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A VARIATIONAL APPROACH TO HISTORY-DEPENDENT THERMO-ELECTRO-ELASTIC CONTACT WITH DAMAGE CONSTRAINTS

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Abstract. This paper develops a novel computational framework for frictional interaction between thermo-piezoelectric materials and deformable thermally and electrically conductive substrates. The formulation integrates a generalized Coulomb friction law with regularized electrical conduction and thermal transfer conditions at the interface. Damage evolution, governed by a nonlinear parabolic differential inclusion, is intrinsically coupled to the contact dynamics. We establish the corresponding variational formulation, proving existence and uniqueness of weak solutions. Spatial discretization employs Galerkin methods, while temporal approximation utilizes Euler schemes, collectively yielding optimal-order error estimates for the numerical solutions.

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1. Introduction

Mechanical contact phenomena represent critically important determinants of performance across industrial applications, with accurate modeling essential for predicting tribological behavior, thermal management, and fatigue life [1]. While foundational work dates to Hertzian contact theory, contemporary approaches employ variational inequalities and finite element methodologies [2, 3, 4]. Despite progress in significant gaps persist in multi-physics scenarios involving piezoelectric interactions [5, 6], thermomechanical behavior [7], viscoelastic memory effects [8, 9], progressive damage evolution [10, 11], and the references therein.

Although contact mechanics has been extensively studied [1], mathematical analysis of thermo-electro-viscoelastic contact with damage remains underdeveloped. Existing literature exhibits notable limitations: simplified isothermal formulations [12], idealized friction laws [13], short-duration interactions omitting long-memory effects [14], and small-strain approximations [15]. Recent damage models [16, 17] fail to address critical scenarios in thermo-electro-viscoelastic systems [18, 19, 20]. More recently, Oultou et al. [21] established numerical methodologies for

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piezoelectric contact mechanics, specifically addressing thermally sensitive locking materials exhibiting strain-dependent stiffness amplification. This framework integrates coupled electro-thermo-mechanical formulations with finite element discretization and rigorous error analysis. A thermo-electro-elastic contact model with temperature-dependent friction is formulated as coupled variational inequalities. Existence and uniqueness of weak solutions was proven under constitutive constraints, resolving field-interaction nonlinearities while preserving thermodynamic consistency [22]. To the best of our knowledge, the quasistatic thermo-electro-elastic frictional contact problem incorporating long-memory viscoelastic effects and progressive damage evolution remains unexplored in the literature. This significant research gap motivates the present work, which establishes the first comprehensive framework for this class of multiphysics systems.

This study resolves these fundamental challenges through the development of an innovative theoretical and computational framework for dynamic frictional contact in thermo-electro-viscoelastic materials with damage evolution. The proposed approach makes the following key contributions:

- **a)** A thermodynamically consistent model unifying mechanical deformation, electric potential, thermal effects, and damage propagation within a single theoretical framework;
- **b)** A rigorous variational formulation of history-dependent constitutive relations, complemented by a complete mathematical analysis proving solution existence and uniqueness;
- **c)** Novel numerical discretization schemes combining hybrid Eulerian approaches with adaptive finite element methods, along with derivation of optimal a priori error estimates for the fully discrete system.

The paper is organized as follows. Section 2 introduces the mechanical problem formulation, including necessary preliminaries, data assumptions, and the variational framework. Section 3 establishes existence and uniqueness of solutions through Banach's fixed point theorem combined with variational and parabolic inequality theory. Section 4 develops semi-discrete spatial approximations with associated error bounds and convergence analysis. Finally, Section 5 constructs a fully discrete scheme via finite element spatial discretization and hybrid Euler temporal integration, deriving corresponding error estimates.

2. Thermo-Electro-Mechanical Contact Framework with Frictional Damage Evolution in Piezoelectric Materials

This section presents a novel model for frictional interaction between a viscoelastic body and a thermally and electrically conductive foundation, deriving its weak formulation under these assumptions.

Consider a viscoelastic body occupying an open, bounded, connected domain $\Omega \subset \mathbb{R}^d$ ($d = 2,3$) with Lipschitz boundary S, partitioned into disjoint segments S_1 ($\text{meas}(S_1) \geq 0$), S_2, and S_3. For electrical constraints, $S_1 \cup S_2$ is divided into measurable subsets S_a ($\text{meas}(S_a) > 0$) and S_b. The body experiences body forces f_0, volume free electric charges q_0, and constant strength heat source q_1. Boundary conditions include: fixed displacement on S_1; surface tractions f_2 and heat flux h_2 on S_2; electrical potential $\varphi = 0$ on S_a; and surface charge density q_b on S_b.

The quasistatic time-dependent frictional contact occurs on Γ_C with a conductive foundation maintained at potential φ_0 and temperature θ_F. This contact incorporates damage effects and allows electric and heat transfer, with normalized gap g. The process evolves over $\mathbb{I} = [0, T]$ ($T \geq 0$).

Let $\mathbb{S}^d$ denote the Banach space of second-order symmetric tensors over $\mathbb{R}^d$. Its associated inner product and induced norm are given by:

$$\mathbf{y} \cdot \mathbf{z} = y_i z_i, \qquad \|\mathbf{y}\|_{\mathbb{R}^d} = (\mathbf{y} \cdot \mathbf{y})^{1/2}, \qquad \forall \mathbf{y}, \mathbf{z} \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij} \tau_{ij}, \qquad \|\sigma\|_{\mathbb{S}^d} = (\sigma \cdot \sigma)^{1/2}, \qquad \forall \sigma, \tau \in \mathbb{S}^d.$$

Boundary components, the normal outward ν and tangential τ are defined by

$$y_\nu = \mathbf{y} \cdot \nu, \qquad \mathbf{y}_\tau = \mathbf{y} - y_\nu \nu,$$
$$\tau_\nu = \tau \nu \cdot \nu, \qquad \tau_\tau = \tau \nu - \tau_\nu \nu.$$

Spatiotemporal domains are given by:

$$\widetilde{\Omega} = \Omega \times \mathbb{I}, \qquad \widetilde{S_1} = S_1 \times \mathbb{I}, \qquad \widetilde{S_2} = S_2 \times \mathbb{I},$$
$$\widetilde{S_3} = S_3 \times \mathbb{I}, \qquad \widetilde{S_a} = S_a \times \mathbb{I}, \qquad \widetilde{S_b} = S_b \times \mathbb{I}.$$

Now, we present the mechanical formulation for the quasistatic frictional contact problem of a thermo-electro-viscoelastic body with long memory.

Problem 2.1. *Determine the displacement field $\mathbf{u} : \widetilde{\Omega} \to \mathbb{R}^d$, electric potential $\varphi : \widetilde{\Omega} \to \mathbb{R}$, temperature field $\theta : \widetilde{\Omega} \to \mathbb{R}$, and damage field $\zeta : \widetilde{\Omega} \to \mathbb{R}$ satisfying:*

$$\begin{aligned}\sigma(\iota) =& \mathbf{A}\varepsilon(\dot{u}(\iota)) + \mathbf{F}\varepsilon(u(\iota)) + \mathbf{P}^*\nabla\varphi(\iota) - \mathbf{M}\theta(\iota) \\ &+ \int_0^\iota \mathscr{C}(\iota - s, \varepsilon(u(s)), \zeta(s))ds \qquad \text{in } \widetilde{\Omega},\end{aligned} \tag{2.1}$$

$$D(\iota) = \mathbf{P}\varepsilon(u(\iota)) - \mathbf{B}\nabla\varphi(\iota) - \mathbf{N}\theta(\iota) \qquad \text{in } \widetilde{\Omega}, \tag{2.2}$$

$$\operatorname{Div}\sigma(\iota) + f_0(\iota) = 0 \qquad \text{in } \widetilde{\Omega}, \tag{2.3}$$

$$\operatorname{div} D(\iota) - q_0(\iota) = 0 \qquad \text{in } \widetilde{\Omega}, \tag{2.4}$$

$$\dot{\theta}(\iota) - \operatorname{div}\mathbf{K}\nabla\theta(\iota) = -\mathbf{R}\varepsilon(u(\iota)) + \mathbf{G}\nabla\varphi(\iota) + q_1(\iota) \qquad \text{in } \widetilde{\Omega}, \tag{2.5}$$

