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DIRECTIONAL DERIVATIVE OF SET-VALUED MAPPINGS INVOLVING GENERALIZED ORIENTED DISTANCE FUNCTIONS AND APPLICATIONS TO SET OPTIMIZATION

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Abstract. This paper focuses on the directional derivative and subdifferential of set-valued mappings via nonlinear scalarizing functions. Firstly, we define the directional derivative and subdifferential of set-valued mappings by using a generalized oriented distance function. Secondly, we systematically investigated the operational rules, positive homogeneity, chain rule, and upper semicontinuity of the directional derivative for set-valued mappings. Thirdly, we examine the convexity and weak* closedness of the subdifferential of set-valued mappings, as well as its relationship with the directional derivative. Finally, the optimality conditions for set optimization problems are established by utilizing the introduced subdifferential.
Keywords. Directional derivative; Nonlinear scalarizing function; Set optimization; Subdifferential.
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1. INTRODUCTION

Scalarization methods play a pivotal role in both the theoretical analysis and solution approaches for set optimization problems. Among these methods, the Gerstewitz's function and oriented distance function are extensively utilized. Due to its superior mathematical properties, the Gerstewitz's function garnered significant attention, and numerous scholars conducted extensive and in-depth research based on this function; see, e.g., [7, 13, 16] and the references therein. The generalized oriented distance function, introduction by Ha [4] in 2018, has garnered significant attention due to its numerous advantageous mathematical properties. Recently, ongoing research continues to uncover its potential characteristics. Han and Yu [10] investigated the translation properties and triangular inequality properties of the generalized oriented distance function, and, they, based on this function, proposed a weighted set order relation. Das et al. [1] studied the existence and connectedness of l-minimal approximate solutions for set-valued optimization problems by using the generalized oriented distance function. In 2021, Han, Huang, and Wen [5] systematically investigated various properties of the generalized oriented distance function, including but not limited to its calculation rules and subadditivity. They further utilized this function to conduct an in-depth analysis of the Dini directional derivative

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of set-valued operators. In this paper, we establish the subadditivity of the generalized oriented distance function. Compared to the approach presented in [5], our proof-process exhibits generality. To the best of our knowledge, no prior research investigated the directional derivative of set-valued mappings by using the generalized oriented distance function. Therefore, We introduce a novel directional derivative of set-valued mappings based on the generalized oriented distance function proposed by Ha [4] and derived several key properties of the directional derivative of set-valued mappings, including its operation rules, positive homogeneity, chain rule, and upper semicontinuity. In 2022, Han [6] conducted a systematic investigation into the Clarke generalized directional derivative of set-valued mappings by using the Gerstewitz's function. Leveraging the operation rules and the positive homogeneity, Han established optimality conditions for set optimization problems. Furthermore, Han highlighted in [6] that further exploration of the Clarke generalized subdifferential for set-valued mappings holds substantial theoretical value. Han in [7] investigated directional derivatives and subdifferentials of cone-convex set-valued mappings based on the Gerstewitz's function. This study yielded several interesting findings, which were subsequently applied to set optimization problems, further perfecting the optimality conditions for set optimization problems. Inspired by the research work of Han [7], we observe that the research on the subdifferential of set-valued mappings based on the generalized oriented distance function remains relatively limited. In this paper, we introduce the concept of the subdifferential of set-valued mappings grounded in the generalized oriented distance function and derive convexity and weak* closedness of the subdifferential of set-valued mappings, as well as its relationship with the directional derivative of set-valued mappings.

The structure of this paper is organized as follows. Section 2 reviews some properties of the generalized oriented distance function and introduces the concepts of semicontinuity and cone-convex set-valued mappings. In Section 3, we employ generalized oriented distance function to examine the directional derivatives of set-valued mappings, thereby deriving several key properties. Section 4 provides an in-depth analysis of subdifferential based on generalized oriented distance function. In Section 5, we investigate the optimality conditions for set optimization problems. Finally, Section 6 ends this paper.

2. PRELIMINARIES

Let X and Y be real-normed linear spaces. K is called a cone in Y if $\lambda x \in K$ for all $x \in K$ and $\lambda \geqslant 0$. The cone K induces a partial order on Y as, for any $x, y \in Y$, $x \preccurlyeq_K y \Longleftrightarrow y - x \in K$. Assume that Y^* is the dual space of Y, and the dual cone K^* of K is defined by $K^* = \{y^* \in Y : \langle y^*, k\rangle \geqslant 0, \forall\, k \in K\}$. Assume that K is a convex, pointed, and closed cone with nonempty interior and U_0 is an unit open ball, while $\bar{U}_0$ is a closed unit ball in Y. We denote the family of nonempty subsets of Y by $P_0(Y)$. $\mathbb{R}^n$ denotes the n dimensional Euclidean space. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_i \geqslant 0, \ i = 1, ..., n\}$. Let $A, B \in P_0(Y)$, $\varepsilon \geqslant 0$, and $e \in intK$. We consider the following set relations on Y, the weak lower relation " $\prec^l_K$ " and the weak ε-lower relation " $\prec^l_{\varepsilon,K}$ ", which are defined as (see [8, 19]):

$$A \prec^l_K B \Longleftrightarrow B \subseteq A + intK, \ A \prec^l_{\varepsilon,K} B \Longleftrightarrow B \subseteq A + intK + \varepsilon e.$$

It is said that a nonempty set $A \subseteq Y$ is K-proper if $A + K \neq Y$, K-bounded if, for each neighbourhood O of zero in Y, there exists some positive number t such that $A \subseteq tO + K$, K-closed if $A + K$

is a closed set, and K-compact if, any cover of A of the form $\{O_\alpha + K \,:\, \alpha \in I,\ O_\alpha \text{ are open}\}$ admits a finite subcover. It was documented if A is K-compact, then A is K-bounded and K-closed (see [20]).

