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THE EXISTENCE OF HYBRID EQUILIBRIA AND WEAK HYBRID EQUILIBRIA FOR MULTI-LEADER-MULTI-FOLLOWER GAMES

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Abstract. This paper introduces a kind of hybrid equilibrium to multi-leader-multi-follower games. The existence of hybrid equilibria of the games is proven via set-valued analysis. Our hybrid equilibria can include both noncooperative equilibria and cooperative equilibria in references as special cases. In addition, a weak hybrid equilibrium is introduced both for normal form games and multi-leader-multi-follower games with infinitely many players. The existence of weak hybrid equilibria is established for infinitely many players. The notion of the weak hybrid equilibrium is a generalization of weak α-core in cooperative games. These equilibrium conceptions and results are new in multi-leader-multi-follower games.

Keywords. Bilevel programming; Hybrid equilibria; Multi-leader-multi-follower games;Nash Equilibrium.

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1. INTRODUCTION

It is known that Nash equilibria focuses the noncooperative behavior of players, while the conceptions of core and other cooperative equilibria focus on the cooperative behavior among players. A classical existence theorem for α-core (see Aumann [1]) in normal form games was given based on the existence of core of characteristic games [2]. The α-core was extended to multi-objective games and TU α-core by Zhao [3, 4], to games with nonordered preferences by Kajii [5], to multi-objective games with continuous set payoffs by Zhang and Sun in [6], to multi-objective games with discontinuous set payoffs in [7] by Song et al, and to population games in [8, 9] (for populations games, see [10]). For the existence results for α-core of discontinuous TU and NTU games, we refer to [11]. Noncooperative equilibria and cooperative equilibria have deep relations. Social coalitional equilibria in [12], defined by Ichiishi, integrate the notion of Nash equilibrium and core. Another closely related conception is the hybrid solution (hybrid equilibrium), introduced for general cooperative games by Zhao [13]. Some structures of solutions of general cooperative games were revealed by Song in [14]. A hybrid equilibrium includes Nash equilibrium points and α-core as special cases in normal form games. Recently, Yang and Yuan generalized the conception of hybrid equilibria to the games with nonordered preferences in [15].

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In the field of multi-leader-multi-follower games ($MLMF\,games$), numerous existence results for noncooperative equilibria were established. Pang and Fukushima [16] formulated a kind of *MLMF* game as a generalized Nash equilibrium problem or a generalized quasi-variational inequality (for the original GQVI, see Yao [17]) . In two-leader-follower games, Yu and Wang [18] provided some Nash equilibrium results. Later, using variational inequalities, Hu and Fukushima [19] obtained the existence and uniqueness of Nash equilibria for *MLMF* games. Recently, for generalized *MLMF* games, Jia et al. [20] demonstrated the existence and stability of weakly Pareto-Nash equilibria. Considering noncompact FC-spaces, the equilibrium existence was proven by Ding [21] for *MLMF* games. Furthermore, it is also known that the original development of bilevel programming is stimulated by the leader-follower games (Stackelberg games); see [22]. In recent years, the study of the existence of cooperative equilibria for *MLMF* games have been arising. In [23], Yang and Yu studied the existence and generic stability of cooperative equilibria, where cooperative equilibria originate from α-core. Yang and Gong, in [24], presented an existence result of weakly cooperative equilibria defined from weak core for normal form games in [25]. Very recently, the continuity of α-core of *MLMF* games was studied in [26]. There are resutls on Nash equilibria and cooperative equilibria in *MLMF* games, however, the hybrid equilibria integrating the noncooperative and cooperative equilibria are still not studied in *MLMF* games. We generalize the notion of the hybrid equilibria in Zhao [13] to *MLMF* games, and study their existence. Inspired by Yang and Yuan [15], we introduce weak hybrid equilibria to *MLMF* games, and give a study of their existence for the case with infinitely many leaders and followers.

The rest is organized as follows. In Section 2, the related setting and the definition of hybrid equilibria are given for *MLMF* games. Section 3 proves the existence of hybrid equilibria for *MLMF* games. In Section 4, the notion of weak hybrid equilibria is introduced to normal form games and *MLMF* games. Then, based on the existence results for normal form games in this paper, the set of weak hybrid equilibria is demonstrated to be nonempty for *MLMF* games.

2. PRELIMINARIES ANDHHYBRID EQUILIBRIA

2.1. **Multi-leader-multi-follower games and hybrid equilibria.** Let $I=\{1,\dots,n\}$ be the set of leaders and $J=\{1,\dots,m\}$ be the set of followers. For $i\in I$, X_i is the strategy set of the leader i and $f_i : X\times Y \longrightarrow \mathbb{R}$ is i's payoff function. For a follower j in J, Y_j is his/her strategy set and $g_j : X\times Y \longrightarrow \mathbb{R}$ is the payoff of the follower j. For convenience, one sets

$$X=\prod_{i\in I}X_i,\ X_{-i}=\prod_{i\neq k,k\in I}X_k,\ X_S=\prod_{\substack{S\subseteq I\\ i\in S}}X_i,\ X_{-S}=\prod_{\substack{S\subseteq I\\ i\notin S}}X_i.$$

Y, Y_{-j}, Y_S, and Y_{-S} are similar with X, X_{-i}, X_S, and X_{-S} for each $j\in J$ and each $S\subseteq J$. Then, $\Gamma=\langle I,J,\{X_i\}_{i\in I},\{Y_j\}_{j\in J},\{f_i\}_{i\in I},\{g_j\}_{j\in J}\rangle$ is called a *multi-leader-multi-follower* ($MLMF$) game with finite players. In a *MLMF* game, the leaders firstly make a decision x from the strategy set X. Secondly, the followers accept the strategy $x\in X$ of leaders and make a corresponding strategy $y\in Y$. In fact, a *MLMF* game is a generalization of usual normal form games. A *normal form game* with a partition p of players is $G=\langle N,p,\{X_i\}_{i\in N},\ \{u_i\}_{i\in N}\rangle$, where p with $p=\{N_1,N_2,\dots,N_k\}$ is a partition of the player set N (that is, $\cup_{r=1}^{k}N_r=N$, $N_r\cap N_{r'}=\emptyset$, $\forall r\neq r'$). For each player $i\in N$, X_i is the strategy set of the player i, and $u_i : X\longrightarrow\mathbb{R}$ denotes the payoff for the player i. In the above game G, if the preference of a player $i \in N$ is denoted by a correspondence $P_i : X \rightrightarrows X$ instead of the payoff u_i, then a game with preferences is $\langle N, p, \{X_i\}_{i\in N}, \{P_i\}_{i\in N}\rangle$.

Definition 2.1. (Zhao [13]) Let G be a normal form game with N being a finite set. A point $x^* = (x^*_{N_1}, x^*_{N_2}, \cdots, x^*_{N_k}) \in X$ is said to be a *hybrid equilibrium* of G if, for any $N_r \in p$ and any $B \subseteq N_r$, there exists no $y_B \in X_B$ such that $u_i(y_B, z_{N_r-B}, x^*_{-N_r}) > u_i(x^*_{N_r}, x^*_{-N_r})$, $\forall i \in B$, $\forall z_{N_r-B} \in X_{N_r-B}$, where $N_r - B = \{i \mid i \in N_r, i \notin B\}$.

Remark 2.1. It should be pointed that if $|p| = 1$, a hybrid equilibrium belongs to α-core, Furthermore, if $|N_r| = 1$ for any $r \in \{1,2,\ldots,k\}$, then hybrid equilibrium is deduced to a Nash equilibrium.

We extend the definition of hybrid equilibria in normal form games to *MLMF* games. For a *MLMF* game with partitions

$$\Gamma = \langle I, J, p, \bar{p}, \{X_i\}_{i\in I}, \{Y_j\}_{j\in J}, \{f_i\}_{i\in I}, \{g_j\}_{j\in J}\rangle, \tag{2.1}$$

where p is a partition of leaders I with $p = \{N_1, N_2, \ldots, N_k\}$ and $|p| = k$; $\bar{p}$ is a partition of followers J with $\bar{p} = \{N'_1, N'_2, \ldots, N'_{k_1}\}$ and $|\bar{p}| = k_1$. Note that $\cup_{r=1}^{k} N_r = I$ and $\cup_{r=1}^{k_1} N'_r = J$, where $N_r \cap N_{r'} = \emptyset$ and $N'_r \cap N'_{r'} = \emptyset$, $\forall r \neq r'$.