$$\dot{\zeta}(\iota) - \kappa \triangle \zeta(t) + \partial\chi_{[0,1]}\zeta(\iota) \ni \mathscr{S}(\varepsilon(u(\iota)), \zeta(\iota)) \qquad \text{in } \widetilde{\Omega}, \tag{2.6}$$

$$u(\iota) = 0 \qquad \text{on } \widetilde{S_1}, \tag{2.7}$$

$$\sigma(\iota).\nu = f_2(\iota) \qquad \text{on } \widetilde{S_2}, \tag{2.8}$$

$$\varphi(\iota) = 0 \qquad \text{on } \widetilde{S_1}, \tag{2.9}$$

$$D(\iota)\cdot\nu = q_b(\iota) \qquad \text{on } \widetilde{S}_2, \tag{2.10}$$

$$\theta(\iota) = 0 \qquad \text{on } \widetilde{S}_1, \tag{2.11}$$

$$q(\iota)\cdot\nu = h_2(\iota) \qquad \text{on } \widetilde{S}_2, \tag{2.12}$$

$$-\sigma_\nu(\iota) = p_\nu(u_\nu(\iota)-g) \qquad \text{on } \widetilde{S}_3, \tag{2.13}$$

$$\begin{cases} |\sigma_\tau(\iota)| \le p_\tau(u_\nu(\iota)-g), \\ |\sigma_\tau(\iota)| \le p_\tau(u_\nu(\iota)-g) \Rightarrow \dot{u}_\tau = 0, \\ |\sigma_\tau(\iota)| = p_\tau(u_\nu(\iota)-g) \Rightarrow \exists\lambda>0 \textit{ such that } \sigma_\tau(\iota) = -\lambda\,\dot{u}_\tau(\iota), \end{cases} \qquad \text{on } \widetilde{S}_3, \tag{2.14}$$

$$D(\iota)\cdot\nu = h_e(u_\nu(\iota)-g)\phi_L(\varphi(\iota)-\varphi_0) \tag{2.15}$$

$$\frac{\partial\, q\,(\iota)}{\partial\nu} = h_c(u_\nu(\iota)-g)\phi_L(\theta(\iota)-\theta_F) \qquad \text{on } \widetilde{S}_3, \tag{2.16}$$

$$\frac{\partial\,\zeta(\iota)}{\partial\nu} = 0 \qquad \text{on } \widetilde{S}_3, \tag{2.17}$$

$$u(0,x) = u_0\,,\ \theta(0,x) = \theta_0\,,\ \zeta(0,x) = \zeta_0 \qquad \text{on } \Omega. \tag{2.18}$$

We now concisely describe the contact model and governing equations. The thermo-electro-viscoelastic constitutive behavior, governed by (2.1)–(2.2), features $\mathbf{A}$ is the viscosity operator and $\mathbf{F}$ is the elasticity operator, $\mathbf{B}$ represents the electric permittivity operator, $\mathbf{P}$ is piezoelectric operator, $\mathbf{M}$ stands for the thermal expansion tensor, $\mathbf{N}$ is pyroelectric tensor, $\mathbf{K}$ is thermal conductivity tensor, and $\mathscr{C}$ describes the relaxation tensor. Equilibrium equations in (2.3)–(2.5) describe stress, electric displacement, and heat conduction balance, where $\operatorname{Div}\sigma$ and $\operatorname{div}\mathbf{D}$ denote divergence operators, while $\mathbf{R}$ and $\mathbf{G}$ characterize thermo-electromechanical coupling. Damage evolution in (2.6) involves the microcrack diffusion constant $\kappa > 0$, damage source operator $\mathscr{S}$, and constraint $\partial\chi_{[0,1]}$ is given by

$$\chi_{[0,1]}(\varepsilon) = \begin{cases} 0 & \varepsilon\in[0,1], \\ +\infty & \text{otherwise,} \end{cases}$$

with subdifferential

$$\partial\chi_{[0,1]}(\varepsilon) = \begin{cases} (-\infty,0] & \varepsilon = 0, \\ \{0\} & \varepsilon\in(0,1), \\ [0,+\infty) & \varepsilon = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Here $\zeta = 1$ corresponds to undamaged material, $\zeta = 0$ to complete damage, and intermediate values to partial damage. Conditions (2.7)–(2.12) describe the mechanical, electrical, and thermal boundary constraints, normal compliance contact with given p_ν is modelled by (2.13). Moreover, (2.14) represents the Coulomb dry friction with bound p_τ. The electrical and thermal conductivity conditions are given by (2.15) and (2.16), respectively.

$$\varphi_L(s) = \begin{cases} \operatorname{sgn}(s)L & |s|>L, \\ s & |s|\le L, \end{cases}$$

where $h_e, h_c > 0$ are conductivity coefficients and $L \gg 1$ is a truncation parameter (practically equivalent to identity for sufficiently large L). A homogeneous Neumann condition for damage appears in (2.17) ($\partial\zeta/\partial\nu$ denotes normal derivative), with initial conditions specified in (2.18) for displacement, temperature, and damage fields.

To deliver the weak formulation of Problem 2.1, we begin by establishing the necessary mathematical framework. We indroduce the following spaces.

$$\mathrm{H} = L^2(\Omega;\mathbb{R}^d), \qquad \mathbb{H}_1 = \{\sigma \in \mathscr{H} \ : \ \mathrm{Div}\,\sigma \in H\}, \qquad Y = L^2(\Omega)$$
$$\mathrm{H}_1 = \left\{u \in H^1(\Omega;\mathbb{R}^d);\ \varepsilon(u) \in \mathscr{H}\right\},\ \mathbb{H} = \left\{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in Y\right\}.$$

These are real Hilbert spaces endowed with the inner products

$$\langle y,z\rangle_{\mathrm{H}} = (y_i,z_i)_Y, \quad \langle y,z\rangle_{\mathrm{H}_1} = \langle y,z\rangle_{\mathrm{H}} + \langle \varepsilon(y),\varepsilon(z)\rangle_{\mathbb{H}},$$
$$\langle \sigma,\tau\rangle_{\mathbb{H}} = (\sigma_{ij}\tau_{ij})_Y,\ \langle \sigma,\tau\rangle_{\mathbb{H}_1} = \langle \sigma,\tau\rangle_{\mathbb{H}} + \langle \mathrm{Div}\,\sigma,\mathrm{Div}\,\tau\rangle_{\mathbb{H}}.$$

The displacement field solution space is defined as

$$\mathbb{V} = \left\{\mathbf{v} \in H^1(\Omega;\mathbb{R}^d) \mid \mathbf{v} = \mathbf{0} \text{ on } S_1\right\}.$$

Equipped with the inner product $\langle \mathbf{u},\mathbf{v}\rangle_{\mathbb{V}} = \langle \varepsilon(\mathbf{u}),\varepsilon(\mathbf{v})\rangle_{\mathbb{H}}$ and induced norm $\|\mathbf{v}\|_{\mathbb{V}} = \|\varepsilon(\mathbf{v})\|_{\mathbb{H}}$, $\mathbb{V}$ constitutes a Hilbert space. Moreover, since $|S_1| > 0$, then Korn's inequality ensures there exists $c_K = c_K(\Omega,S_1) > 0$ such that $c_K\|\mathbf{v}\|_{H^1(\Omega;\mathbb{R}^d)} \leq \|\varepsilon(\mathbf{v})\|_{\mathbb{H}}$ for all $\mathbf{v} \in \mathbb{V}$. Applying the Sobolev trace theorem to obtain $c_0 = c_0(\Omega,S_3,S_1) > 0$ such that $\|v\|_{L^2(\Gamma;\mathbb{R}^d)} \leq c_0\|v\|_{\mathbb{V}}$ for all $v \in \mathbb{V}$. Similarly, we introduce the following function spaces for the thermal, electrical, and damage variables, respectively, as

$$\Theta = \{\theta \in H^1(\Omega;\mathbb{R}) \ : \ \theta = 0 \ \text{ on } \ S_1\},$$
$$\Psi = \{\psi \in H^1(\Omega;\mathbb{R}) \ : \ \psi = 0 \ \text{ on } \ S_a\},$$
$$\Xi = \{\rho \in H^1(\Omega;\mathbb{R}) \ : \ 0 \leq \rho \leq 1 \ \text{ in } \ \Omega\}.$$