Let $A \subseteq Y$. A function $\triangle_A : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by $\triangle_A(y) := d_A(y) - d_{A^c}(y)$, for all $y \in Y$, is said to be an oriented distance function [21], where $d_A(y) := \inf_{a\in A} \| y - a \|$ is the distance function from $y \in Y$ to the set A. Let A, B be nonempty subsets of Y. Recall from [4] that the generalized oriented distance function $D_K : P_0(Y) \times P_0(Y) \to \mathbb{R} \cup \{\pm\infty\}$ is defined by $D_K(A,B) := \sup_{b\in B} \inf_{a\in A} \triangle_{-K}(a-b)$. Let S be a nonempty convex subset of X. A set-valued mapping $F : X \rightrightarrows Y$ is said to be K-convex [3] on S if, for any $x_1, x_2 \in S$ and for any $t \in [0,1]$, $tF(x_1) + (1-t)F(x_2) \subseteq F(tx_1 + (1-t)x_2) + K$. It is clear that if F is K-convex on S, then $F(x)$ is K-convex for any $x \in S$; see [7]. Recall from [3] that a set-valued mapping $F : X \rightrightarrows Y$ is said to be

(i) K-upper semicontinuous (K-$u.s.c.$) at $x_0 \in X$ if, for any neighborhood V of $F(x_0)$, there exists a neighborhood $U(x_0)$ of x_0 such that, for every $x \in U(x_0)$, $F(x) \subseteq V + K$.

(ii) K-lower semicontinuous (K-$l.s.c.$) at $x_0 \in X$ if, for any $y \in F(x_0)$ and any neighborhood V of y, there exists a neighborhood $U(x_0)$ of x_0 such that, for every $x \in U(x_0)$, $F(x) \cap (V - K) \neq \emptyset$.

We define F as (K-$u.s.c.$) and (K-$l.s.c.$) on $S \subseteq X$ if it satisfies (K-$u.s.c.$) and (K-$l.s.c.$) at each point $x \in S$, respectively. We consider that F is K-continuous on S if it is both (K-$u.s.c.$) and (K-$l.s.c.$) on S.

Finally, we present two essential lemmas.

Lemma 2.1. [10, 21] *Let $A \subseteq Y$ be nonempty and $A \neq Y$. Then the following assertions hold:*

(i) $\triangle_A(\cdot)$ is real-valued and 1-Lipschitzian.

(ii) If A is a closed convex cone, then $\triangle_{-A}$ nondecreasing with respect to the ordering induced by A, i.e., if $y_1,\ y_2 \in Y$ and $y_2 - y_1 \in A$, then $\triangle_{-A}(y_1) \leqslant \triangle_{-A}(y_2)$.

(iii) $\triangle_A(-y) = \triangle_{-A}(y)$ for all $y \in Y$.

(iv) If A is a convex cone and int$A \neq \emptyset$, then $\triangle_A(y) := \sup_{y* \in S(A^*)} \langle -y^*, y \rangle$ for all $y \in Y$, where $S(A^*) := \{y^* \in A^* \mid \|y^*\| = 1\}$.

Lemma 2.2. [10, 17] *Let $A, B \in P_0(Y)$. Then the following assertions hold:*

(i) If $e \in K$, $\varepsilon \in R$, $d_{-K}(e) = d_{Y\setminus -K}(-e) = 1$, A and B are K-proper and K-bounded, then $D_K(A + \varepsilon e, B) = D_K(A,B) + \varepsilon$.

(ii) If A is K-proper, then $D_K(A,A) = 0$.

(iii) If B is K-compact and K is solid, then $A \prec^l_K B \iff D_K(A,B) < 0$.

(iv) If $A, B, C \in P_0(Y)$ is K-proper and K-compact, then $D_K(A,B) \leqslant D_K(A,C) + D_K(C,B)$.

3. Directional Derivative of Set-Valued Mappings

In this section, we investigate directional derivative of the generalized oriented distance function. Let S be a nonempty and convex subset of X and $x \in S$. Recall that the directional derivative of f at x in the direction $\mu \in X$, denoted by $f'(x;\mu)$, where $f : X \to \mathbb{R}$ is a function, is defined by $f'(x;\mu) = \lim_{t\to 0^+} \frac{f(x+t\mu) - f(x)}{t}$. Inspired by the results in [7, 11], we define the directional derivative of the set-valued mapping under study by using the generalized oriented distance function introduced by Ha [4].

Let $F : X \rightrightarrows Y$ be a set-valued mapping, S a nonempty and convex subset of x, and $x \in S$ and $\mu \in X$. The directional derivative of the generalized oriented distance function at x in direction

μ is defined by

$$F'(x;\mu) = \lim_{t\to 0^+} \frac{D_K(F(x+t\mu),F(x))}{t}.$$

Remark 3.1. It should be noted that if the limit exists, then the directional derivative $F'(\cdot,\cdot)$ is a generalized real-valued function. Furthermore, it is evident that the directional derivative of the set-valued mapping introduced in this paper differs from the definition presented in [2, 4, 11, 15].

Lemma 3.1. [18] *Let A, $B \in P_0(Y)$. If A is K-proper and B is K-bounded, then $D_K(A,B) \in \mathbb{R}$.*

Remark 3.2. Assume that $F(y)$ is nonempty and K-bounded for any $y \in S$. For each $x \in S$, there exists a function $\xi_x : S \to \mathbb{R}$ such that the directional derivative $\xi_x'(x;\cdot)$ coincides with $F'(x;\cdot)$.

We define the function $\xi_x : S \to \mathbb{R}\cup\{-\infty,+\infty\}$ by $\xi_x(y) = D_K(F(y),F(x))$ for all y in S. It follows from Lemma 3.1 that $-\infty < \xi_x(y) < +\infty$ for any $y \in S$. In view of Lemma 2.2 (ii), we have $D_K(F(x),F(x)) = 0$. Then, for any $\mu \in X$,

$$\begin{aligned}\xi_x'(x;\mu) &= \lim_{t\to 0^+} \frac{\xi_x(x+t\mu)-\xi_x(x)}{t}\\ &= \lim_{t\to 0^+} \frac{D_K(F(x+t\mu),F(x))-D_K(F(x),F(x))}{t}\\ &= \lim_{t\to 0^+} \frac{D_K(F(x+t\mu),F(x))}{t}\\ &= F'(x;\mu).\end{aligned}$$

It is worth noting that $\xi_x'(z;\mu) = F'(x;\mu)$ is only true when $z = x$.

