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A FORWARD-REFLECTED-ANCHORED-BACKWARD SPLITTING ALGORITHM WITH DOUBLE INERTIAL EFFECTS FOR SOLVING NON-MONOTONE INCLUSION PROBLEMS

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Abstract. In this paper, we investigate inclusion problems involving operators that may not be monotone in the classical sense. Specifically, we consider a generalized notion of monotonicity, allowing the modulus of monotonicity to take negative values. This broader assumption extends the applicability of our results to a wider class of operators. To address these non-monotone inclusion problems, we employ the two-step inertial forward–reflected–anchored–backward splitting algorithm proposed in [I. Chinedu, A. Maggie, O.A. Kazeem, Two-step inertial forward–reflected–anchored–backward splitting algorithm for solving monotone inclusion problems, Comput. Appl. Math. 42 (2023), 351] and establish the strong convergence of the generated sequence. Our findings relaxed the assumptions on the operators. We demonstrate the applicability of our approach to various optimization settings, including constrained optimization problems, mixed variational inequalities, and variational inequalities. Finally, we provide a numerical example to illustrate the practical effectiveness of the proposed algorithm.
Keywords. Forward-reflected-anchored-backward algorithm; Monotone inclusion, Two-step inertial.
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1. INTRODUCTION

Let H be a real Hilbert space with the inner product denoted by $\langle \cdot, \cdot \rangle$ and the norm denoted by $\| \cdot \|$. In this paper, we study the following inclusion problem: Find $u^* \in K$ such that

$$0 \in F(u^*) + G(u^*) \tag{1.1}$$

where $F : H \longrightarrow 2^H$ is a set-valued mapping and $G : H \longrightarrow H$ is a single-valued mapping, K is a nonempty closed subset of H. We denote by zer $(F+G)$ the *set of solutions* of problem (1.1).

Problem (1.1) serves as a broad mathematical model that encompasses numerous known problems, including constrained optimization problems (COPs), variational inequalities problems (VIPs), mixed variational inequalities (MVIs), saddle point problems, Nash equilibrium problems in noncooperative games, and fixed point problems. Many of these can be reformulated as special cases of (1.1); see, for instance, [6, 8, 26] and the references therein. For example, we consider a variational inequality problem of finding $u^* \in K$ such that $\langle T(u^*), u - u^* \rangle \geq 0$,

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for all $u \in K$, where K is a nonempty, closed, convex subset of H and $T : H \longrightarrow H$. This problem can be rewritten as the inclusion problem $0 \in F(u)$, where

$$F(x) = \begin{cases} T(u) + N_K(u) & \text{if } u \in K \\ \emptyset & \text{if } u \neq K, \end{cases}$$

with $N_K(u)$ the normal cone to K at u.

In addition, a constrained minimization problem can also be formulated as an inclusion problem. Indeed, given a proper and convex function $g : H \longrightarrow (-\infty, +\infty]$ and K, a nonempty, convex, and closed subset of H, $u^* \in K$ is a solution to the constrained optimization problem (COP): $\min_{u\in K} g(u)$ if and only if it is a solution to the inclusion problem $0 \in \partial g(u) + N_K(u)$, (see, e.g., [3, 21]). Recently, numerous numerical methods were developed to solve the inclusion problems of the form (1.1); see, e.g., [11, 12, 18, 20, 27]. The forward-backward splitting method, introduced in seminal works by Passty [19] and Lions and Mercier [15], has become a standard approach for such problems [1, 3, 9, 16, 24]. Lions and Mercier [15] obtained the weak convergence of the following forward-backward splitting method

$$u_{k+1} = (id + \gamma F)^{-1}(u_k - \gamma G u_k), \quad k \geq 1,$$

which was also proved to strongly convergent under restrictive assumptions [23, 29].

To address these limitations, Tseng [24] introduced an improved algorithm in 2000

$$\begin{cases} v_k = (id + \gamma F)^{-1}(u_k - \gamma B u_k), \\ u_{k+1} = v_k - \gamma G v_k + \gamma G u_k. \end{cases}$$

However, this method incurs additional computational costs due to requiring two forward evaluations of G. To overcome this disadvantage, Malitsky and Tam [16] employed a reflection technique and proposed the following scheme

$$u_{k+1} = (id + \gamma F)^{-1}(u_k - 2\gamma G u_k + \gamma G u_{k-1}), \quad \gamma \in \left(0, \frac{1}{2L}\right).$$

Originating from the discretization of the heavy ball method, the inertial technique has become popular in algorithm design due to its role in accelerating convergence. For example, Tan and Cho [25] introduced the following one-step inertial viscosity-type forward-backward-forward splitting algorithm:

$$\begin{cases} t_k = u_k + \theta_k(u_k - u_{k-1}) \\ v_k = (id + \gamma F)^{-1}(t_k - \gamma G t_k), \\ w_k = v_k - \gamma G v_k + \gamma G u_k, \\ u_{k+1} = \alpha_k f u_k + (1 - \alpha_k) w_k. \end{cases}$$

They obtain the strong convergence of the algorithm in Hilbert spaces.

More recently, two-step inertial method were also successfully incorporated into various algorithms [7, 13], demonstrating significant improvements in convergence rates [13]. In addition, classical assumptions of monotonicity in inclusion problems have been deeply ingrained [4, 5, 28]. Relaxing these conditions is challenging since fundamental results may no longer hold. For instance, if $(G + F)$ lacks strong monotonicity, the inclusion $0 \in (G + F)(x)$ may have no solution. Consequently, the number of algorithms for non-monotone inclusion problems is very limited. Moreover, the monotonicity assumption may restrict the applicability of the results, as operators in real-world applications are often not monotone. Hence, reducing this assumption is a crucial aspect of developing algorithms for these inclusion problems, which serves as the motivation for this research.

In this paper, we extend the concept of monotonicity by allowing a generalized monotonicity framework, where operators may have a negative modulus of monotonicity. This broader perspective enables the study of a wider class of operators beyond the traditional monotone setting. The remainder of the paper is structured as follows. Section 2 revisits fundamental definitions and concepts, and presents several technical lemmas. Specifically, we provide some characterizations for an operator to be maximal generalized monotone and for the sum of two generalized monotone operators to be maximal. Section 3 presents the main results, including an analysis of the strong convergence of the two-step inertial forward-reflected-anchored-backward (FRAB) splitting algorithm. Section 4 discusses some applications of the algorithm to COPs, MVIs, and VIPs. Section 5 presents a numerical example illustrating the effectiveness of the algorithm. Section 6 concludes this paper with some concluding remarks.

2. Preliminaries

In this section, we review essential definitions that are useful in the subsequent discussion.

2.1. **Some notions on convex analysis.** Let $f : H \to (-\infty, +\infty]$ be a convex and lower semicontinuous (l.s.c.) function. Its *domain* is defined as $\operatorname{dom} f = \{x \in H : f(x) < +\infty\}$, and f is said to be *proper* if $\operatorname{dom} f \neq \emptyset$. A proper, convex, and lower semicontinuous function $f : H \to (-\infty, +\infty]$ is said to be *subdifferentiable* at u if its *subdifferential* at u, given by

$$\partial f(u) = \{w \in H : f(v) - f(u) \geq \langle w, v-u\rangle \ \forall v \in H\}$$

is non-empty. Any $w \in \partial f(u)$ is called a *subgradient* of f at u.

For a nonempty, closed, convex subset K of H, the *normal cone* at $u \in K$, $N_K(u)$, is defined as $N_K(u) = \left\{w \in H : \langle w, u-v\rangle \geq 0, \forall v \in K\right\}$, and $N_K(u) = \emptyset$ if $u \notin K$. Recall that the *indicator function* of K,

$$i_K(u) = \begin{cases} 0 & \text{if } u \in K \\ +\infty & \text{otherwise} \end{cases}$$

satisfies $\partial i_K(u) = N_K(u)$ for all $u \in H$. A fundamental tool in inclusion problems is the *metric projection*, defined by, for any $u \in H$, $P_K(u) = \arg\min\left\{\|v-u\| : v \in K\right\}$. Note that when K is nonempty, closed, and convex, $P_K(u)$ exists and is unique.

We now review some useful identities, which are needed for the convergence analysis in the sequel.