We define the following inner products and associated norms on Θ and Ψ as

$$\langle \theta,\eta\rangle_\Theta = \langle \nabla\theta,\nabla\eta\rangle_{\mathrm{H}}, \quad \|\theta\|_\Theta^2 = \langle \theta,\theta\rangle_\Theta \qquad \text{for all } \theta,\,\eta \in \Theta,$$
$$\langle \varphi,\psi\rangle_\Psi = \langle \nabla\varphi,\nabla\psi\rangle_{\mathrm{H}}, \quad \|\varphi\|_\Psi^2 = (\varphi,\varphi)_\Psi \qquad \text{for all } \varphi,\,\psi \in \Psi.$$

It follows from $|S_a| > 0$ and the Friedrichs-Poincaré inequality that

$$\|\psi\|_{H^1(\Omega;\mathbb{R})} \leq c_F\|\nabla\psi\|_{\mathrm{H}} \qquad \text{for all } \psi \in \Psi,$$

where $c_F = c_F(\Omega,S_a) > 0$. Consequently, the norms $\|\cdot\|_\Psi$ and $\|\cdot\|_{H^1(\Omega)}$ are equivalent on Ψ, making $(\Psi,\|\cdot\|_\Psi)$ a real Hilbert space. Furthermore, by the Sobolev trace theorem, there exists $c_1 = c_1(\Omega,S_a,S_3) > 0$ such that $\|\xi\|_{L^2(S_3)} \leq c_1\|\xi\|_\Psi$ for all $\xi \in \Psi$. Similarly, from $|S_1| > 0$, we have, for all $\theta \in \Theta$, $\|\theta\|_{H^1(\Omega;\mathbb{R})} \leq c_R\|\nabla\theta\|_H$ with $c_R = c_R(\Omega,S_1) > 0$. The norms $\|\cdot\|_\Theta$ and $\|\cdot\|_{H^1(\Omega)}$ are thus equivalent on Θ, establishing $(\Theta,\|\cdot\|_\Theta)$ as a real Hilbert space. The trace theorem additionally guarantees the existence of $c_2 = c_2(\Omega,S_1,S_3) > 0$ satisfying $\|\eta\|_{L^2(S_3)} \leq c_2\|\eta\|_\Theta$ for all $\eta \in \Theta$. To analyze Problem 2.1, we require the following hypotheses on the problem data:

The operators $\mathbf{A},\mathbf{F} : \Omega\times\mathbb{S}^d \to \mathbb{S}^d$ and $\mathbf{B},\mathbf{K} : \Omega\times\mathbb{R}^d \to \mathbb{R}^d$ are symmetric, and satisfy

$$\begin{cases} (a) \text{ There exists } m_{\mathbf{A}}, m_{\mathbf{F}}, m_{\mathbf{B}}, m_{\mathbf{K}} > 0 \text{ such that} \\ \qquad \langle \mathbf{A}(x,\varepsilon_1) - \mathbf{A}(x,\varepsilon_2), \varepsilon_1 - \varepsilon_2 \rangle_{\mathbb{H}} \geq m_{\mathbf{A}} \|\varepsilon_1 - \varepsilon_2\|^2_{\mathbb{S}^d}, \\ \qquad \langle \mathbf{F}(x,\varepsilon_1) - \mathbf{F}(x,\varepsilon_2), \varepsilon_1 - \varepsilon_2 \rangle_{\mathbb{H}} \geq m_{\mathbf{F}} \|\varepsilon_1 - \varepsilon_2\|^2_{\mathbb{S}^d}, \\ \qquad \langle \mathbf{B}(x,\xi_1) - \mathbf{B}(x,\xi_2), \xi_1 - \xi_2 \rangle_{\mathrm{H}} \geq m_{\mathbf{B}} \|\xi_1 - \xi_2\|^2_{\mathbb{R}^d}, \\ \qquad \langle \mathbf{K}(x,\xi_1) - \mathbf{K}(x,\xi_2), \xi_1 - \xi_2 \rangle_{\mathrm{H}} \geq m_{\mathbf{K}} \|\xi_1 - \xi_2\|^2_{\mathbb{R}^d}, \\ (b) \text{ There exists } M_{\mathbf{A}}, M_{\mathbf{F}}, M_{\mathbf{B}}, M_{\mathbf{K}} > 0 \text{ such that} \\ \qquad \|\mathbf{A}(x,\varepsilon_1) - \mathbf{A}(x,\varepsilon_2)\|_{\mathbb{S}^d} \leq M_{\mathbf{A}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}, \\ \qquad \|\mathbf{F}(x,\varepsilon_1) - \mathbf{F}(x,\varepsilon_2)\|_{\mathbb{S}^d} \leq M_{\mathbf{F}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}, \\ \qquad \|\mathbf{B}(x,\xi_1) - \mathbf{B}(x,\xi_2)\|_{\mathbb{R}^d} \leq M_{\mathbf{B}} \|\xi_1 - \xi_2\|_{\mathbb{R}^d}, \\ \qquad \|\mathbf{K}(x,\xi_1) - \mathbf{K}(x,\xi_2)\|_{\mathbb{R}^d} \leq M_{\mathbf{K}} \|\xi_1 - \xi_2\|_{\mathbb{R}^d}, \\ \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \, \xi_1, \, \xi_2 \in \mathbb{R}^d. \end{cases} \tag{2.19}$$

The tensors $\mathbf{P}:\Omega\times\mathbb{S}^d\to\mathbb{R}^d$, $\mathbf{M}:\Omega\times\mathbb{R}\to\mathbb{S}^d$, $\mathbf{N}:\Omega\times\mathbb{R}\to\mathbb{R}^d$, $\mathbf{R}:\Omega\times\mathbb{S}^d\to\mathbb{R}$, $\mathbf{G}:\Omega\times\mathbb{R}^d\to\mathbb{R}$ are such that

$$\begin{cases} (a) \ \mathbf{P}, \mathbf{M}, \mathbf{N}, \mathbf{R}, \text{ and } \mathbf{T} \text{ are linear and bounded,} \\ (b) \text{ There exists } M_{\mathbf{P}}, M_{\mathbf{R}}, M_{\mathbf{T}}, M_{\mathbf{M}}, M_{\mathbf{N}} > 0 \text{ such that} \\ \qquad \|\mathbf{P}(x,\varepsilon_1) - \mathbf{P}(x,\varepsilon_2))\|_{\mathbb{R}^d} \leq M_{\mathbf{P}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}, \\ \qquad |\mathbf{R}(x,\varepsilon_1) - \mathbf{R}(x,\varepsilon_2))| \leq M_{\mathbf{R}} \|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d}, \\ \qquad |\mathbf{G}(x,\xi_1) - \mathbf{G}(x,\xi_2)| \leq M_{\mathbf{G}} \|\xi_1 - \xi_2\|_{\mathbb{R}^d}, \\ \qquad \|\mathbf{M}(x,\phi_1) - \mathbf{M}(x,\phi_2))\|_{\mathbb{S}^d} \leq M_{\mathbf{M}} |\phi_1 - \phi_2|, \\ \qquad \|\mathbf{N}(x,\phi_1) - \mathbf{N}(x,\phi_2))\|_{\mathbb{R}^d} \leq M_{\mathbf{N}} |\phi_1 - \phi_2|. \\ \text{for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \, \xi_1, \, \xi_2 \in \mathbb{R}^d, \, \phi_1, \phi_2 \in \mathbb{R}. \end{cases} \tag{2.20}$$

The relaxation operator $\mathscr{C}:\Omega\times\mathbb{I}\times\mathbb{S}^d\times\mathbb{R}\to\mathbb{S}^d$ satisfies:

$$\begin{cases} (a) \ \mathscr{C}(\cdot,\iota,\varepsilon,r) \text{ is measurable on } \Omega \text{ for all } \varepsilon\in\mathbb{S}^d, \, r\in\mathbb{R}, \, \iota\in\mathbb{I}, \\ (b) \ \mathscr{C}(x,\cdot,\varepsilon,r) \text{ is continuous on } \mathbb{I} \text{ for all } x\in\Omega, \, \varepsilon\in\mathbb{S}^d, \, r\in\mathbb{R}, \\ (c) \text{ There exists } L_{\mathscr{C}} > 0 \text{ such that } \forall x\in\Omega, \, \iota\in\mathbb{I}, \, \varepsilon_1,\varepsilon_2\in\mathbb{S}^d, \, r_1,r_2\in\mathbb{R}: \\ \qquad \|\mathscr{C}(x,\iota,\varepsilon_1,r_1) - \mathscr{C}(x,\iota,\varepsilon_2,r_2)\|_{\mathbb{S}^d} \leq L_{\mathscr{C}} \left(\|\varepsilon_1-\varepsilon_2\|_{\mathbb{S}^d} + |r_1-r_2|\right). \end{cases} \tag{2.21}$$

