) + F_i(x_{\varepsilon}, y_{\varepsilon}) = \langle p^{\lambda,\varepsilon}_{1i}, x_{\varepsilon}\rangle + \langle p^{\lambda,\varepsilon}_{2i}, y_{\varepsilon}\rangle \quad i = 1,\ldots,k,$

2) $\beta_\varepsilon^\lambda\left(\hat{f}(x_\varepsilon,y_\varepsilon)+\hat{f}_Y^*(\widetilde{x}_\varepsilon^\lambda,0_{\mathbb{R}^q})-\langle\widetilde{x}_\varepsilon^\lambda,x_\varepsilon\rangle-\mu_\varepsilon^{\lambda\top}\varepsilon\right)=0,$

3) $\inf_{(x,y)\in X\times Y}\left\{\left\langle\begin{pmatrix}\sum_{i=1}^k\lambda_i p_{1i}^{\lambda,\varepsilon}-\beta_\varepsilon^\lambda\widetilde{x}_\varepsilon^\lambda\\ \sum_{i=1}^k\lambda_i p_{2i}^{\lambda,\varepsilon}\end{pmatrix},\begin{pmatrix}x\\ y\end{pmatrix}\right\rangle+\beta_\varepsilon^\lambda\hat{f}(x,y)\right\}=$
$\left\langle\begin{pmatrix}\sum_{i=1}^k\lambda_i p_{1i}^{\lambda,\varepsilon}-\beta_\varepsilon\widetilde{x}_\varepsilon^\lambda\\ \sum_{i=1}^k\lambda_i p_{2i}^{\lambda,\varepsilon}\end{pmatrix},\begin{pmatrix}x_\varepsilon\\ y_\varepsilon\end{pmatrix}\right\rangle+\beta_\varepsilon^\lambda\hat{f}(x_\varepsilon,y_\varepsilon).$

From i), we have

$$\sum_{i=1}^k\lambda_iF_i^*(p_{1i}^{\lambda,\varepsilon},p_{2i}^{\lambda,\varepsilon})+\sum_{i=1}^k\lambda_iF_i(x_\varepsilon,y_\varepsilon)-\left\langle\begin{pmatrix}\sum_{i=1}^k\lambda_i p_{1i}^{\lambda,\varepsilon}\\ \sum_{i=1}^k\lambda_i p_{2i}^{\lambda,\varepsilon}\end{pmatrix},\begin{pmatrix}x_\varepsilon\\ y_\varepsilon\end{pmatrix}\right\rangle=0. \tag{4.2}$$

Summing $i),iii)$, and (4.2), we obtain

$$\sum_{i=1}^k\lambda_iF_i(x_\varepsilon,y_\varepsilon)=\inf_{(x,y)\in X\times Y}\left\{\left\langle\begin{pmatrix}\sum_{i=1}^k\lambda_i p_{1i}^{\lambda,\varepsilon}-\beta_\varepsilon\widetilde{x}_\varepsilon^\lambda\\ \sum_{i=1}^k\lambda_i p_{2i}^{\lambda,\varepsilon}\end{pmatrix},\begin{pmatrix}x\\ y\end{pmatrix}\right\rangle+\beta_\varepsilon^\lambda\hat{f}(x,y)\right.$$
$$\left.-\sum_{i=1}^k\lambda_iF_i^*(p_{1i}^{\lambda,\varepsilon},p_{2i}^{\lambda,\varepsilon})\right\}+\beta_\varepsilon^\lambda\left(\hat{f}_Y^*(\widetilde{x}_\varepsilon^\lambda,0_{\mathbb{R}^q})-\mu_\varepsilon^{\lambda\top}\varepsilon\right). \tag{4.3}$$

From (4.3), the Fenchel-Lagrange duality between $(S_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\widetilde{x}_\varepsilon^\lambda})$ and $(\mathscr{D}_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\widetilde{x}_\varepsilon^\lambda})$, we respectively have $\sum_{i=1}^k\lambda_iF_i(x_\varepsilon,y_\varepsilon)\le\sup(\mathscr{D}_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\widetilde{x}_\varepsilon^\lambda})$ and $\sup(\mathscr{D}_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\widetilde{x}_\varepsilon^\lambda})\le\inf(S_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\widetilde{x}_\varepsilon^\lambda})$.

On the other hand, since $(\widetilde{x}_\varepsilon^\lambda,\mu_\varepsilon^\lambda)\in\mathscr{I}_\varepsilon^\lambda$, then $\inf S_\varepsilon^\lambda=\inf S_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\widetilde{x}_\varepsilon^\lambda}$. Therefore,

$$\sup(\mathscr{D}_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\widetilde{x}_\varepsilon^\lambda})\le\inf(S_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\widetilde{x}_\varepsilon^\lambda})\le\left(\sum_{i=1}^k\lambda_iF_i\right)(x_\varepsilon,y_\varepsilon)\le\sup(\mathscr{D}_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\widetilde{x}_\varepsilon^\lambda}).$$

Since $\sum_{i=1}^k\lambda_iF_i(x_\varepsilon,y_\varepsilon)=\inf\limits_{(x,y)\in X\times Y}\left\{\left\langle\begin{pmatrix}\sum_{i=1}^k\lambda_i p_{1i}^{\lambda,\varepsilon}-\beta_\varepsilon\widetilde{x}_\varepsilon^\lambda\\ \sum_{i=1}^k\lambda_i p_{2i}^{\lambda,\varepsilon}\end{pmatrix},\begin{pmatrix}x\\ y\end{pmatrix}\right\rangle+\beta_\varepsilon^\lambda\hat{f}(x,y)\right.$

$$\left.-\sum_{i=1}^k\lambda_iF_i^*(p_{1i}^{\lambda,\varepsilon},p_{2i}^{\lambda,\varepsilon})\right\}+\beta_\varepsilon^\lambda\left(\hat{f}_Y^*(\widetilde{x}_\varepsilon^\lambda,0_{\mathbb{R}^q})-\mu_\varepsilon^{\lambda\top}\varepsilon\right),$$

we have

a) $\inf(S_\varepsilon^\lambda)=\left(\sum_{i=1}^k\lambda_iF_i\right)(x_\varepsilon,y_\varepsilon),$

b) $\sup(\mathscr{D}_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\widetilde{x}_\varepsilon^\lambda})=\inf_{(x,y)\in X\times Y}\left\{\left\langle\begin{pmatrix}\sum_{i=1}^k\lambda_i p_{1i}^{\lambda,\varepsilon}-\beta_\varepsilon\widetilde{x}_\varepsilon^\lambda\\ \sum_{i=1}^k\lambda_i p_{2i}^{\lambda,\varepsilon}\end{pmatrix},\begin{pmatrix}x\\ y\end{pmatrix}\right\rangle+\beta_\varepsilon^\lambda\hat{f}(x,y)-\right.$
$\left.\sum_{i=1}^k\lambda_iF_i^*(p_{1i}^{\lambda,\varepsilon},p_{2i}^{\lambda,\varepsilon})\right\}+\beta_\varepsilon^\lambda\left(\hat{f}_Y^*(\widetilde{x}_\varepsilon^\lambda,0_{\mathbb{R}^q})-\mu_\varepsilon^{\lambda\top}\varepsilon\right),$

c) $\inf(S_\varepsilon^\lambda)=\sup(\mathscr{D}_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\widetilde{x}_\varepsilon^\varepsilon}).$