Let $h(x)$ be the set of hybrid equilibria of the followers' norm game, $G' = \langle J, \bar{p}, \{Y_j\}_{j\in J}, \{g_j(x,\cdot)\}_{j\in J}\rangle$ for each fixed strategy profile $x \in X$ of leaders. Then, a correspondence $h : X \rightrightarrows Y$ is defined. The point $\bar{y} = (\bar{y}_{N'_1}, \bar{y}_{N'_2}, \ldots, \bar{y}_{N'_{k_1}}) \in h(x)$ implies that $\bar{y}$ is a hybrid equilibrium of G'. That is, for any $N'_r \in \bar{p}\,(r \in \{1,2\ldots,k_1\})$ and $S' \subseteq N'_r$, there exists no $y_{S'} \in Y_{S'}$ such that

$$g_j(x, y_{S'}, w_{N'_r-S'}, \bar{y}_{-N'_r}) > g_j(x, \bar{y}_{N'_r}, \bar{y}_{-N'_r}),\ \forall j \in S',\ \forall w_{N'_r-S'} \in Y_{N'_r-S'}.$$

Definition 2.2. A strategy $\bar{x} = (\bar{x}_{N_1}, \bar{x}_{N_2}, \ldots, \bar{x}_{N_k}) \in X$ is called a *hybrid equilibrium* of a *MLMF* game Γ in (2.1) if, for any $N_r \in p$ and for any $S \subseteq N_r$, there exists no $x_S \in X_S$ such that

$$f_i(x_S, w_{N_r-S}, \bar{x}_{-N_r}, \bar{y}) > f_i(\bar{x}_{N_r}, \bar{x}_{-N_r}, \bar{y}),\ \forall i \in S,\ \forall \bar{y} \in h(\bar{x}),\ \forall w_{N_r-S} \in X_{N_r-S}.$$

Remark 2.2. A hybrid equilibrium $\bar{x}$ of a *MLMF* game requires that, for any hybrid equilibrium $\bar{y} \in h(\bar{x})$ of the followers' game, $\bar{x}$ is still a hybrid equilibrium of leaders' game. Therefore, the hybrid equilibrium in Definition 2.2 is very different from that in Definition 2.1.

The following result needed in the paper is from Yang and Yuan [15] for hybrid equilibria of games without ordered presences.

Lemma 2.1. [15] *Suppose that* $G = \langle N, p, \{X_i\}_{i\in N}, \{P_i\}_{i\in N}\rangle$ *is a game with preferences, where* N *is a finite set,* $p = \{N_1, N_2, \cdots, N_k\}$ *and* $P_i : X \rightrightarrows X$ *for each* $i \in N$ *with* $X = \cup_{i\in N} X_i$. *For each* $i \in N$, *if* G *satisfies the following conditions:*

(1) X_i *is a nonempty, convex, and compact subset of* $\mathbb{R}^{m_i}$*;*

(2) $P_i(\cdot)$ *has convex values with open graph in* $X \times X$ *and* $x \notin P_i(x)$ *for all* x *in* X,

then G *has at least a hybrid equilibrium. That is, for any* $N_r \in p$ *and any* $S \subseteq N_r$, *there exists no* $z_S \in X_S$ *such that* $\{z_S\} \times X_{N_r-S} \times \{\bar{x}_{-N_r}\} \subset P_i(\bar{x}_{N_r}, \bar{x}_{-N_r})$, $\forall i \in S$.

2.2. **On the concept of hybrid equilibria.** The existence of noncooperative equilibria was given in multi-leader-multi-follower games [27]. The original development of bilevel programming is from the leader-follower games (Stackelberg games), see [22]. There are numerous applications of multi-leader-multi-follower games, such as the congestion control in communication networks [28], competitive bidding problems in electricity markets [16], and bidding problems with an arbitrager in electricity markets [16]. For competitive bidding problems in electricity markets, the followers are independent system operators (ISO), and the leaders are the firms to bid for the market power in some regions.

In a normal form game, the concept of cooperative equilibria was used widely [29]. As a generalization of normal form games, it cannot be eliminated cooperation in a multi-leader-multi-follower game. For example, in Blockchain Ecosystems, if one takes independent system operators as followers, and takes the miners as leaders, then it is a *MLMF* game. It was pointed out that there exists cooperation in some miner groups; see [29]. A cooperative equilibrium in *MLMF* games (which was firstly given in [23]) means that it cannot be α-blocked by any coalition, and the meaning of cooperation in miner groups in [29] is also based on α-blockings. Hybrid equilibria bridge the noncooperative equilibria and cooperative equilibria in [3] for normal form games. Hybrid games, as one of important economic aspects, are games to be used to model simultaneous cooperation within firms and competition among firms, such as "island economies"; see [30].

Hybrid equilibria were used to analyze financial problems. Under the framework of Blockchain ecosystems, in mining gap games, all miners (players) p are divided into several disjoint groups with $p = \{N_1, N_2, \cdots, N_k\}$. In each miner group N_j, players cooperate together; players compete among groups. As an applications of hybrid equilibria in [29], consensus equilibria were characterized by hybrid equilibria for mining gap games. Under a given reasonable consensus, the existence of the honest miners keeping "Mining Longest Chain Rules" was guaranteed. In view of the existence of noncooperative equilibria and cooperative equilibria in *MLMF* games and the potential applications of hybrid equilibria, Definition 2.2 gives a concept of hybrid equilibria in *MLMF* games based on the hybrid equilibrium in [13] by Zhao and the cooperative equilibrium in [23] by Yang and Ju.

If $\bar{x}$ is a hybrid equilibrium of a *MLMF* game Γ as (2.1) and there are no followers, then $\bar{x}$ is a hybrid equilibrium (as stated in Definition 2.1) in [13] for a normal form game with partitions. If there has only the coarsest partition, that is, $|p| = |\bar{p}| = 1$, a hybrid equilibrium of a *MLMF* game Γ as (2.1) is a cooperative equilibrium in [23]. If all groups are the finest, that is, $|N_r| \equiv 1$ and $|N'_r| \equiv 1$, then the hybrid equilibrium in Definition 2.2 can reduce to a noncooperative solution (Nash equilibrium) for some special *MLMF* games in references.

Here, we give an *MLMF* game which is a slight variant of the example in [16] by Pang and Fukushima. The follower's problem is to find a best y for a given $x = (x_1, x_2, x_3)$ such that $\max_{y \geq 0} g(x,y) = y(1 - x_1 - x_2) - \frac{1}{2}y^2$. There are three leaders. The first leader is to find the best x_1 of $\max_{x_1 \in [0,1]} f_1(x,y) = -\frac{1}{2}x_1$, the second leader's problem is $\max_{x_2 \in [0,1]} f_2(x,y) = -\frac{1}{2}x_2 x_1 + x_2 y$, and the third leader's problem is $\max_{x_3 \in [0,1]} f_3(x,y) = -\frac{1}{2}x_3 + y$. Let $I = \{1,2,3\}$, $J = \{1\}$, $p = \{\{1\},\{2\},\{3\}\}$, $\bar{p} = \{1\}$, $X_i = [0,1]$ for each $i \in I$, and $Y = [0, +\infty)$. Then, there is a multi-leader-one-follower game with partitions: $\Gamma^1 = \langle I, J, p, \bar{p}, \{X_i\}_{i \in I}, Y, \{f_i\}_{i \in I}, g \rangle$. From the finest partition $\{\{1\},\{2\},\{3\}\}$ of leaders, the hybrid equilibria of Γ^1 in Definition 2.2 are noncoperative equilibria among leaders. The follower's best reaction of a given x is $y \in h(x)$ such that $h(x) = max\{0, 1 - x_1 - x_2\}$. Then, the hybrid equilibria of Γ^1 consisting with competitive equilibria is the singleton set $NE = \{(0,1,0)\} \subset X_1 \times X_2 \times X_3$. For any point in the set NE, there is no coalition $\{i\}$ which can α-block the point. For instance, if the coalition $\{2\}$

can α-block the point $\bar{x} = (0,1,0)$, then there exists an $x_2' \in X_2$ such that $f_2((0,x_2',0),h(\bar{x})) > f_2(\bar{x},h(\bar{x}))$, then it reaches a contradiction. If it is allowed to cooperate between leaders 1 and 2, then the partition of p' of leaders is written as $p' = \{\{1,2\},\{3\}\}$. Then, a multi-leader-one-follower game with partitions is defined as $\Gamma^2 = \langle I,J,p',\bar{p},\{X_i\}_{i\in I},Y,\{f_i\}_{i\in I},g\rangle$. We assert that $x^* = (0,\frac{1}{2},0)$ is in the set of hybrid equilibria of Γ^2, which is a cooperative equilibrium in [23, 24] under [1]. For each coalition S of I, there is no $x_S \in X_S$ which can α-block the strategy x^*. For instance, if $\{2\} \subset \{1,2\}$ can α-block the strategy x^*, then there is an $x_2' \in X_2$ such that $f_2((w_1,x_2',0),h(x^*)) > f_2(x^*,h(x^*)), \forall w_1 \in X_1$. Then, $\frac{1}{2}x_2'(1-w_1) > \frac{1}{4}$ for all w_1 in X_1. There is a contradiction when $w_1 = 1$. Clearly, the cooperative equilibrium point x^* is not in the set NE. Furthermore, it holds that $f_i(x^*,h(x^*)) \geq f_i(x,h(x)), \forall i \in I, \forall x \in NE$. Then, to allow the cooperation for leaders under the α-blocking rule in [1] may benefit leaders in the *MLMF* game. Due to the different partitions of leaders in the above example, it may reduce to competitive equilibria or cooperative equilibria by hybrid equilibria in Definition 2.2. The different partitions of leaders or followers represent the complexity of internal organizations for players. Note the difference between the competitive equilibria and cooperative equilibria of *MLMF* games in the above example. One may ask: what kinds of conditions can bridge the gap between these equilibria? Section 3 aims to find some sufficient conditions. In addition, it is easy to generalize the above example to the case with infinitely many players. For games with infinitely many players, similar with Section 3, Section 4 intends to find some sufficient conditions to bridge the gap between competitive equilibria and cooperative equilibria.