The normal compliance functions $p_l : S_3\times\mathbb{R}\to\mathbb{R}$, $l=\nu,\tau$ verify:

$$\begin{cases} (a) \ \ p_l(\cdot,r) \text{ is measurable on } S_3 \text{ for all } r\in\mathbb{R}, \\ (b) \ \ p_l(x,\cdot) \text{ is continuous on } \mathbb{R}, \text{ and } p_i(x,r)=0 \text{ for all } x\in S_3, \, r\leq 0, \\ (c) \ \ \text{There exists } L_l > 0 \text{ such that:} \\ \qquad |p_l(x,r_1) - p_l(x,r_2)| \leq L_i |r_1-r_2|, \qquad \text{for all } x\in S_3, \, r_1,r_2\in\mathbb{R}, \\ (d) \ \ (p_l(x,r_1) - p_l(x,r_2))(r_1-r_2) \geq 0 \text{ for all } x\in S_3, \, r_1,r_2\in\mathbb{R}, \end{cases} \tag{2.22}$$

The electrical and thermal conductivity functions $h_l : S_3\times\mathbb{R}\to\mathbb{R}^+$, $(l=e,c)$ satisfy:

$$\begin{cases} (a)\ h_k(\cdot, r) \text{ is measurable on } S_3 \text{ for all } r \in \mathbb{R}, \\ (b)\ h_k(x, \cdot) \text{ is continuous on } \mathbb{R} \text{ for all } x \in S_3, \\ (c)\ \text{There exists } L_k > 0 \text{ such that for all } x \in S_3,\, r_1, r_2 \in \mathbb{R}: \\ \qquad |h_k(x, r_1) - h_k(x, r_2)| \leq L_k |r_1 - r_2|, \\ (d)\ |h_k(x, r)| \leq \overline{h}_k \quad \text{for all } x \in S_3,\, r \in \mathbb{R}, \text{ with } \overline{h}_k > 0. \end{cases} \tag{2.23}$$

The damage source operator $\mathscr{S} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies:

$$\begin{cases} (a)\ \mathscr{S}(\cdot, \varepsilon, r) \text{ is measurable on } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d,\, r \in \mathbb{R}, \\ (b)\ \text{There exists } L_{\mathscr{S}} > 0 \text{ such that for all } x \in \Omega,\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\, r_1, r_2 \in \mathbb{R}: \\ \qquad \|\mathscr{S}(x, \varepsilon_1, r_1) - \mathscr{S}(x, \varepsilon_2, r_2)\|_{\mathbb{S}^d} \leq L_{\mathscr{S}}(\|\varepsilon_1 - \varepsilon_2\|_{\mathbb{S}^d} + |r_1 - r_2|), \\ (c)\ \mathscr{S}(x, 0, 0) = 0 \text{ for all } x \in \Omega. \end{cases} \tag{2.24}$$

The functions f_0, f_2, q_0, q_1, h_2, and q_b verify:

$$\begin{aligned} &f_0 \in C(\mathbb{I}; L^2(\Omega; \mathbb{R}^d)), \qquad f_2 \in C(\mathbb{I}; L^2(S_2; \mathbb{R}^d)), \qquad q_0 \in C(\mathbb{I}; L^2(\Omega)), \\ &q_1 \in C(\mathbb{I}; L^2(\Omega)), \qquad h_2 \in C(\mathbb{I}; L^2(S_2)), \qquad q_b \in C(\mathbb{I}; L^2(S_b)). \end{aligned} \tag{2.25}$$

The initial and boundary data satisfy:

$$\begin{aligned} &g \in L^2(S_3), \qquad g \geq 0 \quad \text{a.e., on } S_3, \\ &\varphi_0 \in L^2(\mathbb{I}; L^2(S_3)), \qquad \theta_F \in L^2(0, T; L^2(S_3)), \\ &u_0 \in \mathbb{V}, \qquad \theta_0 \in \Theta, \qquad \zeta_0 \in K. \end{aligned} \tag{2.26}$$

Next, we define the symmetric bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$:

$$a(\zeta, \varsigma) = \kappa \int_\Omega \nabla \zeta \cdot \nabla \varsigma \, dx \qquad \text{for all } \zeta,\, \varsigma \in H^1(\Omega). \tag{2.27}$$

Additionally, define the time-dependent functions $f : \mathbb{I} \to \mathbb{V}$, $h : \mathbb{I} \to \Theta$, $q_e : \mathbb{I} \to \Psi$ and the mappings $J : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$, $\ell : \mathbb{V} \times \Psi \to \Psi$, $\vartheta : \mathbb{V} \times \Theta \to \Theta$ by:

$$\begin{aligned} &\langle f(\iota), v \rangle_{\mathbb{V}} = \langle f_0(\iota), v \rangle_{L^2(\Omega;\mathbb{R}^d)} + \langle f_2(\iota), v \rangle_{L^2(S_2;\mathbb{R}^d)} && \text{for all } v \in \mathbb{V}, \\ &\langle q_e(\iota), \psi \rangle_{\Psi} = \langle q_0(\iota), \psi \rangle_{L^2(\Omega)} - \langle q_b(\iota), \psi \rangle_{L^2(S_b)} && \text{for all } \psi \in \Psi, \\ &\langle h(\iota), \eta \rangle_{\Theta} = \langle q_1(\iota), \xi \rangle_{L^2(\Omega)} - \langle h_2(\iota), \eta \rangle_{L^2(S_2)} && \text{for all } \eta \in \Theta, \\ &\mathscr{J}(u, v) = \int_{S_3} p_\nu(u_\nu - g) v_\nu da + \int_{S_3} p_\tau(u_\tau - g)|v_\tau| da && \text{for all } u, v \in \mathbb{V}, \\ &\langle \ell(u, \varphi), \psi \rangle_{\Psi} = \int_{S_3} h_e(u_\nu - g) \phi_L(\varphi - \varphi_0) \psi da && \text{for all } u \in \mathbb{V}, \varphi, \psi \in \Psi, \\ &\langle \vartheta(u, \theta), \eta \rangle_{\Theta} = \int_{S_3} h_c(u_\nu - g) \phi_L(\theta - \theta_F) . \eta da && \text{for all } u \in \mathbb{V},\, \theta, \eta \in \Theta. \end{aligned}$$

We now derive the weak formulation of mechanical problem 2.1. Assume that $(u,\, \sigma,\, \varphi,\, D,\, \theta,\, q,\, \zeta)$ are regular functions satisfying (2.1)-(2.18), the application of Green's formula and the boundary conditions yields the following variational formulation.