Therefore, $(x_\varepsilon, y_\varepsilon)$ is a solution to (S_ε^λ) and $((p_{1\varepsilon}^\lambda, p_{2\varepsilon}^\lambda), \beta_\varepsilon^\lambda)$ is a solution to $(\mathscr{D}_{\varepsilon,\mu_\varepsilon^\lambda}^{\lambda,\tilde{x}_\varepsilon^\lambda})$. Moreover, strong extended Fenchel-Lagrange duality holds for (S_ε^λ). □

5. Optimality Conditions for Problem (S)

In this section, we provide necessary and sufficient optimality conditions for problem (S). We need the following additional assumptions:

$(\mathscr{H}_1)$ For every $\varepsilon \in int(\mathbb{R}_+^m)$, there exists $(x_\varepsilon, y_\varepsilon) \in intX \times intY$ such that

$$f_i(x_\varepsilon, y_\varepsilon) \leq \inf_{y \in Y} f_i(x_\varepsilon, y) + \varepsilon_i, \forall i \in \{1, ..., m\}.$$

$(\mathscr{H}_2)$ $\exists (\tilde{x}, \tilde{y}) \in \mathbb{R}^p \times \mathbb{R}^q$ s.t. $F_j(\tilde{x}, \tilde{y}) < F_j(x, y)$, $\forall (x, y) \in X \times Y, \forall j \in \{1, ..., k\}$.

$(\mathscr{H}_3)$ $\exists \tilde{y} \in \mathbb{R}^q$ s.t. $f_i(x, \tilde{y}) < f_i(x, y)$, $\forall x \in X, \forall i \in \{1, ..., m\}$.

For $x \in \mathbb{R}^p$, we define the function $f_{i,x}(.)$ on $\mathbb{R}^q$ by $f_{i,x}(y) = f_i(x, y)$.

Remark 5.1. 1) From assumption $(\mathscr{H}_1)$, we have $f_i(x_\varepsilon, y_\varepsilon) \leq \inf_{y \in Y} f_i(x_\varepsilon, y) + \varepsilon_i$ for all $i \in \{1, ..., m\}$.

Then, for all $\mu_i > 0, i = 1, ..., m$ we have $\sum_{i=1}^m \mu_i f_i(x_\varepsilon, y_\varepsilon) \leq \inf_{y \in Y} \sum_{i=1}^m \mu_i f_i(x_\varepsilon, y) + \mu^\top \varepsilon$. Hence, $y_\varepsilon \in \widetilde{\mathscr{M}}_{\mu^\top \varepsilon}^{\mu}(x_\varepsilon)$, i.e., $(x_\varepsilon, y_\varepsilon)$ is a feasible point of $(S_{\varepsilon,\mu}^\lambda)$.

2) Assumptions $(\mathscr{H}_2)$ and $(\mathscr{H}_3)$ imply respectively that

i) $\begin{pmatrix} 0_{\mathbb{R}^p} \\ 0_{\mathbb{R}^q} \end{pmatrix} \notin \partial F_i(x, y)$, $\forall (x, y) \in X \times Y$, $\forall i \in \{1, ..., k\}$,

ii) $\forall x \in X$, $\quad 0_{\mathbb{R}^q} \notin \partial f_{i_x}(y)$, $\forall i \in \{1, ..., m\}$, $\forall y \in Y$.

Example 5.1. Let $X = [0, 1]$ and $Y = [0, 2]$, and let $F_i, f_i, i = 1, 2, 3$ be the functions defined on $\mathbb{R} \times \mathbb{R}$ by

$$\begin{cases} F_1(x, y) = x^2 + y, \\ F_2(x, y) = y, \\ F_3(x, y) = 2x^2 + 2y, \end{cases} \quad \text{and} \quad \begin{cases} f_1(x, y) = x + y, \\ f_2(x, y) = -x + y, \\ f_3(x, y) = 2x + y. \end{cases}$$

Then, X and Y are compact convex sets and F and f are convex functions.

$(\mathscr{H}_1)$ Let $x_\varepsilon = \dfrac{1}{2}$ and $y_\varepsilon = \tilde{\varepsilon}$ such that $0 < \tilde{\varepsilon} < \varepsilon_i, i = 1, 2, 3$. Then, $x_\varepsilon \in intX$ and $y_\varepsilon \in intY$. Moreover, we have $f_i(x_\varepsilon, y_\varepsilon) \leq \inf_{y \in Y} f_i(x_\varepsilon, y) + \varepsilon_i$, $\forall i \in \{1, 2, 3\}$. Then assumption $(\mathscr{H}_1)$ is satisfied.

$(\mathscr{H}_2)$ Let $(\bar{x}, \bar{y}) = (0, -2)$. Then, $\begin{cases} -2 = F_1(\bar{x}, \bar{y}) < F_1(x, y), \\ -2 = F_2(\bar{x}, \bar{y}) < F_2(x, y), \\ -4 = F_3(\bar{x}, \bar{y}) < F_3(x, y), \end{cases}$ for all $(x, y) \in X \times Y$. Then assumption $(\mathscr{H}_2)$ is satisfied.

$(\mathscr{H}_3)$ Let $\tilde{y} = -2 \in \mathbb{R}^q$. Then, $\begin{cases} f_1(x, \tilde{y}) = x - 2 < x + y, \\ f_2(x, \tilde{y}) = -x - 2 < -x + y, \\ f_3(x, \tilde{y}) = 2x - 2 < 2x + y, \end{cases}$ for all $(x, y) \in X \times Y$. Then assumption $(\mathscr{H}_3)$ is satisfied.

5.1. **Necessary optimality conditions for problem** (S)**.** The following theorem gives necessary optimality conditions for the properly efficient solutions of problem (S) which are accumulation points of a sequence of scalarized-regularized solutions.