Lemma 2.1. [7] *Let $x, y, z \in H$ and $a, b, \beta \in \mathbb{R}$. Then*

(a)

$$\begin{aligned}\|(1+a)x-(a-b)y-bz\|^2 = (1+a)\|x\|^2 - (a-b)\|y\|^2 - b\|z\|^2 + (1+a)(a-b)\|x-y\|^2 \\ + b(1+a)\|x-z\|^2 - b(a-b)\|y-z\|^2.\end{aligned} \tag{2.1}$$

(b)

$$\langle x-z, y-x\rangle = \frac{1}{2}\|z-y\|^2 - \frac{1}{2}\|x-z\|^2 - \frac{1}{2}\|y-x\|^2. \tag{2.2}$$

(c)

$$\|\beta x+(1-\beta)y\|^2=\beta\|x\|^2+(1-\beta)\|y\|^2-\beta(1-\beta)\|x-y\|^2.$$

The following lemmas are also essential.

Lemma 2.2. [22] *Let* $\{s_k\}$ *be a sequence with* $s_k\geq 0$ *for all* k*,* $\{\lambda_k\}$ *be a real sequence with* $\lambda_k\in(0,1)$ *for all* k *such that* $\sum_{k=1}^{\infty}\lambda_k=\infty$*, and* $\{a_k\}$ *be a real sequence satisfying* $s_{k+1}\leq(1-\lambda_k)s_k+\lambda_k a_k$*, for all* $k\geq 1$*. Assume further that* $\limsup_{i\to\infty}a_{k_i}\leq 0$ *for each subsequence* $\{a_{k_i}\}$ *of* $\{a_k\}$ *satisfying* $\liminf_{i\to\infty}(a_{k_i+1}-a_{k_i})\geq 0$*. Then* $\lim_{k\to\infty}a_k=0$*.*

Lemma 2.3. [17] *Let* $\{s_k\}$ *and* $\{a_k\}$ *be two nonnegative real sequences,* $\{\lambda_k\}$ *be a sequence in* $(0,1)$*, and* $\{b_k\}$ *be a real sequence such that* $s_{k+1}\leq(1-\lambda_k)s_k+a_k+b_k$*, for all* $k\geq 1$*,* $\sum_{k=1}^{\infty}b_k<\infty$*, and* $a_k\leq\lambda_k C$ *for some* $C\geq 0$*. Then* $\{a_k\}$ *is bounded.*

2.2. **Monotone operators.** In this subsection, we review some notions related to operators, especially the definition of monotonicity. Let $F:H\longrightarrow 2^H$ be a set-valued mapping on H. The *graph* of F is defined as $\mathrm{gr}(F)=\{(x,u)\in H\times H : u\in F(x)\}$. The *domain* and *range* of F are given by $\mathrm{dom}\,F=\{u\in H : F(u)\neq\emptyset\}$ and $\mathrm{ran}\,F=\{y\in H :\ \text{there exists } x\in H, y\in F(x)\}$. We now recall the notion of *generalized monotonicity*, which extends classical monotonicity by allowing the modulus to be negative. This weaker condition enables the study of a broader class of operators.

Definition 2.1. [10] An operator $F:H\longrightarrow 2^H$ is said to be μ_F*-monotone* if there exists $\mu_F\in\mathbb{R}$ such that $\langle x-y,u-v\rangle\geq\mu_F\|x-y\|^2$ for all $x,y\in H, u\in F(x)$, and $v\in F(y)$.

Remark 2.1. Note that, in the definition above, unlike the classical definition, we do not require that $\mu_F\geq 0$. In fact, if $\mu_F<0$, F is said to be *weakly-monotone*. When $\mu_F=0$, μ_F-monotonicity reduces to the classical monotonicity. If $\mu_F>0$, an μ_F-monotone operator F becomes strongly monotone.

Definition 2.2. [10] A μ_F-monotone operator F is said to be *maximal* if there exists no μ_F-monotone operator whose graph strictly contains the graph of F.

Here, we recall an important notion of Lipschitz continuity, which frequently appears in the study of algorithms.

Definition 2.3. An operator $G:H\longrightarrow H$ is said to be *Lipschitz continuous* with constant $L\geq 0$ if $\|G(x)-G(y)\|\leq L\|x-y\|$ for all $x,y\in H$.

The resolvent of an operator is a fundamental tool in the study of inclusion problems. We now recall its definition. The resolvent of an operator F with the parameter γ is given by $J_{\gamma F}=(Id+\gamma F)^{-1}$, where Id is the *identity mapping*.

In the absence of monotonicity, the resolvent may not always return a unique value at a given point. However, the following lemma establishes that for generalized monotone operators, the resolvent remains single-valued under suitable parameter choices. Furthermore, it demonstrates that the resolvent is cocoercive, a property that plays a crucial role in the subsequent analysis.

Lemma 2.4. [3, 10] *Let* $F:H\longrightarrow 2^H$ *be an* μ_F*-monotone operator, and let* $\gamma>0$ *be such that* $1+\gamma\mu_F>0$*. Then,*

(1) $J_{\gamma F}$ *is a singleton;*

(2) ran $J_{\gamma F} = \text{dom } F$;
(3) F *is (maximal)* μ_F*-monotone if and only if* $F' = F - \mu_F id$ *is (maximal) monotone.*

It is known that the sum of two maximal monotone operators is not always maximal. The following lemma provides a criterion for determining when the sum of two operators remains maximal. We begin by recalling a classical result and then extend it to the setting of generalized monotonicity.

Lemma 2.5. [14] *Let* $F : H \longrightarrow 2^H$ *be maximal monotone and* $G : H \longrightarrow H$ *be monotone and Lipschitz continuous on* H*. Then* $F + G$ *is maximally monotone.*

Lemma 2.6. *Let* $F : H \longrightarrow 2^H$ *be maximally* μ_F*-monotone and* $G : H \longrightarrow H$ *be* μ_G*-monotone and Lipschitz continuous on* H*. Then* $F + G$ *is maximally* $(\mu_F + \mu_G)$*-monotone.*

Proof. Let $\gamma > 0$ such that $1 + \gamma.\mu_F > 0$. Because F is maximally μ_F-monotone, it holds that $F' := F - \mu_F id$ is maximally monotone [10]. Since G is μ_G monotone and Lipschitz continuous, it follows that $G' = G - \mu_G id$ is monotone, and Lipschit continuous. By Lemma 2.5, $F' + G' = F + G - (\mu_F + \mu_G)id$ is maximally monotone. It follows from Part (3) of Lemma 2.4 that $F + G$ is maximal $(\mu_F + \mu_G)$-monotone. □

The next lemma provides a characterization when a generalized monotone operator is maximal. This extends the classical result.

Lemma 2.7. *Let* $F : H \longrightarrow 2^H$ *be a* μ_F*-monotone operator. Then* F *is the maximal monotone if and only if*

$$\forall (y, v) \in \text{gr}(F), \langle u - v, x - y \rangle \geq \mu_F \|x - y\|^2 \implies u \in F(x). \tag{2.3}$$

Proof. Suppose that F is maximal μ_F-monotone and $u_0, x_0 \in H$ such that, for all $(y, v) \in \text{gr}F$, $\langle u_0 - v, x_0 - y \rangle \geq 0$. We now suppose contradiction that $u_0 \notin F(x_0)$. Let

$$T(x) = \begin{cases} F(x) & \text{if } x \neq x_0 \\ F(x) \cup \{u_0\} & \text{otherwise} \end{cases}$$

Then T is μ_F-monotone and gr $F \subset gr\ T$, a contradiction to the maximality of F. Hence, $u_0 \in F(x_0)$.

Assume now that $u, x \in H$ satisfies condition (2.3). Let $A : H \longrightarrow 2^H$ be a μ_F-monotone such that $\text{gr}F \subseteq \text{gr}A$. Then, for all $(x, u) \in \text{gr}A$, by the μ_F-monotone of A, we have that $\langle u - v, x - y \rangle \geq \mu_F \|x - y\|^2$ for all $(y, v) \in \text{gr}A$. Since gr $F \subseteq$ gr A, this also holds for all $(y, v) \in \text{gr}F$. By condition (2.3), we derive that $u \in F(x)$ or $(x, u) \in \text{gr}F$. This implies that F is maximal μ_F-monotone. □

For a comprehensive discussion on monotone operators, their applications in optimization problems, and the properties of their resolvent, we refer readers to [2, 3, 10].