Problem 2.2. *Find $u : \mathbb{I} \to \mathbb{V}$, $\varphi : \mathbb{I} \to \Psi$, $\theta : \mathbb{I} \to \Theta$ and $\zeta : \mathbb{I} \to \Xi$ such that*

$$\begin{aligned} &\langle \mathbf{A}\varepsilon(\dot{u}(\iota)), \varepsilon(v) - \varepsilon(\dot{u}(\iota))\rangle_{\mathbb{H}} + \langle \mathbf{F}\varepsilon(u(\iota)), \varepsilon(v) - \varepsilon(\dot{u}(\iota))\rangle_{\mathbb{H}} + \mathscr{J}(u(\iota), v) \\ &+ \langle \mathbf{P}^{*}\nabla\varphi(\iota), \varepsilon(v) - \varepsilon(\dot{u}(\iota))\big)_{\mathbb{H}} - \langle \mathbf{M}\theta(\iota), \varepsilon(v) - \varepsilon(\dot{u}(\iota))\rangle_{\mathbb{H}} - \mathscr{J}(u(\iota), \dot{u}(\iota)) \\ &+ \langle \int_0^{\iota} \mathscr{C}(\iota - s, \varepsilon(u(s)), \zeta(s))\, ds, \varepsilon(v) - \varepsilon(\dot{u}(\iota))\rangle_{\mathbb{H}} \geq \langle f(\iota), v - \dot{u}(\iota)\rangle_{\mathbb{V}}, \end{aligned} \tag{2.28}$$

$$\begin{aligned} &\langle \mathbf{B}\nabla\varphi(\iota), \nabla\psi\rangle_{\mathrm{H}} - \langle \mathbf{P}\varepsilon(u(\iota)), \nabla\psi\rangle_{\mathrm{H}} + \langle \mathbf{N}\theta(\iota), \nabla\psi\rangle_{\mathrm{H}} + \langle \ell(u(\iota), \varphi(\iota)), \psi\rangle_{\Psi} \\ =&\langle q_e(\iota), \psi\rangle_{\Psi}, \end{aligned} \tag{2.29}$$

$$\begin{aligned} &\langle \dot{\theta}(\iota), \eta\rangle_{Y} + \langle \mathbf{K}\nabla\theta(\iota), \nabla\eta\rangle_{\mathrm{H}} + \langle \mathbf{R}\varepsilon(u(\iota)), \eta\rangle_{Y} - \langle \mathbf{G}\nabla\varphi(\iota), \eta\rangle_{Y} \\ &+ \langle \vartheta(u(\iota), \theta(\iota)), \eta\rangle_{\Theta} = \langle h(\iota), \eta\rangle_{\Theta}, \end{aligned} \tag{2.30}$$

$$\langle \dot{\zeta}(\iota), \rho - \zeta(\iota)\rangle_{Y} + a(\zeta(\iota), \rho - \zeta(\iota)) \geq \langle \mathscr{S}(\varepsilon(u(\iota)), \zeta(\iota)), \rho - \zeta(\iota)\rangle_{Y}, \tag{2.31}$$

$$u(0,x) = u_0\ ,\ \theta(0,x) = \theta_0\ ,\ \zeta(0,x) = \zeta_0. \tag{2.32}$$

for all $v \in \mathbb{V}$, $\psi \in \Psi$, $\eta \in \Theta$, $\rho \in \Xi$, and $t \in \mathbb{I}$.

3. Existence and Uniqueness Results

This section is devoted to the existence and uniqueness result for Problem 2.2.

Theorem 3.1. *Let hypotheses (2.19)-(2.26) hold. Then Problem 2.2 admits an only one single solution $(u, \varphi, \theta, \zeta)$ with*

$$\begin{aligned} &u \in C^1(\mathbb{I}; \mathbb{V}), && \varphi \in C(\mathbb{I}; \Psi), \\ &\theta \in C(\mathbb{I}; \Theta), && \zeta \in W^{1,2}(\mathbb{I}; Y) \cap L^2(\mathbb{I}; H^1(\Omega)). \end{aligned}$$

Proof of Theorem 3.1. The proof proceeds in several steps by using variational inequalities and Banach fixed-point arguments [18, 19].

Step 1. Fix $\eta \in C(\mathbb{I}; \mathbb{V})$ and consider the intermediate problem:

Problem 3.1. *Find $u_\eta : \mathbb{I} \to \mathbb{V}$ such that*

$$\begin{aligned} &\langle \mathbf{A}\varepsilon(\dot{u}_\eta(\iota)), \varepsilon(v) - \varepsilon(\dot{u}_\eta(\iota))\rangle_{\mathbb{H}} + \langle \mathbf{F}\varepsilon(\dot{u}_\eta(\iota)), \varepsilon(v) - \varepsilon(u_\eta(\iota))\rangle_{\mathbb{H}} \\ &+ \mathscr{J}(u_\eta(\iota), v) - \mathscr{J}(u_\eta(\iota), \dot{u}_\eta(\iota)) \geq \langle f(\iota) - \eta(\iota), v - \dot{u}_\eta(\iota)\rangle_{\mathbb{V}}, \\ &u_\eta(0) = u_0. \end{aligned}$$

for all $v \in \mathbb{V}$ and $\iota \in \mathbb{I}$

Lemma 3.1. *Under hypotheses (2.19), (2.22), (2.25), and (2.26), Problem 3.1 has a unique solution $u_\eta \in C^1(\mathbb{I}; \mathbb{V})$ for any $\eta \in C(\mathbb{I}; \mathbb{V})$.*

Proof. The proof follows [8] immediately. □

Step 2. Fix $\gamma \in C(\mathbb{I}; Y)$. Using u_η solution of Problem 3.1, we consider

Problem 3.2. *Find* $\theta_{\gamma\eta} : \mathbb{I} \to \Theta$ *such that*

$$\begin{aligned}
&\langle \dot{\theta}_{\gamma\eta}(\iota), \xi \rangle_Y + \langle \mathbf{K}\nabla\theta_{\gamma\eta}(\iota), \nabla\xi \rangle_{\mathrm{H}} + \langle \vartheta(u_\eta(\iota), \theta_{\gamma\eta}(\iota)), \psi \rangle_\Theta \\
=&\langle h(\iota), \xi \rangle_\Theta - \langle \mathscr{R}\varepsilon(u_\eta(\iota)), \xi \rangle_Y + \langle \gamma(\iota), \xi \rangle_Y, \\
&\theta_{\gamma\eta}(0) = \theta_0,
\end{aligned}$$

for all $\xi \in \Theta$, $\iota \in \mathbb{I}$.

Lemma 3.2. *Under assumptions* (2.19)*,* (2.20)*,* (2.22)*,* (2.25)*, and* (2.26)*, Problem* 3.2 *has a unique solution* $\theta_{\gamma\eta} \in C^1(\mathbb{I};\Theta)$.

Proof. The proof follows the same arguments of evolution equations presented in [19, Lemma 3.3]. □

Step 3. Consider the following electrical potential problem:

Problem 3.3. *Find* $\varphi_{\gamma\eta} : \mathbb{I} \to Q$ *satisfying:*

$$\begin{aligned}
&\langle \mathbf{B}\nabla\varphi_{\gamma\eta}(\iota), \nabla\psi \rangle_{\mathbb{H}} + \langle \ell(u_\eta(\iota), \varphi_{\gamma\eta}(\iota)), \psi \rangle_\Psi \\
=&\langle q_e(\iota), \psi \rangle_\Psi + \langle \mathbf{P}\varepsilon(u_\eta(\iota)), \nabla\psi \rangle_{\mathbb{H}} - \langle \mathbf{N}\theta_{\gamma\eta}(\iota), \nabla\psi \rangle_{\mathbb{H}},
\end{aligned}$$

for all $\psi \in \Psi$ and $\iota \in \mathbb{I}$.

Define the operators $\mathscr{P}_{\gamma\eta}$ and $\mathscr{W}_{\gamma\eta}$ as follows

$$\begin{aligned}
\langle \mathscr{P}_{\gamma\eta}(\iota)\varphi, \psi \rangle_\Psi =&\langle \mathbf{B}\nabla\varphi, \nabla\psi \rangle_{\mathbb{H}} + \big(\ell(u_\eta(\iota), \varphi(\iota)), \psi\big)_\Psi, \\
\langle \mathscr{W}_{\gamma\eta}(\iota), \psi \rangle_\Psi =&\langle q_e(\iota), \psi \rangle_\Psi + \langle \mathbf{P}\varepsilon(u_\eta(\iota)), \nabla\psi \rangle_{\mathbb{H}} - \langle \mathbf{N}\theta_{\gamma\eta}(\iota), \nabla\psi \rangle_{\mathbb{H}}.
\end{aligned}$$

Therefore, Problem 3.3 can be rewritten as follows: Find $\varphi_{\gamma\eta} : \mathbb{I} \to Q$ such that

$$\langle \mathscr{P}_{\gamma\eta}(\iota)\varphi_\eta(\iota), \psi \rangle_\Psi = \langle \mathscr{W}_{\gamma\eta}(\iota), \psi \rangle_\Psi \qquad \text{for all } \psi \in \Psi,\ \iota \in \mathbb{I}.$$

Lemma 3.3. *Under conditions* (2.19)*,* (2.20)*,* (2.23)*,* (2.25)*, and* (2.26)*, Problem* 3.3 *has a unique solution* $\varphi_{\gamma\eta} \in C(\mathbb{I};\Psi)$.

