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A COUPLED SYSTEM OF HILFER FRACTIONAL DIFFERENTIAL AND HISTORY-DEPENDENT HEMIVARIATIONAL INEQUALITIES

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Abstract. This paper studies a novel coupled system consisting of a Hilfer fractional differential equation and two dynamic history-dependent hemivariational inequalities. The proposed framework models complex interactions involving memory effects, nonconvex constraints, and coupled evolutionary processes. By applying Rothe's semi-discretization method together with surjectivity results for pseudomonotone operators, we establish the existence of solutions under suitable assumptions. The analysis provides constructive discrete approximations that converge to a solution of the original continuous problem, thereby extending the theory of differential variational inequalities to fractional-order systems with nonsmooth and history-dependent dynamics.
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1. Introduction

Differential variational inequalities (DVIs) constitute an important class of mathematical systems that combine a variational inequality with a differential or partial differential equation. They arise naturally in the modeling of dynamic processes involving inequality constraints, such as contact mechanics, network equilibrium, and optimization problems. The systematic study of DVIs in finite-dimensional spaces was initiated by Pang and Stewart [1]. Since then, significant progress has been made in extending these results to infinite-dimensional settings, leading to important analytical developments and numerous applications. We refer the reader to [2, 3, 4, 5, 6] and the references therein for representative contributions in this direction.

In parallel, hemivariational inequalities (HVIs) were introduced by Panagiotopoulos [7] in order to treat nonconvex and nonsmooth problems involving locally Lipschitz functionals. The theory of HVIs provides a powerful framework for describing a wide range of mechanical and engineering phenomena, particularly those involving nonmonotone and multivalued constitutive laws. Substantial theoretical advances and applications of HVIs were reported in recent years; see, e.g., [8, 9, 10, 11]. As natural extensions of DVIs, differential hemivariational inequalities (DHVIs) and differential variational-hemivariational inequalities (DVHVIs) were also widely

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investigated. These frameworks offer effective analytical tools for studying nonlinear dynamical systems; see e.g., [12, 13] and the references therein.

On the other hand, fractional differential equations (FDEs) have attracted considerable attention due to their ability to model systems with memory and hereditary properties. Such equations appear in numerous applications, including electrical networks, fluid flow, electrochemistry, and viscoelastic materials [14, 15]. In this context, Ke, Van, and Obukhovskii [16] investigated fractional differential variational inequalities (FDVIs) in finite-dimensional spaces. Later, Migórski and Zeng [17] used Rothe's method to analyze fractional differential hemivariational inequalities (FDHVIs) in infinite-dimensional spaces. Hao, Wang, and Lu [18] established existence results for a coupled system involving a fractional differential hemivariational inequality and a fractional hemivariational inequality by combining Rothe's method with surjectivity results for multivalued operators. Subsequently, Oultou et al. [2] studied a class of differential history-dependent variational-hemivariational inequalities motivated by dynamic applications, using Rothe's method together with surjectivity theorems for multivalued operators. More recently, a system involving a fractional history-dependent hemivariational inequality coupled with a nonlocal delay fractional differential equation was investigated in [19]. Further developments include the work of Oultou and Benaissa [20], who analyzed the existence and uniqueness of solutions for history-dependent hemivariational inequalities related to a non-stationary Navier-Stokes problem by applying Rothe's method. In another recent contribution, Oultou and Benaissa [21] studied an impulsive nonlinear fractional equation of Hilfer type coupled with quasi-variational and hemivariational inequalities, proving existence and uniqueness results and applying them to frictional contact problems with impulsive effects. Moreover, Hao, Wang, and Lu [22] investigated the existence of solutions for a coupled system involving fractional differential hemivariational inequalities and fractional hemivariational inequalities. Their results were applied to a quasistatic contact problem governed by fractional Kelvin-Voigt constitutive laws.

Motivated by these developments, we consider a new class of coupled systems involving a Hilfer-type fractional differential equation and two dynamic history-dependent hemivariational inequalities. Let $V_j \subset H_j \subset V_j^*$, $j = 1,2$, be an evolution triple, where $(V_j, \|\cdot\|_{V_j})$ denotes a reflexive and separable Banach space, $(H_j, \|\cdot\|_{H_j})$ is a separable Hilbert space, and $(V_j^*, \|\cdot\|_{V_j^*})$ is the dual space of V_j. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V_j and V_j^*. Let X be a reflexive Banach space. In this setting, we investigate the following abstract system combining a Hilfer fractional differential equation with dual history-dependent hemivariational inequalities.

Problem 1.1. Find $\theta : (0,T) \to X$, $y : (0,T) \to V_1$, and $z : (0,T) \to V_2$ such that

$$^{H}D_0^{\alpha,\beta}\theta(\iota) = h(\iota, \theta(\iota), M_1 z(\iota), M_2 y(\iota)) \text{ for a.e. } \iota \in [0,T],$$

together with the following hemivariational inequalities for y and z

$$\begin{aligned}
&\big(\ddot{y}(\iota), v_1 - \dot{y}(\iota)\big)_{H_1} + \big\langle A_1\dot{y}(\iota), v_1 - \dot{y}(\iota)\big\rangle + \big\langle B_1 y(\iota), v_1 - \dot{y}(\iota)\big\rangle \\
&\quad + \big\langle (\mathfrak{R}_1\dot{y})(\iota), v_1 - \dot{y}(\iota)\big\rangle + \Psi_1^0(M_1 z(\iota), M_2\dot{y}(\iota); M_2 v_1 - M_2\dot{y}(\iota)) \\
&\quad \geq \big\langle f_1(\iota), v_1 - \dot{y}(\iota)\big\rangle \text{ for all } v_1 \in V_1, \text{ and for a.e. } \iota \in [0,T],
\end{aligned}$$

$$\begin{aligned}
&\left(\ddot{z}(\iota), v_2-\dot{z}(\iota)\right)_{H_2}+\left\langle A_2\dot{z}(\iota), v_2-\dot{z}(\iota)\right\rangle+\left\langle B_2 z(\iota), v_2-\dot{y}(\iota)\right\rangle\\
&\quad+\left\langle(\mathfrak{R}_2\dot{z})(\iota), v_2-\dot{z}(\iota)\right\rangle+\Psi_2^0(\theta(\iota), M_2 y(\iota), M_1\dot{z}(\iota); M_1 v_2-M_1\dot{z}(\iota))\\
&\quad\geq\left\langle f_2(\iota), v_2-\dot{z}(\iota)\right\rangle \text{ for all } v_2\in V_2, \text{ and for a.e. } \iota\in[0,T],
\end{aligned}$$

$$y(0)=y_0,\ \dot{y}(0)=\varpi,\ z(0)=\rho,\ \dot{z}(0)=\widetilde{z},\ I_0^{1-\mu}\theta(0)=\theta_0,\ I_0^{1-\mu}\dot{\theta}(0)=\varkappa,$$

where ${}^H D_0^{\alpha,\beta}\theta$ denotes the Hilfer fractional derivative of θ of order $\alpha\in(0,1)$ and type $\beta\in[0,1]$, A_j, $B_j: V_j\to V_j^*$ are given mappings, $f_j\in H_j^2(0,T;V_j^*)$, and $\Psi_1: Y_1\times Y_2\to\mathbb{R}$, $\Psi_2: X\times Y_2\times Y_1\to\mathbb{R}$ are locally Lipschitz functions. Moreover, $h:[0,T]\times X\times Y_1\times Y_2\to X$, $M_1: V_2\to Y_1$, $M_2: V_1\to Y_2$, and $\mathfrak{R}_j:\mathscr{C}(0,T;V_j)\to\mathscr{C}(0,T;V_j^*)$ are history-dependent operators.

The present work generalizes several existing results in the literature, including those in [21, 22]. The main goal is to analyze the solvability of this coupled system involving a Hilfer-type fractional differential equation and dual dynamic history-dependent hemivariational inequalities.

The remainder of the paper is outlined as follows. In Section 2, we recall several definitions, auxiliary results, and notations that are used throughout the paper. Section 3 is devoted to the analysis of the proposed dual dynamical fractional history-dependent hemivariational inequality system. By employing Rothe's method together with a surjectivity theorem for multivalued operators, we establish the existence of solutions to the problem.

2. Preliminaries

This section collects essential notations, definitions, and results that are used throughout the paper. We begin by recalling several definitions of fractional derivatives, including the Riemann-Liouville, Caputo, and Hilfer derivatives. The Hilfer derivative generalizes both the Caputo and Riemann-Liouville derivatives, allowing a unified treatment of fractional-order dynamics.

Definition 2.1 ([23])**.** The Riemann-Liouville fractional integral of order $\alpha>0$ of a function ℓ on $\mathbb{R}^+$ is defined by $I_{0^+}^{\alpha}\ell(s):=\frac{1}{\Gamma(\alpha)}\int_0^s(s-r)^{\alpha-1}\ell(r)\,dr$, where Γ denotes the Gamma function.

Definition 2.2 ([23])**.** The Riemann-Liouville fractional derivative of order $\alpha>0$ is given by

$${}^L D_t^{\alpha}\ell(s)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^s(s-r)^{n-\alpha-1}\ell(r)\,dr,$$

where $n=\lceil\alpha\rceil+1$, and $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Definition 2.3 ([23])**.** The Caputo fractional derivative of order $\alpha>0$ is defined, for $\ell\in\mathscr{C}^n[0,+\infty)$ and $\alpha\in(n-1,n)$, by ${}^C D_t^{\alpha}\ell(s)=\frac{1}{\Gamma(n-\alpha)}\int_0^s(s-r)^{n-\alpha-1}\ell^{(n)}(r)$.

Definition 2.4 ([24])**.** The Hilfer fractional derivative of order $\alpha\in[0,1)$ and type $\beta\in[0,1]$ is defined as ${}^H D_s^{\alpha,\beta}\ell(s)=I_s^{\alpha(1-\beta)}\frac{d}{ds}I_s^{(1-\alpha)(1-\beta)}\ell(s)$, provided that the right-hand side exists pointwise on $[0,\infty)$.

Remark 2.1 ([24])**.** The Hilfer derivative interpolates between the Riemann-Liouville and Caputo derivatives:

(1) If $\alpha = 0$ and $0 < \beta < 1$, then $D_s^{0,\beta}\ell(s) = \frac{d}{ds} I_s^{1-\beta}\ell(s)$, which coincides with the Riemann-Liouville derivative.
(2) If $\alpha = 1$ and $0 < \beta < 1$, then ${}^H D_s^{1,\beta}\ell(s) = I_s^{1-\beta}\frac{d}{ds}\ell(s) = {}^C D_s^{\beta}\ell(s)$, recovering the Caputo derivative.

Lemma 2.1 ([24]). *Let $0 < \alpha < 1$, $0 \le \beta \le 1$, and $\psi \in \mathscr{C}([0,T];W)$. Then*

$$\begin{cases} {}^H D_{0^+}^{\alpha,\beta} w(s) = \psi(s), & s \in [0,T], \\ I_{0^+}^{1-\mu} w(a) = w_a \ \ with & a > 0, \end{cases}$$

is equivalent to the integral equation

$$w(s) = \frac{s^{\mu-1}}{\Gamma(\mu)} w_a - \frac{s^{\mu-1}}{\Gamma(\mu)\Gamma(1-\mu+\alpha)} \int_0^a (s-r)^{\alpha-\mu}\,\psi(r)\,dr + \frac{1}{\Gamma(\alpha)} \int_0^s (s-r)^{\alpha-1}\,\psi(r)\,dr.$$

Next, we recall some function spaces that are essential in the analysis of history-dependent operators and hemivariational inequalities.