3. Algorithm and Convergence Analysis

In this section, we first present an algorithm, proposed in [7], and we then analyze the property of strong convergence for the sequence generated by the algorithm. We emphasize that our operators are assumed to be generalized monotone, which is weaker than classical monotonicity.

Algorithm 3.1. Let $\gamma \in \left(0, \frac{1}{2L}\right)$, $\theta_1 \in [0,1)$, $\theta_2 \le 0$ and take $\{\lambda_k\} \subseteq (0,1)$. For any $w^*, u_{-1}, u_0, u_1 \in H$, let u_k, u_{k-1}, u_{k-2} be given. Set

$$\begin{aligned} u_{k+1} =& J_{\gamma F}\Big(\lambda_k w^* + (1-\lambda_k)\Big(u_k - \theta_1(u_k - u_{k-1}) + \theta_2(u_{k-1} - u_{k-2})\Big) \\ & - \gamma G u_k - \gamma(1-\lambda_k)(Gu_k - Gu_{k-1})\Big), \quad \forall k \ge 1. \end{aligned} \tag{3.1}$$

To achieve strong convergence of algorithm, we impose the following assumptions on the operators and parameters:

Assumption 3.1. (1) F is maximal $-\mu_F$ monotone;
(2) G is μ_G-monotone and Lipschitz continuous with constant $L > 0$;
(3) $\mu_F + \mu_G \ge 0$;
(4) $\text{zer}(F+G) \neq \emptyset$;
(5) θ_1, θ_2 satisfy $0 \le \theta_1 < \frac{1}{3}(1-2\gamma L)$, $\frac{1}{3+4\theta_1}(3\theta_1 - 1 + 2\gamma L) < \theta_2 \le 0$;
(6) $1 + \gamma\mu_F > 0$.

Remark 3.1. It is worth noting that the condition (3) in this assumption is weaker than the monotonicity assumption. Indeed, it may happen that F or G is weakly monotone, while the sum of the two operators is monotone or strongly monotone.

Lemma 3.1. *Assume that Assumption 3.1 holds. Then the sequence $\{u_k\}$ generated by the Algorithm 3.1 is bounded whenever $\lim_{k\to\infty} \lambda_k = 0$.*

Proof. Let $u^* \in zer(F+G)$ and set $z_k = \lambda w^* + (1-\lambda_k)w_k$ with $w_k = u_k + \theta_1(u_k - u_{k-1}) + \theta_2(u_{k-1} - u_{k-2})$. Then

$$-\gamma G u^* \in \gamma F u^* \tag{3.2}$$

and

$$z_k - \gamma G u_k - \gamma(1-\lambda_k)(Gu_k - Gu_{k-1}) - u_{k+1} \in \gamma F u_{k+1}. \tag{3.3}$$

By the μ_F-monotonicity of F, it follows from (3.2) and (3.3) that

$$\langle z_k - \gamma G u_k - \gamma(1-\lambda_k)(Gu_k - Gu_{k-1}) - u_{k+1} + \gamma G u^*, u_{k+1} - u^* \rangle \ge \gamma\mu_F \|u_{k+1} - u^*\|^2.$$

Consequently,

$$\begin{aligned} 2\gamma\mu_F\|u_{k+1} - u^*\|^2 \le & 2\langle u_{k+1} - z_k + \gamma G u_k + \gamma(1-\lambda_k)(Gu_k - Gu_{k-1}) - \gamma G u^*, u^* - u_{k+1}\rangle \\ =& 2\langle u_{k+1} - z_k, u^* - u_{k+1}\rangle + 2\gamma\langle Gu_k - Gu^*, u^* - u_{k+1}\rangle + 2\gamma(1-\lambda_k) \\ & \langle Gu_k - Gu_{k-1}, u^* - u_k\rangle + 2\gamma(1-\lambda_k)\langle Gu_k - G_{k-1}, u_k - u_{k+1}\rangle. \end{aligned}$$

By (2.2), it follows that

$$\begin{aligned} & 2\gamma\mu_F\|u_{k+1} - u^*\|^2 \\ & \le \|z_k - u^*\|^2 - \|u_{k+1} - u^*\| - \|u_{k+1} - z_k\|^2 + 2\gamma\langle Gu_k - Gu^*, u^* - u_{k+1}\rangle \\ & + 2\gamma(1-\lambda_k)\langle Gu_k - Gu_{k-1}, u^* - u_k\rangle + 2\gamma(1-\lambda_k)\langle Gu_k - Gu_{k-1}, u_k - u_{k+1}\rangle. \end{aligned} \tag{3.4}$$

In view of the μ_G-monotonicity of G, one has

$$\begin{aligned} \langle Gu_k - Gu^*, u^* - u_{k+1}\rangle =& \langle Gu_k - Gu_{k+1}, u^* - u_{k+1}\rangle + \langle Gu_{k+1} - Gu^*, u^* - u_{k+1}\rangle \\ \le& \langle Gu_k - Gu_{k+1}, u^* - u_{k+1}\rangle - \mu_G\|u_{k+1} - u^*\|^2. \end{aligned} \tag{3.5}$$

Since G is Lipschit continuous with modulus L, we have

$$2\gamma\langle Gu_k - Gu_{k-1}, u_k - u_{k+1}\rangle \leq 2\gamma L\|u_k - u_{k-1}\|\|u_k - u_{k+1}\| \leq \gamma L\left(\|u_k - u_{k-1}\|^2 + \|u_k - u_{k+1}\|^2\right). \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we arrive at

$$\begin{aligned}&(2\gamma\mu_F + 2\gamma\mu_G)\|u_{k+1} - u^*\|^2 + \|u_{k+1} - u^*\|^2 + 2\gamma\langle Gu_{k+1} - Gu_k, u^* - u_{k+1}\rangle \\ \leq&\|z_k - u^*\|^2 - \|u_{k+1} - z_k\|^2 + 2\gamma(1-\lambda_k)\langle Gu_k - Gu_{k-1}, u^* - u_k\rangle \\ &+ (1-\lambda_k)\gamma L\left(\|u_k - u_{k-1}\|^2 + \|u_k - u_{k+1}\|^2\right), \forall k \geq k_0.\end{aligned} \tag{3.7}$$

By Lemma 2.2, one has

$$\begin{aligned}&\|z_k - u^*\|^2 \\ =& \|w_k - u^*\|^2 + \lambda_k^2\|w_k - w^*\|^2 - 2\lambda_k\langle w_k - u^*, w_k - w^*\rangle \\ =& \|w_k - u^*\|^2 + \lambda_k^2\|w_k - w^*\|^2 - \lambda_k\|w_k - w^*\|^2 - \lambda_k\|w_k - u^*\|^2 + \lambda_k\|w^* - u^*\|^2.\end{aligned} \tag{3.8}$$

Substituting u^* by u_{k+1} into (3.8), one obtains

$$\begin{aligned}\|z_k - u_{k+1}\|^2 =& \|w_k - u_{k+1}\|^2 + \lambda_k^2\|w_k - w^*\|^2 - \lambda_k\|w_k - w^*\|^2 - \lambda_k\|w_k - u_{k+1}\|^2 \\ &+ \lambda_k\|w^* - u_{k+1}\|^2.\end{aligned} \tag{3.9}$$

Difference of (3.8) and (3.9) yields

$$\begin{aligned}&\|z_k - u^*\|^2 - \|w_k - u_{k+1}\|^2 \\ =&(1-\lambda_k)\|w_k - u^*\|^2 + \lambda_k\|w^* - u^*\|^2 - (1-\lambda_k)\|u_{k+1} - w_k\|^2 - \lambda_k\|u_{k+1} - w^*\|.\end{aligned} \tag{3.10}$$