We present the following example to demonstrate the calculation of the directional derivative $F'(\cdot;\cdot)$.

Example 3.1. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $X = \mathbb{R}$, $\bar{y}^* \in K^*$, and $S(K^*) := \{\bar{y}^* \in K^* \mid \|\bar{y}^*\| = 1\}$. Clearly, $K^* = \mathbb{R}^2_+$, $X^* = \mathbb{R}^+$ and $S(K^*)$ is the set of points on the circumference of a quarter of the unit circle. The set valued mapping $F : X \rightrightarrows Y$ is defined by $F(x) = co\{(|x|,|x|),(x^2,x^2)\}$ for all x in X. Clearly, $F(-x) = F(x)$. According to Lemma 2.1 (iii) and (iv), one has

$$\triangle_{-K}(a-b) = \triangle_K(b-a) = \sup_{y^*\in S(K^*)} \langle -y^*, b-a\rangle = \sup_{y^*\in S(K^*)} \langle y^*, a-b\rangle.$$

Let $\bar{x} = 0$. For any $\mu \in X$ and a sufficiently small t, we can obtain $\bar{x}+t\mu \in domF = R$. It is easy to obtain $F(\bar{x}) = F(0) = co\{(0,0),(0,0)\}$ and $F(\bar{x}+t\mu) = F(t\mu) = co\{(|t\mu|,|t\mu|),(t^2\mu^2,t^2\mu^2)\}$, so

$$\begin{aligned}D_K(F(x+t\mu),F(x)) &= \sup_{b\in F(x)} \inf_{a\in F(x+t\mu)} \triangle_{-K}(a-b)\\ &= \sup_{b\in F(x)} \inf_{a\in F(x+t\mu)} \sup_{\bar{y}^*\in S(K^*)} \langle \bar{y}^*, a-b\rangle\\ &= \sup_{b\in F(x)} \inf_{a\in F(x+t\mu)} \langle \frac{a-b}{\|a-b\|}, a-b\rangle\\ &= \langle \frac{(|t\mu|,|t\mu|)-(0,0)}{\|(|t\mu|,|t\mu|)-(0,0)\|}, (|t\mu|,|t\mu|)-(0,0)\rangle\\ &= \sqrt{2}|t\mu|.\end{aligned}$$

Letting $\mu = 1$, we obtain $F'(0;1) = \lim_{t\to 0^+} \frac{\sqrt{2}|t\mu|}{t} = \sqrt{2}$. Therefore, the directional derivative of F at $\bar{x} = 0$ in the direction $\mu = 1$ is $\sqrt{2}$.

Lemma 3.2. [18] *Let $A, B \in P_0(Y)$ and $r \geqslant 0$. Then $D_K(A,B) \leqslant r \Leftrightarrow B \subseteq cl(rU_0 + A + K)$.*

Remark 3.3. If U_0 is a closed set, then the same conclusion holds, $D_K(A,B) \leqslant r \Leftrightarrow B \subseteq cl(r\bar{U}_0 + A + K)$.

Theorem 3.1. *Let $x, \mu \in X$, and let $F(\cdot)$ be K-convex on X with nonempty K-compact values. Then, for any $t, r \in \mathbb{R}$ with $0 < t \leqslant r$, $\frac{D_K(F(x+t\mu),F(x))}{t} \leqslant \frac{D_K(F(x+r\mu),F(x))}{r}$.*

Proof. Let $t, r \in R$ with $0 < t \leqslant r$. Since $F(\cdot)$ is K-convex on X, one has

$$\frac{r-t}{r} F(x) + \frac{t}{r} F(x + r\mu) \subseteq F(x + t\mu) + K. \tag{3.1}$$

Since $F(x + r\mu)$ and $F(x)$ are nonempty, K-proper and K-bounded, we obtain from Lemma 3.1 that $-\infty < D_K(F(x + r\mu), F(x)) < +\infty$. Let $\eta := D_K(F(x + r\mu), F(x))$. By Remark 3.3, we have $F(x) \subseteq cl\left(\eta\bar{U}_0 + F(x + r\mu) + K\right)$. For any $y \in F(x)$, there exist sequences

$$\{y_\eta^{(n)}\} \subseteq \eta\bar{U}_0, \quad \{z_r^{(n)}\} \subseteq F(x + r\mu), \quad \{k_0^{(n)}\} \subseteq K, \text{ for all } n \in \mathbb{N},$$

such that

$$y = \lim_{n\to\infty} \left(y_\eta^{(n)} + z_r^{(n)} + k_0^{(n)}\right). \tag{3.2}$$

For each n, by equation (3.1), there exist $z_t^{(n)} \in F(x + t\mu)$ and $\bar{k}^{(n)} \in K$ satisfying

$$\frac{r-t}{r}\left(y_\eta^{(n)} + z_r^{(n)} + k_0^{(n)}\right) + \frac{t}{r} z_r^{(n)} = z_t^{(n)} + \bar{k}^{(n)}, \text{ for all } n \in \mathbb{N}. \tag{3.3}$$

Thanks to (3.2) and (3.3), we obtain

$$y = \frac{r-t}{r} y + \frac{t}{r} y = \lim_{n\to\infty}\left(z_t^{(n)} + \frac{t}{r} y_\eta^{(n)} + \bar{k}^{(n)} + \frac{t}{r} k_0^{(n)}\right) \in cl\left(F(x + t\mu) + \frac{t\eta}{r}\bar{U}_0 + K\right).$$

By the arbitrariness of y, one has

$$F(x) \subseteq cl\left(\frac{t\eta}{r}\bar{U}_0 + F(x + t\mu) + K\right),$$

which together with Remark 3.3 implies that $D_K(F(x+t\mu), F(x)) \leqslant \frac{t}{r}\eta = \frac{t}{r} D_K(F(x+r\mu), F(x))$, so

$$\frac{D_K(F(x + t\mu), F(x))}{t} \leqslant \frac{D_K(F(x + r\mu), F(x))}{r}.$$

This completes the proof. □

Let $F : X \rightrightarrows Y$ be a nonempty set-valued mapping and S_1 and S_2 be two nonempty subsets of X. We define $\tau : S_1 \times S_2 \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\tau(\lambda, \mu) = D_K(F(\lambda), F(\mu)) = \sup_{b\in F(\mu)} \inf_{a\in F(\lambda)} \triangle_{-K}(a - b), \quad \forall\, (\lambda, \mu) \in S_1 \times S_2.$$

From [17, Proposition 3.6], we can draw the following corollary.