Let $\bigsqcup_l$ be a finite partition of $[0,T]$ into disjoint subintervals $d_i = (a_i, b_i)$ such that $[0,T] = \bigcup_{i=1}^n \overline{d}_i$, and let $\digamma$ denote the family of all such partitions. For $1 \le n < \infty$ and a vector-valued function $x : [0,T] \to X$, define the seminorm

$$\|x\|_{BV^n([0,T];X)} = \sup_{\bigsqcup_l \in \digamma} \left(\sum_{d_i \in \bigsqcup_l} \|x(b_i) - x(a_i)\|_X^n \right),$$

and the space

$$BV^n([0,T];X) = \left\{ x : [0,T] \to X \ : \ \|x\|_{BV^n([0,T];X)} < \infty \right\}.$$

Furthermore, let X and Z be Banach spaces with a continuous embedding $X \hookrightarrow Z$, and let $1 \le m,n < \infty$. We define the Banach space $\mathscr{M}^{m,n}([0,T];X,Z) = L^m(0,T;X) \cap BV^n([0,T];Z)$, endowed with the norm $\|x\|_{\mathscr{M}^{m,n}([0,T];X,Z)} = \|x\|_{L^m([0,T];X)} + \|x\|_{BV^n([0,T];Z)}$.

3. Well-Posedness Results for a Dual Hilfer Fractional Problem

Let $V_j \subset H_j \subset V_j^*$, $j = 1,2$, be an evolution triple of spaces, where $(V_j, \|\cdot\|V_j)$ denotes a reflexive and separable Banach space, $(H_j, \|\cdot\|H_j)$ is a separable Hilbert space, and $(V_j^*, \|\cdot\|_{V_j^*})$ is the dual space of V_j. We denote by $\langle \cdot,\cdot \rangle$ the duality pairing between V_j and V_j^*. Let X be a reflexive Banach space.

Within this framework, we consider an abstract system consisting of a Hilfer fractional differential equation coupled with two dynamic history-dependent hemivariational inequalities.

Problem 3.1. Find $\theta : (0,T) \to X$, $y : (0,T) \to V_1$, and $z : (0,T) \to V_2$ such that

$${}^H D_0^{\alpha,\beta}\theta(\iota) = h(\iota, \theta(\iota), M_1 z(\iota), M_2 y(\iota)) \ \text{ for a.e. } \ \iota \in [0,T], \tag{3.1}$$

together with the following dynamic inequalities for y and z

$$\begin{aligned} &\big(\ddot{y}(\iota), v_1 - \dot{y}(\iota)\big)_{H_1} + \big\langle A_1\dot{y}(\iota), v_1 - \dot{y}(\iota)\big\rangle + \big\langle B_1 y(\iota), v_1 - \dot{y}(\iota)\big\rangle \\ &+ \big\langle (\mathfrak{R}_1\dot{y})(\iota), v_1 - \dot{y}(\iota)\big\rangle + \Psi_1^0\big(M_1 z(\iota), M_2\dot{y}(\iota); M_2 v_1 - M_2\dot{y}(\iota)\big) \\ &\quad \geq \big\langle f_1(\iota), v_1 - \dot{y}(\iota)\big\rangle \text{ for all } v_1 \in V_1, \text{ and for a.e. } \iota \in [0,T], \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\big(\ddot{z}(\iota), v_2 - \dot{z}(\iota)\big)_{H_2} + \big\langle A_2\dot{z}(\iota), v_2 - \dot{z}(\iota)\big\rangle + \big\langle B_2 z(\iota), v_2 - \dot{z}(\iota)\big\rangle \\ &+ \big\langle (\mathfrak{R}_2\dot{z})(\iota), v_2 - \dot{z}(\iota)\big\rangle + \Psi_2^0\big(\theta(\iota), M_2 y(\iota), M_1\dot{z}(\iota); M_1 v_2 - M_1\dot{z}(\iota)\big) \\ &\quad \geq \big\langle f_2(\iota), v_2 - \dot{z}(\iota)\big\rangle \text{ for all } v_2 \in V_2, \text{ and for a.e. } \iota \in [0,T], \end{aligned} \tag{3.3}$$

and with the initial conditions

$$y(0) = y_0\,,\ \dot{y}(0) = \varpi\,,\ z(0) = \rho\,,\ \dot{z}(0) = \widetilde{z},\ I_0^{1-\mu}\theta(0) = \theta_0\,,\ I_0^{1-\mu}\dot{\theta}(0) = \varkappa. \tag{3.4}$$

Here ${}^{H}D_0^{\alpha,\beta}$ denotes the Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$. The operators A_j, $B_j : V_j \to V_j^*$ $(j = 1,2)$ are given linear mappings, $f_j \in H_j^2(0,T;V_j^*)$, and $\Psi_1 : Y_1 \times Y_2 \to \mathbb{R}$ and $\Psi_2 : X \times Y_2 \times Y_1 \to \mathbb{R}$ are locally Lipschitz functions. Moreover, $h : [0,T] \times X \times Y_1 \times Y_2 \to X$, $M_1 : V_2 \to Y_1$, $M_2 : V_1 \to Y_2$, and $\mathfrak{R}_j : \mathscr{C}(0,T;V_j) \to \mathscr{C}(0,T;V_j^*)$ are history-dependent operators.

To analyze coupled system (3.1)-(3.4), we impose the following assumptions.

$$\begin{cases} \text{For } j = 1,2, \text{ operator } A_j \in \mathscr{L}(V_j, V_j^*) \text{ satisfies} \\ \text{(a) there exists } m_{A_j} > 0 \text{ such that } \langle A_j v_j, v_j\rangle \geq m_{A_j}\|v_j\|^2 \text{ for all } v_j \in V_j, \\ \text{(b) for all } u_j, v_j \in V_j,\ \langle A_j v_j, u_j\rangle = \langle v_j, A_j u_j\rangle. \end{cases} \tag{3.5}$$

$$\begin{cases} \text{For } j = 1,2,\ B_j \in \mathscr{L}(V_j, V_j^*) \text{ satisfies} \\ \text{(a) there exists } m_{B_j} > 0 \text{ such that } \langle B_j v_j, v_j\rangle \geq m_{B_j}\|v_j\|^2 \text{for all } v_j \in V_j, \\ \text{(b) for all } u_j, v_j \in V_j,\ \langle B_j v_j, u_j\rangle = \langle v_j, B_j u_j\rangle. \end{cases} \tag{3.6}$$

$$M_1 \in \mathscr{L}(V_2, Y_1) \text{ and } M_2 \in \mathscr{L}(V_1, Y_2) \text{ are compact.} \tag{3.7}$$

$$\begin{cases} \Psi_1 : Y_1 \times Y_2 \to \mathbb{R} \text{ satisfies} \\ \text{(a) } \Psi_1(y_1, \cdot) \text{ is locally Lipschitz on } Y_2, \text{ for all } y_1 \in Y_1, \\ \text{(b) there exists } c_{\Psi_1} > 0 \text{ such that } \|\partial\Psi_1(y_1,y_2)\|_{Y_2^*} \leq c_{\Psi_1}\big(1 + \|y_2\|_{Y_2}\big),\ \forall y_j \in Y_j, \\ \text{(c) for all } y_2 \in Y_2,\ \Psi_1^0(\cdot,\cdot;y_2) \text{ is upper semicontinuous on } Y_1 \times Y_2, \\ \text{(d) there exists } m_{\Psi_1} > 0 \text{ such that } \langle \xi_1 - \zeta_1, y_1 - z_1\rangle \geq -m_{\Psi_1}\|y_1 - z_1\|_{Y_1}^2,\ \forall y_1, z_1 \in Y_1 \\ \qquad \text{where } \xi_1 \in \partial\Psi_1(y_1,y_2) \text{ and } \zeta_1 \in \partial\Psi_1(z_1,z_2) \text{ for all } y_2, z_2 \in Y_2. \end{cases} \tag{3.8}$$

$$\begin{cases}\Psi_2 : X\times Y_2\times Y_1\to\mathbb{R} \text{ satisfies}\\ \text{(a) } \Psi_2(\theta,y_2,\cdot) \text{ is locally Lipschitz on } Y_2, \text{ for all } \theta\in X,\, y_2\in Y_2,\\ \text{(b) there exists } m_{\Psi_2}>0 \text{ such that } \|\partial\Psi_2(\theta,y_2,y_1)\|_{Y_1^*}\le m_{\Psi_2}(1+\|y_1\|_{Y_1}),\ \forall y_j\in Y_j,\, \theta\in X,\\ \text{(c) for all } \mu\in Y_1,\, \Psi_2^0(\cdot,\cdot,\cdot;\mu) \text{ is upper semicontinuous on } X\times Y_2\times Y_1,\\ \text{(d) there exists } m_{\Psi_2}>0 \text{ such that } \langle\xi_2-\zeta_2,y_2-z_2\rangle\ge -m_{\Psi_2}\|y_2-z_2\|^2_{Y_2},\ \forall y_2,z_2\in Y_2,\\ \qquad \text{where } \xi_2\in\partial\Psi_2(\theta,y_2,y_1),\zeta_2\in\partial\Psi_2(\theta,z_2,z_1) \text{ for all } y_1,z_1\in Y_1.\end{cases} \tag{3.9}$$

$$\begin{cases} h:[0,T]\times X\times Y_1\times Y_2\to X \text{ satisfies}\\ \text{(a) } h(\cdot,y_1,y_2,x) \text{ is continuous for all } y_j,\,z_j\in Y_j,\, x\in X,\\ \text{(b) there exists } c_h>0 \text{ such that for all } \iota\in[0,T], \text{ one has}\\ \qquad \|h(\iota,y_1,y_2,x_1)-h(\iota,z_1,z_2,x_2)\|\\ \qquad \le c_h(\|y_1-y_2\|+\|z_1-z_2\|+\|x_1-x_2\|),\ \forall y_j,\,z_j\in Y_j,\, x_j\in X,\\ \text{(c) there exists } \rho\in L^{\frac{1}{p}}(Q,\mathbb{R}_+)\ (1<p<q<2) \text{ such that}\\ \qquad \|h(\iota,y_1,y_2,\theta)\|_X\le\rho(\iota),\ \forall\iota\in[0,T],\, y_1\in Y_1,\, y_2\in Y_2,\, \theta\in X.\end{cases} \tag{3.10}$$

$$\begin{cases} f_j\in H_j^2([0,T];V_j^*)\ (j=1,2) \text{ satisfies}\\ \qquad H_j^2([0,T];V_j^*)=\left\{v_j\in L^2([0,T];V_j^*)\,:\ \dot v_j\in\mathbb{V}_j,\ \ddot v_j\in\mathbb{V}_j\right\}.\end{cases} \tag{3.11}$$

$$\begin{cases} \text{(a) There exist parameters } a,b \text{ with } 1<a<\alpha<2 \text{ and } b>0 \text{ such that}\\ \qquad \frac{c_hT^{(p+1)(2-a)}}{\Gamma(\alpha)(p+1)^{2-a}}\left(\frac{a-1}{b}\right)^{a-1}<1, \quad \text{with } p=\frac{\alpha-1}{2-a}\in(0,1).\\ \text{(b) There exist } S_1\in\mathscr{L}(H_1;Y_2) \text{ and } S_2\in\mathscr{L}(H_2;Y_1) \text{ such that}\\ \qquad S_1(\ell_1(y))=M_2(y), \text{ for } y\in V_1, \text{ and } S_2(k_2(z))=M_1(z) \text{ for } z\in V_2.\end{cases} \tag{3.12}$$

$$\begin{cases}\text{The operators } R_j\in\mathscr{L}(V_j,V_j^*)\ (j=1,2) \text{ are linear and continuous.}\end{cases} \tag{3.13}$$

$$\begin{cases}\text{The maps } q_j:[0,T]\times[0,T]\to\mathscr{L}(V_j,V_j^*) \text{ are continuous and satisfy}\\ \qquad \|q_j(r,\iota)-q_j(s,\iota)\|\le L_{q_j}|r-s|,\ \forall r,\,s,\,\iota\in[0,T] \ \text{ with } \ L_{q_j}>0,\end{cases} \tag{3.14}$$

$$\begin{cases}\text{The mappings } \mathfrak{R}_j:\mathscr{C}(0,T;V_j)\to\mathscr{C}(0,T;V_j) \text{ given by}\\ \qquad (\mathfrak{R}_jv)(t):=R_j\Big(a_{\mathfrak{R}_j}+\int_0^\iota q_j(\iota,s)\,v(s)\,ds\Big),\\ \text{are linear, bounded, and satisfying for } L_{\mathfrak{R}_j}=L_{q_j}\|R_j\|>0 \text{ such that}\\ \qquad \big\|(\mathfrak{R}_ju_j)(\iota)-(\mathfrak{R}_jv_j)(\iota)\big\|\le L_{\mathfrak{R}_j}\int_0^\iota\|u(r)-v(r)\|_{V_j}\,dr,\\ \qquad\qquad \forall u_j,\,v_j\in V_j,\,\iota\in[0,T].\end{cases} \tag{3.15}$$

Under these assumptions, Problem 3.1 can be rewritten in the equivalent form.