Combining (3.10) and (3.7), we see that

$$\begin{aligned}&(2\gamma\mu_F + 2\gamma\mu_G)\|u_{k+1} - u^*\|^2 + \|u_{k+1} - u^*\|^2 + 2\gamma\langle Gu_{k+1} - Gu_k, u^* - u_{k+1}\rangle \\ \leq&(1-\lambda_k)\|w_k - u^*\|^2 + \lambda_k\|w^* - u^*\|^2 - (1-\lambda_k)\|u_{k+1} - w_k\|^2 - \lambda_k\|u_{k+1} - w^*\|^2 \\ &+ 2\gamma(1-\lambda)\langle Gu_k - Gu_{k-1}, u^* - u_k\rangle + (1-\lambda_k)\gamma L\left(\|u_k - u_{k-1}\|^2 + \|u_k - u_{k+1}\|^2\right)\end{aligned} \tag{3.11}$$

for all $k \geq k_0$. It follows from (2.1) that

$$\begin{aligned}\|w_k - u^*\|^2 =& \|(1+\theta_1)(u_k - u^*) - (\theta_1 - \theta_2)(u_{k-1} - u^*) - \theta_2(u_{k-2} - u^*\|^2 \\ &+ (1+\theta_1)(\theta_1 - \theta_2)\|u_k - u_{k-1}\|^2 + \theta_2(1+\theta_1)\|u_k - u_{k-2}\|^2.\end{aligned} \tag{3.12}$$

In addition, by (2.2), it holds that

$$\begin{aligned}\|u_{k+1} - w_k\|^2 =& \|u_{k+1} - u_k\| - 2\theta_1\langle u_{k+1} - u_k, u_k - u_{k-1}\rangle \\ &- 2\theta_2\langle u_{k+1} - u_k, u_{k-1} - u_{k-2}\rangle + \theta_1^2\|u_k - u_{k-1}\|^2 \\ &+ 2\theta_2\theta_1\langle u_k - u_{k-1}, u_{k-1} - u_{k-2}\rangle + \theta_2^2\|u_{k-1} - u_{k-2}\|^2.\end{aligned} \tag{3.13}$$

Moreover, note that

$$\begin{aligned}-2\theta_1\langle u_{k+1} - u_k, u_k - u_{k-1}\rangle \geq& -2\theta_1\|u_{k+1} - u_k\|\|u_k - u_{k-1}\| \\ \geq& -\theta_1\|u_{k+1} - u_k\|^2 - \theta_1\|u_k - u_{k-1}\|^2,\end{aligned} \tag{3.14}$$

$$-2\theta_2\langle u_{k+1}-u_k, u_{k-1}-u_{k-2}\rangle \geq -2|\theta_2|\|u_{k+1}-u_k\|\|u_{k-1}-u_{k-2}\|$$
$$\geq -|\theta_2|\|u_{k+1}-u_k\|^2-|\theta_2|\|u_{k-1}-u_{k-2}\|^2, \quad (3.15)$$

and

$$2\theta_2\theta_1\langle u_k-u_{k-1}, u_{k-1}-u_{k-2}\rangle \geq -2|\theta_2||\theta_1|\|u_k-u_{k-1}\|u_{k-1}-u_{k-2}\|$$
$$\geq -|\theta_2||\theta_1|\|u_k-u_{k-1}\|^2-|\theta_2||\theta_1|\|u_{k-1}-u_{k-2}\|^2. \quad (3.16)$$

Plugging (3.14), (3.15), and (3.16) into (3.13) yields

$$\|u_{k+1}-w_k\|^2 \geq (1-\theta_1-|\theta_2|)\|u_{k+1}-u_k\|^2+(\theta_1^2-\theta_1-\theta_1|\theta_2|)\|u_k-u_{k-1}\|^2$$
$$+(\theta_2^2-|\theta_2|-\theta_1|\theta_2|)\|u_{k-1}-u_{k-2}\|^2. \quad (3.17)$$

Substituting (3.12) and (3.17) into (3.11) obtains

$$(2\gamma\mu_F+2\gamma\mu_G)\|u_{k+1}-u^*\|^2+\|u_{k+1}-u^*\|^2+2\gamma\langle Gu_{k+1}-Gu_k, u^*-u_{k+1}\rangle$$
$$\leq (1-\lambda_k)\Big[(1+\theta_1)\|u_k-u^*\|^2-(\theta_1-\theta_2)\|u_{k-1}-u^*\|^2-\theta_2\|u_{k-2}-u^*\|^2$$
$$(1+\theta_1)(\theta_1-\theta_2)\|u_k-u_{k-1}\|^2+\theta_2(1+\theta_1)\|u_k-u_{k-2}\|^2-\theta_2(\theta_1-\theta_2)\|u_{k-1}-u_{k-2}\|^2\Big]$$
$$+\lambda_k\|w^*-u^*\|^2-(1-\lambda_k)\Big[(1-\theta_1-|\theta_2|)\|u_{k-1}-u_k\|^2+(\theta_1^2-\theta_1-\theta_1|\theta_2|)\|u_k-u_{k-1}\|^2$$
$$+(\theta_2^2-|\theta_2|-\theta_1|\theta_2|)\|u_{k-1}-u_{k-2}\|^2\Big]-\lambda_k\|u_{k+1}-w^*\|^2$$
$$+2\gamma(1-\lambda_k)\langle Gu_k-Gu_{k-1}, u^*-u_k\rangle+(1-\lambda_k)\gamma L\left(\|u_k-u_{k-1}\|^2+\|u_{k+1}-u_k\|^2\right)$$
$$\leq (1-\lambda_k)\Big[(1+\theta_1)\|u_k-u^*\|^2-(\theta_1-\theta_2)\|u_{k-1}-u^*\|^2-\theta_2\|u_{k-2}-u^*\|^2$$
$$+(2\theta_1-\theta_2-\theta_1\theta_2+\theta_1|\theta_2|)\|u_k-u_{k-1}\|^2+(|\theta_2|+|\theta_2|\theta_1-\theta_2\theta_1)\|u_{k-1}-u_{k-2}\|^2$$
$$-(1-\theta_1-|\theta_2|)\|u_{k+1}-u_k\|^2+2\gamma\langle Gu_k-Gu_{k-1}, u^*-u_k\rangle\Big]$$
$$+\lambda_k\|w^*-u^*\|^2+(1-\lambda_k)\gamma L\left(\|u_k-u_{k-1}\|^2+\|u_{k+1}-u_k\|^2\right), \forall k \geq k_0.$$

It follows that

$$(1+2\gamma\mu_F+2\gamma\mu_G)\|u_{k+1}-u^*\|^2-\theta_1\|u_k-u^*\|^2-\theta_2\|u_{k-1}-u^*\|^2$$
$$+2\gamma\langle Gu_{k+1}-Gu_k, u^*-u_{k+1}\rangle+(1-|\theta_2|-\theta_1-\gamma L)\|u_{k+1}-u_k\|^2$$
$$\leq (1-\lambda_k)\Bigg[\|u_k-u^*\|^2-\theta_1\|u_{k-1}-u^*\|^2-\theta_2\|u_{k-2}-u^*\|^2+2\gamma\langle Gu_k-Gu_{k-1}, u^*-u_k\rangle$$
$$+(1-|\theta_2|-\theta_1-\gamma L)\|u_k-u_{k-1}\|^2-(2\gamma L+3\theta_1-1+(1+\theta_1)(|\theta_2|-\theta_2)\Big(\|u_{k-1}-u_{k-2}\|^2$$
$$-\|u_k-u_{k-1}\|^2\Big)\Bigg]+\lambda_k\|w^*-u^*\|^2-(1-\lambda_k)\Bigg[-\Big(2\gamma L+3\theta_1-1+(1+\theta_1)(|\theta_2|-\theta_2)\Big)$$
$$-(|\theta_2|+|\theta_2|\theta_1-\theta_2\theta_1)\Bigg]\|u_{k-1}-u_{k-2}\|^2. \quad (3.18)$$

Set

$$a_1=-(2\gamma L+3\theta_1-1+(1+\theta_1)(|\theta_2|-\theta_2)),$$

$$a_2 = 1 - 3\theta_1 - 2\gamma L - 2|\theta_2| - 2\theta_1|\theta_2| + \theta_2 + 2\theta_1\theta_2,$$

and

$$q_k = (1 + 2\gamma\mu_F + 2\gamma\mu_G)\|u_k - u^*\|^2 - \theta_1\|u_{k-1} - u^*\|^2 - \theta_2\|u_{k-2} - u^*\| \\ + 2\gamma\langle Gu_k - Gu_{k-1}, u^* - u_k\rangle + (1 - |\theta_2| - \theta_1 - \gamma L)\|u_k - u_{k-1}\|^2 + a_1\|u_{k-1} - u_{k-2}\|^2.$$

Then (3.18) reads as

$$\begin{aligned} q_{k+1} &\leq (1-\lambda_k)q_k + \lambda_k\|u^* - w^*\|^2 - (1-\lambda_k)a_2\|u_{k-1} - u_{k-2}\|^2 \\ &\quad - (1-\lambda_k)(2\gamma\mu_F + 2\gamma\mu_G)\|u_k - u^*\|^2 \\ &\leq (1-\lambda_k)q_k + \lambda_k\|u^* - w^*\|^2 - (1-\lambda_k)a_2\|u_{k-1} - u_{k-2}\|^2 \end{aligned} \tag{3.19}$$

for all $k \geq k_0$. The last inequality holds since $\lambda_k \in (0,1), \gamma > 0$, and the condition (3) in Assumption 3.1.