Theorem 5.1. *Let $\varepsilon_n \searrow 0^+_{\mathbb{R}^m}$. Let assumptions $(\mathscr{H}_1)-(\mathscr{H}_3)$ be satisfied. Let $\lambda \in int(\mathbb{R}^k_+)$ and (x_n, y_n) be a feasible point of (S^λ_n) given by assumption $(\mathscr{H}_1)$ for ε_n. Assume that there exists $(\mu_{\varepsilon_n}, \widetilde{x}_n) \in \mathscr{I}^\lambda_{\varepsilon_n}$ and a feasible point $(p_{1,n}, p_{2,n}, \beta_n) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}$ of the dual problem $(\mathscr{D}^{\lambda,\widetilde{x}^\lambda_n}_{\varepsilon_n,\mu^\lambda_{\varepsilon_n}})$ of $(S^{\lambda,\widetilde{x}^\lambda_n}_{\varepsilon_n,\mu^\lambda_{\varepsilon_n}})$ that satisfy together with (x_n, y_n) the conditions (i)-(iii) in Theorem 4.3. Let $(\bar{x}, \bar{y})$ be an accumulation point of the sequence (x_n, y_n) . Then, $(\bar{x}, \bar{y})$ is a properly efficient solution of problem (S) and there exist $((\bar{p}_1, \bar{p}_2), \bar{\beta})$ with $\bar{p}_1 = (\bar{p}_{11}, ..., \bar{p}_{1k}) \in \mathbb{R}^p \times ... \times \mathbb{R}^p$, $\bar{p}_2 = (\bar{p}_{21}, ..., \bar{p}_{2k}) \in \mathbb{R}^q \times ... \times \mathbb{R}^q$, $\bar{\beta} \in \mathbb{R}^*_+, \bar{\mu} \in int(\mathbb{R}^m_+)$ and $\widetilde{x} \in \mathbb{R}^p$ s.t.*

$$i)\ \begin{pmatrix} \bar{p}_{1i} \\ \bar{p}_{2i} \end{pmatrix} \in \partial F_i(\bar{x}, \bar{y}),\ \forall (x,y) \in X \times Y,\ \forall i \in \{1, ..., k\},$$

$$ii)\ \left(\sum_{j=1}^m \bar{\mu}_j f_j\right)^* (\widetilde{x}, 0_{\mathbb{R}^q}) + \left(\sum_{j=1}^m \bar{\mu}_j f_j\right)(\bar{x}, \bar{y}) = \langle \widetilde{x}, \bar{x} \rangle,$$

$$\text{iii)}\quad \begin{pmatrix} \widetilde{x} - \dfrac{\sum_{i=1}^k \lambda_i p_{1i}}{\bar{\beta}} \\ -\dfrac{\sum_{i=1}^k \lambda_i p_{2i}}{\bar{\beta}} \end{pmatrix} \in \partial \left(\textstyle\sum_{j=1}^m \bar{\mu}_j f_j\right)(\bar{x}, \bar{y}).$$

Proof. first of all, let us show that $\beta_n > 0$ for large $n \in \mathbb{N}$. Assume that there exists an infinite subset $\mathscr{N}^*$ of $\mathbb{N}$ such that $\beta_n = 0$ for all $n \in \mathscr{N}^*$. Let $n \in \mathscr{N}^*$. In our case, properties $i) - iii)$ in Theorem 4.3 are written as follows:

$$\text{a)}\ F_i^*(p^n_{1i}, p^n_{2i}) + F_i(x_n, y_n) = \langle p^n_{1i}, x_n \rangle + \langle p^n_{2i}, y_n \rangle,\ \forall i \in \{1, ..., k\},$$

$$\text{b)}\ \beta_n(\hat{f}(x_n, y_n) + \hat{f}^*_Y(\widetilde{x}_n, 0_{\mathbb{R}^q}) - \langle \widetilde{x}_n, x_n \rangle - \mu_n^\top \varepsilon_n) = 0,$$

$$\text{c)}\ \inf_{(x,y)\in X\times Y} \left\{ \left\langle \begin{pmatrix} \sum_{i=1}^k \lambda_i p^n_{1i} - \beta_n \widetilde{x}_n \\ \sum_{i=1}^k \lambda_i p^n_{2i} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle + \beta_n \hat{f}(x,y) \right\} = \sum_{i=1}^k \lambda_i \left\langle \begin{pmatrix} p^n_{1i} - \beta_n \widetilde{x}_n \\ p^n_{2i} \end{pmatrix}, \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\rangle + \beta_n \hat{f}(x_n, y_n).$$

From Remark 4.2, the property (c) is written as

$$\begin{pmatrix} \beta_n \widetilde{x} - \sum_{i=1}^k \lambda_i p^n_{1i} \\ -\sum_{i=1}^k \lambda_i p^n_{2i} \end{pmatrix} \in \partial(\beta_n \hat{f})(x_n, y_n) + \mathscr{N}_{X\times Y}(x_n, y_n). \tag{5.1}$$

From assumption $(\mathscr{H}_1)$, we have $(x_n, y_n) \in \mathrm{int}(X \times Y)$, which implies that $\mathscr{N}_{X\times Y}(x_n, y_n) = \begin{pmatrix} 0_{\mathbb{R}^p} \\ 0_{\mathbb{R}^q} \end{pmatrix}$. Hence property (5.1) becomes $\begin{pmatrix} \beta_n \widetilde{x}_n - \sum_{i=1}^{k} \lambda_i p_{1i}^n \\ -\sum_{i=1}^{k} \lambda_i p_{2i}^n \end{pmatrix} \in (\beta_n \hat{f})(x_n, y_n)$. That is,

$$\beta_n \hat{f}(x,y) \geq \beta_n \hat{f}(x_n, y_n) + \left\langle \begin{pmatrix} \beta_n \widetilde{x}_n - \sum_{i=1}^{k} \lambda_i p_{1i}^n \\ -\sum_{i=1}^{k} \lambda_i p_{2i}^n \end{pmatrix}, \begin{pmatrix} x - x_n \\ y - y_n \end{pmatrix} \right\rangle, \quad \forall (x,y) \in \mathbb{R}^p \times \mathbb{R}^q.$$

Since $\beta_n = 0$, then $\left\langle \begin{pmatrix} -\sum_{i=1}^{k} \lambda_i p_{1i}^n \\ -\sum_{i=1}^{k} \lambda_i p_{2i}^n \end{pmatrix}, \begin{pmatrix} x - x_n \\ y - y_n \end{pmatrix} \right\rangle \leq 0,\ (x,y) \in \mathbb{R}^p \times \mathbb{R}^q$. Hence,

$$\left\langle \begin{pmatrix} -\sum_{i=1}^{k} \lambda_i p_{1i}^n \\ -\sum_{i=1}^{k} \lambda_i p_{2i}^n \end{pmatrix}, \begin{pmatrix} x - x_n \\ y - y_n \end{pmatrix} \right\rangle \leq 0,\ (x,y) \in X \times Y.$$