3. THE EXISTENCE OF HYBRID EQUILIBRIA FOR *MLMF* GAMES WITH FINITE PLAYERS

The correspondence h in Section 2 has the property in Lemma 3.1.

Lemma 3.1. *Suppose that a MLMF game Γ as (2.1) satisfies*
(1) for each $i \in I$, X_i is a nonempty convex compact subset of $\mathbb{R}^{m_i}$;
(2) for each $j \in J$, Y_j is a nonempty convex compact subset of $\mathbb{R}^{m_j}$;
(3) for each $j \in J$, g_j is continuous on $X \times Y$.
Then, $h : X \rightrightarrows Y$ is an upper semi-continuous correspondence with compact values.

Proof. By the closed Graph Theorem in [31], it suffices to show that the graph of h is closed. Let $\{(x^n,\bar{y}^n)\}$ be a sequence in $X \times Y$ with $(x^n,\bar{y}^n) \longrightarrow (x,\bar{y}) \in X \times Y$ and $\bar{y}^n \in h(x^n)$. Suppose that $\bar{y} \notin h(x)$. Then, there exists $N_r' \in \bar{p}$, $S' \subseteq N_r'$, and $u_{S'} \in Y_{S'}$ such that

$$g_j(x,u_{S'},w_{N_r'-S'},\bar{y}_{-N_r'}) > g_j(x,\bar{y}_{N_r'},\bar{y}_{-N_r'}),\ \forall j \in S',\ \forall w_{N_r'-S'} \in Y_{N_r'-S'}.$$

Because g_j is continuous on $X \times Y$ and $Y_{N_r'-S'}$ is a compact subset, it holds that

$$\min_{w_{N_r'-S'} \in Y_{N_r'-S'}} g_j(x,u_{S'},w_{N_r'-S'},\bar{y}_{-N_r'}) > g_j(x,\bar{y}_{N_r'},\bar{y}_{-N_r'}),\ \forall j \in S'. \tag{3.1}$$

By the Berge Maximum Theorem in [31], $(x,u_{S'},\bar{y}_{-N_r'}) \longrightarrow \min_{w_{N_r'-S'} \in Y_{N_r'-S'}} g_j(x,u_{S'},w_{N_r'-S'},\bar{y}_{-N_r'})$ is continuous. Therefore, from (3.1), there exists $n_0 > 0$ such that, when $n > n_0$,

$$\min_{w_{N_r'-S'} \in Y_{N_r'-S'}} g_j(x^n,u_{S'},w_{N_r'-S'},\bar{y}^n_{-N_r'}) > g_j(x^n,\bar{y}^n_{N_r'},\bar{y}^n_{-N_r'}),\ \forall j \in S'.$$

Then, for $N_r' \in \bar{p}$ and $S' \subseteq N_r'$, there exists $u_{S'} \in Y_{S'}$ such that

$$g_j(x^n,u_{S'},w_{N_r'-S'},\bar{y}^n_{-N_r'}) > g_j(x^n,\bar{y}^n_{N_r'},\bar{y}^n_{-N_r'}),\ \forall j \in S', \forall w_{N_r'-S'} \in Y_{N_r'-S'}.$$

It is contradictory to the fact $\bar{y}^n \in h(x^n)$. □

From Definition 2.2, Lemma 3.1, and Lemma 2.1, we establish the following existence of hybrid equilibria in *MLMF* games.

Theorem 3.1. *Suppose that a MLMF game* Γ *as* (2.1) *satisfies the following conditions:*
(1) I and J are finite sets;
(2) for each $i \in I$ *and each* $j \in J$, X_i *and* Y_j *are two nonempty, convex, and compact subsets with* $X_i \subset \mathbb{R}^{m_i}$ *and* $Y_j \subset \mathbb{R}^{m_j}$*;*
(3) for each $i \in I$ *and each* $j \in J$, f_i *and* g_j *are continuous on* $X \times Y$*;*
(4) for each $i \in I$, $f_i(\,\cdot\,, x_{-N_r}, y)$ *is quasi-concave on* X_{N_r}*;*
(5) for each $j \in J$, $g_j(x, \,\cdot\,, y_{-N'_r})$ *is quasi-concave on* $Y_{N'_r}$*.*
Then, Γ *has at least a hybrid equilibrium.*

Proof. The proof is divided into two steps.

Step 1. Given an $x \in X$, for any $N'_r \in \bar{p}$ and $j \in N'_r$, we define a preference correspondence $P_j^{F_{r'}}(x,\cdot) : Y \rightrightarrows Y$ for the follower j by

$$P_j^{F_{r'}}(x,y) = \{(u_{N'_r}, y_{-N'_r}) \in Y \mid g_j(x, u_{N'_r}, y_{-N'_r}) > g_j(x, y_{N'_r}, y_{-N'_r})\}. \tag{3.2}$$

Since each member in $\bar{p}$ corresponds to a $N'_r \in \bar{p}$ and a $j \in N'_r$, we have a preference $P_j^{F_{r'}}$ for each member in $\bar{p}$. For convenience, let $J' = \{(N'_r, j)_{j \in N'_r} \mid r \in \{1, 2, \cdots, k_1\}\}$. Clearly, there exists a bijection between J' and J. Then, given an $x \in X$, the followers' game can be denoted by $\langle J, \bar{p}, \{Y_t\}_{t \in J}, \{P_j^{F_{r'}}(x,\cdot)\}_{(N'_r, j) \in J'}\rangle$. Clearly, $y \notin P_j^{F_{r'}}(x,y)$ for any $y \in Y$ and $j \in N'_r$. Note that $\bar{p}$ with $\bar{p} = \{N'_1, N'_2, \ldots, N'_{k_1}\}$ is a partition of J. Therefore, for any $j \in J$, it holds that $y \notin P_j^{F_{r'}}(x,y)$ for a given $x \in X$. Given a point $x \in X$, by the quasi-concave condition (5), from [31, Lemma 7.73], it obtains directly that $P_j^{F_{r'}}(x,\cdot)$ is convex for each $j \in J$. Next, from condition (3), g_j is continuous. For any $x \in X$ and $j \in N'_r$, it holds that $Graph(P_j^{F_{r'}}(x,\cdot))$ is open in $Y \times Y$. Thus, from Lemma 2.1, the followers game $\langle J, \bar{p}, \{Y_t\}_{t \in J}, \{P_j^{F_{r'}}(x,\cdot)\}_{(N'_r, j) \in J'}\rangle$ exists a hybrid equilibrium $\bar{y} = (\bar{y}_{N'_1}, \bar{y}_{N'_2}, \ldots, \bar{y}_{N'_{k_1}}) \in Y$. This means that, for any $N'_r \in \bar{p}$ and any $S' \subseteq N'_r$, there exists no $u_{S'} \in Y_{S'}$ such that $\{x\} \times \{u_{S'}\} \times Y_{N'_r - S'} \times \{\bar{y}_{N'_r}\} \subset P_j^{F_{r'}}(x, \bar{y})$, $\forall j \in S'$. According to (3.2), for the strategy $\bar{y} = (\bar{y}_{N'_1}, \bar{y}_{N'_2}, \ldots, \bar{y}_{N'_{k_1}})$ in Y, for any $N'_r \in \bar{p}$ and $S' \subseteq N'_r$, there exists no $u_{S'} \in Y_{S'}$ such that $g_j(x, u_{S'}, w_{N'_r - S'}, \bar{y}_{-N'_r}) > g_j(x, \bar{y}_{N'_r}, \bar{y}_{-N'_r})$, $\forall j \in S'$, $\forall w_{N'_r - S'} \in Y_{N'_r - S'}$.

Step 2. For the partition p of I with $p = \{N_1, N_2, \ldots, N_k\}$, for any $N_r \in p$ and $i \in N_r$, we define the preference correspondence $P_i^{L_r} : X \rightrightarrows X$ for the leader i by

$$P_i^{L_r}(x) = \{(z_{N_r}, x_{-N_r}) \in X \mid f_i(z_{N_r}, x_{-N_r}, y) > f_i(x_{N_r}, x_{-N_r}, y),\ \forall y \in h(x)\}. \tag{3.3}$$

Let $I' = \{(N_r, i)_{i \in N_r} \mid r \in \{1, 2, \cdots, k\}\}$. Obviously, there exists a bijection between I' and I. Then, the leaders' game can be denoted by $\langle I, p, \{X_t\}_{t \in I}, \{P_i^{L_r}\}_{(N_r, i) \in I'}\rangle$. Clearly, $x \notin P_i^{L_r}(x)$ for any $i \in I$ and $x \in X$. For any $x \in X$ and $i \in I$, $P_i^{L_r}(x)$ is convex. This can be deduced directly from [31, Lemma 7.73] by noting the quasi-concave condition (4).