Problem 3.2. Find $\theta : [0,T] \to X$, $y : [0,T] \to V_1$, and $z : [0,T] \to V_2$ such that

$$
\begin{aligned}
&{}^H D_0^{\alpha,\beta} \theta(\iota) = h(\iota, \theta(\iota), M_2 y(\iota), M_1 z(\iota)),\\
&\ell_1^* \ell_1 \ddot{y}(\iota) + A_1 \dot{y}(\iota) + B_1 y(\iota) + (\mathfrak{R}_1 \dot{y})(\iota) + M_2^* \xi_1(\iota) = f_1(\iota),\\
&\ell_2^* \ell_2 \ddot{z}(\iota) + A_2 \dot{z}(\iota) + B_2 z(\iota) + (\mathfrak{R}_2 \dot{z})(\iota) + M_1^* \xi_2(\iota) = f_2(\iota),
\end{aligned}
$$

together with the initial conditions

$$
I_0^{1-\mu} \theta(0) = \theta_0\,,\ I_0^{1-\mu} \dot{\theta}(0) = \varkappa\,,\ y(0) = y_0\,,\ \dot{y}(0) = \varpi\,,\ z(0) = z_0\,,\ \dot{z}(0) = \rho,
$$

where $\xi_1(\iota) \in \partial \Psi_1(M_1 z(\iota), M_2 \dot{y}(\iota))$ and $\xi_2(\iota) \in \partial \Psi_2(\theta(\iota), M_2 y(\iota), M_1 \dot{z}(\iota))$.

The solution of Problem 3.2 is understood in the following weak sense.

Definition 3.1. A triplet (θ, y, z) with $\theta \in \mathscr{C}([0,T];X)$, $y \in \mathscr{C}([0,T];V_1)$ and $z \in \mathscr{C}([0,T];V_2)$, is called a solution of Problem 3.2 if the fractional equation holds in its integral form and the evolution inclusions for y and z are satisfied

$$
\begin{aligned}
&\theta(\iota) = \frac{\iota^{\mu-1}}{\Gamma(\alpha)} \theta_0 + \frac{\iota^{\mu}}{\Gamma(\alpha)} \varkappa + \frac{1}{\Gamma(\alpha)} \int_0^{\iota} (\iota - r)^{\alpha-1} h(r, M_1 z(r), M_2 y(r), \theta(r))\, dr,\\
&\ell_1^* \ell_1 \ddot{y}(\iota) + A_1 \dot{y}(\iota) + B_1 y(\iota) + (\mathfrak{R}_1 \dot{y})(\iota) + M_2^* \xi_1(\iota) = f_1(\iota),\\
&\ell_2^* \ell_2 \ddot{z}(\iota) + A_2 \dot{z}(\iota) + B_2 z(\iota) + (\mathfrak{R}_2 \dot{z})(\iota) + M_1^* \xi_2(\iota) = f_2(\iota),
\end{aligned}
$$

together with the initial conditions

$$
y(0) = y_0\,,\ \dot{y}(0) = \varpi\,,\ z(0) = z_0\,,\ \dot{z}(0) = \rho,
$$

where $\xi_1(\iota) \in \partial \Psi_1(M_1 z(\iota), M_2 \dot{y}(\iota))$ and $\xi_2(\iota) \in \partial \Psi_2(\theta(\iota), M_2 y(\iota), M_1 \dot{z}(\iota))$ for a.e. $\iota \in [0,T]$. Moreover, $\ell_j : V_j \to H_j$ denotes the canonical embedding from the evolution triple, and ℓ_j^* its adjoint.

To investigate the solvability of Problem 3.2, we employ Rothe's method, which is based on a time discretization of the interval $[0,T]$. The idea is to construct a sequence of semi-discrete problems and then pass to the limit as the time step tends to zero in order to recover a solution to Problem 3.2.

To this end, let $N \in \mathbb{N}^*$ be a fixed integer and define the time step

$$
\delta = \frac{T}{N}\,,\ \iota_k = k\,\delta \ \text{ with } \ k = 0, 1, \cdot, N.
$$

For $j = 1, 2$, we introduce the averaged values of the functions f_j by

$$
f_{j\delta}^k = \frac{1}{\delta} \int_{\iota_{k-1}}^{\iota_k} f_j(r) dr \; k = 1, 2, \ldots, N.
$$

With these notations, the semi-discrete approximation of Problem 3.2 is given by the following.

Problem 3.3. Find sequences $\{y_\delta^k\}_{k=1}^N \subset V_1$, $\{z_\delta^k\}_{k=1}^N \subset V_2$, $\{\xi_{1\delta}^k\}_{k=1}^N \subset Y_2^*$ and $\{\xi_{2\delta}^k\}_{k=1}^N \subset Y_1^*$ and a function $\theta_\delta \in \mathscr{C}([0,T];X)$ such that $y_\delta^0 = y_0$, $z_\delta^0 = z_0$, $I_0^{1-\mu} \theta(0) = \theta_0$, and $I_0^{1-\mu} \dot{\theta}(0) = \varkappa$

are satisfied, and, for $k = 1,\dots,N$, the following relations hold:

$$\theta_\delta(\iota) = \frac{\iota^{\mu-1}}{\Gamma(\alpha)}\theta_0 + \frac{\iota^{\mu}}{\Gamma(\alpha)}\varkappa + \frac{1}{\Gamma(\alpha)}\int_0^\iota (\iota-r)^{\alpha-1} h(r,\theta_\delta(r),M_1 z_\delta(r),M_2 y_\delta(r))\,dr,$$

$$\frac{\ell_1^*\ell_1}{\delta^2}(y_\delta^k - 2y_\delta^{k-1} + y_\delta^{k-2}) + \frac{1}{\delta}A_1(y_\delta^k - y_\delta^{k-1}) + B_1 y_\delta^k + \eta_{1\delta}^k + M_2^*\xi_{1\delta}^k = f_{1\delta}^k,$$

$$\frac{\ell_1^*\ell_1}{\delta^2}(y_\delta^k - 2y_\delta^{k-1} + y_\delta^{k-2}) + \frac{1}{\delta}A_2(y_\delta^k - y_\delta^{k-1}) + B_2 y_\delta^k + \eta_{2\delta}^k + M_1^*\xi_{2\delta}^k = f_{2\delta}^k,$$

where

$$\xi_{1\delta}^k \in \partial\Psi_1\Big(M_1 z_\delta^k, M_2\Big(\frac{y_\delta^k - y_\delta^{k-1}}{\delta}\Big)\Big),\ \xi_{2\delta}^k \in \partial\Psi_2\Big(\theta_\delta(\iota_k), M_2 y_\delta^k, M_1\Big(\frac{z_\delta^k - z_\delta^{k-1}}{\delta}\Big)\Big).$$

Moreover, for $j = 1,2$, $\eta_{j\delta}^k$ are defined by

$$\eta_{1\delta}^k = R_1\Big(a_{\mathfrak{R}_1} + \sum_{i=1}^k \mathrm{I}_i^1\Big(\frac{y_\delta^i - y_\delta^{i-1}}{\delta}\Big)\Big),\ \mathrm{I}_i^1(y_\delta^i) = \int_{\iota_{i-1}}^{\iota_i} q_1(\iota_k,s)\,y_\delta^i\,ds,$$

$$\eta_{2\delta}^k = R_2\Big(a_{\mathfrak{R}_2} + \sum_{i=1}^k \mathrm{I}_i^2\Big(\frac{z_\delta^i - z_\delta^{i-1}}{\delta}\Big)\Big),\ \mathrm{I}_i^2(z_\delta^i) = \int_{\iota_{i-1}}^{\iota_i} q_2(\iota_k,s)\,z_\delta^i\,ds.$$

For simplicity, we introduce the discrete velocity and acceleration variables:

$$v_{1\delta}^k = \frac{y_\delta^k - y_\delta^{k-1}}{\delta},\ w_{1\delta}^k = \frac{v_{1\delta}^k - v_{1\delta}^{k-1}}{\delta},\ v_{2\delta}^k = \frac{z_\delta^k - z_\delta^{k-1}}{\delta},\ w_{2\delta}^k = \frac{v_{2\delta}^k - v_{2\delta}^{k-1}}{\delta}.$$

Moreover, the discrete approximations of y and z are defined by

$$y_\tau^k = y_0 + \tau\sum_{i=1}^k v_{1\delta}^i,\ z_\delta^k = z_0 + \delta\sum_{i=1}^k v_{2\delta}^i,$$

with the initial values

$$y_\tau^0 = y_0,\ v_{1\delta}^0 = \varpi,\ z_\tau^0 = z_0,\ v_{2\delta}^0 = \rho.$$

Using these notations, Problem 3.3 can be rewritten in the following equivalent form.

Problem 3.4. Find sequences $\{v_{1\delta}^k\}_{k=1}^N \subset V_1$, $\{v_{2\delta}^k\}_{k=1}^N \subset V_2$, $\{\xi_{1\delta}^k\}_{k=1}^N \subset Y_2^*$ and $\{\xi_{2\delta}^k\}_{k=1}^N \subset Y_1^*$ and a function $\theta_\delta \in \mathscr{C}([0,T];X)$ such that $v_{1\delta}^0 = \varpi$, $v_{2\delta}^0 = \rho$, $I_0^{1-\mu}\theta(0) = \theta_0$, and $I_0^{1-\mu}\dot\theta(0) = \varkappa$, are satisfied, and for $k = 1,\dots,N$ the following relations hold:

$$\theta_\delta(\iota) = \frac{\iota^{\mu-1}}{\Gamma(\alpha)}\theta_0 + \frac{\iota^{\mu}}{\Gamma(\alpha)}\varkappa + \frac{1}{\Gamma(\alpha)}\int_0^\iota (\iota-r)^{\alpha-1} h(r,\theta_\delta(r),M_1\hat v_{2\delta}(r),M_2\hat v_{1\delta}(r))dr, \tag{3.16}$$

$$\frac{\ell_1^*\ell_1 v_{1\delta}^k}{\delta} + A_1 v_{1\delta}^k + \delta B_1 v_{1\delta}^k + R_1(\mathrm{I}_k(v_{1\delta}^k)) + M_2^*\xi_{2\delta}^k = F_{1\delta}^k, \tag{3.17}$$

$$\frac{\ell_2^*\ell_2 v_{2\delta}^k}{\delta} + A_2 v_{2\delta}^k + \delta B_2 v_{2\delta}^k + R_2(\mathrm{I}_k(v_{2\delta}^k)) + M_1^*\xi_{1\delta}^k = F_{2\delta}^k, \tag{3.18}$$

where $\xi_{2\delta}^k \in \partial\Psi_1(M_1 z_\delta^k, M_2 v_{1\delta}^k)$ and $\xi_{1\delta}^k \in \partial\Psi_2(\theta_\delta(\iota_k), M_2 y_\delta^k, M_1 v_{2\delta}^k)$. Moreover, for $j = 1,2$, the terms $F_{j\delta}^k$ and $\hat{v}_{j\delta}$ are given by

$$F_{j\delta}^k = f_{j\delta}^k + \frac{\ell^*\ell v_{j\delta}^{k-1}}{\delta} - R_j\Big(a_{\mathfrak{R}_j} + \sum_{i=1}^{k-1} \mathrm{I}_i(v_{j\delta}^i)\Big) - B_1\Big(y_0 + \delta\sum_{i=1}^{k-1} v_{j\delta}^i\Big), \tag{3.19}$$

and

$$\tilde{v}_{j\delta}(\iota) = \begin{cases} v_{j\delta}^{k-1} + \frac{\iota-\iota_k}{\tau}(v_{j\delta}^{k-1} + v_{j\delta}^{k-2}) & \iota \in (\iota_{k-1}, \iota_k], \\ 0 & \iota = 0. \end{cases} \tag{3.20}$$

To establish the existence and uniqueness of solutions for Problem 3.4, we first prove the following auxiliary result concerning the fractional integral equation.