We now prove that $a_1 > 0, a_2 > 0$, and $q_k \geq 0$ for all $k = 1, 2, \ldots$. By condition (5) in Assumption 3.1, we have $3\theta_1 - 1 + 2\gamma L < 0$. Therefore,

$$\frac{1}{2 + 2\theta_1}(3\theta_1 - 1 + 2\gamma L) < \frac{1}{3 + 4\theta_1}(3\theta_1 - 1 + 2\gamma L) < \theta_2,$$

which implies $3\theta_1 - 1 + 2\gamma L - 2\theta_2 - 2\theta_1\theta_2 < 0$. Because $|\theta_2| = -\theta_2$, it holds that

$$3\theta_1 - 1 + 2\gamma L + |\theta_2| - \theta_2 + \theta_1|\theta_2| - \theta_1\theta_2 < 0. \tag{3.20}$$

As a result, $a_1 > 0$. From the assumption that $\frac{1}{3 + 4\theta_1}(3\theta_1 - 1 + 2\gamma L) < \theta_2$, we see that

$$1 - 3\theta_1 - 2\gamma L + 3\theta_2 + 4\theta_2\theta_1 > 0.$$

Again, because $|\theta_2| = -\theta_2$ it holds that

$$1 - 3\theta_1 - 2\gamma L - 2|\theta_2| + \theta_2 - 2|\theta_2|\theta_1 + 2\theta_2\theta_1 > 0, \tag{3.21}$$

which implies that $a_2 > 0$. To show that $q_k \geq 0$ for all k, we see that $\theta_2 \leq 0$ and $a_1 > 0$. Hence, for all $k \geq k_0$, we have

$$\begin{aligned} q_k \geq& (1 + 2\gamma\mu_F + 2\gamma\mu_G)\|u_k - u^*\|^2 - \theta_1\|u_{k-1} - u^*\|^2 + 2\gamma\langle Gu_k - Gu_{k-1}, u^* - u_k\rangle \\ &+ (1 - |\theta_2| - \theta_1 - \gamma L)\|u_k - u_{k-1}\|^2 \\ \geq& (1 + 2\mu_F + 2\mu_G)\|u_k - u^*\|^2 - \theta_1\|u_{k-1} - u^*\|^2 - \gamma L(\|u_k - u_{k-1}\|^2 + \|u_k - u^*\|^2) \\ &+ (1 - |\theta_2| - \theta_1 - \gamma L)\|u_k - u_{k-1}\|^2 \\ \geq& (1 + 2\gamma\mu_F + 2\gamma\mu_G - \gamma L)\|u_k - u^*\|^2 - \theta_1(2\|u_k - u_{k-1}\|^2 + 2\|u_k - u^*\|^2) \\ &+ (1 - |\theta_2| - \theta_1 - 2\gamma L)\|u_k - u_{k-1}\|^2 \\ \geq& (1 + 2\gamma\mu_F + 2\gamma\mu_G - 3\theta_1 - \gamma L)\|u_k - u^*\|^2 + (1 - |\theta_2| - \theta_1 - 2\gamma L)\|u_k - u_{k-1}\|^2. \end{aligned} \tag{3.22}$$

Because $\theta_1 < \frac{1}{3}(1 - 2\gamma L)$ and $\mu_F + \mu_G > 0, \gamma > 0$, one has $3\theta_1 - 1 + 2\gamma L - 2\gamma\mu_F - 2\gamma\mu_G < 0$. It follows that $1 + 2\gamma\mu_F + 2\gamma\mu_G - 3\theta_1 - \gamma L > 1 - 3\theta_1 - \gamma L > 0$ and

$$3\theta_1 - 1 + 2\gamma L < \frac{1}{3 + 4\theta_1}(3\theta_1 - 1 + 2\gamma L) < \theta_2.$$

Thus $-|\theta_2|-3\theta_1+1-2\gamma L>0$. From (3.22), we derive $q_k\geq 0$ for all $k\geq k_0$. In addition, in view of $a_2>0$ and $\lambda_k\in(0,1)$, we obtain from (3.19) that $q_{k+1}\leq(1-\lambda_k)q_k+\lambda_k\|u^*-w^*\|^2$. Thus Lemma 2.3 can be invoked with $b_k=0$ for all k and $a_k=\lambda\|u^*-w^*\|^2$. Hence $\{q_k\}$ is bounded. Therefore, from (3.22), we derive that $\{u_k\}$ is also bounded as asserted. □

Theorem 3.1. *Let Assumption 3.1 hold. Suppose that* $\lim_{k\to\infty}\lambda_k=0$ *and* $\sum_{k=1}^{\infty}\lambda_k=\infty$. *Then the sequence* $\{u_k\}$ *generated by Algorithm 3.1 converges strongly to* $P_{\text{zer}(F+G)}w^*$.

Proof. Let $u^*=P_{\text{zer}(F+G)}(w^*)$. Using (2.2), we have

$$\begin{aligned}\|z_k-u^*\| =&\|\lambda_k(w^*-u^*)+(1-\lambda_k)(w_k-u^*)\|^2\\ =&\lambda_k^2\|w^*-u^*\|^2+(1-\lambda_k)^2\|w_k-u^*\|^2+2\lambda_k(1-\lambda_k)\langle w^*-u^*,w_k-u^*\rangle.\end{aligned} \tag{3.23}$$

By (2.2), we have

$$\begin{aligned}&\|z_k-u_{k+1}\|^2\\ &=\lambda_k^2\|w^*-u_{k+1}\|^2+(1-\lambda_k)^2\|w_k-u_{k+1}\|^2+2\lambda_K(1-\lambda_k)\langle w^*-u_{k+1},w_k-u_{k+1}\rangle\\ &\geq\lambda_k^2\|u_{k+1}-w^*\|^2+(1-\lambda_k)^2\|u_{k+1}-w_k\|^2-2\lambda_k(1-\lambda_k)\|u_{k+1}-w^*\|\|u_{k+1}-w_k\|\\ &\geq\lambda_k^2\|u_{k+1}-w^*\|^2+(1-\lambda_k)^2\|u_{k+1}-w_k\|^2-2\lambda_k(1-\lambda_k)\Gamma\|u_{k+1}-w_k\|,\end{aligned}$$

where $\Gamma=\sup_{k\geq 1}\|u_{k+1}-w^*\|$. Note that this supremum exists because $\{u_k\}$ is bounded due to Lemma 3.1. Putting (3.23) and (3.24) into (3.7), we have

$$\begin{aligned}&(1+2\gamma\mu_F+2\gamma\mu_G)\|u_{k+1}-u^*\|^2+2\gamma\langle Gu_{k+1}-Gu_k,u^*-u_{k+1}\rangle\\ \leq&\lambda_k^2\|u^*-w^*\|^2+(1-\lambda_k)^2\|w_k-u^*\|^2+2\lambda_k(1-\lambda_k)\langle w^*-u^*,w_k-u^*\rangle\\ &-(\lambda_k^2\|u_{k+1}-w^*\|^2+(1-\lambda_k)^2\|u_{k+1}-w_k\|^2-2\lambda_k(1-\lambda_k)\Gamma\|u_{k+1}-w_k\|)\\ &+2\gamma(1-\lambda_k)\langle Gu_k-Gu_{k-1},u^*-u_k\rangle\\ &+(1-\lambda_k)\gamma L(\|u_k-u_{k-1}\|^2+\|u_{k+1}-u_k\|^2)\\ \leq&(1-\lambda_k)\left(\|w_k-u^*\|^2+2\gamma\langle Gu_k-Gu_{k-1},u^*-u_k\rangle\right)\\ &+\lambda_k(\lambda_k\|u^*-w^*\|^2+2(1-\lambda_k)\langle w^*-u^*,w_k-u^*\rangle+2(1-\lambda_k)\Gamma\|u_{k+1}-w_k\|)\\ &-(1-\lambda_k)^2\|u_{k+1}-w_k\|^2+(1-\lambda_k)\gamma L(\|u_k-u_{k-1}\|^2+\|u_{k+1}-u_k\|^2),\forall k\geq k_0.\end{aligned} \tag{3.24}$$