We next show that the $Graph(P_i^{L_r})$ is open in $X \times X$ for each $i \in N_r$. For each $i \in N_r$, suppose that $(z^n_{N_r}, x^n_{-N_r}) \notin P_i^{L_r}(x^n)$ with $(x^n, z^n_{N_r}, x^n_{-N_r}) \longrightarrow (x, z_{N_r}, x_{-N_r}) \in X \times X$. It suffices to prove that $(z_{N_r},\, x_{-N_r}) \notin P_i^{L_r}(x)$. As $(z^n_{N_r}, x^n_{-N_r}) \notin P_i^{L_r}(x^n)$ for each $n = 1, 2, \ldots$, we have $f_i(z^n_{N_r}, x^n_{-N_r}, y^n) \leq$

$f_i(x^n_{N_r}, x^n_{-N_r}, y^n)$, for some $y^n \in h(x^n)$. From Lemma 3.1, h is an upper semi-continuous correspondence with nonempty compact values. From [31], one also sees that there exists a subsequence $\{y^{n_k}\}$ of y^n such that $\{y^{n_k}\} \longrightarrow y^0 \in h(x)$. Therefore, as k tends to infinity, we have $f_i(z_{N_r}, x_{-N_r}, y^0) \le f_i(x_{N_r}, x_{-N_r}, y^0)$ and $y^0 \in h(x)$. It implies that $(z_{N_r}, x_{-N_r}) \notin P_i^{L_r}(x)$. Thus, for each $i \in N_r$, for each $r \in \{1, 2, \cdots, k\}$, the graph of $P_i^{L_r}$ is open in $X \times X$. From Lemma 2.1, the game $\langle I, p, \{X_t\}_{t\in I}, \{P_i^{L_r}\}_{(N_r,i)\in I'}\rangle$ exists at least an $\bar{x} = (\bar{x}_{N_1}, \bar{x}_{N_2}, \ldots, \bar{x}_{N_k}) \in X$ such that for any $N_r \in p$ and any $S \subseteq N_r$, there exists no $z_S \in X_S$ for which $\{z_{S_i}\} \times X_{N_r - S} \times \{\bar{x}_{-N_r}\} \subset P_i^{L_r}(\bar{x})$, $\forall i \in S$. By (3.3), for any $N_r \in p$ and any $S \subseteq N_r$, there exists no $z_S \in X_S$ such that

$$f_i(z_S, w_{N_r-S}, \bar{x}_{-N_r}, \bar{y}) > f_i(\bar{x}_{N_r}, \bar{x}_{-N_r}, \bar{y}),\ \forall i \in S,\ \forall \bar{y} \in h(\bar{x}),\ \forall w_{N_r-S} \in X_{N_r-S}.$$

Finally, we have that the strategy profile $\bar{x} \in X$ is a hybrid equilibrium of Γ. The proof is completed. □

Remark 3.1.

(a) If the set J of followers is a singleton, Theorem 3.1 shows the existence of hybrid equilibria in multi-leader-single-follower games. If there are no followers in Γ, a hybrid equilibrium of Γ is a hybrid equilibrium of a normal form game with a partition p, which is a kind of general cooperative game in Zhao [13].

(b) For partitions p and $\bar{p}$, if $|N_r| = 1$ and $|N'_r| = 1$, the existence of hybrid equilibria is deduced to the existence of noncooperative (Nash) equilibria in *MLMF* games. On the other hand, if $|\ p\ | = |\ \bar{p}\ | = 1$, the existence of a hybrid equilibrium $\bar{x}$ with $\bar{y} \in h(\bar{x})$ is the existence of cooperative equilibria ($\bar{x}$ and $\bar{y}$ are α-core equilibria for leaders and followers, respectively) in *MLMF* games in Yang [23].

Example 3.1. Consider a two-leader-two-follower game

$$\Gamma = \langle I, J, p, \bar{p}, \{X_i\}_{i\in I}, \{Y_j\}_{j\in J}, \{f_i\}_{i\in I}, \{g_j\}_{j\in J}\rangle,$$

where $I = \{1, 2\}$, $J = \{1, 2\}$, $X_1 = X_2 = [0, 1] \in \mathbb{R}$, $Y_1 = Y_2 = [0, 1] \in \mathbb{R}$, and $f_1(x_1, x_2, y_1, y_2) = x_1 y_1 y_2$, $f_2(x_1, x_2, y_1, y_2) = x_2 y_2$, $g_1(x_1, x_2, y_1, y_2) = 1$, and $g_2(x_1, x_2, y_1, y_2) = y_2$ for any $(x_1, x_2, y_1, y_2) \in X \times Y$.

Case 1. For partitions $p = \{\{1\}, \{2\}\}$ and $\bar{p} = \{\{1\}, \{2\}\}$, Γ satisfies all the conditions of Theorem 3.1.

Let $\aleph(x)$ be the set of Nash equilibria in the parametric followers' game. It is clear that $\aleph(x) = [0, 1] \times \{1\} \subset Y$ for any $x \in X$. Therefore, the set of hybrid equilibria is $\{1\} \times \{1\}$ for the two-leader-two-follower game, where $\{1\} \times \{1\} \subset X$ is actually the Nash equilibrium set of the parametric leaders' game.

Case 2. For $p = \{\{1, 2\}\}$, $\bar{p} = \{\{1, 2\}\}$, Γ satisfies all the conditions of Theorem 3.1, and it is obvious that the hybrid equilibria of the two-leader-two-follower game are the same as α-core for leaders and followers.

Let $h(x)$ be the set of α-core for the parametric followers' game with the leaders' strategy $x \in X$. We can obtain that $h(x) = [0, 1] \times \{1\} \subset Y$ for any $x \in X$. Therefore, it can be checked that the set of hybrid equilibria of Γ is $[0, 1] \times \{1\}$. Furthermore, it can be found that the hybrid equilibrium are different in the above two cases.

Corollary 3.1. *Let $h : X \rightrightarrows Y$ be upper semi-continuous and nonempty compact valued. If a normal form game $\langle I, p, \{X_i\}_{i\in I}, \{f_i\}_{i\in I}\rangle$ with a partition p satisfies the conditions (1)-(4) of*

Theorem 3.1, then we say that the game has at least one hybrid equilibrium with respect to h. That is, there exists $\bar{x} \in X$ such that, for any $N_r \in p$ and $S \subseteq N_r$, there exists no $z_S \in X_S$ such that

$$f_i(z_S, w_{N_r-S}, \bar{x}_{-N_r}, \bar{y}) > f_i(\bar{x}_{N_r}, \bar{x}_{-N_r}, \bar{y}),\ \forall i \in S,\ \forall \bar{y} \in h(\bar{x}),\ \forall w_{N_r-S} \in X_{N_r-S}.$$

Proof. By the step 2 in Theorem 3.1, the existence of hybrid equilibria in the parametric leaders' game implies the desired result. □

4. The Existence of Weak Hybrid Equilibria for *MLMF* Games with Infinitely Players

Before introducing weak hybrid equilibria of *MLMF* games and proving their existence, we need to introduce weak hybrid equilibria to normal form games and give a deep study for their existence.

Given a normal form game

$$G = \langle N, p, \{X_i\}_{i\in N}, \{v_i\}_{i\in N}\rangle, \tag{4.1}$$

where $p = \{N_r \mid r \in R\}$ is a partition of the player set N (R is an index set), and $v_i : X \longrightarrow \mathbb{R}$ is the payoff function of a player $i \in N$. Recall that $X = X_N = \prod_{i\in N} X_i$. Define a set

$$\hat{\Omega} = \{(N_r, S) \mid S \subseteq N_r,\ N_r \in p\}.$$

Definition 4.1. For a normal form game G as (4.1), a strategy $\bar{x} \in X$ is said to be *hybrid-blocked* by (N_r, S) of $\hat{\Omega}$ if there exists $x_S \in X_S$ such that $v_i(x_S, z_{N_r-S}, \bar{x}_{-N_r}) - v_i(\bar{x}_{N_r}, \bar{x}_{-N_r}) > 0,\ \forall i \in S,\ \forall z_{N_r-S} \in X_{N_r-S}$. A strategy $\bar{x} \in X$ is *strongly hybrid-blocked* by (N_r, S) of $\hat{\Omega}$ if there exists $x_S \in X_S$ and $\varepsilon > 0$ such that $v_i(x_S, z_{N_r-S}, \bar{x}_{-N_r}) - v_i(\bar{x}_{N_r}, \bar{x}_{-N_r}) > \varepsilon,\ \forall i \in S,\ \forall z_{N_r-S} \in X_{N_r-S}$. A strategy profile $\bar{x} \in X$ is called a *weak hybrid equilibrium* of G, if it cannot be strongly hybrid-blocked by any $(N_r, S) \in \hat{\Omega}$.

Remark 4.1. (a) The hybrid-blocking concept for normal form games in Definition 4.1 is from the hybrid-blocking concept in games without ordered preferences in Yang and Yuan [15]. The hybrid-blocking concept was defined by the $\alpha-$blocking concept, see [15, 25]. The strongly hybrid-blocking in Definition 4.1 is inspired by the strongly hybrid-blocking for games without ordered preferences in [15], and the relation between them needs to be studied in the future.

(b) If a strategy $\bar{x}$ cannot be hybrid-blocked, which means that, for each group N_r, each coalition S in N_r, and the fixed strategy $\bar{x}_{-N_r}$ of the other groups, $\bar{x}_{N_r}$ cannot be $\alpha-$blocked by S. That is, each $\bar{x}_{N_r}$ is in the $\alpha-$core of the game $\langle N_r, (X_i)_{i\in N_r}, v_i(\cdot, \bar{x}_{-N_r})_{i\in N_r}\rangle$.