Lemma 3.1. *Assume that conditions* (3.10) *and* (3.12) *are satisfied. Then, for any given functions $v_{1\delta} \in \mathbb{V}_1$, $v_{2\delta} \in \mathbb{V}_2$ and for a.e. $\iota \in [0,T]$, the fractional integral equation*

$$\theta_\delta(\iota) = \frac{\iota^{\mu-1}}{\Gamma(\alpha)}\,\theta_0 + \frac{\iota^\mu}{\Gamma(\alpha)}\,\theta_1 + \frac{1}{\Gamma(\alpha)}\int_0^\iota (\iota-r)^{\alpha-1} h(r, \theta_\delta(r), M_1 v_{2\delta}, M_2 v_{1\delta})\,dr,$$

admits a unique solution $\theta \in \mathscr{C}([0,T];X)$.

Proof. The proof relies on the Banach fixed-point theorem. To this end, we define the operator $\Lambda : (\mathscr{C}([0,T];X), \|\cdot\|) \to (\mathscr{C}([0,T];X), \|\cdot\|)$ by

$$\Lambda\theta(\iota) = \frac{\iota^{\mu-1}}{\Gamma(\alpha)}\,\theta_0 + \frac{\iota^\mu}{\Gamma(\alpha)}\,\varkappa + \frac{1}{\Gamma(\alpha)}\int_0^\iota (\iota-r)^{\alpha-1} h(r, \theta_\delta(r), M_1 v_{2\delta}, M_2 v_{1\delta})\,dr.$$

We prove that the operator Λ admits a unique fixed point in $\mathscr{C}([0,T];X)$. The proof is divided into two steps.

Step 1. Λ maps $\mathscr{C}([0,T];X)$ into itself. Letting $\theta \in \mathscr{C}([0,T];X)$ and $\kappa > 0$, we estimate

$$\begin{aligned}
&\|\Lambda\theta(\iota+\kappa) - \Lambda\theta(\iota)\|_X \\
&\le \big|(\iota+\kappa)^{\alpha-1} - \iota^{\alpha-1}\big|\,\frac{\|\theta_0\|}{\Gamma(\alpha)} + \big|(\iota+\kappa)^\alpha - \iota^\alpha\big|\,\frac{\|\varkappa\|}{\Gamma(\alpha)} \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^\iota \big|(\iota-r)^{\alpha-1} - (\iota+\kappa-r)^{\alpha-1}\big|\,\|h(r,\theta_\delta(r), M_1 v_{2\delta}, M_2 v_{1\delta})\|_X\,dr \\
&+ \frac{1}{\Gamma(\alpha)}\int_\iota^{\iota+\kappa} (\iota+\kappa-r)^{\alpha-1}\|h(r,\theta_\delta(r), M_1 v_{2\delta}, M_2 v_{1\delta})\|_X\,dr \\
&\le \kappa\,(\alpha-1)\,\iota^{\alpha-2}\,\|\theta_0\| + (\alpha\iota^{\alpha-1}\kappa + \kappa^\alpha)\,\|\varkappa\| + \frac{1}{\Gamma(\alpha)}\int_\iota^{\iota+\kappa}(\iota+\kappa-r)^{\alpha-1}\rho(r)\,dr \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^\iota \big((\iota-r)^{\alpha-1} - (\iota+\kappa-r)^{\alpha-1}\big)\,\rho(r)\,dr
\end{aligned}$$

$$
\begin{aligned}
&\leq \kappa(\alpha-1)\,\iota^{\alpha-2}\,\|\theta_0\| + (\alpha\iota^{\alpha-1}\kappa+\kappa^{\alpha})\,\|\varkappa\| \\
&+ \frac{1}{\Gamma(\alpha)}\left(\int_0^{\iota}\left((\iota-r)^{\alpha-1}-(\iota+\kappa-r)^{\alpha-1}\right)^{\frac{1}{2-q}}ds\right)^{2-q}\left(\int_0^{\iota}\rho(r)^{\frac{1}{q-1}}\,dr\right)^{q-1} \\
&+ \frac{1}{\Gamma(\alpha)}\left(\int_{\iota}^{\iota+\kappa}\left((\iota+\kappa-r)^{\alpha-1}\right)^{\frac{1}{2-q}}dr\right)^{2-q}\left(\int_{\iota}^{\iota+\kappa}\rho(r)^{\frac{1}{q-1}}\,dr\right)^{q-1} \\
&\leq \kappa(\alpha-1)\,\iota^{\alpha-2}\|\theta_0\| + (\alpha\iota^{\alpha-1}\kappa+\kappa^{\alpha})\,\|\varkappa\| \\
&+ \frac{\kappa^{(\varsigma+1)(2-q)}}{\Gamma(\alpha)\,(\varsigma+1)^{2-\delta}}\left(\int_0^{\iota}\rho(r)^{\frac{1}{q-1}}\,dr\right)^{q-1} \\
&+ \frac{1}{\Gamma(\alpha)}\left(\int_0^{\iota}\left((\iota-r)^{\frac{\alpha-1}{2-q}}-(\iota+\kappa-r)^{\frac{\alpha-1}{2-q}}\right)ds\right)^{2-p}\left(\int_0^{\iota}\rho(r)^{\frac{1}{q-1}}\,dr\right)^{q-1} \\
&\leq \kappa(\alpha-1)\,\iota^{\alpha-2}\|\theta_0\| + (\alpha\iota^{\alpha-1}\kappa+\kappa^{\alpha})\|\varkappa\| + \frac{3\,\kappa^{(\varsigma+1)(2-q)}}{\Gamma(\alpha)\,(\varsigma+1)^{2-q}}\,\|\rho\|_{L^{\frac{1}{q-1}}(0,\iota)}.
\end{aligned}
$$

Using the continuity assumptions on h and standard estimates for fractional kernels, we obtain that this difference tends to zero as $\kappa\to 0$. Consequently, we find $\Lambda\theta\in\mathscr{C}([0,T];X)$.
Step 2. Λ is a contraction. We consider on $\mathscr{C}([0,T];X)$ the equivalent norm

$$
\|f\|_b = \sup_{\iota\in[0,T]} e^{-b\iota}\,\|f(\iota)\|_X,\; b>0.
$$

Let θ_1, $\theta_2\in\mathscr{C}([0,T];X)$. Using assumption (3.10) and Hölder's inequality, we find

$$
\begin{aligned}
&\|\Lambda\theta_1(\iota)-\Lambda\theta_2(\iota)\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^{\iota}(\iota-r)^{\alpha-1}\|h(r,\theta_1(r),M_1v_{2\delta},M_2v_{1\delta})-h(r,\theta_2(r),M_1v_{2\delta},M_2v_{1\delta})\|\,dr \\
&\leq \frac{c_h}{\Gamma(\alpha)}\int_0^{\iota}(\iota-r)^{\alpha-1}\|\theta_1-\theta_2\|_b\,e^{br}\,dr \\
&\leq \frac{c_h\,\|\theta_1-\theta_2\|_b}{\Gamma(\alpha)}\int_0^{\iota}(\iota-r)^{\alpha-1}e^{br}\,dr \\
&\leq \frac{c_h\,\|\theta_1-\theta_2\|_b}{\Gamma(\alpha)}\left(\int_0^{\iota}(\iota-r)^{\frac{\alpha-1}{2-a}}\,dr\right)^{2-a}\left(\int_0^{\iota}e^{\frac{br}{a-1}}\,dr\right)^{a-1} \\
&\leq \frac{c_h\,T^{(p+1)(2-a)}}{\Gamma(\alpha)\,(p+1)^{2-a}}\left(\frac{a-1}{b}\right)^{a-1}\|\theta_1-\theta_2\|_b\,e^{b\iota}.
\end{aligned}
$$

By multiplying each side by $e^{-b\iota}$, we derive

$$
\|\Lambda\theta_1(\iota)-\Lambda\theta_2(\iota)\|_b \leq \frac{c_h\,T^{(p+1)(2-a)}}{\Gamma(\alpha)(p+1)^{2-a}}\left(\frac{a-1}{b}\right)^{a-1}\|\theta_1-\theta_2\|_b,\ \forall\,\iota\in[0,T].
$$

Therefore, using condition (3.12)(a), we obtain that Λ is a contraction mapping, and the Banach fixed-point theorem ensures the existence of a unique fixed point $\theta \in \mathscr{C}([0,T];X)$. This completes the proof. □

Lemma 3.2. *Assume that hypotheses* (3.5)-(3.15) *hold and that*

$$m_{B_j} > 3\,\|R_j\|\, m_{q_j},\ \ j=1,2. \tag{3.21}$$

Then, there exists $\delta_0>0$ such that, for every $\delta\in(0,\delta_0)$, Problem 3.4 has at least one solution.

Proof. Let y^0_δ, $v^0_{1\delta},\ldots,v^{k-1}_{1\delta}$ and z^0_δ, $v^0_{2\delta},\ldots,v^{k-1}_{2\delta}$ be given. Assume that there exist $v^k_{j\delta}\in V_j$, $\xi^k_{j\delta}\in Y^*_j$ $(j=1,2)$, and $\theta_\delta\in C([0,\iota_k];X)$ satisfying relations (3.16)-(3.20). We introduce the operators $P_1:V_1\to V^*_1$ and $P_2:V_2\to V^*_2$, defined by

$$P_1y=\frac{\ell^*_1\ell_1}{\delta}y+A_1y+\delta B_1y+R_1(\mathrm{I}^1_k(y))+M^*_2\partial\Psi_1(M_1z^k_\delta,M_2y),$$

$$P_2z=\frac{\ell^*_2\ell_2}{\delta}z+A_2z+\delta B_2z+R_2(\mathrm{I}^2_k(z))+M^*_1\partial\Psi_2(\theta_\delta(\iota_k),M_2y^k_\delta,M_1z).$$

We prove that P_1 and P_2 are surjective, and we do that in several steps.

Step 1. Pseudomonotonicity. From the definition of the operators ℓ_l $(l=1,2)$ and assumptions (3.5), (3.7), and (3.13)-(3.15), the mappings

$$V_1\ni y\mapsto\frac{\ell^*_1\ell_1}{\delta}y+A_1y+\delta B_1y+R_1(\mathrm{I}^1_k(y))\in V^*_1$$

and

$$V_2\ni z\mapsto\frac{\ell^*_2\ell_2}{\delta}z+A_2z+\delta B_2z+R_2(\mathrm{I}^2_k(z))\in V^*_2$$

are continuous, bounded and monotone. Therefore, by Theorem 3.96 in [12], they are pseudomonotone. Moreover, assumptions (3.8) and (3.9) together with [9, Proposition 5.6] imply that the operators $y\mapsto M^*_2\partial\Psi_1(M_1z^k_\delta,M_2y)$, and $z\mapsto M^*_1\partial\Psi_2(\theta_\delta(\iota_k),M_2y^k_\delta,M_1z)$ are pseudomonotone. Consequently, by Proposition 3.59(ii) in [12], the operators P_1 and P_2, being sums of pseudomonotone operators, are themselves pseudomonotone.