Now plugging (3.12) and (3.17) into (3.24), we obtain

$$\begin{aligned}&(1+2\gamma\mu_F+2\gamma\mu_G)\|u_{k+1}-u^*\|^2+2\gamma\langle Gu_{k+1}-Gu_k,u^*-u_{k+1}\rangle\\ \leq&(1-\lambda_k)\Big[(1+\theta_1)\|u_k-u^*\|^2-(\theta_1-\theta_2)\|u_{k-1}-u^*\|^2-\theta_2\|u_{k-2}-u^*\|^2\\ &+(1+\theta_1)(\theta_1-\theta_2)\|u_k-u_{k-1}\|^2+\theta_2(1+\theta_1)\|u_k-u_{k-2}\|^2\\ &-\theta_2(\theta_1-\theta_2)\|u_{k-1}-u_{k-2}\|^2+2\gamma\langle Gu_k-Gu_{k-1},u^*-u_k\rangle\Big]\\ &+\lambda_k(\lambda_k(\|u^*-w^*\|^2+2(1-\lambda_k)\langle w^*-u^*,w_k-u^*\rangle+2(1-\lambda_k)\Gamma\|u_{k+1}-w_k\|)\\ &-(1-\lambda_k)^2\Big[(1-\theta_1-|\theta_2|)\|u_{k+1}-u_k\|^2+(\theta_1^2-\theta_1-\theta_1|\theta_2|)\|u_k-u_{k-1}\|^2\\ &+(\theta_2^2-|\theta_2|-\theta_1|\theta_2|)\|u_{k-1}-x_{k-2}\|^2\Big]+(1-\lambda_k)\gamma L(\|u_k-u_{k-1}\|^2+\|u_{k+1}-u_k\|^2).\end{aligned}$$

It follows that

$$
\begin{aligned}
&(1+2\gamma\mu_F+2\gamma\mu_G)\|u_{k+1}-u^*\|^2-\theta_1\|u_k-u^*\|^2-\theta_2\|u_{k_1}-u^*\|^2\\
&+2\gamma\langle Gu_{k+1}-Gu_k,u^*-u_{k+1}\rangle+(1-|\theta_2|-\theta_1-\gamma L)\|u_{k+1}-u_k\|^2\\
\leq&(1-\lambda_k)\Big[\|u_k-u^*\|^2-\theta_1\|u_{k-1}-u^*\|^2-\theta_2\|u_{k-2}-u^*\|^2+2\gamma\langle Gu_k-Gu_{k-1},u^*-u_k\rangle\\
&+(1-|\theta_2|-\theta_1-\gamma L)\|u_k-u_{k-1}\|^2\Big]\\
&+\lambda_k\Big(\lambda_k\|u^*-w^*\|^2+2(1-\lambda_k)\langle w^*-u^*,w_k-u^*\rangle+2(1-\lambda_k)\Gamma\|u_{k+1}-w_k\|\Big)\\
&+(1-\lambda_k)\Big[2\gamma L+2\theta_1-\theta_2-\theta_1\theta_2-1+|\theta_2|+\theta_1^2-(1-\lambda_k)(\theta_1^2-\theta_1-\theta_1|\theta_2|)\Big] \qquad (3.25)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|u_k-u_{k-1}\|^2+(1-\lambda_k)\Big[\theta_2^2-\theta_2\theta_1-(1-\lambda_k)(\theta_2^2-|\theta_2|-\theta_1|\theta_2|\Big]\|u_{k-1}-u_{k-2}\|^2\\
=&(1-\lambda_k)\Bigg[\|u_k-u^*\|^2-\theta_1\|u_{k-1}-u^*\|^2-\theta_2\|u_{k-2}-u^*\|^2+2\gamma\langle Gu_k-Gu_{k-1},u^*-u_k\rangle\\
&+(1-|\theta_2|-\theta_1-\gamma L)\|u_k-u_{k-1}\|^2\\
&-\Big(2\gamma L+2\theta_1-\theta_2-\theta_1\theta_2-1+|\theta_2|+\theta_1^2-(1-\lambda_k)(\theta_1^2-\theta_1-\theta_1|\theta_2|)\Big)\\
&\Big(\|u_{k-1}-u_{k-2}\|^2-\|u_k-u_{k-1}\|^2\Big)\Bigg]\\
&+\lambda_k\Big(\lambda_k(\|u^*-w^*\|^2+2(1-\lambda_k)(w^*-u^*,w_k-u^*\rangle+2(1-\lambda_k)\Gamma\|u_{k+1}-w_k\|\Big)\\
&-(1-\lambda_k)\Big[1-2\theta_1-2\gamma L+\theta_2+2\theta_1\theta_2-|\theta_2|-\theta_2^2-\theta_1^2+(1-\lambda_k)](\theta_1^2-\theta_1-\theta_1|\theta_2|)\\
&+(1-\lambda_k)(\theta_2^2-|\theta_2|-\theta_1|\theta_2|)\Big]\|u_{k-1}-u_{k-2}\|^2, \quad \forall k\geq k_0. \qquad (3.26)
\end{aligned}
$$

Set

$$
\begin{aligned}
r_k=&-\Big(2\gamma L+2\theta_1-\theta_2-\theta_1\theta_2-1+|\theta_2|+\theta_1^2-(1-\lambda_k)(\theta_1^2-\theta_1-\theta_1|\theta_2|)\Big),\\
a_k=&\lambda_k\|w^*-u^*\|^2+2(1-\lambda_k)\langle w^*-u^*,w_k-u^*\rangle+2(1-\lambda_k)\Gamma\|u_{k+1}-w_k\|,\\
s_{k+1}=&(1+2\gamma\mu_F+2\gamma\mu_G)\|u_k-u^*\|^2-\theta_1\|u_{k-1}-u^*\|^2-\theta_2\|u_{k-2}-u^*\|^2\\
&+2\gamma\langle Gu_k-Gu_{k-1},u^*-u_k\rangle+(1-|\theta_2|-\theta_1-\gamma L)\|u_k-u_{k-1}\|^2+r_k\|u_{k-1}-u_{k-2}\|^2,\\
q_k=&1-2\theta_1-2\gamma L+\theta_2+2\theta_1\theta_2-|\theta_2|-\theta_2^2-\theta_1^2+(1-\lambda_k)(\theta_1^2-\theta_1-\theta_1|\theta_2|)\\
&+(1-\lambda_k)(\theta_2^2-|\theta_2|-\theta_1|\theta_2|).
\end{aligned}
$$

From (3.25) and (3.26), we derive

$$\begin{aligned} s_{k+1} \leq &(1-\lambda_k)s_k + \lambda_k a_k - (1-\lambda_k)q_k\|u_{k-1}-u_{k-2}\|^2 - (1-\lambda_k)(2\gamma\mu_F + 2\gamma\mu_G)\|u_{k+1}-u^*\|^2 \\ \leq &(1-\lambda_k)s_k + \lambda_k a_k - (1-\lambda_k)q_k\|u_{k-1}-u_{k-2}\|^2, \forall k \geq k_0. \end{aligned} \tag{3.27}$$

By (3.20), one has $1-3\theta_1-2\gamma L-|\theta_2|+\theta_2-\theta_1|\theta_2|+\theta_1\theta_2>0$, which implies that

$$\begin{aligned} \lim_{k\to\infty} r_k = &\lim_{k\to\infty} -\Big(2\gamma L + 2\theta_1 - \theta_2 - \theta_1\theta_2 - 1 + |\theta_2| + \theta_1^2 - (1-\lambda_k)(\theta_1^2 - \theta_1 - \theta_1|\theta_2|)\Big) \\ = &1-3\theta_1-2\gamma L-|\theta_2|+\theta_2-\theta_1|\theta_2|+\theta_1\theta_2 > 0. \end{aligned}$$