Assumption 4.1. N is a nonempty compact subset of a Hausdorff topological space.

Assumption 4.2. For each $i \in N$, X_i is a nonempty convex compact subset of $\mathbb{R}^{m_i}$.

Assumption 4.3. For each $i \in N$, v_i is continuous on X, and v_i is quasi-concave on X_{N_r} for each $N_r \in p$.

From Definition 4.1, it is obvious that the following lemma holds.

Lemma 4.1. *Under Assumptions 4.1-4.3, if a strategy $x \in X$ can be strongly hybrid-blocked by a member (N_r, S) in $\hat{\Omega}$, then x can be hybrid-blocked by (N_r, S).*

Lemma 4.2. *Under Assumptions 4.1-4.3, for each $(N_r,S)\in\hat{\Omega}$, define $F(N_r,S)=\{x\in X\mid x$ cannot be strongly hybrid-blocked by $(N_r,S)\}$. Then $F(N_r,S)$ is closed on X.*

Proof. It suffices to show that $F(N_r,S)^c$ is open on X. If $x\in F(N_r,S)^c$, then there exists $y_S\in X_S$ and $\varepsilon>0$ such that $v_i(y_S,z_{N_r-S},x_{-N_r})-v_i(x_{N_r},x_{-N_r})>\varepsilon$, $\forall i\in S$, $\forall z_{N_r-S}\in X_{N_r-S}$. Since N is a nonempty compact set, clS is compact on N. By Assumption 4.3, we have

$$\min_{i\in clS}\min_{z_{N_r-S}\in X_{N_r-S}}[v_i(y_S,z_{N_r-S},x_{-N_r})-v_i(x_{N_r},x_{-N_r})]\geq\varepsilon>\tfrac{\varepsilon}{2}.$$

Then,

$$(x_{N_r},x_{-N_r})\to\min_{i\in clS}\min_{z_{N_r-S}\in X_{N_r-S}}[v_i(y_S,z_{N_r-S},x_{-N_r})-v_i(x_{N_r},x_{-N_r})]$$

is continuous on X. Then there exists an open neighborhood $U(x)$ of x in X such that

$$\min_{i\in clS}\min_{z_{N_r-S}\in X_{N_r-S}}[v_i(y_S,z_{N_r-S},x'_{-N_r})-v_i(x'_{N_r},x'_{-N_r})]>\frac{\varepsilon}{2},\ \forall x'\in U(x).$$

Therefore, for any $x'\in U(x)$, we obtain that $v_i(y_S,z_{N_r-S},x'_{-N_r})-v_i(x'_{N_r},x'_{-N_r})>\frac{\varepsilon}{2}$. That is, $U(x)\subset F(N_r,S)^c$. The proof is completed. □

Lemma 4.3. *Under Assumptions 4.1-4.3, for any $\{(N_r,S_{r,i})_{i=1}^{n(r)}\mid S_{r,i}\subseteq N_r,\ \forall i=1,\dots,n(r)\}_{r=1}^{r_0}$ of $\hat{\Omega}$, there exists $x\in X$ such that x cannot be hybrid-blocked by $(N_r,S_{r,i})$, $\forall i=1,\dots,n(r)$, $\forall r=1,\dots,r_0$.*

Proof. Without loss of generality, we assume that $N_r-\bigcup_{i=1}^{n(r)}S_{r,i}\neq\emptyset$, $N-\bigcup_{r=1}^{r_0}N_r\neq\emptyset$. Let

$$S_{r,n(r)+1}=N_r-\bigcup_{i=1}^{n(r)}S_{r,i},\ \forall r=1,\dots,r_0,\ N_{r_0+1}=N-\bigcup_{r=1}^{r_0}N_r,\ S_{r_0+1,1}=N_{r_0+1},$$

$\bar{n}(r)=n(r)+1$, and $\bar{n}(r_0+1)=1$. Thus, we obtain a family $\{(N_r,S_{r,i})_{i=1}^{\bar{n}(r)}\mid S_{r,i}\subseteq N_r,\ \forall i=1,\dots,\bar{n}(r)\}_{r=1}^{r_0+1}$. For any $\{S_{r,i}\subseteq N_r\mid i=1,\dots,\bar{n}(r)\}$, $r=1,\dots,r_0+1$, there definitely exists a family $\{K_{r,j}\subseteq N_r\mid j=1,\dots,m(r)\}$ such that

$$N_r=\bigcup_{i=1}^{\bar{n}(r)}S_{r,i}=\bigcup_{j=1}^{m(r)}K_{r,j}\text{ with }K_{r,a}\cap K_{r,b}=\emptyset,\ \forall a\neq b.$$

Obviously, each $S_{r,i}$ is a union of some sets $K_{r,j}$ for any $r=1,\dots,r_0+1$. In fact, $\{K_{r,j}\mid j=1,\dots,m(r)\}$ becomes a partition of N_r.

We next construct a finite-player normal form game $\langle I,\bar{p},\{Y_q\}_{q\in I},\{\phi_q\}_{q\in I}\rangle$ with a partition $\bar{p}$ as follows:

(1) $\bar{p}=\{I_1,I_2,\dots,I_{r_0}\}$, $I_r=\{(r,j(r))\mid j(r)=1,\dots,m(r)\}$, $\forall\ r=1,\dots,r_0$.
(2) $I=\cup_{r=1}^{r_0}I_r=\cup_{r=1}^{r_0}\{(r,j(r))\mid j(r)=1,\dots,m(r)\}$. In addition, for convenience, denote $\cup_{j\in B}K_{r,j}$ by $K_{r,B}$, and let $(r,B)=\{(r,j):j\in B\}\subset I_r$, where $r\in\{1,\dots,r_0\}$ and $B\subseteq\{1,\dots,m(r)\}$. Let $K_I=\{K_{r,j}:(r,j)\in I\}$. It is clear that $N-K_I=N_{r_0+1}\neq\varnothing$.
(3) For any player $q=(r,j(r))\in I$, Y_q is the player's nonempty convex compact strategy set with $Y_q=\prod_{i\in K_q}X_i\subset\prod_{i\in K_q}\mathbb{R}^{m_i}$.

(4) For each player $q=(r,j(r))\in I$, the payoff function $\phi_q: Y_I(Y_I=\prod_{l\in I}Y_l=\prod_{l\in I}\prod_{i\in K_l}X_i=X_{K_I})\to\mathbb{R}$ is defined by $\phi_q(y_I)=v_{t(q)}(y_{I_r},y_{I-I_r},x^0_{-K_I})$, $\forall y_I\in Y_I$, where $t(q)$ is picked and fixed in K_q, and $x^0_{-K_I}$ is picked and fixed with $x^0_{-K_I}\in X_{-K_I}=X_{N-K_I}=X_{N_{r_0+1}}$.

Firstly, it is easy to verify that ϕ_q is continuous on Y_I. Next, since $v_{t(q)}$ is quasi-concave on X_{N_r} from Assumption 4.3, we have that ϕ_q is quasi-concave on Y_{I_r} with $Y_{I_r}=\prod_{l\in I_r}Y_l=\prod_{l\in I_r}\prod_{i\in K_l}X_i=X_{N_r}$, $\forall r=1,2,\ldots,r_0$. Therefore, the game $\langle I,\bar{p},\{Y_q\}_{q\in I},\{\phi_q\}_{q\in I}\rangle$ satisfies all the conditions of the theorem 2 in Zhao [13]. Hence, there exists a hybrid equilibrium $y^0_I\in\prod_{l\in I}Y_l$ such that, for any $I_r\in\bar{p}$ and any $B\subseteq\{1,\ldots,m(r)\}$, there exists no $y_{(r,B)}\in\prod_{l\in(r,B)}Y_l$ such that

$$\phi_q(y_{(r,B)},z_{I_r-(r,B)},y^0_{I-I_r})>\phi_q(y^0_{I_r},y^0_{I-I_r}),\forall q\in(r,B),\ \forall z_{I_r-(r,B)}\in\prod_{l\in I_r-(r,B)}Y_l.$$