Step 2. Coercivity. Let $y\in V_1$. Using (3.5)-(3.9), (3.12)(b), (3.13) and (3.14), we obtain

$$\begin{aligned}\langle P_1y,y\rangle &\ge \Big\langle\frac{\ell^*_1\ell_1}{\delta}y+A_1y+\delta B_1y+R_1(\mathrm{I}^1_k(y))+M^*_2\partial\Psi_1(M_1z^k_\delta,M_2y),y\Big\rangle\\ &\ge\frac{1}{\delta}\,\|y\|^2_{H_1}+m_{A_1}\|y\|^2+\delta m_{B_1}\|y\|^2-c_{\Psi_1}\big(1+\|M_2y\|_{Y_2}\big)\,\|M_2y\|_{Y_2}-\delta\,\|R_1\|\,m_{q_1}\|y\|^2\\ &\ge\Big(\frac{1}{\delta}-c_{\Psi_1}\|S_1\|^2\Big)\|y\|^2_{H_1}+m_{A_1}\|y\|^2+\delta\big(m_{B_1}-\|R_1\|\,m_{q_1}\big)\|y\|^2-c_{\Psi_1}\,\|S_1\|\,\|y\|_{H_1}.\end{aligned}$$

Choosing $\delta_1=\frac{1}{c_{\Psi_1}\|S_1\|^2}>0$, and using (3.21), we deduce

$$\langle P_1y,y\rangle\ge m_{A_1}\,\|y\|^2_{V_1}+\delta_1\,\|y\|^2_{V_1}-c_{\Psi_1}\|S_1\|\,\|y\|_{H_1}.$$

A similar argument yields, for $z\in V_2$, that

$$\langle P_2z,z\rangle\ge m_{A_2}\,\|z\|^2_{V_2}+\delta_2\,\|z\|^2_{V_2}-c_{\Psi_2}\|S_2\|\,\|z\|_{H_2},$$

with $\delta_2 = 1/(c_{\Psi_2}\|S_2\|^2)$. Let $\delta_0 = \min\{\delta_1, \delta_2\}$. Then, for any $\delta \in (0, \delta_0)$,

$$\frac{\langle P_1 y, y\rangle}{\|y\|} \to \infty \ \text{ as } \ \|y\| \to \infty,$$

and

$$\frac{\langle P_2 z, z\rangle}{\|z\|} \to \infty \ \text{ as } \ \|z\| \to \infty.$$

Hence, P_1 and P_2 are coercive.

Finally, since P_1 and P_2 are pseudomonotone, bounded and coercive, Lemma 2.2 in [25] ensures that they are surjective. Therefore, for every $\delta \in (0, \delta_0)$ there exist $v_{1\delta}^k$ and $v_{2\delta}^k$ satisfying relations (3.16)-(3.20). Consequently, Problem 3.4 admits at least one solution for all $\delta \in (0, \delta_0)$, which completes the proof. □

Lemma 3.3. *Assume that the hypotheses of Lemma 3.1 hold. Then, there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, the following estimates hold.*

$$\max_{1\le k\le N} \|y_\delta^k\|_{V_1} \le \mathfrak{C}, \tag{3.22}$$

$$\max_{1\le k\le N} \|z_\delta^k\|_{V_2} \le \mathfrak{C}, \tag{3.23}$$

$$\max_{1\le k\le N} \|v_{j\delta}^k\|_{V_j} \le \mathfrak{C}, \ j = 1,2, \tag{3.24}$$

$$\max_{1\le k\le N} \|w_{j\delta}^k\|_{H_j} \le \mathfrak{C}, \ j = 1,2, \tag{3.25}$$

$$\sum_{k=1}^{N} \|v_{j\delta}^k - v_{j\delta}^{k-1}\|_{V_j} \le \mathfrak{C}, \ j = 1,2, \tag{3.26}$$

$$\sum_{k=1}^{N} \|w_{j\delta}^k - w_{j\delta}^{k-1}\|_{H_j} \le \mathfrak{C}, \ j = 1,2, \tag{3.27}$$

where $\mathfrak{C} > 0$ is a constant independent of δ and may change from line to line.

Proof. Replacing k with $k-1$ in (3.17) and (3.18), we obtain

$$\frac{\ell_1^*\ell_1 v_{1\delta}^{k-1}}{\delta} + A_1 v_{1\delta}^{k-1} + \delta B_1 v_{1\delta}^{k-1} + R_1(\mathrm{I}_k^1(v_{1\delta}^{k-1})) + M_2^*\xi_{2\delta}^{k-1} = F_{1\delta}^{k-1}, \tag{3.28}$$

$$\frac{\ell_2^*\ell_2 v_{2\delta}^{k-1}}{\delta} + A_2 v_{2\delta}^{k-1} + \delta B_2 v_{2\delta}^{k-1} + R_2(\mathrm{I}_k^2(v_{2\delta}^{k-1})) + M_1^*\xi_{1\delta}^{k-1} = F_{2\delta}^{k-1}. \tag{3.29}$$

Subtracting (3.28) from (3.17) and (3.29) from (3.18), we deduce

$$\ell_1^*\ell_1(w_{1\delta}^k - w_{1\delta}^{k-1}) + \delta A_1 w_{1\delta}^k + \delta B_1 v_{1\delta}^k + R_1(\mathrm{I}_k^1(v_{1\delta}^k)) + M_2^*(\xi_{2\delta}^k - \xi_{2\delta}^{k-1}) = f_{1\delta}^k - f_{1\delta}^{k-1}, \tag{3.30}$$

$$\ell_2^*\ell_2(w_{2\delta}^k - w_{2\delta}^{k-1}) + \delta A_2 w_{2\delta}^k + \delta B_2 v_{2\delta}^k + R_2(\mathrm{I}_k^2(v_{2\delta}^k)) + M_1^*(\xi_{1\delta}^k - \xi_{1\delta}^{k-1}) = f_{2\delta}^k - f_{2\delta}^{k-1}. \tag{3.31}$$

Taking the duality pairing of (3.30) with $-w^k_{1\delta}$ and of (3.31) with $-w^k_{2\delta}$, we obtain

$$\begin{aligned}&(w^k_{1\delta}-w^{k-1}_{1\delta},w^k_{1\delta})_{H_1}+\langle A_1w^k_{1\delta},w^k_{1\delta}\rangle+\delta\langle B_1v^k_{1\delta},v^k_{1\delta}-v^{k-1}_{1\delta}\rangle\\ &\le\langle f^k_{1\delta}-f^{k-1}_{1\delta},w^k_{1\delta}\rangle+\frac{1}{\delta}\langle\xi^k_{2\delta}-\xi^{k-1}_{2\delta},M_2(v^k_{1\delta}-v^{k-1}_{1\delta})\rangle+\langle R_1(\mathrm{I}^1_k(v^k_{1\delta})),w^k_{1\delta}\rangle,\end{aligned}\tag{3.32}$$

$$\begin{aligned}&(w^k_{2\delta}-w^{k-1}_{2\delta},w^k_{2\delta})_{H_2}+\delta\langle A_2w^k_{2\delta},w^k_{2\delta}\rangle+\delta\langle B_2v^k_{2\delta},v^k_{2\delta}-v^{k-1}_{2\delta}\rangle\\ &\le\langle f^k_{2\delta}-f^{k-1}_{2\delta},w^k_{2\delta}\rangle+\frac{1}{\delta}\langle\xi^k_{1\delta}-\xi^{k-1}_{1\delta},M_1(v^k_{2\delta}-v^{k-1}_{2\delta})\rangle+\langle R_2(\mathrm{I}^2_k(v^k_{2\delta})),w^k_{2\delta}\rangle.\end{aligned}\tag{3.33}$$

Using

$$(w^k_{j\delta}-w^{k-1}_{j\delta},w^k_{j\delta})_{H_j}=\frac{1}{2}\left(\|w^k_{j\delta}\|^2_{H_j}-\|w^{k-1}_{j\delta}\|^2_{H_j}+\|w^k_{j\delta}-w^{k-1}_{j\delta}\|^2_{H_j}\right),$$

and

$$\langle B_jv^k_{j\delta},v^k_{j\delta}-v^{k-1}_{j\delta}\rangle=\frac{1}{2}\big(\langle B_jv^k_{j\delta},w^k_{j\delta}\rangle-\langle B_jv^{k-1}_{j\delta},v^{k-1}_{j\delta}\rangle+\langle B_j(v^k_{j\delta}-v^{k-1}_{j\delta}),v^k_{j\delta}-v^{k-1}_{j\delta}\rangle\big),$$

and conditions (3.5)-(3.9), (3.13), (3.14) and (3.15), we see that

$$\begin{aligned}&\|w^k_{1\delta}\|^2_{H_1}+\|w^k_{1\delta}-w^{k-1}_{1\delta}\|^2_{H_1}+m_{A_1}\|w^k_{1\delta}\|^2+m_{B_1}\|v^k_{1\delta}\|^2+m_{B_1}\|v^k_{1\delta}-v^{k-1}_{1\delta}\|^2\\ &\le 2\delta m_{\Psi_1}\|S_1\|^2\|w^k_{1\delta}\|^2_{H_1}+\frac{2}{\delta}\langle f^k_{1\delta}-f^{k-1}_{1\delta},v^k_{1\delta}-v^{k-1}_{1\delta}\rangle\\ &+2m_{q_1}\|R_1\|\|v^k_{1\delta}\|\|v^k_{1\delta}-v^{k-1}_{1\delta}\|+\|w^{k-1}_{1\delta}\|^2_{H_1}+m_{B_1}\|v^{k-1}_{1\delta}\|^2,\end{aligned}\tag{3.34}$$

and

$$\begin{aligned}&\|w^k_{2\delta}\|^2_{H_2}+\|w^k_{2\delta}-w^{k-1}_{2\delta}\|^2_{H_2}+m_{A_2}\|w^k_{2\delta}\|^2+m_{B_2}\|v^k_{2\delta}\|^2+m_{B_2}\|v^k_{2\delta}-v^{k-1}_{2\delta}\|^2\\ &\le 2\delta m_{\Psi_2}\|S_2\|^2\|w^k_{2\delta}\|^2_{H_2}+\frac{2}{\delta}\langle f^k_{2\delta}-f^{k-1}_{2\delta},v^k_{2\delta}-v^{k-1}_{2\delta}\rangle\\ &+2m_{q_2}\|R_2\|\|v^k_{2\delta}\|\|v^k_{2\delta}-v^{k-1}_{2\delta}\|+\|w^{k-1}_{2\delta}\|^2_{H_2}+m_{B_2}\|v^{k-1}_{2\delta}\|^2.\end{aligned}\tag{3.35}$$

Replacing k by l and summing from 1 to n,

$$\begin{aligned}&\|w^k_{j\delta}\|^2_{H_j}+\sum_{l=1}^k\|w^l_{j\delta}-w^{l-1}_{j\delta}\|^2_{H_j}+m_{B_j}\|v^k_{j\delta}\|^2+m_{B_j}\sum_{l=1}^k\|v^l_{j\delta}-v^{l-1}_{j\delta}\|^2\\ &\le 2\delta m_{\Psi_j}\|S_j\|^2\sum_{l=1}^k\|w^l_{j\delta}\|^2_{H_j}+\frac{2}{\delta}\sum_{l=1}^k\langle f^l_{j\delta}-f^{l-1}_{j\delta},v^l_{j\delta}-v^{l-1}_{j\delta}\rangle\\ &+2m_{q_j}\|R_j\|\sum_{l=1}^k\|v^l_{j\delta}\|\|v^l_{j\delta}-v^{l-1}_{j\delta}\|+\|w^0_{j\delta}\|^2_{H_j}+m_{B_j}\|v^0_{j\delta}\|^2,\ j=1,2.\end{aligned}\tag{3.36}$$

We extend the functions f_j, $j=1,2$, from the interval $[0,T]$ to $[-\delta,T]$ by defining

$$f_j(\iota)=\begin{cases}f_j(\iota) & \text{for }\ \iota\in[0,T],\\ f_j(0) & \text{for }\ \iota\in[-\delta,0].\end{cases}$$

Consequently, it follows that $f_j\in L^\infty([-\delta,T];V^*_j)$, $j=1,2$. Next, we introduce the discrete difference

$$\tilde{f}^k_{j\delta}=\frac{1}{\delta}(f^k_{j\delta}-f^{k-1}_{j\delta}),\ k=1,2,\dots,N,\ j=1,2.$$

Observe that

$$\begin{aligned}\|\tilde{f}^k_{j\delta}\|_{V_j^*} &= \frac{1}{\delta^2}\Big\|\int_{\iota_{k-1}}^{\iota_k}\big(f_j(r)-f_j(r-\delta)\big)\,dr\Big\|_{V_j^*}\\ &= \frac{1}{\delta^2}\Big\|\int_{\iota_{k-1}}^{\iota_k}\int_{r-\delta}^{r}\dot{f}_j(s)\,ds\,dr\Big\|_{V_j^*}\\ &\le \frac{1}{\delta}\int_{\iota_{k-1}}^{\iota_k}\|\dot{f}_j(r)\|_{V_j^*}\,dr.\end{aligned}$$