Then, $\begin{pmatrix} -\sum_{i=1}^{k} \lambda_i p_{1i}^n \\ -\sum_{i=1}^{k} \lambda_i p_{2i}^n \end{pmatrix} \in \mathscr{N}_{X\times Y}(x_n, y_n) = \begin{pmatrix} 0_{\mathbb{R}^p} \\ 0_{\mathbb{R}^q} \end{pmatrix}$. Hence, $-\sum_{i=1}^{k} \lambda_i p_{1i}^n = 0$ and $-\sum_{i=1}^{k} \lambda_i p_{2i}^n = 0$. Since $\lambda \in \mathrm{int}(\mathbb{R}_+^k)$, then $\lambda_i > 0,\ i \in \{1, ..., k\}$. Hence, $p_{1i}^n = 0$ and $p_{2i}^n = 0,\ i \in \{1, ..., k\}$. On the other hand, we have from (a) that $\begin{pmatrix} p_{1i}^n \\ p_{2i}^n \end{pmatrix} \in \partial F_i(x_n, y_n),\ i \in \{1, ..., k\}$, i.e., $\begin{pmatrix} 0_{\mathbb{R}^p} \\ 0_{\mathbb{R}^q} \end{pmatrix} \in \partial F_i(x_n, y_n), i \in \{1, ..., k\}$. From assumption $(\mathscr{H}_2)$, we have $\begin{pmatrix} 0_{\mathbb{R}^p} \\ 0_{\mathbb{R}^q} \end{pmatrix} \notin \partial F_i(x_n, y_n),\ i \in \{1, ..., k\}$, which gives a contradiction. Hence $\beta_n > 0$ for large $n \in \mathbb{N}$. i.e., $\exists n_0 \in \mathbb{N}$, $n \geq n_0$, $\beta_n > 0$. Now, let us show that the accumulation point $(\bar{x}, \bar{y})$ is a properly efficient solution of problem (S). Let $\mathscr{N}$ be an infinite subset of $\mathbb{N}$ such that $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to +\infty$, $n \geq n_0$. Then, Theorem 4.5 implies that (x_n, y_n) solves problem (S_n^λ) and $((p_{1n}, p_{2n}), \beta_n)$ solves $(\mathscr{D}_{\varepsilon_n, \mu_n}^{\lambda, x_n^*})$. It follows from Theorem 3.1 that the accumulation point $(\bar{x}, \bar{y})$ is a properly efficient solution of the original bilevel problem (S). In order to show properties $i) - iii)$. we set $\mathscr{N}_1 = \mathscr{N} \cap \{n \in \mathbb{N}\ /\ n \geq n_0\}$.

Property $i)$: For $n \in \mathscr{N}_1$, we have

$$\begin{pmatrix} p_{1i}^n \\ p_{2i}^n \end{pmatrix} \in \partial F_i(x_n, y_n) \subset \bigcup_{(x,y) \in X \times Y} \partial F_i(x,y) \qquad \forall i \in \{1, ..., k\}.$$

Since $X \times Y \subset \mathrm{int}(\mathrm{dom} F) = \mathbb{R}^p \times \mathbb{R}^p$ and $X \times Y$ is compact, then $\bigcup_{(x,y) \in X \times Y} \partial F_i(x,y)$ is compact (Theorem 2.1). Hence, there exists an infinite subset $\mathscr{N}_2$ of $\mathscr{N}_1$ such that the sequence

$(p^n_{1i}, p^n_{2i})_{n\in\mathcal{N}_2}$ converges to $(\bar{p}_{1i}, \bar{p}_{2i})$. On the other hand, we have

$$F_i(x,y) \geq F_i(x_n,y_n) + \left\langle \begin{pmatrix} p^n_{1i} \\ p^n_{2i} \end{pmatrix}, \begin{pmatrix} x-x_n \\ y-y_n \end{pmatrix} \right\rangle,\ \forall (x,y) \in \mathbb{R}^p \times \mathbb{R}^q. \tag{5.2}$$

Passing to the limit in (5.2) as $n \to +\infty$, we deduce that $\begin{pmatrix} \bar{p}_{1i} \\ \bar{p}_{2i} \end{pmatrix} \in\ \partial F_i(\bar{x},\bar{y}),\ i \in \{1,...,k\}$. Hence, property $i)$ is satisfied.

Property $ii)$: Let $n \in \mathcal{N}_2$. Since $\beta_n > 0$, then (b) becomes $\hat{f}^*_Y(\widetilde{x}_n,,0_{\mathbb{R}^q}) + \hat{f}(x_n,y_n) - \langle \widetilde{x}_n, x_n\rangle - \mu_n^\top \varepsilon_n = 0$. Hence,

$$\begin{aligned} \hat{f}^*_Y(\widetilde{x}_n, 0_{\mathbb{R}^q}) &= \langle \widetilde{x}_n, x_n\rangle - \hat{f}(x_n,y_n) + \mu_n^\top \varepsilon_n \\ &= \sup_{(x,y)\in\mathbb{R}^p\times Y} \left\{ \left\langle \begin{pmatrix} x^*_n \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle - \hat{f}(x,y) \right\}. \end{aligned}$$

Therefore, for all $(x,y) \in \mathbb{R}^p \times Y$

$$\hat{f}(x,y) \geq \hat{f}(x_n,y_n) + \langle \widetilde{x}_n, x - x_n\rangle - \mu_n^\top \varepsilon_n. \tag{5.3}$$

Then, for all $x \in \mathbb{R}^p$, we have $\inf_{y\in Y} \hat{f}(x,y) = v_{\mu_n}(x) \geq v_{\mu_n}(x_n) + \langle \widetilde{x}_n, x - x_n\rangle - \mu_n^\top \varepsilon_n$, i.e., $\widetilde{x}_n \in \partial_{\mu_n^\top \varepsilon_n} v_{\mu_n}(x_n)$. Note $\tilde{\varepsilon}_n = \mu_n^\top \varepsilon_n$. Hence $\widetilde{x}_n \in \partial_{\tilde{\varepsilon}_n} v_{\mu_n}(x_n)$. Let $\varepsilon^* \in \mathrm{int}(\mathbb{R}^+_k)$. Since $\varepsilon_n \searrow 0^+$, then $\tilde{\varepsilon}_n \searrow 0^+$, $n \in \mathcal{N}_2$. On the other hand, we have $\varepsilon^*_i > 0, \forall i \in \{1,...,k\}$. Then, there exists $n_1 \in \mathcal{N}_2$ such that $\tilde{\varepsilon}_{ni} < \varepsilon^*_i$, $n \geq n_1, n \in \mathcal{N}_2, i = 1,...,k$. Hence $\partial_{\tilde{\varepsilon}_n} v_{\mu_n}(x_n) \subset \partial_{\varepsilon^*} v_{\mu_n}(x_n)$, $n \geq n_1, n \in \mathcal{N}_2$. Since $\widetilde{x}_n \in \partial_{\tilde{\varepsilon}_n} v_{\mu_n}(x_n) \subset \bigcup\limits_{x\in X} \partial_{\varepsilon^*} v_{\mu_n}(x_n)$, $n \geq n_1, n \in \mathcal{N}_2$ which is compact, then, there exists an infinite subset $\mathcal{N}_3$ of $\mathcal{N}_2$ such that $\widetilde{x}_n \to \widetilde{x}$, as $n \to \infty, n \in \mathcal{N}_3$. On the other hand, we have $\mu^n_j > 0, j = 1,...,m$ and $\sum_{j=1}^m \mu^n_j = 1$. Hence $\mu^n_j \in [0,1]$ compact. Then, there exists $\mathcal{N}_4 \subset \mathcal{N}_3$ such that $\mu^n_j \to \bar{\mu}_j$, $n \to +\infty, n \in \mathcal{N}_4$. From (5.3), we have $\sum_{j=1}^m \mu^n_j f_j(x,y) \geq \sum_{j=1}^m \mu^n_j f_j(x_n,y_n) + \langle \widetilde{x}_n, x - x_n\rangle - \mu_n^\top \varepsilon_n$. Passing to the limit as $n \to +\infty, n \in \mathcal{N}_4$, we obtain

$$\sum_{j=1}^m \bar{\mu}_j f_j(x,y) \geq \sum_{j=1}^m \bar{\mu}_j f_j(\bar{x},\bar{y}) + \langle \widetilde{x}, x - \bar{x}\rangle, \quad \forall (x,y) \in \mathbb{R}^p \times Y.$$