That is, for any $r=1,\ldots,r_0$ and $B\subseteq\{1,\ldots,m(r)\}$, there exists no $y_{(r,B)}\in\prod_{l\in(r,B)}Y_l$ such that

$$v_{t(q)}(y_{(r,B)},z_{I_r-(r,B)},y^0_{I-I_r},x^0_{-K_I})>v_{t(q)}(y^0_{I_r},y^0_{I-I_r},x^0_{-K_I}),\tag{4.2}$$

for any $z_{I_r-(r,B)}\in\prod_{l\in I_r-(r,B)}Y_l$, and for any $q=(r,j(r))\in(r,B)$ (recall that $t(q)$ is fixed in K_q). Since $\prod_{l\in(r,B)}Y_l=\prod_{l\in(r,B)}\prod_{i\in K_l}X_i=X_{K_{r,B}}$, it is written $y_{(r,B)}\in\prod_{l\in(r,B)}Y_l$ as $y_{K_{r,B}}\in X_{K_{r,B}}$. Since $Y_{I_r}=X_{N_r}$, it holds that $\prod_{l\in I_r-(r,B)}Y_l=\prod_{l\in I_r-(r,B)}\prod_{i\in K_l}X_i=X_{N_r-K_{r,B}}$. We will write $z_{I_r-(r,B)}\in\prod_{l\in I_r-(r,B)}Y_l$ as $z_{N_r-K_{r,B}}\in X_{N_r-K_{r,B}}$, and write $y^0_{I_r}\in Y_{I_r}$ as $y^0_{N_r}\in X_{N_r}$. In addition, since $Y_{I-I_r}=X_{K_I-N_r}=X_{N-N_{r_0+1}-N_r}$, it can be written $y^0_{I-I_r}\in Y_{I-I_r}$ as $y^0_{K_I-N_r}\in X_{K_I-N_r}$. Then, from (4.2), for any $r=1,\ldots,r_0$, $B\subseteq\{1,\ldots,m(r)\}$, and $y_{K_{r,B}}\in X_{K_{r,B}}$, there exist some $z_{N_r-K_{r,B}}\in X_{N_r-K_{r,B}}$, and $q=(r,j(r))\in(r,B)$ such that $v_{t(q)}(y_{K_{r,B}},z_{N_r-K_{r,B}},y^0_{K_I-N_r},x^0_{-K_I})\le v_{t(q)}(y^0_{N_r},y^0_{K_I-N_r},x^0_{-K_I})$. Observe that, for any fixed $r=\{1,\ldots,r_0\}$ and fixed $i\in\{1,\ldots,n(r)\}$, there exists $B\subseteq\{1,\ldots,m(r)\}$, such that $S_{r,i}=K_{r,B}$. Therefore, for any $y_{S_{r,i}}=y_{K_{r,B}}\in X_{K_{r,B}}=X_{S_{r,i}}$, there exist some $z_{N_r-S_{r,i}}\in X_{N_r-S_{r,i}}$, and $q=(r,j(r))\in(r,B),t(q)\in K_{r,j(r)}\subset S_{r,i}$ such that

$$v_{t(q)}(y_{S_{r,i}},z_{N_r-S_{r,i}},y^0_{K_I-N_r},x^0_{-K_I})\le v_{t(q)}(y^0_{N_r},y^0_{k_I-N_r},x^0_{-K_I}).\tag{4.3}$$

Let $x^0_{-N_r}=(y^0_{K_I-N_r},x^0_{-K_I})$. Then, Eq. (4.3) becomes $v_{t(q)}(y_{S_{r,i}},z_{N_r-S_{r,i}},x^0_{-N_r})\le v_{t(q)}(y^0_{N_r},x^0_{-N_r})$. Since $y^0_{N_r}\in X_{N_r}$, $y^0_{k_I-N_r}\in X_{K_I-N_r}=X_{N-N_{r_0+1}-N_r}$, and $x^0_{-K_I}\in X_{-K_I}=X_{N_{r_0}+1}$, we have $x^0_{-N_r}=(y^0_{k_I-N_r},x^0_{-K_I})\in X_{-N_r}$. Hence, $\hat{x}=(y^0,x^0_{-K_I})=(y^0_{N_r},x^0_{-N_r})\in X$. It shows that $\hat{x}\in X$ cannot be hybrid-blocked by $(N_r,S_{r,i})$ for each $r=1,\ldots,r_0$ and $i=1,\ldots,n(r)$. This completes the proof. □

Theorem 4.1. *If a normal form game G as (4.1) satisfies Assumptions 4.1-4.3, then G at least has a weak hybrid equilibrium.*

Proof. By Lemma 4.3, for any finite family $\hat{\Omega}'$ of members of $\hat{\Omega}$, there exists $x\in X$ such that x cannot be hybrid-blocked by any $(N_r,S)\in\hat{\Omega}'$. Further, by Lemma 4.1, x cannot be strongly hybrid-blocked by any $(N_r,S)\in\hat{\Omega}'$. That is, $x\in\cap_{(N_r,S)\in\hat{\Omega}'}F(N_r,S)$. Note that $F(N_r,S)$ is closed by Lemma 4.2. In view of the compactness of X, one sees there exists $x\in X$ such that $x\in\cap_{(N_r,S)\in\hat{\Omega}}F(N_r,S)$. Then, x is a weak hybrid equilibrium of G. The proof is completed. □

Based on the above analysis and methods of weak hybrid equilibria for normal form games, we can introduce the notion of the weak hybrid equilibria for *MLMF* games with infinitely many players and give their existence results.

Give a *MLMF* game

$$\Gamma = \langle I, J, p, \bar{p}, \{X_i\}_{i\in I}, \{Y_j\}_{j\in J}, \{f_i\}_{i\in I}, \{g_j\}_{j\in J}\rangle \tag{4.4}$$

with a partition $p = \{N_r \subseteq I \mid r \in R\}$ of the set I of leaders and a partition $\bar{p} = \{N'_r \subseteq J \mid r \in R'\}$ of the set J of followers, where R and R' are index sets. Note that $\cup_{r\in R} N_r = I$ and $\cup_{r\in R'} N'_r = J$, where $N_r \cap N_{r'} = \emptyset$ and $N'_r \cap N'_{r'} = \emptyset$, $\forall r \neq r'$. We define the set Ω and Ω' by

$$\Omega = \{(N_r, S) \mid S \subseteq N_r,\ N_r \in p\},\ \Omega' = \{(N'_r, S') \mid S' \subseteq N'_r,\ N'_r \in \bar{p}\}.$$

Given a point $x \in X$, we say that $\bar{y} \in Y$ is *strongly hybrid-blocked* by (N'_r, S') of Ω', if there exists $y_{S'} \in Y_{S'}$ and $\varepsilon' > 0$ such that

$$g_j(x, y_{S'}, w_{N'_r - S'}, \bar{y}_{-N'_r}) - g_j(x, \bar{y}_{N'_r}, \bar{y}_{-N'_r}) > \varepsilon',\ \forall j \in S',\ \forall w_{N'_r - S'} \in Y_{N'_r - S'}.$$

Let $h_s(x)$ be the set of weak hybrid equilibria of the followers' normal form game $\langle J, \bar{p}, \{Y_j\}_{j\in J}, \{g_j(x,\cdot)\}_{j\in J}\rangle$ with the parameter $x \in X$, which yields a correspondence $h_s : X \rightrightarrows Y$. That is, $\bar{y} \in h_S(x)$ means that $\bar{y}$ cannot be strongly hybrid-blocked by any (N'_r, S') in Ω'.

Definition 4.2. A point $\bar{x} \in X$ is *strongly hybrid-blocked* by (N_r, S) of Ω if there exists $x_S \in X_S$ and $\varepsilon > 0$ such that

$$f_i(x_S, w_{N_r - S}, \bar{x}_{-N_r}, \bar{y}) - f_i(\bar{x}_{N_r}, \bar{x}_{-N_r}, \bar{y}) > \varepsilon,\ \forall i \in S,\ \forall \bar{y} \in h_s(\bar{x}),\ \forall w_{N_r - S} \in X_{N_r - S}.$$

A point $\bar{x} \in X$ is a *weak hybrid equilibrium* of a *MLMF* game Γ, if $\bar{x}$ cannot be strongly hybrid-blocked by any (N_r, S) in Ω.

We need the following assumptions and lemmas.

Assumption 4.4. I and J are nonempty and compact subsets of Hausdorff topological space.

Assumption 4.5. For each leader $i \in I$ (each follower $j \in J$), the strategy set $X_i(Y_j)$ is a nonempty, convex, and compact subset of $\mathbb{R}^{m_i}$ $(\mathbb{R}^{m_j})$.

Assumption 4.6. For each $i \in I$, f_i is continuous on $X \times Y$, and $f_i(\,\cdot\,, x_{-N_r}, y)$ is quasi-concave on X_{N_r}.

Assumption 4.7. For each $j \in J$, g_j is continuous on $X \times Y$, and $g_j(x,\,\cdot\,, y_{-N'_r})$ is quasi-concave on $Y_{N'_r}$.

Lemma 4.4. *Under Assumptions 4.4-4.7, $h_s : X \rightrightarrows Y$ is an upper semi-continuous correspondence with nonempty compact values.*

Proof. Given a fixed $x \in X$, Assumptions 4.4, 4.5, and 4.7 imply that the parametric followers' normal form game $G = \langle J, \bar{p}, \{Y_j\}_{j\in J}, \{g_j(x,\cdot)\}_{j\in J}\rangle$ satisfies the conditions of Theorem 4.1. Then $h_s(x) \neq \emptyset$. Since Y is compact, it suffices to prove that $Graph(h_s)$ is closed. Suppose that $\{(x^n, y^n)\}$ is a sequence in $X \times Y$ with $(x^n, y^n) \longrightarrow (x, y) \in X \times Y$ and $y^n \in h_s(x^n)$. It needs to show that $y \in h_s(x)$. If $y \notin h_s(x)$, then there exists $(N'_r, S') \in \Omega'$, $z_{S'} \in Y_{S'}$ and $\varepsilon' > 0$ such that

$$g_j(x, z_{S'}, w_{N'_r - S'}, y_{-N'_r}) - g_j(x, y_{N'_r}, y_{-N'_r}) > \varepsilon' > 0,\ \forall j \in S',\ \forall w_{N'_r - S'} \in Y_{N'_r - S'_i}.$$

From Assumption 4.5 and Assumption 4.7, we have

$$\min_{w_{N'_r - S'} \in Y_{N'_r - S'}} [g_j(x, z_{S'}, w_{N'_r - S'}, y_{-N'_r}) - g_j(x, y_{N'_r}, y_{-N'_r})] > \frac{\varepsilon'}{2} > 0,\ \forall j \in S'. \tag{4.5}$$

Then, $(x, z_{S'}, y_{N'_r}, y_{-N'_r}) \longrightarrow \min_{w_{N'_r-S'} \in Y_{N'_r-S'}} [g_j(x, z_{S'}, w_{N'_r-S'}, y_{-N'_r}) - g_j(x, y_{N'_r}, y_{-N'_r})]$ is continuous. Therefore, from (4.5), there exists $n_0 > 0$ such that, when $n > n_0$, it holds that, for each $j \in S'$,

$$\min_{w_{N'_r-S'} \in Y_{N'_r-S'}} [g_j(x^n, z_{S'}, w_{N'_r-S'}, y^n_{-N'_r}) - g_j(x^n, y^n_{N'_r}, y^n_{-N'_r})] > \frac{\varepsilon'}{2} > 0,$$

which contradicts $y^n \in h_s(x^n)$. □

Similar to Lemma 4.2, one can obtain the following result. For completeness, we give the proof.