Since $W^{1,2}([0,T];V_j^*)\hookrightarrow\mathscr{C}([0,T];V_j^*)$ is continuous, assumption (3.11) implies that, for all $r\in[0,T]$, $\|\dot{f}_j(r)\|_{V_j^*}\le c_{f_j}$. Hence, we obtain

$$\sum_{l=1}^{k}\Big\langle\frac{f^l_{j\delta}-f^{l-1}_{j\delta}}{\delta},v^l_{j\delta}-v^{l-1}_{j\delta}\Big\rangle=\langle\tilde{f}^k_{j\delta},v^k_{j\delta}\rangle-\langle\tilde{f}^1_{j\delta},v^1_{j\delta}\rangle-\delta\sum_{l=2}^{k}\Big\langle\frac{\tilde{f}^l_{j\delta}-\tilde{f}^{l-1}_{j\delta}}{\delta},v^{l-1}_{j\delta}\Big\rangle.$$

We now estimate the quantity $\big(\tilde{f}^k_{j\delta}-\tilde{f}^{k-1}_{j\delta}\big)/\delta$. Indeed,

$$\begin{aligned}\frac{\tilde{f}^k_{j\delta}-\tilde{f}^{k-1}_{j\delta}}{\delta}=\frac{f^k_{j\delta}-2f^{k-1}_{j\delta}+f^{k-2}_{j\delta}}{\delta^2}&=\frac{1}{\delta^3}\int_{\iota_{k-1}}^{\iota_k}\big(f_j(s)-2f_j(s-\delta)+f_j(s-2\delta)\big)\,ds\\ &=\frac{1}{\delta^3}\int_{\iota_{k-1}}^{\iota_k}\int_{r-\delta}^{r}\int_{s-\delta}^{s}\ddot{f}_j(\tau)\,d\tau\,ds\,dr.\end{aligned}$$

Applying Hölder's inequality yields

$$\begin{aligned}\Big\|\frac{\tilde{f}^k_{j\delta}-\tilde{f}^{k-1}_{j\delta}}{\delta}\Big\|_{V_j^*}&\le\frac{1}{\delta^3}\int_{\iota_{k-1}}^{\iota_k}\int_{r-\delta}^{r}\int_{s-\delta}^{s}\|\ddot{f}_j(\tau)\|_{V_j^*}\,d\tau\,ds\,dr\\ &\le\frac{1}{\delta^3}\left(\int_{\iota_{k-3}}^{\iota_k}\frac{(\iota_k-s)^4}{4}\,ds\right)^{1/2}\left(\int_{\iota_{k-3}}^{\iota_k}\|\ddot{f}_j(s)\|^2_{V_j^*}\,ds\right)^{1/2}\\ &\le\frac{\sqrt{3}}{2\sqrt{5}}\left(\int_{\iota_{k-3}}^{\iota_k}\|\ddot{f}_j(s)\|^2_{V_j^*}\,ds\right)^{1/2}.\end{aligned}$$

Using Young's inequality $ab\le\varepsilon_j a^2+\frac{1}{4\varepsilon_j}b^2$ ($j=1,2$), we deduce

$$\begin{aligned}&\Big|\sum_{l=1}^{k}\Big\langle\frac{f^l_{j\delta}-f^{l-1}_{j\delta}}{\delta},v^l_{j\delta}-v^{l-1}_{j\delta}\Big\rangle\Big|\\ &\le\|\tilde{f}^k_{j\delta}\|\,\|v^k_{j\delta}\|+\|\tilde{f}^1_{1\delta}\|\,\|v^0_{1\delta}\|+\delta\sum_{l=2}^{k}\Big\|\frac{\tilde{f}^l_{j\delta}+\tilde{f}^{l-1}_{j\delta}}{\delta}\Big\|_{V_j^*}\|v^{k-1}_{j\delta}\|_{V_l}\\ &\le\varepsilon_j\|v^k_{j\delta}\|^2+\frac{1}{4\varepsilon_j}\|\tilde{f}^k_{j\delta}\|^2+\|\tilde{f}^1_{j\delta}\|\,\|v^0_{1\delta}\|+\delta\,\varepsilon_j\sum_{l=1}^{k-1}\|v^l_{j\delta}\|^2+\frac{\delta}{4\varepsilon_j}\sum_{l=1}^{k}\Big\|\frac{\tilde{f}^l_{j\delta}+\tilde{f}^{l-1}_{j\delta}}{\delta}\Big\|_{V_j^*}\\ &\le\varepsilon_j\|v^k_{j\delta}\|^2+\frac{\|\tilde{f}^k_{j\delta}\|^2}{4\varepsilon_j}+\|\tilde{f}^1_{j\delta}\|\,\|v^0_{j\delta}\|+\delta\left(\frac{\sqrt{2}}{8\varepsilon_j\sqrt{5}}\|\ddot{f}_j\|_{L^2([-\delta,T];V_j^*)}+\varepsilon_j\sum_{l=1}^{k-1}\|v^l_{j\delta}\|^2\right).\end{aligned}\tag{3.37}$$

Using assumptions (3.13)–(3.15), we obtain

$$2m_{q_j}\|R_j\|\sum_{l=1}^{k}\|v^l_{j\delta}\|\|w^l_{j\delta}\| \le 3m_{q_j}\|R_j\|\|v^k_{j\delta}\|^2 + 4m_{q_j}\|R_j\|\sum_{l=1}^{k-1}\|v^l_{j\delta}\|^2. \tag{3.38}$$

Combining inequalities (3.36), (3.37), and (3.38), we obtain

$$\begin{aligned}
&\left(1-2\delta\, m_{\Psi_j}\|S_j\|^2\right)\left\|w^k_{j\delta}\right\|^2_{H_j} + \sum_{l=1}^{k}\left\|w^l_{j\delta}-w^{l-1}_{j\delta}\right\|^2_{H_j} + m_{B_j}\sum_{l=1}^{k}\left\|v^l_{j\delta}-v^{l-1}_{j\delta}\right\|^2\\
&+\left(m_{B_j}-3m_{q_j}\|R_j\|-\varepsilon_j\right)\left\|v^k_{j\delta}\right\|^2\\
&\le 2\delta\, m_{\Psi_j}\left\|S_j\right\|^2\sum_{l=1}^{k-1}\left\|w^l_{j\delta}\right\|^2_{H_j} + \left\|w^0_{j\delta}\right\|^2_{H_j} + \frac{1}{4\varepsilon_j}\left\|\tilde f^k_{j\delta}\right\|^2 + \left\|\tilde f^1_{j\delta}\right\|\left\|v^0_{j\delta}\right\|\\
&+4m_{q_j}\|R_j\|\sum_{l=1}^{k-1}\left\|v^l_{j\delta}\right\|^2 + \delta\,\frac{\sqrt{2}}{8\varepsilon_j\sqrt{5}}\left\|\ddot f_j\right\|_{L^2([-\delta,T];V_j^*)} + \delta\,\frac{\varepsilon_j}{2}\sum_{l=1}^{k-1}\left\|v^l_{j\delta}\right\|^2.
\end{aligned}$$

Choosing

$$\varepsilon_j = \frac{m_{B_j}-3m_{q_j}\|R_j\|}{2}\ (j=1,2),$$

and using assumption (3.21), we deduce

$$\begin{aligned}
&\left(1-2\delta\, m_{\Psi_j}\|S_j\|^2\right)\left\|w^k_{j\delta}\right\|^2_{H_j} + \sum_{l=1}^{k}\left\|w^l_{j\delta}-w^{l-1}_{j\delta}\right\|^2_{H_j}\\
&+m_{B_j}\sum_{l=1}^{k}\left\|v^l_{j\delta}-v^{l-1}_{j\delta}\right\|^2 + \frac{m_{B_j}-3m_{q_j}\|R_j\|}{2}\|v^k_{j\delta}\|^2\\
&\le 2\delta\, m_{\Psi_j}\|S_j\|^2\sum_{l=1}^{k-1}\left\|w^l_{j\delta}\right\|^2_{H_j} + \left\|w^0_{j\delta}\right\|^2_{H_j} + \frac{1}{2(m_{B_j}-3m_{q_j}\|R_j\|)}\left\|\tilde f^k_{j\delta}\right\|^2\\
&+\left\|\tilde f^1_{j\delta}\right\|\left\|v^0_{j\delta}\right\| + 4m_{q_j}\|R_j\|\sum_{l=1}^{k-1}\left\|v^l_{j\delta}\right\|^2 + \frac{\left(m_{B_j}-3m_{q_j}\|R_j\|\right)\delta}{4}\sum_{l=1}^{k-1}\left\|v^l_{j\delta}\right\|^2\\
&+\frac{\delta\sqrt{2}}{4\sqrt{5}\left(m_{B_j}-3m_{q_j}\|R_j\|\right)}\left\|\ddot f_j\right\|_{L^2([-\delta,T];V_j^*)}.
\end{aligned}$$

Let $\delta_{0j} = \frac{1}{2m_{\Psi_j}\|S_j\|^2}$ $(j=1,2)$, and $\delta_0 = \min\{\delta_{01},\delta_{02}\}$. Then, for $\delta\in(0,\delta_0)$, we obtain

$$\begin{aligned}
&\|w^k_{j\delta}\|^2_{H_j} + \|v^k_{j\delta}\|^2 + \sum_{l=1}^{k}\|w^l_{j\delta}-w^{l-1}_{j\delta}\|^2_{H_j} + \sum_{l=1}^{k}\|v^l_{j\delta}-v^{l-1}_{j\delta}\|^2\\
&\le \mathfrak{C} + \delta\,\mathfrak{C} + \delta\mathfrak{C}\sum_{l=1}^{k-1}\left(\left\|w^l_{j\delta}\right\|^2_{H_j} + \left\|v^l_{j\delta}\right\|^2\right).
\end{aligned} \tag{3.39}$$

Applying the discrete Grönwall inequality to (3.39), we obtain $\max_{1\le k\le N}\left(\|w^k_{j\delta}\|^2_{H_j} + \|v^k_{j\delta}\|^2\right) \le \mathfrak{C}$. Hence, estimates (3.24) and (3.25) hold, and consequently (3.26) and (3.27) follow.

Finally, from the definitions of y_δ^k and z_δ^k, we obtain

$$\|y_\delta^k\|_{V_1} \le \|y_\delta^0\| + \delta \sum_{l=1}^{k} \|v_{1\delta}^l\| \le \|y_\delta^0\| + k\delta\,\mathfrak{C} \le \mathfrak{C},$$

$$\|z_\delta^k\|_{V_2} \le \|z_\delta^0\| + \delta \sum_{l=1}^{k} \|v_{2\delta}^l\| \le \|z_\delta^0\| + k\delta\,\mathfrak{C} \le \mathfrak{C}.$$

Thus, (3.22) and (3.23) follow. Moreover, using (3.8), (3.9), and (3.24), we deduce

$$\|\xi_{1\delta}^k\|_{Y_1^*} \le c_{\Psi_2}\left(1 + \|M_1\|\|v_{2\delta}^k\|\right) \le \mathfrak{C},$$

$$\|\xi_{2\delta}^k\|_{Y_2^*} \le c_{\Psi_1}\left(1 + \|M_2\|\|v_{1\delta}^k\|\right) \le \mathfrak{C}.$$

The proof is therefore complete. □

Next, for a given $\delta > 0$, we introduce the following piecewise affine interpolants associated with the discrete sequences $\{y_\delta^k\}$, $\{z_\delta^k\}$, $\{v_{j\delta}^k\}$, $\{w_{j\delta}^k\}$. For $\iota \in (\iota_{k-1}, \iota_k]$, $k = 1, \ldots, N$ and $j = 1, 2$, we define

$$y_\delta(\iota) = y_\delta^k + \frac{\iota - \iota_k}{\delta}\left(y_\delta^k - y_\delta^{k-1}\right),\ z_\delta(\iota) = z_\delta^k + \frac{\iota - \iota_k}{\delta}\left(z_\delta^k - z_\delta^{k-1}\right),$$

$$v_{j\delta}(\iota) = v_{j\delta}^k + \frac{\iota - \iota_k}{\delta}\left(v_{j\delta}^k - v_{j\delta}^{k-1}\right),\ w_{j\delta}(\iota) = w_{j\delta}^k + \frac{\iota - \iota_k}{\delta}\left(w_{j\delta}^k - w_{j\delta}^{k-1}\right).$$