Then,

$$\left\langle \begin{pmatrix} \widetilde{x} \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle - \sum_{j=1}^m \bar{\mu}_j f_j(x,y) \leq \left\langle \begin{pmatrix} \widetilde{x} \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right\rangle - \textstyle\sum_{j=1}^m \bar{\mu}_j f_j(\bar{x},\bar{y}) \quad \forall (x,y) \in \mathbb{R}^p \times Y.$$

Then, for all $(x,y) \in \mathbb{R}^p \times Y$,

$$\sup_{(x,y)\in\mathbb{R}^p\times Y} \left\{ \left\langle \begin{pmatrix} \widetilde{x} \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle - \sum_{j=1}^m \bar{\mu}_j f_j(x,y) \right\} = \left\langle \begin{pmatrix} \widetilde{x} \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right\rangle - \textstyle\sum_{j=1}^m \bar{\mu}_j f_j(\bar{x},\bar{y}).$$

Hence,

$$\left(\sum_{j=1}^m \bar{\mu}_j f_j \right)^*_Y (\widetilde{x}, 0_{\mathbb{R}^q}) = \left\langle \begin{pmatrix} \widetilde{x} \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \right\rangle - \sum_{j=1}^m \bar{\mu}_j f_j(\bar{x},\bar{y}).$$

That is, Property $ii)$ is satisfied.

Property $iii)$: Let $n \in \mathcal{N}_4$. From property (c) and the fact that $(x_n, y_n) \in \mathrm{int}(X \times Y)$, we have $\begin{pmatrix} \beta_n \widetilde{x}_n - \sum_{i=1}^k \lambda_i p_{1i}^n \\ -\sum_{i=1}^k \lambda_i p_{2i}^n \end{pmatrix} \in \partial(\beta_n \hat{f})(x_n, y_n)$. That is, for all $(x,y) \in \mathbb{R}^p \times \mathbb{R}^q$,

$$\hat{f}(x,y) \geq \hat{f}(x_n, y_n) + \left\langle \begin{pmatrix} \widetilde{x}_n - \dfrac{\sum_{i=1}^k \lambda_i p_{1i}^n}{\beta_n} \\ -\dfrac{\sum_{i=1}^k \lambda_i p_{2i}^n}{\beta_n} \end{pmatrix}, \begin{pmatrix} x - x_n \\ y - y_n \end{pmatrix} \right\rangle. \tag{5.4}$$

Moreover, $\partial \hat{f}(x_n, y_n) \subset \bigcup_{(x,y) \in X \times Y} \partial \hat{f}(x,y)$. Since $\bigcup_{(x,y) \in X \times Y} \partial \hat{f}(x,y)$ is compact, then there exist $(r_1, r_2) \in \mathbb{R}^p \times \mathbb{R}^q$ and an infinite subset $\mathcal{N}_5$ of $\mathcal{N}_4$ such that $r_{1n} = \widetilde{x}_n - \frac{\sum_{i=1}^k \lambda_i p_{1i}^n}{\beta_n} \to r_1$ as $n \to +\infty$, $n \in \mathcal{N}_5$ and $r_{2n} = \frac{\sum_{i=1}^k \lambda_i p_{2i}^n}{\beta_n} \to r_1$ as $n \to +\infty$, $n \in \mathcal{N}_5$. Passing to the limit in (5.4) as $n \to +\infty, n \in \mathcal{N}_5$, we have

$$\sum_{j=1}^m \bar{\mu}_j f_j(x,y) \geq \sum_{j=1}^m \bar{\mu}_j f_j(\bar{x}, \bar{y}) + \left\langle \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ y - \bar{x} \end{pmatrix} \right\rangle, \forall (x,y) \in \mathbb{R}^p \times \mathbb{R}^q. \tag{5.5}$$

Setting $x = \bar{x}$ in (5.5), we deduce that $r_2 \in \partial \left(\sum_{j=1}^m \bar{\mu}_j f_j \right)_{\bar{x}} (\bar{y})$. Moreover, assumption $(\mathcal{H}_3)$ implies that $0_{\mathbb{R}^q} \notin \partial \left(\sum_j^m \bar{\mu}_j f_j \right)_{\bar{x}} (\bar{y})$. Then $r_2 \neq 0_{\mathbb{R}^q}$. We have $\|r_{2n}\| \to \|r_2\|$ as $n \to +\infty, n \in \mathcal{N}_5$. Since $r_2 \neq 0_{\mathbb{R}^q}$, then there exists $n_2 \in \mathcal{N}_5$ such that $\|r_{2n}\| > 0,\ n \geq n_2,\ n \in \mathcal{N}_5$. Hence, for all $n \geq n_2, n \in \mathcal{N}_5$, we have $\beta_n = \frac{\|\sum_{i=1}^k \lambda_i p_{1i}^n\|}{\|r_{2n}\|}$. Since $p_{2i}^n \to \bar{p}_{2i}, n \to +\infty, n \in \mathcal{N}_5,\ i \in \{1, ..., k\}$, then

$$\beta_n = \frac{\|\sum_{i=1}^k \lambda_i p_{1i}^n\|}{\|r_{2n}\|} \to \bar{\beta} = \frac{\|\sum\limits_{i=1}^k \lambda_i \bar{p}_{1i}\|}{\|r_2\|}, n \to +\infty, n \in \mathcal{N}_5.$$

Thus

$$\begin{pmatrix} \widetilde{x}_n - \dfrac{\sum\limits_{i=1}^k \lambda_i p_{1i}^n}{\beta_n} \\ -\dfrac{\sum\limits_{i=1}^k \lambda_i p_{2i}^n}{\beta_n} \end{pmatrix} \to \begin{pmatrix} \widetilde{x} - \dfrac{\sum\limits_{i=1}^k \lambda_i \bar{p}_{1i}}{\bar{\beta}} \\ -\dfrac{\sum\limits_{i=1}^k \lambda_i \bar{p}_{2i}}{\bar{\beta}} \end{pmatrix} \text{ as } n \to +\infty, n \in \mathcal{N}_5.$$