Corollary 3.1. *If $\mu_0 \in S_2$ and $F(\cdot)$ is K-compact and K-convex, then $\tau(\cdot, \mu_0)$ is a convex function.*

Based on the convexity of the $\tau(\cdot, \mu)$, we can establish the following theorem.

Theorem 3.2. *Let $x, \mu \in X$, and let $F(\cdot)$ be K-convex on X with nonempty K-compact values. Then, for any $\delta, \beta > 0$,*

$$\frac{D_K(F(x), F(x-\beta\mu))}{\beta} \leqslant \frac{D_K(F(x+\delta\mu), F(x))}{\delta}.$$

Proof. Let $\delta, \beta > 0$.Then

$$x = \frac{\delta}{\delta+\beta}(x-\beta\mu) + \frac{\beta}{\delta+\beta}(x+\delta\mu). \tag{3.4}$$

We define the function $\tau_1 : S \to \mathbb{R}$ by, for all y in S, $\tau_1(y) = D_K(F(y), F(x-\beta\mu))$. It follows from Lemma 3.1 that $-\infty < \tau_1(y) < +\infty$ for any $y \in S$. By Lemma 2.2 (ii), we have $D_K(F(x-\beta\mu), F(x-\beta\mu)) = 0$, so $\tau_1(x-\beta\mu) = 0$. It follows from Corollary 3.1 that τ_1 is a convex function on S. Combining this with (3.4) and $\tau_1(x-\beta\mu) = 0$, we obtain

$$\begin{aligned}\frac{D_K(F(x), F(x-\beta\mu))}{\beta} &= \frac{\tau_1(x) - \tau_1(x-\beta\mu)}{\beta} = \frac{\tau_1(\frac{\delta}{\delta+\beta}(x-\beta\mu) + \frac{\beta}{\delta+\beta}(x+\delta\mu))}{\beta} \\ &\leqslant \frac{\frac{\delta}{\delta+\beta}\tau_1(x-\beta\mu) + \frac{\beta}{\delta+\beta}\tau_1(x+\delta\mu)}{\beta} = \frac{\tau_1(x+\delta\mu)}{\delta+\beta}.\end{aligned}$$

Further, we can also obtain

$$\frac{\tau_1(x)}{\beta} \leqslant \frac{\tau_1(x+\delta\mu) - \tau_1(x)}{\delta}. \tag{3.5}$$

Thanks to Lemma 2.2 (iv), we have

$$\begin{aligned}\tau_1(x+\delta\mu) - \tau_1(x) &= D_K(F(x+\delta\mu), F(x-\beta\mu)) - D_K(F(x), F(x-\beta\mu)) \\ &\leqslant D_K(F(x+\delta\mu), F(x)).\end{aligned} \tag{3.6}$$

We can derive the desired conclusion based on Inequalities (3.5) and (3.6) immediately. □

Based on the above theorem, we derive the operational rule of the directional derivative of set-valued mappings.

Theorem 3.3. *Assume that $F(\cdot)$ is K-convex on X with nonempty K-compact values. If the directional derivative $F'(x;\mu)$ exists, for any positive number $\delta > 0$, then*

$$F'(x;\mu) = \inf_{0<t\leqslant\delta} \frac{D_K(F(x+t\mu), F(x))}{t}.$$

Proof. Together with Theorems 3.1 and 3.2, we have

$$\frac{D_K(F(x), F(x-\beta\mu))}{\beta} \leqslant \frac{D_K(F(x+t\mu), F(x))}{t}, \quad \forall\, t \in (0, \delta],$$

which indicates that $\inf\limits_{0<t\leqslant\delta} \frac{D_K(F(x+t\mu),F(x))}{t}$ exists and

$$\frac{D_K(F(x), F(x-\beta\mu))}{\beta} \leqslant \inf_{0<t\leqslant\delta} \frac{D_K(F(x+t\mu), F(x))}{t}.$$

Let $\tau_2 = \inf\limits_{0<t\leqslant\delta} \frac{D_K(F(x+t\mu),F(x))}{t}$. For any $\varepsilon > 0$, there exists $\tilde{t} \in (0, \delta]$ such that

$$\frac{D_K(F(x+\tilde{t}\mu), F(x))}{\tilde{t}} < \tau_2 + \varepsilon.$$

Then, for any $t \in (0,\tilde{t})$, we conclude from Theorem 3.1 that

$$\tau_2 - \varepsilon < \tau_2 \leqslant \frac{D_K(F(x+t\mu),F(x))}{t} \leqslant \frac{D_K(F(x+\tilde{t}\mu),F(x))}{\tilde{t}} < \tau_2 + \varepsilon.$$

Hence, we have

$$F'(x;\mu) = \lim_{t\to 0^+} \frac{D_K(F(x+t\mu),F(x))}{t} = \tau_2 = \inf_{0<t\leqslant\delta} \frac{D_K(F(x+t\mu),F(x))}{t}.$$

The proof is completed. □

In addition to the operational properties of the directional derivative of set-valued mappings, we can also establish its positive homogeneity.

Theorem 3.4. *Let $x,\mu \in X$, and let $F(\cdot)$ be K-convex on X with nonempty and K-compact values. Then the following assertions hold:*

(i) $F'(x;0) = 0$.

(ii) $F'(x;\lambda\mu) = \lambda F'(x;\mu)$ for all $\lambda \geqslant 0$.