Lemma 4.5. *Give a MLMF game Γ as (4.4). For each $(N_r, S) \in \Omega$, let $T(N_r, S) = \{\bar{x} \in X \mid \bar{x}$ cannot be strongly hybrid-blocked by $(N_r, S)\}$. Then, under Assumptions 4.4-4.7, $T(N_r, S)$ is closed in X.*

Proof. It suffices to prove that $T(N_r, S)^c$ is open on X. Take an $\bar{x} \in T(N_r, S)^c$. According the notion of strong hybrid-blocking, there exists $x_S \in X_S$ and $\varepsilon > 0$ such that

$$f_i(x_S, w_{N_r-S}, \bar{x}_{-N_r}, \bar{y}) - f_i(\bar{x}_{N_r}, \bar{x}_{-N_r}, \bar{y}) > \varepsilon,\ \forall i \in S,\ \forall \bar{y} \in h_s(\bar{x}),\ \forall w_{N_r-S} \in X_{N_r-S}.$$

By Lemma 4.4, $h_s(\bar{x})$ is compact valued and upper semi-continuous. From Assumption 4.5 and Assumption 4.6, we have

$$\min_{\bar{y} \in h_s(\bar{x})} \min_{i \in clS} \min_{w_{N_r-S} \in X_{N_r-S}} [f_i(x_S, w_{N_r-S}, \bar{x}_{-N_r}, \bar{y}) - f_i(\bar{x}_{N_r}, \bar{x}_{-N_r}, \bar{y})] \geq \varepsilon > \frac{\varepsilon}{2},$$

and

$$(\bar{x}_{N_r}, \bar{x}_{-N_r}) \longrightarrow \min_{\bar{y} \in h_s(\bar{x})} \min_{i \in clS} \min_{w_{N_r-S} \in X_{N_r-S}} [f_i(x_S, w_{N_r-S}, \bar{x}_{-N_r}, \bar{y}) - f_i(\bar{x}_{N_r}, \bar{x}_{-N_r}, \bar{y})]$$

is lower semi-continuous on X. Therefore, there exists an open neighborhood $V(\bar{x})$ of $\bar{x}$ in X such that, for any $x' \in V(\bar{x})$, it holds that

$$\min_{\bar{y} \in h_s(x')} \min_{i \in clS} \min_{w_{N_r-S} \in X_{N_r-S}} [f_i(x_S, w_{N_r-S}, x'_{-N_r}, \bar{y}) - f_i(x'_{N_r}, x'_{-N_r}, \bar{y})] > \frac{\varepsilon}{2}.$$

That is, for any $x' \in V(\bar{x})$,

$$f_i(x_S, w_{N_r-S}, x'_{-N_r}, \bar{y}) - f_i(x'_{N_r}, x'_{-N_r}, \bar{y}) > \frac{\varepsilon}{2}, \forall i \in S, \forall \bar{y} \in h_s(x'), \forall w_{N_r-S} \in X_{N_r-S}.$$

Then, $V(\bar{x}) \subset T(N_r, S)^c$. Hence, $T(N_r, S)$ is closed on X. □

Using partially the methods in Lemma 4.3, we have the following lemma.

Lemma 4.6. *Give a MLMF game Γ as (4.4). Under Assumptions 4.4-4.7, for any finite set*

$$\{(N_r, S_{r,i})_{i=1}^{n(r)} \mid S_{r,i} \subseteq N_r, \forall i = 1, \ldots, n(r)\}_{r=1}^{r_0}$$

of Ω, there exists $\bar{x} \in X$ such that $\bar{x}$ cannot be hybrid-blocked by $(N_r, S_{r,i})$, $\forall r = 1, \ldots, r_0$, $\forall i = 1, \ldots, n(r)$.

Proof. Without loss of generality, we assume that

$$N_r - \bigcup_{i=1}^{n(r)} S_{r,i} \neq \emptyset,\ I - \bigcup_{r=1}^{r_0} N_r \neq \emptyset.$$

Let

$$S_{r,n(r)+1} = N_r - \bigcup_{i=1}^{n(r)} S_{r,i},\ \forall r = 1,\dots,r_0,\ N_{r_0+1} = I - \bigcup_{r=1}^{r_0} N_r,\ S_{r_0+1,1} = N_{r_0+1},$$

$\bar{n}(r) = n(r)+1$, and $\bar{n}(r_0+1) = 1$. Thus, we obtain a family $\{(N_r, S_{r,i})_{i=1}^{\bar{n}(r)} \mid S_{r,i} \subseteq N_r,\ \forall i = 1,\dots,\bar{n}(r)\}_{r=1}^{r_0+1}$. For any $r = 1,\dots,r_0$, there must exist a family $\{K_{r,j} \subseteq N_r \mid j = 1,\dots,m(r)\}$ such that

$$\bigcup_{i=1}^{\bar{n}(r)} S_{r,i} = \bigcup_{j=1}^{m(r)} K_{r,j},\ K_{r,a} \cap K_{r,b} = \emptyset,\ \forall a \neq b.$$

Obviously, each $S_{r,i}$ is a union of some sets $K_{r,j}$. Then, it holds that $\{K_{r,j} \mid j = 1,\dots,m(r)\}$ is a partition of N_r for each $r = 1,\dots,r_0$. It will construct a finite-player normal form game $\langle M, \widetilde{p}, \{U_q\}_{q\in M}, \{\varphi_q\}_{q\in M}\rangle$ with a partition $\widetilde{p}$ of the set M of players and a correspondence $H : \prod_{l\in M} U_l \rightrightarrows Y$ as follows.

(1) $M = \cup_{r=1}^{r_0}\{(r,j) \mid j = 1,\dots,m(r)\}$; for convenience, let $K_M = \cup_{q\in M} K_q$; then $I - K_M = N_{r_0+1} \neq \emptyset$; for each $r \in \{1,\dots,r_0\}$ and $B \subseteq \{1,\dots,m(r)\}$, let $K_{r,B} = \cup_{l\in B} K_{r,l}$, and $(r,B) = \{(r,j) : j \in B\}$;
(2) $\widetilde{p} = \{M_1, M_2, \dots, M_{r_0}\}$, $M_r = \{(r,j) \mid j = 1,\dots,m(r)\}, \forall r = 1,\dots,r_0$; it is clear that $M = \cup_{r=1}^{r_0} M_r$;
(3) for any player $q = (r, j(r)) \in M$, the strategy set of the player, U_q with $U_q = \prod_{i\in K_q} X_i$, is a nonempty convex compact subset of $\prod_{i\in K_q} \mathbb{R}^{m_i}$;
(4) Y is the joint strategy set of followers of Γ with $Y = \prod_{j\in J} Y_j$, a nonempty convex compact subset of $\prod_{j\in J} R^{m_j}$;
(5) for each player $q = (r, j(r)) \in M$, the payoff function $\varphi_q : \prod_{l\in M} U_l \times Y$ (clearly, $\prod_{l\in M} U_l \times Y = X_{K_M} \times Y) \longrightarrow \mathbb{R}$ is defined by

$$\varphi_q(u,y) = f_{t(q)}(u, z'_{N_{r_0}+1}, y), \forall u = (u_{M_1}, u_{M_2}, \dots, u_{M_{r_0}}) \in \prod_{l\in M} U_l,\ \forall y \in Y,$$

where $t(q)$ is picked and fixed in K_q, and $z'_{N_{r_0}+1}$ is picked and fixed in $X_{N_{r_0}+1}$ with $X_{N_{r_0}+1} = X_{I-K_M}$. It is true that $(u, z'_{N_{r_0}+1}) \in X = X_I$;
(6) the correspondence $H : \prod_{l\in M} U_l \rightrightarrows Y$ is defined by

$$H(u) = h_s(u, z'_{N_{r_0}+1}),\ \forall u = (u_{M_1}, u_{M_2}, \dots, u_{M_{r_0}}) \in \prod_{l\in M} U_l.$$

where h_s is from Definition 4.2 and $z'_{N_{r_0}+1}$ is the same as that in (5).