Thus there exists $k_1 \geq k_0$ such that $r_k > 0$ for all $k \geq k_1$. Moreover, from (3.21), one has

$$1-3\theta_1-2\gamma L-2|\theta_2|+\theta_2-2|\theta_2|\theta_1+2\theta_2\theta_1 > 0.$$

It follows that

$$\begin{aligned} \lim_{k\to\infty} q_k = \lim_{k\to\infty} \Big(&1-2\theta_1-2\gamma L+\theta_2+2\theta_1\theta_2-|\theta_2|-\theta_2^2-\theta_1^2 \\ &+(1-\lambda_k)(\theta_1^2-\theta_1-\theta_1|\theta_2|)+(1-\lambda_k)(\theta_2^2-\theta_2-\theta_1|\theta_2|)\Big) \\ = &1-3\theta_1-2\gamma L-2|\theta_2|+\theta_2-2|\theta_2|\theta_1+2\theta_2\theta_1 > 0. \end{aligned}$$

Hence, there exists $k_2 \geq k_0$ such that $q_k > 0$ for all $k \geq k_2$. Then, it follows from(3.27) that

$$s_{k+1} \leq (1-\lambda_k)s_k + \lambda_k a_k, \forall k \geq k_2. \tag{3.28}$$

Let s_{k_i} be the sequence satisfying that $\liminf_{i\to\infty}(s_{k_i+1}-s_{k_i}) \geq 0$. Then (3.27) implies that

$$\begin{aligned} &\limsup_{i\to\infty}\Big((1-\lambda_{k_i})q_{k_i}\|u_{k_i-1}-u_{k_i-2}\|^2\Big) \\ &\leq \limsup_{i\to\infty}\Big((s_{k_i}-s_{k_i+1})+\lambda_{k_i}(q_{k_i}-s_{k_i})\Big) \\ &\leq -\liminf_{i\to\infty}(\Big(s_{k_i+1}-s_{k_i}) \leq 0. \end{aligned}$$

Because $\lim_{i\to\infty}(1-\lambda_{k_i})q_{k_i} > 0$, it holds that

$$\lim_{i\to\infty}\|u_{k_i-1}-u_{k_i-2}\| = 0 = \lim_{i\to\infty}\|u_{k_i+1}-u_{k_i}\|. \tag{3.29}$$

So,

$$\lim_{i\to\infty}\|w_{k_i}-u_{k_i}\| = \lim_{i\to\infty}\|\theta_1(u_{k_i}-u_{k_i-1})+\theta_2(u_{k_i-1}-u_{k_i-2})\| = 0. \tag{3.30}$$

By (3.29) and (3.30), we have

$$\lim_{i\to\infty}\|u_{k_i+1}-w_{k_i}\| = 0. \tag{3.31}$$

Since $\lim_{k\to\infty}\lambda_k = 0$, it follows that

$$\lim_{i\to\infty}\|z_{k_i}-w_{k_i}\| = \lim_{i\to\infty}\lambda_{k_i}\|w^*-w_{k_i}\| = 0. \tag{3.32}$$

(3.31) and (3.32) imply

$$\lim_{i\to\infty}\|z_{k_i}-u_{k_i+1}\| = 0. \tag{3.33}$$

In view of (3.29) and the Lipschit continuity of G, we have

$$\lim_{i\to\infty}\|Gu_{k_i+1}-Gu_{k_i}\|=0. \tag{3.34}$$

Due to Lemma 3.1, we see that $\{u_{k_i}\}$ is bounded. Hence, there exists a subsequence $\{u_{k_{i_j}}\}$ of $\{u_{k_i}\}$ which converges weakly to $\bar{u}\in H$, and

$$\limsup_{i\to\infty}\langle w^*-u^*,u_{k_i}-u^*\rangle=\lim_{j\to\infty}\langle w^*-u^*,u_{k_{i_j}}-u^*\rangle=\langle w^*-u^*,\bar{u}-u^*\rangle. \tag{3.35}$$

Let $(x,y)\in\mathrm{gr}(F+G)$. Then $\gamma(y-Gx)\in\gamma Fx$. Due to (3.3) and the μ_F-monotone of F, we have that

$$\langle\gamma(y-Gx)-z_{k_{i_j}}+\gamma Gu_{k_{i_j}}+\gamma(1-\lambda_{k_{i_j}})(Gu_{k_{i_j}}-Gu_{k_{i_j}-1})+u_{k_{i_j}+1},x-u_{k_{i_j}+1}\rangle\geq\gamma\mu_F\|u_{k_{i_j}+1}-x\|^2.$$

Combining this fact with the μ_G-monotonicity of G, we have

$$\begin{aligned}
&\langle y,x-u_{k_{i_j}+1}\rangle\geq\frac{1}{\gamma}\langle\gamma Gx+z_{k_{i_j}}-\gamma Gu_{k_{i_j}}-\gamma(1-\lambda_{k_{i_j}})(Gu_{k_{i_j}}-Gu_{k_{i_J}-1})-u_{k_{i_j}+1},x-u_{k_{i_j}+1}\rangle\\
&\quad+\mu_F\|u_{k_{i_j}+1}-x\|^2\\
&=\langle Gx-Gu_{k_{i_j}+1},x-u_{k_{i_j}+1}\rangle+\langle Gu_{k_{i_j}+1}-Gu_{k_{i_j}},x-u_{k_{i_j}+1}\rangle\\
&\quad+(1-\lambda_{k_{i_j}})\langle Gu_{k_{i_j}-1}-Gu_{k_{i_j}},x-u_{k_{i_j}+1}\rangle+\frac{1}{\gamma}\langle z_{k_{i_j}}-u_{k_{i_j}+1},x-u_{k_{i_j}+1}\rangle+\mu_F\|u_{k_{i_j}+1}-x\|^2\\
&\quad\geq\langle Gu_{k_{i_j}+1}-Gu_{k_{i_j}},x-u_{k_{i_j}+1}\rangle+(1-\lambda_{k_{i_j}})\langle Gu_{k_{i_j}-1}-Gu_{k_{i_j}},x-u_{k_{i_j}+1}\rangle\\
&\quad+\frac{1}{\gamma}\langle z_{k_{i_j}}-u_{k_{i_j}+1},x-u_{k_{i_j}+1}\rangle+\mu_G\|x-u_{k_{i_j}+1}\|^2+\mu_F\|u_{k_{i_j}+1}-x\|^2\\
&=\langle Gu_{k_{i_j}+1}-Gu_{k_{i_j}},x-u_{k_{i_j}+1}\rangle+(1-\lambda_{k_{i_j}})\langle Gu_{k_{i_j}-1}-Gu_{k_{i_j}},x-u_{k_{i_j}+1}\rangle\\
&\quad+\frac{1}{\gamma}\langle z_{k_{i_j}}-u_{k_{i_j}+1},x-u_{k_{i_j}+1}\rangle+(\mu_G+\mu_F)\|x-u_{k_{i_j}+1}\|^2\\
&\geq\langle Gu_{k_{i_j}+1}-Gu_{k_{i_j}},x-u_{k_{i_j}+1}\rangle+(1-\lambda_{k_{i_j}})\langle Gu_{k_{i_j}-1}-Gu_{k_{i_j}},x-u_{k_{i_j}+1}\rangle\\
&\quad+\frac{1}{\gamma}\langle z_{k_{i_j}}-u_{k_{i_j}+1},x-u_{k_{i_j}+1}\rangle.
\end{aligned}\tag{3.36}$$

The last inequality holds by condition (3) in Assumption 3.1. As $j\to\infty$ in (3.36), using (3.33) and (3.34), we have $\langle y,x-\bar{u}\rangle\geq 0$. Due to Lemma 2.6, $F+G$ is maximal $(\mu_F+\mu_G)$-monotone. Hence $\bar{u}\in\mathrm{zer}(F+G)$ by Lemma 2.7. In view of $u^*=P_{\mathrm{zer}(F+G)}w^*$, (3.35) and the characterization of the metric projection imply that

$$\limsup_{i\to\infty}\langle w^*-u^*,u_{k_i}-u^*\rangle=\langle w^*-u^*,\bar{u}-u^*\rangle\leq 0. \tag{3.37}$$

By (3.30), (3.31), and (3.37), we see $\limsup_{i\to\infty}a_{k_i}\leq 0$. Hence, the condition $\sum_{k=1}^{\infty}\lambda_k=0$, Lemma 2.2, and (3.28) obtain $\lim_{k\to\infty}s_k=0$. From the fact and (3.22), we conclude that $\{u_k\}$ converges strongly to $u^*=P_{\mathrm{zer}(F+G)}w^*$, as claimed. □

4. Applications

In this section, we investigate special cases of problem (1.1).