We also define the function $\vartheta_{l\delta}$ for $l = 1, 2$ by

$$\vartheta_{l\delta}(\iota) = \begin{cases} a_{\mathfrak{R}_1} + \displaystyle\sum_{i=1}^{k} \int_{\iota_{i-1}}^{\iota_i} q_l(\iota_k, r)\, v_{l\delta}^i\, dr & \text{for } \iota \in (\iota_{k-1}, \iota_k], \\ a_{\mathfrak{R}_1} & \text{for } \iota = 0. \end{cases}$$

Moreover, for $j = 1, 2$, we introduce the following piecewise constant interpolants

$$\bar{y}_\delta(\iota) = \begin{cases} y_\delta^k, & \iota \in (\iota_{k-1}, \iota_k], \\ y_\delta^0, & \iota = 0, \end{cases} \quad \bar{z}_\delta(\iota) = \begin{cases} z_\delta^k, & \iota \in (\iota_{k-1}, \iota_k], \\ z_\delta^0, & \iota = 0, \end{cases}$$

$$\bar{v}_{j\delta}(\iota) = \begin{cases} v_{j\delta}^k, & \iota \in (\iota_{k-1}, \iota_k], \\ v_{j\delta}^0, & \iota = 0, \end{cases} \quad \bar{w}_{j\delta}(\iota) = \begin{cases} w_{j\delta}^k, & \iota \in (\iota_{k-1}, \iota_k], \\ w_{j\delta}^0, & \iota = 0, \end{cases}$$

$$\bar{\xi}_{j\delta}(\iota) = \begin{cases} \xi_{j\delta}^k, & \iota \in (\iota_{k-1}, \iota_k], \\ \xi_{j\delta}^0, & \iota = 0, \end{cases} \quad \bar{f}_{j\delta}(\iota) = \begin{cases} f_{j\delta}^k, & \iota \in (\iota_{k-1}, \iota_k], \\ f_j(0), & \iota = 0. \end{cases}$$

Moreover, we need to consider the following Bochner-Lebesgue spaces

$$\mathbb{V}_j = L^2(0, T; V_j), \quad \mathbb{H}_j = L^2(0, T; H_j), \quad \text{and } \mathbb{Y}_j^* = L^2(0, T; Y_j^*),$$

corresponding to V_j, H_j and Y_j $(j = 1, 2)$, respectively.

Lemma 3.4. *Under the assumptions of Lemma 3.3, there exist constants $\delta_0 > 0$ and $\mathfrak{C} > 0$, independent of δ, such that, for all $\delta \in (0, \delta_0)$, the following estimates hold:*

$$\|\bar{y}_\delta\|_{L^\infty(0,T;V_1)} \le \mathfrak{C}, \tag{3.40}$$

$$\|\bar{z}_\delta\|_{L^\infty(0,T;V_2)} \le \mathfrak{C}, \tag{3.41}$$

$$\|y_\delta\|_{\mathscr{C}(0,T;V_1)} \le \mathfrak{C},\ \|z_\delta\|_{\mathscr{C}(0,T;V_2)} \le \mathfrak{C},\ \|\dot{y}_\delta\|_{\mathbb{V}_1} \le \mathfrak{C},$$

$$\|\dot{z}_\delta\|_{\mathbb{V}_2} \le \mathfrak{C}, \tag{3.42}$$

$$\|\bar{\xi}_{j\delta}\|_{\mathbb{Y}_j^*} \le \mathfrak{C}\ (j = 1,2), \tag{3.43}$$

$$\|\bar{v}_{j\delta}\|_{L^\infty(0,T;V_j)} \le \mathfrak{C}\ (j = 1,2), \tag{3.44}$$

$$\|v_{j\delta}\|_{\mathscr{C}(0,T;V_j)} \le \mathfrak{C}\ (j = 1,2), \tag{3.45}$$

$$\|\dot{v}_{j\delta}\|_{\mathbb{H}_j} \le \mathfrak{C}\ (j = 1,2),\ \|\bar{v}_{j\delta}\|_{M^{2,2}(0,T;V_j,V_j^*)} \le \mathfrak{C}\ (j = 1,2).$$

The proof of Lemma 3.4 is based on the same argument as in [2].

Theorem 3.1. *Assume that (3.5)-(3.15) hold. Let $\{\delta\}$ be a sequence such that $\delta \to 0$. Then, there exists a subsequence, still denoted by $\{\delta\}$, such that*

$$\bar{y}_\delta \rightharpoonup y \ \text{ weakly in } \ \mathbb{V}_1, \tag{3.46}$$

$$\bar{z}_\delta \rightharpoonup z \ \text{ weakly in } \ \mathbb{V}_2, \tag{3.47}$$

$$\bar{v}_{1\delta} \rightharpoonup \dot{y} \ \text{ weakly in } \ \mathbb{V}_1, \tag{3.48}$$

$$\bar{v}_{2\delta} \rightharpoonup \dot{z} \ \text{ weakly in } \ \mathbb{V}_2, \tag{3.49}$$

$$\bar{v}_{1\delta} \to \dot{y} \ \text{ strongly in } \ \mathbb{H}_1, \tag{3.50}$$

$$\bar{v}_{2\delta} \to \dot{z} \ \text{ strongly in } \ \mathbb{H}_2, \tag{3.51}$$

$$\bar{\xi}_{j\delta} \rightharpoonup \xi_j \ \text{ weakly in } \ \mathbb{Y}_j^*, \quad j = 1,2, \tag{3.52}$$

$$\dot{v}_{1\delta} \rightharpoonup \ddot{y} \ \text{ weakly in } \ \mathbb{H}_1, \tag{3.53}$$

$$\dot{v}_{2\delta} \rightharpoonup \ddot{z} \ \text{ weakly in } \ \mathbb{H}_2, \tag{3.54}$$

$$\vartheta_\delta \to \vartheta \ \text{ in } \ \mathscr{C}(0,T;X), \tag{3.55}$$

where the triplet $(y, z, \theta) \in \mathbb{V}_1 \times \mathbb{V}_2 \times \mathscr{C}(0,T;X)$ is a solution of Problem 3.1.

Proof. From estimates (3.40), (3.41), and (3.44), together with the reflexivity of the spaces $\mathbb{V}_j$ $(j = 1,2)$, there exist elements y, $v_1 \in \mathbb{V}_1$, and z, $v_2 \in \mathbb{V}_2$ such that, up to a subsequence,

$$\bar{y}_\delta \rightharpoonup y,\ \bar{v}_{1\delta} \rightharpoonup v_1 \ \text{ weakly in } \ \mathbb{V}_1,$$

$$\bar{z}_\delta \rightharpoonup z,\ \bar{v}_{2\delta} \rightharpoonup v_2 \ \text{ weakly in } \ \mathbb{V}_2.$$

Hence, (3.46) and (3.47) hold. Moreover, by the definition of the interpolants, we obtain

$$\|\bar{y}_\delta - y_\delta\|_{\mathbb{V}_1}^2 = \int_{\iota_{k-1}}^{\iota_k} \|\bar{y}_\delta - y_\delta\|^2 \, d\iota = \sum_{k=1}^{N} \int_{\iota_{k-1}}^{\iota_k} (\iota_k - \iota)^2 \left\| \frac{y_\delta^k - y_\delta^{k-1}}{\delta} \right\|^2 d\iota \le C\delta^2.$$

Therefore, $\left\|\bar{y}_\delta - y_\delta\right\|_{\mathbb{V}_1} \to 0$ as $\delta \to 0$. By the same argument, we also obtain $\left\|\bar{z}_\delta - z_\delta\right\|_{\mathbb{V}_2} \to 0$ as $\delta \to 0$. Consequently, $y_\delta \rightharpoonup y$ in $\mathbb{V}_1$ and $z_\delta \rightharpoonup z$ in $\mathbb{V}_2$.

Next, from estimate (3.45) and Proposition 8 in [26], we deduce that $\bar{v}_{j\delta} \to \tilde{v}_j$ strongly in $\mathbb{H}_j$, $(j=1,2)$. Then, Proposition 4 in [26] yields $\tilde{v}_1 = \dot{y}$, and $\tilde{v}_2 = \dot{z}$, and Hence, (3.48)-(3.51) hold. Furthermore, from estimate (3.42) and the reflexivity of $\mathbb{Y}_j^*$, we obtain $\bar{\xi}_{j\delta} \rightharpoonup \xi_j$ weakly in $\mathbb{Y}_j^*$, $(j=1,2)$, which gives (3.52). Using arguments similar to those in [2, Theorem 2.5], we also deduce the convergences (3.53) and (3.54).

Next, from (3.43), (3.45), (3.48), (3.49), (3.53), and (3.54), we infer that $\dot{y}, \tilde{v}_{1\delta} \in W^{1,2}(0,T;V_1)$ and $\dot{z}, \tilde{v}_{2\delta} \in W^{1,2}(0,T;V_2)$. Since $W^{1,2}(0,T;V_j) \hookrightarrow \mathscr{C}(0,T;V_j)$ $(j=1,2)$ are continuous, we derive $\dot{y}, \tilde{v}_{1\delta} \in \mathscr{C}(0,T;V_1)$ and $\dot{z}, \tilde{v}_{2\delta} \in \mathscr{C}(0,T;V_2)$. Then, using the compactness of M_1 and M_2, we obtain

$$M_2\tilde{v}_{1\delta} \to M_2\dot{y} \text{ strongly in } \mathscr{C}(0,T;Y_2), \tag{3.56}$$

$$M_1\tilde{v}_{2\delta} \to M_1\dot{z} \text{ strongly in } \mathscr{C}(0,T;Y_1). \tag{3.57}$$

Since $\dot{y} \in \mathbb{V}_1$ and $\dot{z} \in \mathbb{V}_2$, Lemma 3.1 ensures the existence of a unique $\theta \in \mathscr{C}(0,T;X)$ satisfying, for $\iota \in [0,T]$, the following fractional integral equation

$$\theta(\iota) = \frac{\iota^{\mu-1}}{\Gamma(\alpha)}\theta_0 + \frac{\iota^{\mu}}{\Gamma(\alpha)}\varkappa + \frac{1}{\Gamma(\alpha)}\int_0^\iota (\iota-r)^{\alpha-1} h\big(r,\theta(r),M_1\dot{z},M_2\dot{y}\big)\,dr.$$

Using assumption (3.10), we infer

$$\begin{aligned}
&\left\|\theta_\delta(\iota)-\theta(\iota)\right\|_X \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^\iota (\iota-r)^{\alpha-1}\left\|h\big(r,\theta_\delta(r),M_1\tilde{v}_{2\delta},M_2\tilde{v}_{1\delta}\big)-h\big(r,\theta(r),M_1\dot{z},M_2\dot{y}\big)\right\|dr \\
&\leq \frac{c_h}{\Gamma(\alpha)}\int_0^\iota (\iota-r)^{\alpha-1}[\|\theta_\delta(r)-\theta(r)\|+\|M_1\tilde{v}_{2\delta}-M_1\dot{z}\|+\|M_2\tilde{v}_{1\delta}-M_2\dot{y}\|]dr \\
&\leq \frac{c_h}{\Gamma(\alpha)}\Big(\left\|M_1\tilde{v}_{2\delta}-M_1\dot{z}\right\|+\left\|M_2\tilde{v}_{1\delta}-M_2\dot{y}\right\|\Big)\int_0^\iota (\iota-r)^{\alpha-1}\,dr \\
&\quad+\frac{c_h}{\Gamma(\alpha)}\int_0^\iota (\iota-r)^{\alpha-1}\left\|\theta_\delta(r)-\theta(r)\right\|_X dr \\
&\leq \frac{T^\alpha c_h}{\Gamma(\alpha+1)}\Big(\left\|M_1\tilde{v}_{2\delta}-M_1\dot{z}\right\|_{\mathscr{C}(0,T;Y_1)}+\left\|M_2\tilde{v}_{1\delta}-M_2\dot{y}\right\|_{\mathscr{C}(0,T;Y_2)}\Big) \\
&\quad+\frac{c_h}{\Gamma(\alpha)}\int_0^\iota (\iota-r)^{\alpha-1}\left\|\theta_\delta(r)-\theta(r)\right\|_X dr.
\end{aligned}$$

Applying Grönwall's inequality yields the existence of constant $C>0$ satisfying

$$\|\theta_\delta-\theta\|_{\mathscr{C}(0,T;X)} \leq C\Big(\left\|M_1\tilde{v}_{2\delta}-M_1\dot{z}\right\|_{\mathscr{C}(0,T;Y_1)}+\left\|M_2\tilde{v}_{1\delta}-M_2\dot{y}\right\|_{\mathscr{C}(0,T;Y_2)}\Big).$$

Therefore, by (3.56) and (3.57), we conclude that $\theta_\delta \to \theta$ in $\mathscr{C}(0,T;X)$, which proves (3.55).