Proof. (i) If $\mu = 0$, then $F(x+t\mu) = F(x)$. It follows from Lemma 2.2 (ii) that $D_K(F(x+t\mu),F(x)) = 0$. Thus, $F'(x;0) = 0$.

(ii) For $\lambda = 0$, it is obvious from (i). For $\lambda > 0$, we have

$$F'(x;\lambda\mu) = \lim_{t\to 0^+} \frac{D_K(F(x+t\lambda\mu),F(x))}{t} = \lambda \lim_{t\to 0^+} \frac{D_K(F(x+t\lambda\mu),F(x))}{t\lambda} = \lambda F'(x;\mu).$$

The proof is completed. □

Subsequently, we investigate the subadditivity property of the generalized oriented distance function. Compared with the method presented in [5], our method is more intuitive.

Theorem 3.5. *Assume that A,B,C and D are nonempty and K-bounded. Then, $D_K(A+C,B+D) \leqslant D_K(A,B)+D_K(C,D)$.*

Proof. Let $\alpha = D_K(A,B)$ and $\beta = D_K(C,D)$. For any $\varepsilon \geqslant 0$, it is clear that $D_K(A,B) \leqslant \alpha+\varepsilon$ and $D_K(C,D) \leqslant \beta+\varepsilon$. Thanks to Lemma 3.2, we have $B \subseteq cl((\alpha+\varepsilon)U_0+A+K)$ and $D \subseteq cl((\beta+\varepsilon)U_0+C+K)$. Then, $B+D \subseteq cl((\alpha+\beta+2\varepsilon)U_0+A+C+K)$. Combining this with Lemma 3.2, we find $D_K(A+C,B+D) \leqslant \alpha+\beta+2\varepsilon$. By the arbitrariness of $\varepsilon \geqslant 0$, we obtain $D_K(A+C,B+D) \leqslant \alpha+\beta = D_K(A,B)+D_K(C,D)$. This completes the proof. □

Furthermore, based on the above subadditivity, we can derive the chain rule for directional derivatives of set-valued mappings.

Theorem 3.6. *Let $x,\mu \in X$, and let $F_1(\cdot),F_2(\cdot)$ be K-convex and K-compact on X with nonempty values. Then $(F_1+F_2)'(x;\mu) \leqslant F_1'(x;\mu)+F_2'(x;\mu)$.*

Proof. Note that $(F_1+F_2)(x) = F_1(x)+F_2(x)$ for any $x \in X$. From Theorem 3.3, for any $\delta > 0$, we obtain

$$F_i'(x;\mu) = \inf_{0<t\leqslant\delta} \frac{D_K(F_i(x+t\mu),F_i(x))}{t}, \quad i=1,2.$$

Thus, for any $\varepsilon > 0$, there exists $t_i \in (0,\delta]$ such that

$$\frac{D_K(F_i(x+t_i\mu),F_i(x))}{t_i} \leqslant F_i'(x;\mu)+\varepsilon, \quad i=1,2. \tag{3.7}$$

Let $\tilde{t} = \min\{t_1, t_2\} > 0$. It follows from Theorem 3.1 and inequality (3.7) that

$$\frac{D_K(F_i(x+\tilde{t}\mu), F_i(x))}{\tilde{t}} \leqslant \frac{D_K(F_i(x+t_i\mu), F_i(x))}{t_i} < F_i'(x;\mu) + \varepsilon, \quad i = 1,2. \tag{3.8}$$

Furthermore, according to Theorem 3.5, one has

$$\begin{aligned} D_K(F_1(x+\tilde{t}\mu) + F_2(x+\tilde{t}\mu), F_1(x) + F_2(x)) \leqslant\, & D_K(F_1(x+\tilde{t}\mu), F_1(x)) \\ & + D_K(F_2(x+\tilde{t}\mu), F_2(x)). \end{aligned} \tag{3.9}$$

In view of Theorem 3.3, (3.7), (3.8), and (3.9), we have

$$\begin{aligned} (F_1+F_2)'(x;\mu) &= \inf_{0<t\leqslant\delta} \frac{D_K(F_1(x+t\mu) + F_2(x+t\mu), F_1(x) + F_2(x))}{t} \\ &\leqslant \frac{D_K(F_1(x+\tilde{t}\mu) + F_2(x+\tilde{t}\mu), F_1(x) + F_2(x))}{\tilde{t}} \\ &\leqslant \frac{D_K(F_1(x+\tilde{t}\mu), F_1(x)) + D_K(F_2(x+\tilde{t}\mu), F_2(x))}{\tilde{t}} \\ &< F_1'(x;\mu) + F_2'(x;\mu) + 2\varepsilon. \end{aligned}$$

By the arbitrariness of $\varepsilon > 0$, we see that $(F_1+F_2)'(x;\mu) \leqslant F_1'(x;\mu) + F_2'(x;\mu)$. The proof is completed. □

The following corollary establishes the continuity of the generalized oriented distance function.

Corollary 3.2. [14] *Let $F : X \rightrightarrows Y$ be a nonempty set-valued mapping. If $F(\cdot)$ is K-continuous and K-compact values, then $\tau(\cdot,\cdot)$ is continuous on $S_1 \times S_2$.*

On account of the continuity of the generalized oriented distance function, it can be deduced that the directional derivative exhibits upper semicontinuity.