Hence, passing to the limit in (5.4), we obtain, for all $(x,y) \in \mathbb{R}^p \times \mathbb{R}^p$,

$$\sum_{j=1}^m \bar{\mu}_j f_j(x,y) \geq \sum_{j=1}^m \bar{\mu}_j f_j(\bar{x}, \bar{y}) + \left\langle \begin{pmatrix} \widetilde{x} - \dfrac{\sum_{i=1}^k \lambda_i \bar{p}_{1i}}{\bar{\beta}} \\ -\dfrac{\sum_{i=1}^k \lambda_i \bar{p}_{2i}}{\bar{\beta}} \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ y - \bar{x} \end{pmatrix} \right\rangle.$$

Then

$$\begin{pmatrix} \widetilde{x} - \dfrac{\sum_{i=1}^{k} \lambda_i \bar{p}_{1i}}{\bar{\beta}} \\ -\dfrac{\sum_{i=1}^{k} \lambda_i \bar{p}_{2i}}{\bar{\beta}} \end{pmatrix} \in \partial \left(\sum_{j=1}^{m} \bar{\mu}_j f_j \right) (\bar{x}, \bar{y}).$$

That is, property (iii) is satisfied. □

5.2. **Sufficient optimality conditions for problem** (S)**.** In this subsection, we provide sufficient optimality conditions for solving problem (S).

Theorem 5.2. *Let $(\bar{x}, \bar{y}) \in X \times Y$. Assume that there exists $\widetilde{x} \in \mathbb{R}^p$, $\bar{\mu} \in int\mathbb{R}^m_+, \lambda \in int\mathbb{R}^k_+$ and $((\bar{p}_1, \bar{p}_2), \bar{\beta})$ s.t., $\bar{p}_1 = (\bar{p}_{11}, ..., \bar{p}_{1k}) \in (\mathbb{R}^p)^k, \bar{p}_2 = (\bar{p}_{21}, ..., \bar{p}_{2k}) \in (\mathbb{R}^q)^k, \bar{\beta} \in \mathbb{R}^*_+$ and*

(i) $\begin{pmatrix} \bar{p}_{1i} \\ \bar{p}_{2i} \end{pmatrix} \in \partial F_i(\bar{x}, \bar{y}),\ \forall i \in \{1, ..., k\},$

(ii) $0 \in \partial \left(\sum_{j=1}^{m} \bar{\mu}_j f_{j_{\bar{x}}} \right) (\bar{y}) + \mathcal{N}_Y(\bar{y}),$

(iii) $\begin{pmatrix} \widetilde{x} - \dfrac{\sum_{i=1}^{k} \lambda_i \bar{p}_{1i}}{\bar{\beta}} \\ -\dfrac{\sum_{i=1}^{k} \lambda_i \bar{p}_{2i}}{\bar{\beta}} \end{pmatrix} \in \partial \left(\sum_{j=1}^{m} \bar{\mu}_j f_j \right) (\bar{x}, \bar{y}),$

(iv) $(\bar{x}, \bar{y})$ *solves* $\max_{(x,y) \in X \times Y} \left\{ \sum_{j=1}^{m} \bar{\mu}_j f_j(x, y) - \langle \widetilde{x}, x \rangle \right\}.$

Then, $(\bar{x}, \bar{y})$ is a properly efficient solution to problem (S).

Proof. Feasibility: Let us show that $\bar{y} \in \mathcal{M}(\bar{x})$, i.e., $\bar{y}$ is a properly efficient solution of $\mathscr{P}(\bar{x})$. Since $\mathscr{P}(\bar{x})$ is convex, then it is equivalent to show that there exists $\mu \in int\mathbb{R}^m_+$ such that $\bar{y}$ solves $\min_{y \in Y} \sum_{j=1}^{m} \mu_j f_j(\bar{x}, y)$. From property (ii), we have,

$$\exists \bar{\mu} \in int\mathbb{R}^m_+ \quad \text{s.t.,} \quad 0 \in \partial \left(\sum_{j}^{m} \bar{\mu}_j f_{j_{\bar{x}}} \right) (\bar{y}) + \mathcal{N}_Y(\bar{y}).$$

Hence, $\bar{y} \in \mathcal{M}(\bar{x})$.

Optimality: Let us show that $(\bar{x}, \bar{y})$ is a properly efficient solution of (S).

From Theorem 2.3, we show that there exists $\lambda \in \text{int}\mathbb{R}^m_+$ such that $(\bar{x}, \bar{y})$ is an optimal solution to

$$(S^{\lambda}) \qquad \min_{\substack{x \in X \\ y \in \mathcal{M}(x)}} \sum_{i=1}^{k} \lambda_i F_i(x, y),$$

i.e., $\sum_{i=1}^{k} \lambda_i F_i(\bar{x}, \bar{y}) \leq \sum_{i=1}^{k} \lambda_i F_i(x, y)$ for all $(x, y) \in X \times \mathcal{M}(x)$. Let $(x, y) \in X \times Y$ such that $y \in \mathcal{M}(x)$. From property (i), we have

$$F_i(x', y') \geq F_i(\bar{x}, \bar{y}) + \left\langle \begin{pmatrix} \bar{p}_{1i} \\ \bar{p}_{2i} \end{pmatrix}, \begin{pmatrix} x' - \bar{x} \\ y' - \bar{y} \end{pmatrix} \right\rangle, \quad \forall (x', y') \in \mathbb{R}^p \times \mathbb{R}^q, \forall i \in \{1, ..., k\}.$$

Then, for $x^{'} = x$ and $y^{'} = y$, we obtain, $\forall i \in \{1,...,k\}$, $F_i(x,y) \geq F_i(\bar{x},\bar{y}) + \left\langle \begin{pmatrix} \bar{p}_{1i} \\ \bar{p}_{2i} \end{pmatrix}, \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} \right\rangle$.
Likewise, property (iii) is written as

$$\sum_{j=1}^{m} \bar{\mu}_j f_j(x^{'},y^{'}) \geq \sum_{j=1}^{m} \bar{\mu}_j f_j(\bar{x},\bar{y}) + \left\langle \begin{pmatrix} \widetilde{x} - \dfrac{\sum_{i=1}^{k} \lambda_i \bar{p}_{1i}}{\bar{\beta}} \\ -\dfrac{\sum_{i=1}^{k} \lambda_i \bar{p}_{2i}}{\bar{\beta}} \end{pmatrix}, \begin{pmatrix} x^{'}-\bar{x} \\ y^{'}-\bar{x} \end{pmatrix} \right\rangle, \ \forall (x^{'},y^{'}) \in \mathbb{R}^p \times \mathbb{R}^p.$$