4.1. **Application to COP.** Consider the constrained optimization problem (COP)

$$\min_{u\in H} f(u)+g(u) \tag{4.1}$$

where $g : H \longrightarrow \mathbb{R}$ is continuous differential and convex, and $f : H \longrightarrow \mathbb{R}$ is a proper, l.s.c. convex real value function. Note that f may lack differentiability. when $f \equiv 0$, problem (4.1) is reduced to an unconstrained optimization problem. As mentioned earlier, this problem can be reformulated as the inclusion problem (1.1) with $F = \partial f$ and $G = \nabla g$. If $F = \partial f$, then $J_{\gamma F}(u) = \text{prox}_{\gamma f}(u)$. Thus (3.1) is reduced to the following

$$\begin{aligned} u_{k+1} =& \text{prox}_{\gamma f}(\lambda_k w^* + (1-\lambda_k)\Big(u_k - \theta_1(u_k - u_{k-1}) + \theta_2(u_{k-1} - u_{k-2}) \\ & - \gamma\nabla g u_k - \gamma(1-\lambda_k)(\nabla g u_k - \nabla g u_{k-1})\Big), \quad \forall k \geq 1. \end{aligned} \tag{4.2}$$

We make the following assumption on function g: g is convex and ∇g is Lipschitz continuous with constant $L > 0$, and $\gamma \in \left(0, \dfrac{1}{2L}\right)$. Observe that the conditions related to the operators in Assumption 3.1 hold. Consequently, if θ_1 and θ_2 are chosen to satisfy condition (5), then the sequence generated by (4.2) strongly converges to the solution of the COP.

4.2. **Application to MVIP.** Now we examine the mixed variational inequality problem (MVIP):

$$\text{Find } u^* \in H \text{ such that } \langle T(u^*), u - u^* \rangle + f(u^*) - f(u) \geq 0 \text{ for all } u \in H, \tag{4.3}$$

where $T : H \longrightarrow H$ is a vector-valued operator and $f : H \longrightarrow \mathbb{R}$ is a proper, l.s.c. convex function. Once again, this problem can be rewritten as an inclusion problem of the form (1.1) with $G = T$ and $F = \partial f$. Similarly to the previous case, since $F = \partial f$, we have $J_{\gamma F}(u) = \text{prox}_{\gamma f}(u)$. Hence (3.1) reduces to the following

$$\begin{aligned} u_{k+1} =& \text{prox}_{\gamma f}(\lambda_k w^* + (1-\lambda_k)\Big(u_k - \theta_1(u_k - u_{k-1}) + \theta_2(u_{k-1} - u_{k-2}) \\ & - \gamma T u_k - \gamma(1-\lambda_k)(T u_k - T u_{k-1})\Big), \quad \forall k \geq 1. \end{aligned} \tag{4.4}$$

If f is convex, T is monotone and L-Lipschitz continuous, and $\gamma \in (0, \frac{1}{2L})$, then all the conditions related to the operators in Assumption 3.1 hold. As a result, the sequence generated by (4.4) converges strongly to the solution of problem (4.3) with appropriate values of the parameters.

4.3. **Application to VIP.** We now consider a special case of problem 4.3 where $f = 0$. This problem is known as the variational inequality problem and can be stated as follows

$$\text{Find } u^* \in C \text{ such that } \langle T(u^*), u - u^* \rangle \geq 0, \quad \forall u \in C, \tag{4.5}$$

where C is a closed and convex subset of H, and $T : C \longrightarrow H$ is an operator. We denote this problem by VIP(T, C).

The variational inequality problems (VIPs) in (4.5) can be reformulated as an inclusion problem of the form $0 \in (F+G)(u)$ with $G = T$ and $F = N_C$. Again, in this case, we have $J_{\gamma F}(u) = P_C(u)$. Hence Algorithm 3.1 reads as

$$\begin{aligned} u_{k+1} =& P_C(\lambda_k w^* + (1-\lambda_k)\Big(u_k - \theta_1(u_k - u_{k-1}) + \theta_2(u_{k-1} - u_{k-2}) \\ & - \gamma T u_k - \gamma(1-\lambda_k)(T u_k - T u_{k-1})\Big), \quad \forall k \geq 1. \end{aligned} \tag{4.6}$$

If T is monotone and Lipschitz continuous with constant $L > 0$, then all the conditions in Assumption 3.1 related to the operators also hold. Hence, the sequence generated by (4.6) converges strongly to the solution of problem (4.5) for appropriate values of the parameters.

5. NUMERICAL EXAMPLE

In this section, we present an illustrative example of the algorithm's effectiveness. The code was written in Python and executed on Google Colab using an HP ProBook 430 G6 laptop running Windows 11 Home Single Language, equipped with an Intel Core i5-8265U CPU at 1.60 GHz and 4 GB of RAM.

Example 5.1. Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $F(u) = Au$ for all $u \in \mathbb{R}^2$. Note that the eigenvalues of A are $\lambda = 2, \lambda = 4$. Hence, F is maximally 2-monotone. We define $G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $G(u) = -u$ for all u in $\mathbb{R}^2$. Then it is easy to see that G is -1-monotone and 1-Lipschitz continuous. Observe also that $0 = (0,0)$ is the unique solution to the inclusion problem $0 \in F(u) + G(u)$. We run the algorithm with the following values of parameters:

$$\gamma = 0.2, \theta_1 = 0.1, \theta_2 = -0.05, \lambda_k = 0.5 \text{ for all } k = 1,2,3,\ldots$$

and use initial iterates

$$u_{-1} = (500, -500), u_0 = (300, 200), u_1 = (-500, 600), w^* = (0,0).$$

A straightforward verification shows that all the conditions in Assumption 3.1 are satisfied by the chosen parameter values. After 200 iterations, Algorithm 3.1 produces an approximate solution

$$u_{200} = (-6.20227189e - 58, 6.20227189e - 58),$$

with residual

$$\|u_{200}\| = 8.7713370294568e - 58.$$

The results of running the algorithm with inertial terms and the chosen parameter values are demonstrated in Figure 1.

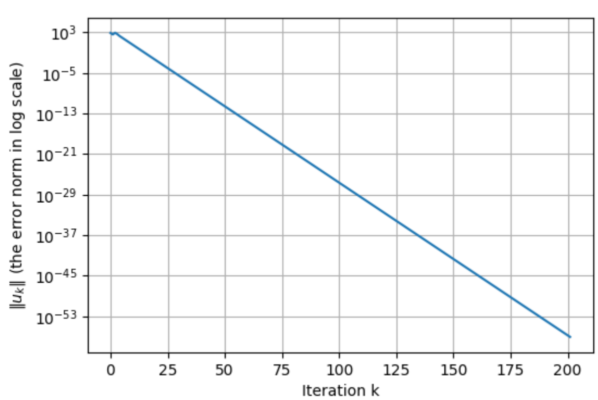

FIGURE 1. Convergence rate of Algorithm 3.1.

6. CONCLUSION

In this paper, We established the strong convergence of a sequence generated by a two-step inertial forward–reflected–anchored–backward splitting algorithm for solving the non-monotone inclusion problem (1.1) in a real Hilbert space. Our result improves the results in [7] by relaxing the assumption on the operators from monotonicity to non-monotonicity. Additionally, we provided a characterization when a generalized monotone operator becomes maximal. Moreover, we present the conditions that ensure the maximality of the sum of two generalized monotone operators. Finally, we discussed some applications to related problems, including constrained optimization, mixed variational inequality problems, and variational inequality problems.

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