Theorem 3.7. *Let $S \subseteq X$ be a nonempty and convex set with nonempty interior and $F(\cdot)$ be K-convex on X with nonempty and K-compact values. Then, $F'(\cdot;\cdot)$ is an upper semicontinuous function on $S \times X$.*

Proof. Let $\{(x_n, \mu_n)\} \subseteq intS \times X$ with $(x_n, \mu_n) \to (x_0, \mu_0) \in intS \times X$. It suffices to show that $\limsup_{n\to\infty} F'(x_n, \mu_n) \leqslant F'(x_0, \mu_0)$. Suppose that $\limsup_{n\to\infty} F'(x_n, \mu_n) > F'(x_0, \mu_0)$. Then, there exists $\delta \in \mathbb{R}$ such that

$$\limsup_{n\to\infty} F'(x_n, \mu_n) > \delta > F'(x_0, \mu_0) \tag{3.10}$$

Due to $x_0 \in intS$, there exists a neighborhood O of $0 \in X$ such that $x_0 + O \subseteq S$. In view of $(x_n, \mu_n) \to (x_0, \mu_0)$, we can find $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that

$$\forall\, n \geqslant n_0 : x_n + \alpha\mu_n \in x_0 + O \subseteq S. \tag{3.11}$$

Note that $\mu_0 \in X$, there exists $\eta > 0$ such that $x_0 + \eta\mu_0 \in S$. It follows from (3.10) and Theorem 3.3 that $F'(x_0;\mu) = \inf_{0<t\leqslant\eta} \frac{D_K(F(x+t\mu), F(x))}{t} < \delta$. Then, there exists $t_0 \in (0, \eta]$ such that

$$\frac{D_K(F(x_0 + t_0\mu_0), F(x_0))}{t_0} < \delta. \tag{3.12}$$

Let $\beta := \min\{t_0, \alpha\} > 0$. In view of (3.11), we have

$$\forall\, n \geqslant n_0 : x_n + \beta \mu_n \in x_0 + O \subseteq S. \tag{3.13}$$

Thanks to (3.12) and Theorem 3.1, we have

$$\frac{D_K(F(x_0+\beta\mu_0), F(x_0))}{\beta} \leqslant \frac{D_K(F(x_0+t_0\mu_0), F(x_0))}{t_0} < \delta. \tag{3.14}$$

Due to $x_n + \beta\mu_n \to x_0 + \beta\mu_0$, $x_n \to x_0$ and Corollary 3.2, we have

$$\frac{D_K(F(x_n+\beta\mu_n), F(x_n))}{\beta} \to \frac{D_K(F(x_0+\beta\mu_0), F(x_0))}{\beta}$$

This together with (3.13), (3.14), and Theorem 3.3 implies that

$$F'(x_n;\mu_n) = \inf_{0<t\leqslant\beta} \frac{D_K(F(x_n+t\mu_n), F(x_n))}{t} \leqslant \frac{D_K(F(x_n+\beta\mu_n), F(x_n))}{\beta} < \delta.$$

for n large enough. This means that $\limsup_{n\to\infty} F'(x_n,\mu_n) \leqslant \delta$, which contradicts (3.10). This completes the proof. □

4. Subdifferential of Set-Valued Mappings

In this section, we provide a rigorous characterization of the subdifferential of set-valued mappings with respect to the generalized oriented distance function. Assume that $f : X \to \mathbb{R}$ is a real valued function. The Clarke generalized directional derivative is a classical directional derivative, and it is expressed as $f^\circ(x;\mu) = \limsup_{\substack{y\to x\\ t\to 0^+}} \frac{f(y+t\mu)-f(y)}{t}$. The generalized subdifferential of f at x, denoted by $\partial_0 f(x)$, is the subset of the dual space X^* defined as follows:

$$\partial_0 f(x) = \{y^* \in X^* : f^\circ(x;\mu) \geqslant \langle y^*, \mu\rangle,\ \forall\, \mu \in X\}.$$

Inspired by the work in [7, 6], we provide the formal definition of the subdifferential of set-valued mappings for the generalized oriented distance function.

Definition 4.1. Let S be a nonempty and convex subset of X and $F : X \rightrightarrows Y$ be a set valued mapping. Assume that $S \subseteq domF$ and $x \in S$. The subdifferential of the generalized oriented distance function at x on S, denoted by $\partial_S F(x)$, is the subset of the dual space X^* defined as $\partial_S F(x) = \{y^* \in X^* : D_K(F(z), F(x)) \geqslant \langle y^*, z-x\rangle, \forall z \in S\}$.

In the following, we present several fundamental characteristics of subdifferential of the generalized oriented distance function.

Theorem 4.1. *Let S be a nonempty and convex subset of X and $F : X \rightrightarrows Y$ be a set valued mapping. Assume that $S \subseteq domF$ and $x \in S$. The following assertions hold:*
(i) If $D \subseteq S$, then $\partial_S F(x) \subseteq \partial_D F(x)$.
(ii) $\partial_S F(x)$ is convex.
(iii) $\partial_S F(x)$ is weak* closed

Proof. (i) Let $y^* \in \partial_S F(x)$. According to Definition 4.1, we have $D_K(F(z), F(x)) \geqslant \langle y^*, z-x\rangle$ for any $z \in S$. Since $D \subseteq S$, any point in D is also in S. Therefore, for any $z \in D$, $D_K(F(z), F(x)) \geqslant \langle y^*, z-x\rangle$.

(ii) Let $\bar{y}^* = \lambda y_1^* + (1-\lambda) y_2^*$, where $y_1^*,\ y_2^* \in \partial_S F(x)$ and $\lambda \in [0,1]$. It suffices to show that $D_K(F(z), F(x)) \geqslant \langle \bar{y}^*, z-x\rangle$ for all $z \in S$. Due to $\bar{y}^* = \lambda y_1^* + (1-\lambda) y_2^*$, we obtain $\langle \bar{y}^*, z-x\rangle =$

$\langle \lambda y_1^* + (1-\lambda)y_2^*, z-x\rangle = \lambda\langle y_1^*, z-x\rangle + (1-\lambda)\langle y_2^*, z-x\rangle$. Because of $y_1^*,\ y_2^* \in \partial_S F(x)$, we can obtain $D_K(F(z),F(x)) \geqslant \langle y_1^*, z-x\rangle$ and $D_K(F(z),F(x)) \geqslant \langle y_2^*, z-x\rangle$ for all $z \in S$. Further, we obtain

$$\lambda D_K(F(z),F(x)) \geqslant \lambda\langle y_1^*, z-x\rangle \tag{4.1}$$

and

$$(1-\lambda)D_K(F(z),F(x)) \geqslant (1-\lambda)\langle y_1^*, z-x\rangle. \tag{4.2}$$

By adding (4.1) and (4.2), we can obtain $D_K(F(z),F(x)) \geqslant \lambda\langle y_1^*, z-x\rangle + (1-\lambda)\langle y_1^*, z-x\rangle = \langle \bar{y}^*, z-x\rangle$.