Proof. The proof follows from strong monotonicity of $\mathscr{P}_{\gamma\eta}(\iota)$ and standard variational equality theory. □

Step 4: Fix $\mu \in C(\mathbb{I};Y)$. Consider the damage problem:

Problem 3.4. *Find* $\zeta_\mu : \mathbb{I} \to H^1(\Omega)$ *with* $\zeta_\mu(t) \in \Xi$ *satisfying:*

$$\begin{aligned}
&\langle \dot{\zeta}_\mu(\iota), \rho - \zeta_\mu(\iota) \rangle_Y + a(\zeta_\mu(\iota), \rho - \zeta_\mu(\iota)) \geq \langle \mu(\iota), \rho - \zeta_\mu(\iota) \rangle_Y, \\
&\zeta_\mu(0) = \zeta_0.
\end{aligned}$$

for all $\rho \in \Xi$ and $\iota \in \mathbb{I}$.

Lemma 3.4. *Under conditions* (2.19)*,* (2.26) *and* (2.27)*, Problem* 3.4 *has a unique solution* $\zeta_\mu \in W^{1,2}(\mathbb{I};Y) \cap L^2(\mathbb{I};H^1(\Omega))$.

Proof. The proof follows from properties of the bilinear form $a(\cdot,\cdot)$, and Theorem 2.28 in [23]. □

Step 5. Consider a space $Z = \mathbb{V} \times Y \times Y$ and define the operator

$$\Im : C(\mathbb{I}; Z) \to C(\mathbb{I}; Z)$$

by

$$\Im(\eta,\gamma,\mu)(\iota) = \big(\Im_1(\eta,\gamma,\mu)(\iota), \Im_2(\eta,\gamma,\mu)(\iota), \Im_3(\eta,\gamma,\mu)(\iota)\big),$$

where

$$\Im_1(\eta,\gamma,\mu)(\iota) = \mathbf{P}^*\nabla\varphi_{\gamma\eta}(\iota) - \mathbf{M}\theta_{\gamma\eta}(\iota) + \int_0^\iota \mathscr{C}(\iota - s, \varepsilon(u_\eta(s)), \zeta_\mu(s))ds,$$

$$\Im_2(\eta,\gamma,\mu)(\iota) = \mathbf{G}\nabla\varphi_{\gamma\eta}(\iota),$$

and

$$\Im_3(\eta,\gamma,\mu)(\iota) = \mathscr{S}(\varepsilon(u_\eta(\iota)), \zeta_\mu(\iota)).$$

Here u_η, $\varphi_{\gamma\eta}$, $\theta_{\gamma\eta}$, ζ_μ are solutions from Problems 3.1, 3.2, 3.3, and 3.4, respectively. For pairs (η_1,γ_1,μ_1) and (η_2,γ_2,μ_2), denote corresponding solutions of $(u_1,\varphi_1,\theta_1,\zeta_1)$ and $(u_1,\varphi_1,\theta_1,\zeta_1)$, respectively. Under (2.19)-(2.26), there exists $c > 0$ such that

$$\begin{aligned}
&\left\|\Im(\eta_1,\gamma_1,\mu_1)(t) - \Im(\eta_2,\gamma_2,\mu_2)(t)\right\|_Z^2 \\
\leq& c\int_0^\iota \|\eta_1(s) - \eta_2(s)\|_V^2 ds + c\int_0^t \|\gamma_1(s) - \gamma_2(s)\|_Y^2 ds + c\int_0^\iota \|\mu_1(s) - \mu_2(s)\|_Y^2 ds.
\end{aligned}$$

Lemma 3.5. *$\Im$ has a unique fixed point $(\eta^*,\gamma^*,\mu^*) \in C(\mathbb{I}; Z)$.*

Proof follows from fixed-point theorems [24].

Let u^*, φ^*, θ^*, and ζ^* be solutions to Problems 3.1, 3.2, 3.3, and 3.4, respectively. Then $(u^*,\varphi^*,\theta^*,\zeta^*)$ solves Problem 2.2. Uniqueness follows from Banach's fixed-point theorem. □

4. Finite Element Approximation

We introduce a finite element algorithm for approximating solutions to Problem 2.2. For spatial discretization, assume that Ω is a polygonal/polyhedral domain with $\{\mathscr{T}^h\}$ being a regular family of triangular/tetrahedral meshes partitioning $\overline{\Omega}$ compatible with boundary subdivisions. Define finite-dimensional subspaces:

$$\begin{aligned}
\mathbb{V}^h &= \left\{ v^h \in C(\overline{\Omega})^d : v^h|_{T_r} \in \mathbb{P}_1(T_r)^d \quad \text{for all } T_r \in \mathscr{T}^h,\ v^h = 0 \text{ on } S_1 \right\}, \\
\Psi^h &= \left\{ \psi^h \in C(\overline{\Omega}) : \psi^h|_{T_r} \in \mathbb{P}_1(T_r) \quad \text{for all } T_r \in \mathscr{T}^h,\ \psi^h = 0 \text{ on } S_1 \right\}, \\
\Theta^h &= \left\{ \xi^h \in C(\overline{\Omega}) : \xi^h|_{T_r} \in \mathbb{P}_1(T_r) \quad \text{for all } T_r \in \mathscr{T}^h,\ \xi^h = 0 \text{ on } S_1 \cap S_2 \right\}, \\
\Xi^h &= \left\{ \rho^h \in C(\overline{\Omega}) : \rho^h|_{T_r} \in \mathbb{P}_1(T_r) \quad \text{for all } T_r \in \mathscr{T}^h,\ 0 \leq \rho^h \leq 1 \text{ in } \Omega \right\},
\end{aligned}$$

where $h > 0$ is the spatial discretization parameter, and $\mathbb{P}_1(T_r)$ denotes linear polynomials on T_r. Our semi-discrete approximation is as follows.

Problem 4.1. *Find $u^h : \mathbb{I} \to \mathbb{V}^h$, $\varphi^h : \mathbb{I} \to \Psi^h$, $\theta^h : \mathbb{I} \to \Theta^h$, and $\zeta^h : \mathbb{I} \to \Xi^h$ such that*

$$\begin{aligned}
&\langle \mathbf{A}\varepsilon(\dot{u}^h(\iota)), \varepsilon(v^h) - \varepsilon(\dot{u}^h(\iota))\rangle_{\mathbb{H}} + \langle \mathbf{F}\varepsilon(u^h(\iota)), \varepsilon(v^h) - \varepsilon(\dot{u}^h(\iota))\rangle_{\mathbb{H}} \\
&+ \langle \mathbf{P}^*\nabla\varphi^h(\iota), \varepsilon(v^h) - \varepsilon(\dot{u}^h(\iota))\rangle_{\mathbb{H}} - \langle \mathbf{M}\theta^h(\iota), \varepsilon(v^h) - \varepsilon(\dot{u}^h(\iota))\rangle_{\mathbb{H}} \\
&+ \langle \int_0^\iota \mathscr{C}(\iota - s, \varepsilon(u^h(s)), \zeta^h(s))\, ds, \varepsilon(v^h) - \varepsilon(\dot{u}^h(\iota))\rangle_{\mathbb{H}} \\
&+ \mathscr{J}(u^h(\iota), v^h) - \mathscr{J}(u^h(\iota), \dot{u}^h(\iota)) \geq \langle f(\iota), v^h - \dot{u}^h(\iota)\rangle_{\mathbb{V}},
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&\langle \mathbf{B}\nabla\varphi^h(\iota), \nabla\psi^h\rangle_{\mathrm{H}} - \langle \mathbf{P}\varepsilon(u^h(\iota)), \nabla\psi^h\rangle_{\mathrm{H}} + \langle \mathbf{N}\theta^h(\iota), \nabla\psi^h\rangle_{\mathrm{H}} \\
=&\langle q_e(\iota), \psi^h\rangle_{\Psi} - \langle \ell(u^h(\iota), \varphi^h(\iota)), \psi^h\rangle_{\Psi},
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\langle \dot{\theta}^h(\iota), \xi^h\rangle_Y + \langle \mathbf{K}\nabla\theta^h(\iota), \nabla\xi^h\rangle_{\mathrm{H}} + \langle \mathbf{R}\varepsilon(\dot{u}^h(\iota)) - \mathbf{G}\nabla\varphi^h(\iota), \xi^h\rangle_Y \\
=&\big(h(\iota), \xi^h\big)_{\Theta} - \big(\vartheta(u^h(\iota), \theta^h(\iota)), \xi^h\big)_{\Theta},
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&\big(\dot{\zeta}^h(\iota), \rho^h - \zeta^h(\iota)\big)_Y + a(\zeta^h(\iota), \rho^h - \zeta^h(\iota)) \\
\geq&\big(\mathscr{S}(\varepsilon(u^h(\iota)), \zeta^h(\iota)), \rho^h - \zeta^h(\iota)\big)_Y,
\end{aligned} \tag{4.4}$$