For $x^{'} = x$ and $y^{'} = y$, one has

$$\left\langle \begin{pmatrix} \frac{\sum_{i=1}^{k} \lambda_i \bar{p}_{1i}}{\bar{\beta}} \\ -\frac{\sum_{i=1}^{k} \lambda_i \bar{p}_{2i}}{\bar{\beta}} \end{pmatrix}, \begin{pmatrix} x-\bar{x} \\ y-\bar{x} \end{pmatrix} \right\rangle \geq \sum_{j=1}^{m} \bar{\mu}_j f_j(\bar{x},\bar{y}) - \sum_{j=1}^{m} \bar{\mu}_j f_j(x,y) + \left\langle \begin{pmatrix} \widetilde{x} \\ 0_{\mathbb{R}^q} \end{pmatrix}, \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} \right\rangle. \tag{5.6}$$

On the other hand, from property (ii), we have $\sum_{j=1}^{m} \bar{\mu}_j f_j(\bar{x},\bar{y}) - \langle \widetilde{x}, \bar{x} \rangle \geq \sum_{j=1}^{m} \bar{\mu}_j f_j(x,y) - \langle \widetilde{x}, x \rangle$.
Hence, from (5.6) and $\bar{\beta} > 0$, we see that

$$\left\langle \begin{pmatrix} \sum_{i=1}^{k} \lambda_i \bar{p}_{1i} \\ \sum_{i=1}^{k} \lambda_i \bar{p}_{2i} \end{pmatrix}, \begin{pmatrix} x-\bar{x} \\ y-\bar{x} \end{pmatrix} \right\rangle \geq 0,$$

which implies

$$\sum_{i=1}^{k} \lambda_i F_i(x,y) \geq \sum_{i=1}^{k} \lambda_i F_i(\bar{x},\bar{y}) + \sum_{i=1}^{k} \lambda_i \left\langle \begin{pmatrix} \bar{p}_{1i} \\ \bar{p}_{2i} \end{pmatrix}, \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} \right\rangle.$$

Hence, we see that there exists $\lambda \in \text{int}\mathbb{R}_+^k$ such that $\sum_{i=1}^{k} \lambda_i F_i(x,y) \geq \sum_{i=1}^{k} \lambda_i F_i(\bar{x},\bar{y})$. Therefore, $(\bar{x},\bar{y})$ is a properly efficient solution to (S). □

Example 5.2. Let $X = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $Y = [0,1] \times [-1,1]$, and let F_i and f_i, $i = 1,2,3$ be the functions defined on $\mathbb{R} \times \mathbb{R}^2$ by

$$\begin{cases} F_1(x,y) = -2|x| - 2y_1 + y_2^2 - 2y_2, \\ F_2(x,y) = -2x - 2y_1, \\ F_3(x,y) = -2|x| - 2y_1 + 1, \end{cases} \quad \text{and} \quad \begin{cases} f_1(x,y) = x^2 + y_1, \\ f_2(x,y) = y_1, \\ f_3(x,y) = x^2 + 2y_1. \end{cases}$$

Then, X and Y are compact convex sets and F and f are convex functions. Let us determine a point $(\bar{x},\bar{y}) \in X \times Y$ that satisfies the sufficient conditions in Theorem 5.2 Then, we are led to verify if there exists $\widetilde{x} \in \mathbb{R}$, $\bar{p}_1 = (\bar{p}_{11}, \bar{p}_{12}, \bar{p}_{13}) \in \mathbb{R}^3$, $\bar{p}_2 = (\bar{p}_{21}, \bar{p}_{22}, \bar{p}_{23}) \in (\mathbb{R}^2)^3, \bar{p}_{2i} = (\bar{p}_{2i}^1, \bar{p}_{2i}^2) \in \mathbb{R}^2, i = 1,2,3$ $\bar{\beta} \in \mathbb{R}_+^*$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)^\top \in int\mathbb{R}_+^3, \bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3)^\top \in int\mathbb{R}_+^3$ such that the following optimality conditions are satisfied

(i) $\begin{pmatrix} \bar{p}_{1i} \\ \bar{p}_{2i} \end{pmatrix} \in \partial F_i(\bar{x},\bar{y}), \forall i \in \{1,2,3\}$, (ii) $0 \in \partial \left(\sum_{j=1}^{3} \bar{\mu}_j f_j \right)_{\bar{x}} (\bar{y}) + \mathcal{N}_Y(\bar{y})$.

(iii) $\begin{pmatrix} \widetilde{x} - \frac{\sum_{i=1}^{3} \lambda_i \bar{p}_{1i}}{\bar{\beta}} \\ -\frac{\sum_{i=1}^{3} \lambda_i \bar{p}_{2i}}{\bar{\beta}} \end{pmatrix} \in \partial \left(\sum_{j=1}^{3} \bar{\mu}_j f_j \right) (\bar{x},\bar{y})$.

(iv) $(\bar{x},\bar{y})$ solves $\max_{(x,y)\in X\times Y}\left\{\sum_{j=1}^{3}\bar{\mu}_j f_j(x,y)-\langle\widetilde{x},x\rangle\right\}$.
For $x\in X$ and $y\in Y$, we have

$$\partial F_1(x,y)=\begin{cases}\{-2\}\times\left\{(-2,2y_2-2)^\top\right\} & \text{if } x>0,\ y\in Y,\\ [-2,2]\times\left\{(-2,2y_2-2)^\top\right\} & \text{if } x=0,\ y\in Y,\\ \{2\}\times\left\{(-2,2y_2-2)^\top\right\} & \text{if } x<0,\ y\in Y,\end{cases}$$

$$\partial F_2(x,y)=\{-2\}\times\left\{\begin{pmatrix}-2\\0\end{pmatrix}\right\},\ \partial F_3(x,y)=\begin{cases}\{-2\}\times\left\{(-2,0)^\top\right\} & \text{if } x>0,\ y\in Y,\\ [-2,2]\times\left\{(-2,0)^\top\right\} & \text{if } x=0,\ y\in Y,\\ \{2\}\times\left\{(-2,0)^\top\right\} & \text{if } x<0,\ y\in Y.\end{cases}$$

If $\bar{x}>0$, then we obtain from i) that

$$\begin{pmatrix}\bar{p}_{11}\\ \bar{p}_{21}\end{pmatrix}=\begin{pmatrix}-2\\-2\\2y_2-2\end{pmatrix},\ \begin{pmatrix}\bar{p}_{12}\\ \bar{p}_{22}\end{pmatrix}=\begin{pmatrix}-2\\-2\\0\end{pmatrix},\ \begin{pmatrix}\bar{p}_{13}\\ \bar{p}_{23}\end{pmatrix}=\begin{pmatrix}-2\\-2\\0\end{pmatrix}. \tag{5.7}$$

Note that ii) implies that $(\bar{y}_1,\bar{y}_2)\in\mathscr{M}_\mu(\bar{x})$, where $\mathscr{M}_\mu(\bar{x})$ is the solution set of the problem

$$\min_{y\in[0,1]\times[-1,1]}\bar{x}^2(\mu_1+\mu_3)+y_1(\mu_1+\mu_2+2\mu_3).$$

Then, $\mathscr{M}_\mu(\bar{x})=\{0\}\times[-1,1]$. Hence, $\bar{y}_1=0$ and $\bar{y}_2\in[-1,1]$.