(iii) There exists a sequence $y_n^* \in \partial_S F(x)$ such that y_n^* weak* converges to y^*. According to the properties of weak* convergence, for any $x,z \in S$, one has $\langle y_n^*, z-x\rangle \to \langle y^*, z-x\rangle$, so $D_K(F(z),F(x)) \geqslant \langle y_n^*, z-x\rangle$. Taking the limit of the aforementioned expression yields

$$D_K(F(z),F(x)) \geqslant \lim_{n\to\infty}\langle y_n^*, z-x\rangle = \langle y^*, z-x\rangle.$$

This completes the proof. □

Subsequently, we provide an illustrative example to elucidate the concept of the subdifferential of the generalized oriented distance function.

Example 4.1. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $X = [0,2]$, $\bar{y}^* \in K^*$, and $S(K^*) := \{\bar{y}^* \in K^* \mid \|\bar{y}^*\| = 1\}$. Clearly, $K^* = \mathbb{R}^2_+$, $X^* = \mathbb{R}^+$, and $S(K^*)$ is the set of points on the circumference of a quarter of the unit circle. The set valued mapping $F : X \rightrightarrows Y$ is defined by $F(x) = (x, x+1) + U$ for all x in X, where $U = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1\}$. According to Lemma 2.1 (iii) and (iv), one has

$$\triangle_{-K}(a-b) = \triangle_K(b-a) = \sup_{y^*\in S(K^*)}\langle -y^*, b-a\rangle = \sup_{y^*\in S(K^*)}\langle y^*, a-b\rangle.$$

Let $S = X$ and $x \in S$. For any $z \in S$, we obtain

$$\begin{aligned} D_K(F(z),F(x)) &= \sup_{b\in F(x)}\inf_{a\in F(z)}\triangle_{-K}(a-b)\\ &= \sup_{b\in F(x)}\inf_{a\in F(z)}\sup_{\bar{y}^*\in S(K^*)}\langle \bar{y}^*, a-b\rangle\\ &= \sup_{b\in F(x)}\inf_{a\in F(z)}\langle \frac{a-b}{\|a-b\|}, a-b\rangle\\ &= \langle \frac{(0,1)-(3,4)}{\|(0,1)-(3,4)\|}, (0,1)-(3,4)\rangle = 3\sqrt{2}. \end{aligned}$$

Because $3\sqrt{2} \geqslant \langle y^*, z-x\rangle$ is satisfied for any $z-x \in [-2,2]$,we can see that $0 \leqslant y^* \leqslant \frac{3\sqrt{2}}{2}$ for any $y^* \in \partial_S F(x)$. Then, it is easy to obtain that $\partial_S F(x) = [0, \frac{3\sqrt{2}}{2}]$.

The following theorem elucidates the intrinsic relationship between the directional derivative and the subdifferential of set-valued mappings.

Theorem 4.2. *Let $S \subseteq X$ be a nonempty and convex set with nonempty interior and $F(\cdot)$ be K-convex on X with nonempty and K-compact values. Then, $y^* \in \partial_S F(x)$ if and only if $F'(x;\mu) \geqslant \langle y^*, \mu\rangle$ for any $\mu \in X$.*

Proof. Let $y^* \in \partial_S F(x)$. Then, $D_K(F(z),F(x)) \geqslant \langle y^*, z-x\rangle$ for all z in S. There exists $\delta > 0$ such that $x+t\mu \in S$ for all $t \in (0,\delta]$. Consequently, it follows that

$$D_K(F(x+t\mu),F(x)) \geqslant \langle y^*, x+t\mu - x\rangle = t\langle y^*,\mu\rangle,\ \forall\, t \in (0,\delta],$$

so $\frac{D_K(F(x+\lambda\mu),F(x))}{t} \geqslant \langle y^*,\mu\rangle$ for all t in $(0,\delta]$. Theorem 3.3 yields

$$F'(x;\mu) = \inf_{0<t\leqslant\delta} \frac{D_K(F(x+t\mu),F(x))}{t} \geqslant \langle y^*,\mu\rangle.$$

Conversely, suppose that $F'(x;\mu) \geqslant \langle y^*,\mu\rangle$ for all $\mu \in X$. For any $z \in S$, let $\mu = z-x$. It then follows from Theorems 3.1 and 3.3 that

$$\begin{aligned} D_K(F(z),F(x)) = D_K(F(x+\mu),F(x)) &\geqslant \frac{D_K(F(x+t\mu),F(x))}{t} \\ &\geqslant \inf_{0<t\leqslant 1} \frac{D_K(F(x+t\mu),F(x))}{t} = F'(x;\mu) \geqslant \langle y^*, z-x\rangle. \end{aligned}$$

This completes the proof. □

Theorem 4.3. *Let $x \in S$ and U_x be a convex neighborhood of x such that $U_x \subseteq S$. Let $S \subseteq X$ be a nonempty and convex set with nonempty interior and $F(\cdot)$ be K-convex on X with nonempty and K-compact values. Then, $\partial_S F(x) = \partial_{U_x} F(x)$.*

Proof. By $U_x \subseteq S$ and Theorem 4.1 (i), we obtain that $\partial_S F(x) \subseteq \partial_{U_x} F(x)$. All we need to do is to prove that $\partial_{U_x} F(x) \subseteq \partial_S F(x)$ holds. Letting $\bar{y}^* \in \partial_{U_x} F(x)$, one has

$$D_K(F(z),F(x)) \geqslant \langle \bar{y}^*, z-x\rangle, \forall\, z \in U_x. \tag{4.3}$$

Due to the convexity of U_x, there exists $\lambda \in [0,1]$ such that $(1-\lambda)x+\lambda y \in U_x$ holds for any $y \in S$. From inequality (4.3) we have

$$D_K(F((1-\lambda)x+\lambda y),F(x)) \geqslant \langle \bar{y}^*, (1-\lambda)x+\lambda y - x\rangle = \lambda\langle \bar{y}^*, y-x\rangle. \tag{4.4}$$