$$u^h(0) = u_0^h,\ \theta^h(0) = \theta_0^h,\ \zeta^h(0) = \zeta_0^h.$$

for all $v^h \in \mathbb{V}$, $\psi^h \in \Psi^h$, $\xi \in \Theta^h$, $\rho^h \in \Xi^h$ and all $\iota \in \mathbb{I}$.

Proposition 4.1. *Under assumptions (2.19)-(2.26), the solution $(u^h, \varphi^h, \theta^h, \zeta^h)$ of Problem 4.1 is uniformly bounded in $\mathbb{V} \times \Psi \times \Theta \times \Xi$ independently of h.*

Proof. Set $\psi^h = \varphi^h$ in (4.2). Using the ellipticity of $\mathbf{B}$, we obtain that

$$\begin{aligned}
m_{\mathbf{B}}\|\nabla\varphi^h(\iota)\|_{\mathrm{H}}^2 \leq&\big|\langle \mathbf{P}\varepsilon(u^h(\iota)), \nabla\varphi^h(\iota)\rangle_{\mathrm{H}}\big| + \big|\langle \mathbf{N}\theta^h(\iota), \nabla\varphi^h(\iota)\rangle_{\mathrm{H}}\big| \\
&+ \big|\langle q_e(\iota), \varphi^h(\iota)\rangle_{\Psi}\big| + \big|\big(\ell(u^h(\iota), \varphi^h(\iota)), \varphi^h(\iota)\big)_{\Psi}\big|.
\end{aligned}$$

By (2.19), (2.20), (2.23), and Young's inequality, we see that

$$\|\varphi^h(\iota)\|_{\Psi} \leq c_0\left(\|q_e(\iota)\|_{\Psi^*} + \|u^h(\iota)\|_{\mathbb{V}} + \|\theta^h(\iota)\|_{\Theta}\right). \tag{4.5}$$

Next, setting $\xi^h = \theta^h$ in (4.3) and using the ellipticity of $\mathbf{K}$, we conclude that

$$\begin{aligned}
m_{\mathbf{K}}\|\nabla\theta^h(\iota)\|_{\mathrm{H}}^2 \leq&|\langle \mathbf{G}\nabla\varphi^h(\iota), \theta^h(\iota)\rangle_Y| + |\langle \vartheta(u^h(\iota), \theta^h(\iota)), \theta^h(\iota)\rangle_{\Theta}| \\
&+ |\langle h(\iota), \theta^h(\iota)\rangle_{\Theta}| + |\langle \mathbf{R}\varepsilon(\dot{u}^h(\iota)), \theta^h(\iota)\rangle_Y|.
\end{aligned}$$

By (2.19), (2.20), and (2.23), we obtain that

$$\|\theta^h(\iota)\|_{\Theta} \leq c_1\left(\|h(\iota)\|_{\Theta^*} + \|\dot{u}^h(\iota)\|_{\mathbb{V}} + \|\varphi^h(\iota)\|_{\Psi}\right). \tag{4.6}$$

On other hand, setting $\rho^h = 0$ in (4.4) and using condition (2.24), we have

$$(\dot{\zeta}^h(\iota), \zeta^h(\iota))_Y + \kappa\|\nabla\zeta^h(\iota)\|_Y^2 \leq \big|\big(\mathscr{S}(\varepsilon(u^h(\iota)), \zeta^h(\iota)), \zeta^h(\iota)\big)_Y\big|.$$

By (2.25) , the boundedness of $\mathscr{S}$, and Poincaré's inequality, we further have

$$\|\zeta^h(\iota)\|_{H^1(\Omega)} \leq c_2\|u^h(\iota)\|_{\mathbb{V}}. \tag{4.7}$$

Now, by taking $v^h = 0$ in (4.1), using (2.21) and (2.22), we find that

$$\begin{aligned}
&m_{\mathbf{A}}\|\varepsilon(\dot{u}^h(\iota))\|_{\mathbb{H}}^2 + m_{\mathbf{F}}\|\varepsilon(u^h(\iota))\|_{\mathbb{H}}^2 \\
\leq &\left|\langle f(\iota), \dot{u}^h(\iota)\rangle_{\mathbb{V}}\right| - \langle \mathbf{M}\theta^h(\iota), \varepsilon(u^h(\iota))\rangle_{\mathbb{H}} - \langle \mathbf{P}^*\nabla\varphi^h(\iota), \varepsilon(u^h(\iota))\rangle_{\mathbb{H}} \\
&- \langle \mathbf{A}\varepsilon(\dot{u}^h(\iota)), \varepsilon(u^h(\iota))\rangle_{\mathbb{H}} + \mathscr{J}(u^h(\iota), \dot{u}^h(\iota)) - \mathscr{J}(u^h(\iota), \dot{u}^h(\iota)) \\
&- \left(\int_0^\iota \mathscr{C}(\iota - r, \varepsilon(u^h(r)), \zeta^h(r))\, dr, \varepsilon(\dot{u}^h(\iota))\right)_{\mathbb{H}}.
\end{aligned}$$

Gronwall's inequality yields

$$\|\dot{u}^h(\iota)\|_{\mathbb{V}}^2 + \|u^h(\iota)\|_{\mathbb{V}}^2 \leq C\left(1 + \int_0^\iota \|u^h(r)\|_{\mathbb{V}}^2 dr + \int_0^\iota \|\zeta^h(r)\|^2 dr\right) \tag{4.8}$$

Combining (4.5)–(4.8) and using Gronwall's lemma yields

$$\begin{aligned}
&\|u^h\|_{L^\infty(V)} + \|\varphi^h\|_{L^\infty(\Psi)} + \|\theta^h\|_{L^\infty(\Theta)} + \|\zeta^h\|_{L^\infty(H^1(\Omega))} \\
\leq &C\left(\|f\|_{V^*} + \|q_e\|_{\Psi^*} + \|h\|_{\Theta^*}\right),
\end{aligned}$$

where C depends only on the problem data. □

Theorem 4.1. *Lett conditions (2.19)-(2.26) hold. Then, Problem 4.1 admits a unique solution $(u^h, \varphi^h, \theta^h, \zeta^h)$ verifying:*

$$u^h \in C^1(\mathbb{I}; \mathbb{V}^h), \quad \varphi^h \in C(\mathbb{I}; \Psi^h), \quad \theta^h \in W^{1,2}(\mathbb{I}; \Theta^h), \quad \zeta^h \in L^2(\mathbb{I}; H^1(\Omega)).$$