Finally, we prove that the triplet (y,z,θ) solves Problem 3.1. Let us first, introduce the Nemitskii operators $\mathscr{A}_j$, $\mathscr{B}_j$, $\mathscr{S}_j$, $\tilde{\mathscr{S}}_j$, $\mathscr{M}_1$, and $\mathscr{M}_2$ associated to A_j, B_j, $\mathfrak{R}_j$, M_1, and M_2, respectively. $\mathscr{S}_j$ are weakly continuous. From (3.48)-(3.49), together with (3.13)-(3.15), we infer from $\delta \to 0$ that

$$\langle \mathscr{S}_1 \bar{v}_{1\delta}, v_1 - \bar{v}_{1\delta} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1} \longrightarrow \langle \mathscr{S}_1 \dot{y}, v_1 - \dot{y} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1} \text{ for all } v_1 \in \mathbb{V}_1,$$
$$\langle \mathscr{S}_2 \bar{v}_{2\delta}, v_2 - \bar{v}_{2\delta} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} \longrightarrow \langle \mathscr{S}_2 \dot{z}, v_2 - \dot{z} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} \text{ for all } v_2 \in \mathbb{V}_2.$$

Moreover, for $j = 1,2$, it yields $\tilde{\mathscr{S}}_j(\vartheta_{j\delta} - a_{\mathfrak{R}_j}) - \mathscr{S}_j(\bar{v}_{j\delta}) \longrightarrow 0$ strongly in $\mathbb{V}_j^*$. Consequently, for all $v_1 \in \mathbb{V}_1$, we deduce

$$\begin{aligned}
&\lim_{\delta \to 0} \langle \tilde{\mathscr{S}}_1 \vartheta_{1\delta}, v_1 - \bar{v}_{1\delta} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1} \\
&= \lim_{\delta \to 0} \Big(\langle \tilde{\mathscr{S}}_1(\vartheta_{1\delta} - a_{\mathfrak{R}_1}) - \mathscr{S}_1 \bar{v}_{1\delta}, v_1 - \bar{v}_{1\delta} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1} \\
&\quad + \langle \mathscr{S}_1 \bar{v}_{1\delta}, v_1 - \bar{v}_{1\delta} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1} + \langle \tilde{\mathscr{S}}_1(a_{\mathfrak{R}_1}), v_1 - \bar{v}_{1\delta} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1} \Big) \\
&= \langle \mathscr{S}_1 \dot{y}, v_1 - \dot{y} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1} + \langle \tilde{\mathscr{S}}_1(a_{\mathfrak{R}_1}), v_1 - \dot{y} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1}.
\end{aligned} \tag{3.58}$$

Similarly, for all $v_2 \in \mathbb{V}_2$, we derive

$$\begin{aligned}
\lim_{\delta \to 0} \langle \tilde{\mathscr{S}}_2 \vartheta_{2\delta}, v_2 - \bar{v}_{2\delta} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} &= \lim_{\delta \to 0} \Big(\langle \tilde{\mathscr{S}}_2(\vartheta_{2\delta} - a_{\mathfrak{R}_2}) - \mathscr{S}_2 \bar{v}_{2\delta}, v_2 - \bar{v}_{2\delta} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} \\
&\quad + \langle \mathscr{S}_2 \bar{v}_{2\delta}, v_2 - \bar{v}_{2\delta} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} + \langle \tilde{\mathscr{S}}_2(a_{\mathfrak{R}_2}), v_2 - \bar{v}_{2\delta} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} \Big) \\
&= \langle \mathscr{S}_2 \dot{z}, v_2 - \dot{z} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} + \langle \tilde{\mathscr{S}}_2(a_{\mathfrak{R}_2}), v_2 - \dot{z} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2}.
\end{aligned} \tag{3.59}$$

On the other hand, using assumptions (3.5)-(3.7), together with (3.46)-(3.55), one has

$$\langle \dot{v}_{1\delta}, v_1 - v_{1\delta} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1} \longrightarrow \langle \ddot{y}, v_1 - \dot{y} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1}, \text{ as } \delta \to 0,$$
$$\langle \dot{v}_{2\delta}, v_2 - v_{2\delta} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} \longrightarrow \langle \ddot{z}, v_2 - \dot{z} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2}, \text{ as } \delta \to 0,$$

$$\langle A_1 \bar{v}_{1\delta}, v_1 - v_{1\delta} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1} \longrightarrow \langle A_1 \dot{y}, v_1 - \dot{y} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1}, \text{ as } \delta \to 0,$$
$$\langle A_2 \bar{v}_{2\delta}, v_2 - v_{2\delta} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} \longrightarrow \langle A_2 \dot{z}, v_2 - \dot{z} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2}, \text{ as } \delta \to 0,$$

$$\langle \xi_{1\delta}, M_2 v_1 - M_2 v_{1\delta} \rangle_{\mathbb{Y}_2^* \times \mathbb{Y}_2} \longrightarrow \langle \xi_1, M_2 v_1 - M_2 \dot{y} \rangle_{\mathbb{Y}_2^* \times \mathbb{Y}_2}, \text{ as } \delta \to 0,$$
$$\langle \xi_{2\delta}, M_1 v_2 - M_1 v_{2\delta} \rangle_{\mathbb{Y}_1^* \times \mathbb{Y}_1} \longrightarrow \langle \xi_2, M_1 v_2 - M_1 \dot{z} \rangle_{\mathbb{Y}_1^* \times \mathbb{Y}_1}, \text{ as } \delta \to 0,$$

$$\langle B_1 y_{1\delta}, v_1 - v_{1\delta} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1} \longrightarrow \langle B_1 y, v_1 - \dot{y} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1}, \text{ as } \delta \to 0,$$
$$\langle B_2 z_{2\delta}, v_2 - v_{2\delta} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} \longrightarrow \langle B_2 z, v_2 - \dot{z} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2}, \text{ as } \delta \to 0,$$

$$\langle f_{1\delta}, v_1 - v_{1\delta} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1} \longrightarrow \langle f_1, v_1 - \dot{y} \rangle_{\mathbb{V}_1^* \times \mathbb{V}_1}, \text{ as } \delta \to 0,$$
$$\langle f_{2\delta}, v_2 - v_{2\delta} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} \longrightarrow \langle f_2, v_2 - \dot{z} \rangle_{\mathbb{V}_2^* \times \mathbb{V}_2}, \text{ as } \delta \to 0.$$

Combining with (3.58)-(3.59), the discrete inequalities imply

$$\begin{aligned}&\langle \ddot{y}, v_1-\dot{y}\rangle_{\mathbb{V}_1^*\times\mathbb{V}_1}+\langle A_1\dot{y}+B_1y, v_1-\dot{y}\rangle_{\mathbb{V}_1^*\times\mathbb{V}_1}+\langle \xi_1, M_2v_1-M_2\dot{y}\rangle_{\mathbb{Y}_2^*\times\mathbb{Y}_2}\\&\quad+\langle \mathscr{S}_1\dot{y}+\tilde{\mathscr{S}}_1(a_{\mathfrak{R}_1}), v_1-\dot{y}\rangle_{\mathbb{V}_1^*\times\mathbb{V}_1}=\langle f_1, v_1-\dot{y}\rangle_{\mathbb{V}_1^*\times\mathbb{V}_1},\ \forall v_1\in\mathbb{V}_1,\end{aligned}\tag{3.60}$$

and

$$\begin{aligned}&\langle \ddot{z}, v_2-\dot{z}\rangle_{\mathbb{V}_2^*\times\mathbb{V}_2}+\langle A_2\dot{z}+B_2z, v_2-\dot{z}\rangle_{\mathbb{V}_2^*\times\mathbb{V}_2}+\langle \xi_2, M_1v_2-M_1\dot{z}\rangle_{\mathbb{Y}_1^*\times\mathbb{Y}_1}\\&\quad+\langle \mathscr{S}_2\dot{z}+\tilde{\mathscr{S}}_2(a_{\mathfrak{R}_2}), v_2-\dot{z}\rangle_{\mathbb{V}_2^*\times\mathbb{V}_2}=\langle f_2, v_2-\dot{z}\rangle_{\mathbb{V}_2^*\times\mathbb{V}_2},\ \forall v_2\in\mathbb{V}_2.\end{aligned}\tag{3.61}$$

Now, we prove that for $\iota\in[0,T]$, one has

$$\xi_1(\iota)\in\partial\Psi_1(M_1z(\iota),M_2\dot{y}(\iota))\ \text{ and }\ \xi_2(\iota)\in\partial\Psi_2(\theta(\iota),M_2y(\iota),M_1\dot{z}(\iota)).$$

To this end, the converse Lebesgue dominated convergence theorem yields

$$\bar{y}_\delta(\iota)\rightharpoonup y(\iota),\ \bar{v}_{1\delta}(\iota)\rightharpoonup\dot{y}(\iota)\ \text{ weakly in } V_1\ \text{ for a.e. } \iota\in[0,T],$$
$$\bar{z}_\delta(\iota)\rightharpoonup z(\iota),\ \bar{v}_{2\delta}(\iota)\rightharpoonup\dot{z}(\iota)\ \text{ weakly in } V_2\ \text{ for a.e. } \iota\in[0,T].$$

Furthermore, since the operators M_1 and M_2 are compact, the above weak convergences imply, for almost every $\iota\in[0,T]$, the following strong convergences

$$M_2\bar{y}_\delta(\iota)\to M_2y(\iota)\ \text{ in } Y_2,\ M_1\bar{z}_\delta(\iota)\to M_1z(\iota)\ \text{ in } Y_1,$$

and

$$M_2\bar{v}_{1\delta}(\iota)\to M_2\dot{y}(\iota)\ \text{ in } Y_2,\ M_1\bar{v}_{2\delta}(\iota)\to M_1\dot{z}(\iota)\ \text{ in } Y_1.$$

Moreover, from (3.55), we have $\theta_\delta(\iota)\to\theta(\iota)$ in X for all $\iota\in[0,T]$. Furthermore, the multifunctions $\partial\Psi_1$ and $\partial\Psi_2$ have weakly compact and convex values for all $(y,z)\in Y_1\times Y_2$ and $(\theta,y,z)\in X\times Y_2\times Y_1$, respectively. Therefore, using condition (3.52) together with [27, Theorem 77], we deduce

$$\xi_1(\iota)\in\partial\Psi_1(M_1z(\iota),M_2\dot{y}(\iota))\ \text{ and }\ \xi_2(\iota)\in\partial\Psi_2(\theta(\iota),M_2y(\iota),M_1\dot{z}(\iota))$$

for a.e. $\iota\in[0,T]$. Finally, combining these inclusions with (3.60) and (3.61), we conclude that $(y,z,\theta)\in\mathbb{V}_1\times\mathbb{V}_2\times\mathscr{C}([0,T];X)$ is a solution of Problem 3.1, and this completes the proof. □

4. Conclusion

This paper introduced and analyzed a novel coupled system consisting of a Hilfer fractional differential equation and two dynamic history-dependent hemivariational inequalities. Under a comprehensive set of assumptions, we established the existence of solutions by combining Rothe's semi-discretization method with surjectivity results for pseudomonotone operators. The framework extends the theory of differential variational inequalities to incorporate fractional-order memory effects, nonconvexity, and nonsmooth constraints in a coupled setting. Future work will apply these abstract results to the analysis of frictional contact problems between viscoelastic bodies exhibiting long-memory effects and adhesion.

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