It can be checked that φ_q is continuous on $\prod_{l\in M} U_l \times Y$ and quasi-concave on U_{M_r} by Assumption 4.6 (note that $U_{M_r} = \prod_{l\in M_r} U_l = \prod_{l\in M_r}\prod_{i\in K_l} X_i = X_{N_r}$). And H is upper semi-continuous and has nonempty compact values by Lemma 4.4.

Obviously, the game $\langle M, \widetilde{p}, \{U_q\}_{q\in M}, \{\varphi_q\}_{q\in M}\rangle$ with Y and the correspondence H satisfies all conditions of Corollary 3.1. Thus, there exists $\bar{x} \in \prod_{l\in M} U_l$ such that $\bar{x}$ cannot be hybrid-blocked by any $M_r \in \widetilde{p}$ and any coalitions in M_r. That is, for any $M_r \in \widetilde{p}$ and any $B \subseteq \{1,\dots,m(r)\}$, there exists no $z_{(r,B)} \in \prod_{l\in(r,B)} U_l$, such that, for any $q = (r,j) \in (r,B)$, $w_{M_r-(r,B)} \in \prod_{l\in M_r-(r,B)} U_l$, and $y \in H(\bar{x})$, it holds that $\varphi_q(z_{(r,B)}, w_{M_r-(r,B)}, \bar{x}_{-M_r}, y) > \varphi_q(\bar{x}_{M_r}, \bar{x}_{-M_r}, y)$. Hence, for any $r =$

$1,\ldots,r_0$ and any $B \subseteq \{1,\ldots,m(r)\}$, there exists no $z_{(r,B)} \in \prod_{l\in(r,B)} U_l$ such that, for any $q = (r, j(r)) \in (r,B)$, $w_{M_r-(r,B)} \in \prod_{l\in M_r-(r,B)} U_l$, and $y \in H(\bar{x})$, it is true that

$$f_{t(q)}(z_{(r,B)}, w_{M_r-(r,B)}, \bar{x}_{M-M_r}, z'_{N_{r_0}+1}, y) > f_{t(q)}(\bar{x}_{M_r}, \bar{x}_{M-M_r}, z'_{N_{r_0}+1}, y).$$

It means that, for any $r = 1,\ldots,r_0$, $B \subseteq \{1,\ldots,m(r)\}$, and $z_{(r,B)} \in \prod_{l\in(r,B)} U_l$, there exist some $w_{M_r-(r,B)} \in \prod_{l\in M_r-(r,B)} U_l$, $y \in h_s(\bar{x}, z'_{N_{r_0}+1})$, and $q = (r, j) \in (r,B)$, such that

$$f_{t(q)}(z_{(r,B)}, w_{M_r-(r,B)}, \bar{x}_{M-M_r}, z'_{N_{r_0}+1}, y) \leq f_{t(q)}(\bar{x}_{M_r}, \bar{x}_{M-M_r}, z'_{N_{r_0}+1}, y). \tag{4.6}$$

Since $\prod_{l\in(r,B)} U_l = \prod_{l\in(r,B)} \prod_{i\in K_l} X_i = X_{K_{r,B}}$, we can write $z_{(r,B)} \in \prod_{l\in(r,B)} U_l$ as $z_{K_{r,B}} \in X_{K_{r,B}}$. Note that $\prod_{l\in M_r-K_{r,B}} U_l = X_{N_r-K_{r,B}}$. Then, $w_{M_r-(r,B)} \in \prod_{l\in M_r-(r,B)} U_l$ will be written as $w_{N_r-K_{r,B}} \in X_{N_r-K_{r,B}}$. Since $\prod_{l\in M_r} U_l = X_{N_r}$, it can be written $\bar{x}_{M_r} \in \prod_{l\in M_r} U_l$ as $\bar{x}_{N_r} \in X_{N_r}$ for each $r = 1,2,\ldots,r_0$. Therefore, (4.6) can be expressed as: for any $r \in \{1,\ldots,r_0\}$, $B \subseteq \{1,\ldots,m(r)\}$, and $z_{K_{r,B}} \in X_{K_{r,B}}$, there exist some $w_{N_r-K_{r,B}} \in X_{N_r-K_{r,B}}$, $y \in h_s(\bar{x}, z'_{N_{r_0}+1})$, and $q = (r, j) \in (r,B)$, such that $f_{t(q)}(z_{K_{r,B}}, w_{N_r-K_{r,B}}, \bar{x}_{I-N_{r_0+1}-N_r}, z'_{N_{r_0+1}}, y) \leq f_{t(q)}(\bar{x}_{N_r}, \bar{x}_{I-N_{r_0+1}-N_r}, z'_{N_{r_0+1}}, y)$. Note that, for any $r = \{1,\ldots,r_0\}$ and $i \in \{1,\ldots,n(r)\}$, there exists $B \subseteq \{1,\ldots,m(r)\}$, such that $S_{r,i} = K_{r,B}$. Thus, for any $z_{S_{r,i}} = z_{K_{r,B}} \in X_{S_{r,i}} = X_{K_{r,B}}$ with $r \in \{1,\ldots,r_0\}$ and $i \in \{1,\ldots,n(r)\}$, there exists $w_{N_r-S_{r,i}} \in X_{N_r-S_{r,i}}$, $y \in h_s(\bar{x}, z'_{N_{r_0}+1})$, and $q = (r, j) \in (r,B)$ with $t(q) \in K_{r,j} \subset S_{r,i}$, such that

$$f_{t(q)}(z_{S_{r,i}}, w_{N_r-S_{r,i}}, \bar{x}_{I-N_{r_0+1}-N_r}, z'_{N_{r_0+1}}, y) \leq f_{t(q)}(\bar{x}_{N_r}, \bar{x}_{I-N_{r_0+1}-N_r}, z'_{N_{r_0+1}}, y). \tag{4.7}$$

Let $x'_{-N_r} = (\bar{x}_{I-N_{r_0+1}-N_r}, z'_{N_{r_0+1}}) \in X_{-N_r}$. Then, (4.7) is reduced to $f_{t(q)}(z_{S_{r,i}}, w_{N_r-S_{r,i}}, x'_{-N_r}, y) \leq f_{t(q)}(\bar{x}_{N_r}, x'_{-N_r}, y)$, which implies that $\hat{x} = (\bar{x}, z'_{N_{r_0+1}})$ cannot be hybrid-blocked by $(N_r, S_{r,i})$ for any $r = \{1,\ldots,r_0\}$ and $i \in \{1,\ldots,n(r)\}$. The proof is finished. □

Based on the above lemmas, we can establish the following existence theorem for weak hybrid equilibria in *MLMF* games with infinitely many leaders and many followers.

Theorem 4.2. *If a MLMF game* Γ *as* (4.4) *satisfies Assumptions 4.4-4.7, then the game has at least a weak hybrid equilibrium.*

Proof. The proof is divided into two steps.

Step 1. By Lemma 4.4, we know that, for any $x \in X$, $h_s(x) \neq \emptyset$. That is, for each strategy profile x in X, there exists $\bar{y} \in h_s(x)$ such that, for any $(N'_r, S') \in \Omega'$, there exists no $y_{S'} \in Y_{S'}$ and $\varepsilon' > 0$ for which $g_j(x, y_{S'}, w_{N'_r-S'}, \bar{y}_{-N'_r}) - g_j(x, \bar{y}_{N'_r}, \bar{y}_{-N'_r}) > \varepsilon'$, $\forall j \in S'$, $\forall w_{N'_r-S'} \in Y_{N'_r-S'}$.

Step 2. From Lemma 4.6 and Lemma 4.5, any finite members of $\{T(N_r,S) \mid (N_r,S) \in \Omega\}$ have a nonempty intersection. Since X is compact, and $T(N_r,S)$ is closed by Lemma 4.5, there exists $\bar{x} \in X$ such that $\bar{x} \in \bigcap_{(N_r,S)\in\Omega} T(N_r,S)$ That is, $\bar{x}$ is a weak hybrid equilibrium of the *MLMF* game Γ. □

Remark 4.2. For a *MLMF* game Γ as (4.4), if $|\ p\ | = |\ \bar{p}\ | = 1$, the existence of weak hybrid equilibria of Γ is reduced to the existence of a weak α-core in the *MLMF* game Γ.

5. Concluding Remarks

Hybrid equilibria and weak hybrid equilibria were introduced for *MLMF* games. By constructing nonordered preferences for each player, the results of Yang and Yuan in [15] and Lemma 3.1 were employed to prove the existence of hybrid equilibria of *MLMF* games. Theorem 3.1 includes a hybrid equilibrium of a normal form game with partitions in Zhao [13]. If

$|p| = |\bar{p}| = 1$, the existence of a hybrid equilibrium implies the existence of cooperative equilibria in Yang and Ju [23]. If all partitions only have one player, a hybrid equilibrium in Theorem 3.1 can reduce to a noncooperative solution in [19] by Hu and Fukushima. For proposed weak hybrid equilibria of $MLMF$ games, the results in [15] cannot be directly used to prove their existence. In this paper, by proving the existence of weak hybrid equilibria of normal form games with partitions in Theorem 4.1, and combing the existence of hybrid equilibria in Theorem 3.1, the existence of weak hybrid equilibria for $MLMF$ games was proved.

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