Thanks to (4.4) and Theorem 3.1, we have

$$D_K(F(y),F(x)) \geqslant \frac{D_K(F((1-\lambda)x+\lambda y),F(x))}{\lambda} \geqslant \langle \bar{y}^*, y-x\rangle.$$

This means that $\bar{y}^* \in \partial_S F(x)$. This completes the proof. □

5. Applications to Set Optimization Problems

In this section, we leverage the results established in the preceding sections to derive necessary and sufficient optimality conditions for set optimization problems. We consistently assume that F possesses a directional derivative $F'(x;\mu)$ at $x \in S$ in the direction $\mu \in X$. Moreover, the solution set of the optimization problem is nonempty. Let $F : X \rightrightarrows Y$ be a set-valued mapping and S be a nonempty subset in X. We consider the following set optimization problems (SOP):

$$(SOP) \quad \begin{cases} \min & F(x) \\ s.t. & x \in S. \end{cases}$$

In the following, we review the concept of the solutions for the problem (SOP) with regard to the set order relation "$\prec^l_K$" and "$\preccurlyeq^l_K$".

Definition 5.1. [8] For $\varepsilon \geqslant 0$, an element $x_0 \in S$ is said to be

(i) weak l-minimal solution of (SOP) if, for $x \in S$, $F(x) \prec^l_K F(x_0)$ implies $F(x_0) \prec^l_K F(x)$.

(ii) weak l-minimal approximate solution of (SOP) if, for $x \in S$, $F(x) \prec^l_{\varepsilon,K} F(x_0)$ implies $F(x_0) \prec^l_{\varepsilon,K} F(x)$.

$W_l(F,S)$ and $W_l(\varepsilon,F,S)$ are defined as the weak l-minimal solution set and weak l-minimal approximate solution set of (SOP), respectively.

Lemma 5.1. [9, 12] *Assume that $x_0 \in S$ and $F(x_0)$ is K-compact.*

(i) $x_0 \in W_l(F,S)$ if and only if there does not exist $y \in S$ satisfying $F(y) \prec^l_K F(x_0)$.

(ii) If $\varepsilon \geqslant 0$, then $x_0 \in W_l(\varepsilon,F,S)$ if and only if there does not exist $y \in S$ satisfying $F(y) \prec^l_{\varepsilon,K} F(x_0)$.

In the following, we present the optimality conditions for the set optimization problem.

Theorem 5.1. *Assume that S is convex and $F(\cdot)$ is K-convex on S with nonempty and K-compact values. Then $x_0 \in W_l(F,S)$ if and only if $0 \in \partial_S F(x_0)$.*

Proof. Let $x_0 \in W_l(F,S)$. It follows from Lemma 5.1 that $F(y) \not\prec^l_K F(\bar{x})$ for any $y \in S$. By considering the converse of statement (iii) in Lemma 2.2, we can infer $D_K(F(z),F(x_0)) \geqslant 0 = \langle 0, z - x_0\rangle$ for all z in S. This means that $0 \in \partial_S F(x_0)$.

Conversely, due to the existence of $0 \in \partial_S F(x_0)$, we have $D_K(F(z),F(x_0)) \geqslant 0 = \langle 0, z - x_0\rangle$ for all z in S. In view of Lemma 2.2 (iii) and Lemma 5.1, we have $x_0 \in W_l(F,S)$. This completes the proof. □

Theorem 5.2. *Let $S \subseteq X$ be a nonempty and convex set and $x_0 \in S$. Suppose that $F(\cdot)$ is K-convex on S with nonempty and K-compact values. Then $x_0 \in W_l(F,S)$ if and only if $F'(x_0;\mu) \geqslant 0$ for all $\mu \in \{\mu \in X : \exists\, t > 0\ , x_0 + t\mu\ \in S\}$.*

Proof. Let $x_0 \in W_l(F,S)$. It follows from Lemma 5.1 that $F(y) \not\prec^l_K F(x_0)$ for any $y \in S$. By considering the converse of statement (iii) in Lemma 2.2, we can deduce that $D_K(F(y),F(x_0)) \geqslant 0$. Since $\mu \in \{\mu \in X : \exists\, t > 0\ , x_0 + t\mu\ \in S\}$, then $D_K(F(x_0+t\mu),F(x_0)) \geqslant 0$, which implies that

$$\lim_{t \to 0^+} \frac{D_K(F(x_0+t\mu),F(x_0))}{t} \geqslant 0.$$

i.e., $F'(x_0;\mu) \geqslant 0$. Conversely, due to the presence of $F'(x_0;\mu) \geqslant 0$, one has

$$F'(x_0;\mu) = \lim_{t \to 0^+} \frac{D_K(F(x_0+t\mu),F(x_0))}{t} \geqslant 0.$$

Thus, $D_K(F(x_0+t\mu),F(x_0)) \geqslant 0$. Putting $y = x_0 + t\mu$, we obtain $D_K(F(y),F(x_0)) \geqslant 0$. By considering the converse of statement (iii) in Lemma 2.2 and Lemma 5.1 we can infer that $x_0 \in W_l(F,S)$. This completes the proof. □

Remark 5.1. We aimed to investigate whether weak l-minimal approximate solution set of (SOP) defined in Definition 5.1 (ii) can yield similar results by using Lemma 2.2 (iii). Regrettably, this investigation is only meaningful under condition $\varepsilon = 0$, which effectively returns us to the context of Theorem 5.2.

6. CONCLUSIONS

In this paper, We defined the directional derivative and subdifferential by using the generalized oriented distance function and derived several key properties of the directional derivative of set-valued mappings, including its operation rules, positive homogeneity, chain rule, and upper semicontinuity. Additionally, we investigated the convexity and weak* closedness of the subdifferential of set-valued mappings, as well as its relationship with the directional derivative of set-valued mappings. As an application, we derived the necessary and sufficient optimality conditions for set optimization problems. However, the conditions under which the directional derivative exists have not yet been fully explored, and this will be a key focus of our future research.

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