On the other hand, iii) implies that

$$\begin{pmatrix}\bar{\beta}\widetilde{x}-\sum_{i=1}^{k}\lambda_i\bar{p}_{1i}\\ -\sum_{i=1}^{k}\lambda_i\bar{p}_{2i}^1\\ -\sum_{i=1}^{k}\lambda_i\bar{p}_{2i}^2\end{pmatrix}\in\bar{\beta}\partial\left(\bar{\mu}_1 f_1(\bar{x},\bar{y})+\bar{\mu}_2 f_2(\bar{x},\bar{y})+\bar{\mu}_3 f_3(\bar{x},\bar{y})\right).$$

Then, we obtain $\begin{cases}\bar{\beta}\widetilde{x}-\lambda_1\bar{p}_{11}-\lambda_2\bar{p}_{12}-\lambda_3\bar{p}_{13}=2\bar{\beta}(\bar{\mu}_1+\bar{\mu}_3)\bar{x},\\ -\lambda_1\bar{p}_{21}^1-\lambda_2\bar{p}_{22}^1-\lambda_3\bar{p}_{23}^1=\bar{\beta}(\bar{\mu}_1+\bar{\mu}_2+2\bar{\mu}_3),\\ -\lambda_1\bar{p}_{21}^2-\lambda_2\bar{p}_{22}^2-\lambda_3\bar{p}_{23}^2=0,\end{cases}$ which together with (5.7) yields

$$\begin{cases}\bar{\beta}\widetilde{x}-2\lambda_1+2\lambda_2+2\lambda_3=2\bar{\beta}(\bar{\mu}_1+\bar{\mu}_3)\bar{x}, & (5.8)\\ 2\lambda_1+2\lambda_2+2\lambda_3=\bar{\beta}(\bar{\mu}_1+\bar{\mu}_2+2\bar{\mu}_3), & (5.9)\\ \lambda_1(2-2y_2)\bar{p}_{21}^2=0, & (5.10)\end{cases}$$

Let $\lambda=(1,1,1)$. From (5.10), we have $2-2y_2=0$. Thus $y_2=1$. Since $\mu\in\operatorname{int}(\mathbb{R}^3_+)$, we find from (5.9) that

$$\bar{\beta}=\frac{2}{\bar{\mu}_1+\bar{\mu}_2+2\bar{\mu}_3}. \tag{5.11}$$

From (5.8), we obtain

$$\frac{2}{\bar{\mu}_1+\bar{\mu}_2+2\bar{\mu}_3}\widetilde{x}+2=\frac{4(\bar{\mu}_1+2\bar{\mu}_3)}{\bar{\mu}_1+\bar{\mu}_2+2\bar{\mu}_3}\bar{x}. \tag{5.12}$$

(iv) implies that $(\bar{x}, \bar{y})$ solves

$$\max_{(x,y)\in X\times Y}\{\bar{\mu}_1 f_1(x,y)+\bar{\mu}_2 f_2(x,y)+\bar{\mu}_3 f_3(x,y)-\widetilde{x}x\}.$$

Let $h(x,y)=\bar{\mu}_1 f_1(x,y)+\bar{\mu}_2 f_2(x,y)+\bar{\mu}_3 f_3(x,y)-\widetilde{x}x$. We need to solve the problem

$$\max_{(x,y)\in X\times Y_1} h(x,y).$$

We see that $X\times Y_1=[-\frac{1}{2},\frac{1}{2}]\times[0,1]$ is convex compact and h is convex. Thne the maximum is attained at an extreme point of $X\times Y_1$. Let us calculate the images of the extreme points of $X\times Y_1$ by h. Since $\bar{y}_1=0$, we need to compare the values

$$h\left(-\frac{1}{2},0\right)=\frac{1}{4}(\bar{\mu}_1+\bar{\mu}_3)+\frac{1}{2}\widetilde{x} \quad \text{and} \quad h\left(\frac{1}{2},0\right)=\frac{1}{4}(\bar{\mu}_1+\bar{\mu}_3)-\frac{1}{2}\widetilde{x}.$$

In view of $\bar{x}>0$, we conclude that the maximum is attained at $(\bar{x},\bar{y}_1)=(\frac{1}{2},0)$. By letting $(\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3)=(1,1,1)$, we obtain from (5.11) that $\bar{\beta}=\frac{1}{2}$ and we also obtain from (5.12) that $\widetilde{x}=-2$. We conclude that $(\bar{x},\bar{y}_1,\bar{y}_2)^\top=(\frac{1}{2},0,1)^\top$ is a properly efficient solution to (S). Letting $\lambda=\bar{\mu}=(1,1,1)$ we obtain $\widetilde{x}=-2, \bar{\beta}=\frac{1}{2}, \bar{p}_1=(\bar{p}_{11},\bar{p}_{12},\bar{p}_{13})=(-2,-2,-2)$, and $\bar{p}_2=(\bar{p}_{21},\bar{p}_{22},\bar{p}_{23})=((-2,0),(-2,0),(-2,0))$.

6. Conclusion

Most current multiobjective bilevel problems in the literature are problems where exactly one level is vectorial. In order to investigate the class of strong multiobjective bilevel programming where both levels are vectorial, as (S), we provided necessary and sufficient optimality conditions. These results were obtained based on four operations: regularization, scalarization, decomposition, and conjugate duality. They need to use these operations stems from the fact that problem (S) is not convex and does not satisfy the classical slater condition. In order to avoid this situation, we proceeded by scalarizing, then regularizing the scalarized problem. As a stability result, we demonstrated that any accumulation point of a sequence of scalarized-regularized solutions solves the original bilevel problem (S). Then, we decomposed $(S^\lambda_\varepsilon), \lambda\in \mathrm{int}(\mathbb{R}^k_+)$ according to the second variable into a family of subproblems $(S^\lambda_{\varepsilon,\mu}), \mu\in\mathrm{int}(\mathbb{R}^m_+)$ that satisfies the slater condition. However, in general, we have the same problem with $(S^\lambda_{\varepsilon,\mu}), \mu\in\mathrm{int}(\mathbb{R}^m_+)$ concerning the lack of convexity. We gave a decomposition of the problem $(S^\lambda_{\varepsilon,\mu})$ by a family of convex programming subproblems $(S^{\lambda,\widetilde{x}}_{\varepsilon,\mu}), \widetilde{x}\in\mathbb{R}^p$. Under a constraint qualification, we gave its Fenchel-Lagrange dual. Thanks to the decomposition and this duality, we defined an extended Fenchel-Lagrange duality for the scalarized-regularized problem (S^λ_ε). Under appropriate assumptions, we demonstrated that strong extended Fenchel-Lagrange duality holds for (S^λ_ε) and provided optimality conditions for it. Finally, we established necessary and sufficient optimality conditions for problem (S). We here mention that our results extend the ones given in [2] from the scalar case to the multiobjective one.

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