Moreover, if the initial approximations satisfy:

$$\begin{aligned}
&\lim_{h\to 0}\|u_0 - u_0^h\|_{\mathbb{V}} = 0, \quad \lim_{h\to 0}\|\varphi_0 - \varphi_0^h\|_{\Psi} = 0, \\
&\lim_{h\to 0}\|\theta_0 - \theta_0^h\|_{\Theta} = 0, \quad \lim_{h\to 0}\|\zeta_0 - \zeta_0^h\|_{Y} = 0,
\end{aligned}$$

and the exact solution has the following regularity

$$\begin{aligned}
&u \in H^2(\mathbb{I}; \mathbb{V}) \cap C^1(\mathbb{I}; H^2(\Omega)^d), \\
&\varphi \in C(\mathbb{I}; H^2(\Omega)), \\
&\theta \in H^1(\mathbb{I}; Y) \cap C(\mathbb{I}; H^2(\Omega)), \quad \dot{\theta} \in L^2(\mathbb{I}; H^1(\Omega)), \\
&\zeta \in H^1(\mathbb{I}; Y) \cap C(\mathbb{I}; H^2(\Omega)), \quad \dot{\zeta} \in L^2(\mathbb{I}; H^1(\Omega)),
\end{aligned}$$

then the semi-discrete solution converges strongly to the exact solution

$$\begin{aligned}
&\max_{\iota\in\mathbb{I}}\|u(\iota) - u^h(\iota)\|_{\mathbb{V}} \to 0, \quad \max_{\iota\in\mathbb{I}}\|\varphi(\iota) - \varphi^h(\iota)\|_{\Psi} \to 0, \\
&\max_{\iota\in\mathbb{I}}\|\theta(\iota) - \theta^h(\iota)\|_{Y} \to 0, \quad \max_{\iota\in\mathbb{I}}\|\zeta(\iota) - \zeta^h(\iota)\|_{Y} \to 0 \quad \text{as} \quad h \to 0.
\end{aligned} \tag{4.9}$$

Proof. The existence and uniqueness solution comes from arguments similar to Theorem 3.1. For the convergence, let $(u, \varphi, \theta, \zeta)$ and $(u^h, \varphi^h, \theta^h, \zeta^h)$ be solutions to Problems 2.2 and 4.1 respectively. Define errors:

$$e_u = u - u^h, \quad e_\varphi = \varphi - \varphi^h, \quad e_\theta = \theta - \theta^h, \quad e_\zeta = \zeta - \zeta^h.$$

First, we combine (2.28) and (4.1) with $v = \dot{u}^h$ and $v^h = \Pi_{V^h}\dot{u}$ to find

$$
\begin{aligned}
&m_{\mathscr{A}}\|\varepsilon(\dot{e}_u)\|_{\mathscr{H}}^2 + m_{\mathscr{F}}\|\varepsilon(e_u)\|_{\mathscr{H}}^2 \\
&\le c\Big(\|\nabla e_\varphi\|_{\mathscr{H}}^2 + \|e_\theta\|_Y^2 + \int_0^t \big[\|\varepsilon(e_u(s))\|_{\mathscr{H}}^2 + \|e_\zeta(s)\|_Y^2\big]\,ds \\
&\qquad + \|\dot{u} - \Pi_{\mathbb{V}^h}\dot{u}\|_{\mathbb{V}}^2 + \|\dot{u} - \Pi_{\mathbb{V}^h}\dot{u}\|_{\mathbb{V}}\Big).
\end{aligned}
$$

Next, from (2.29) and (4.2), we obtain

$$
m_{\mathbf{B}}\|\nabla e_\varphi\|_{\mathscr{H}}^2 \le c\Big(\|\varepsilon(e_u)\|_{\mathscr{H}}^2 + \|e_\theta\|_Y^2 + \|\varphi - \Pi_{\Psi^h}\varphi\|_\Psi^2 + \|\varphi - \Pi_{\Psi^h}\varphi\|_\Psi\Big).
$$

From (2.30) and (4.3), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e_\theta\|_Y^2 + m_{\mathscr{K}}\|\nabla e_\theta\|_H^2 \\
&\le c\Big(\|\varepsilon(\dot{e}_u)\|_{\mathscr{H}}^2 + \|\nabla e_\varphi\|_H^2 + \|\theta - \Pi_{\Theta^h}\theta\|_\Theta^2 + \|\theta - \Pi_{\Theta^h}\theta\|_\Theta + \|e_\theta\|_Y^2\Big).
\end{aligned}
$$

From (2.31) and (4.4), we obtain

$$
\frac{1}{2}\frac{d}{dt}\|e_\zeta\|_Y^2 + m_a\|\nabla e_\zeta\|_Y^2 \le c\Big(\|\varepsilon(e_u)\|_{\mathscr{H}}^2 + \|e_\zeta\|_Y^2 + \|\zeta - \Pi_{\Xi^h}\zeta\|_Y^2\Big).
$$

Combining all above estimates and using Gronwall's inequality yield

$$
\begin{aligned}
&\|e_u(t)\|_{\mathbb{V}}^2 + \|e_\varphi(t)\|_\Psi^2 + \|e_\theta(t)\|_Y^2 + \|e_\zeta(t)\|_Y^2 \\
&+ \int_0^t \big[\|\dot{e}_u\|_{\mathbb{V}}^2 + \|\nabla e_\theta\|_H^2 + \|\nabla e_\zeta\|_Y^2\big]\,ds \\
\le\, & ce^{cT}\Big(\|e_u(0)\|_{\mathbb{V}}^2 + \|e_\varphi(0)\|_\Psi^2 + \|e_\theta(0)\|_\Theta^2 + \|e_\zeta(0)\|_Y^2 \\
&+ \int_0^T \big[\|\dot{u} - \Pi_{V^h}\dot{u}\|_{\mathbb{V}}^2 + \|\varphi - \Pi_{\Psi^h}\varphi\|_\Psi^2 + \|\theta - \Pi_{\Theta^h}\theta\|_\Theta^2 + \|\zeta - \Pi_{\Xi^h}\zeta\|_Y^2\big]\,ds\Big).
\end{aligned}
$$

Using interpolation estimates for $\Pi_{\mathbb{V}^h}, \Pi_{\Theta^h}, \Pi_{\Psi^h}, \Pi_{\Xi^h}$, and [2, Ch. 3] to obtain

$$
\begin{aligned}
\|\dot{u} - \Pi_{\mathbb{V}^h}\dot{u}\|_{\mathbb{V}} &\le ch\|\dot{u}\|_{H^2(\Omega)^d}, \\
\|\varphi - \Pi_{\Psi^h}\varphi\|_\Psi &\le ch\|\varphi\|_{H^2(\Omega)}, \\
\|\theta - \Pi_{\Theta^h}\theta\|_\Theta &\le ch\|\theta\|_{H^2(\Omega)}, \\
\|\zeta - \Pi_{\Xi^h}\zeta\|_Y &\le ch\|\zeta\|_{H^2(\Omega)},
\end{aligned}
$$

and initial error convergence, we obtain the strong convergence result (4.9). □

5. Fully Discrete Approximation and Error Estimates

We introduce a fully discrete scheme for Problem 2.2 and derive error estimates. Using the finite element spaces $V^h \subset V$, $\Psi^h \subset \Psi$, $\Theta^h \subset \Theta$, and $\Xi^h \subset \Xi$ defined in Section 4, we partition $[0,T]$ uniformly: $0 = t_0 < t_1 < \cdots < t_K = T$ with time step $\tau = T/K$. For a continuous function f, we denote $f_n = f(t_n)$, and for a sequence $\{w_n\}_{n=0}^K$, we define the divided difference $\delta w_\tau = (w_k - w_{k-1})/\tau$. The fully discrete scheme combines forward and backward Euler methods:

Problem 5.1. *Find* $\{u_k^{h,\tau}\} \subset V^h$, $\{\varphi_k^{h,\tau}\} \subset \Psi^h$, $\{\theta_k^{h,\tau}\} \subset \Theta^h$, $\{\zeta_k^{h,\tau}\} \subset \Xi^h$ *such that:*

$$\begin{aligned}
&\Big(\mathscr{A}\varepsilon(\delta u_k^{h,\tau}), \varepsilon(v^h) - \varepsilon(\delta u_k^{h,\tau})\Big)_{\mathscr{H}} + \big(\mathscr{F}\varepsilon(u_k^{h,\tau}), \varepsilon(v^h) - \varepsilon(\delta u_k^{h,\tau})\big)_{\mathscr{H}} \\
&+ \big(\mathscr{P}^{\mathsf{T}}\nabla\varphi_{k-1}^{h,\tau}, \varepsilon(v^h) - \varepsilon(\delta u_k^{h,\tau})\big)_{\mathscr{H}} - \big(\mathscr{M}\theta_{k-1}^{h,\tau}, \varepsilon(v^h) - \varepsilon(\delta u_k^{h,\tau})\big)_{\mathscr{H}} \\
&+ J(u_k^{h,\tau}, v^h) - J(u_k^{h,\tau}, \delta u_k^{h,\tau}) \\
\geq & \big(f_k, v^h - \delta u_k^{h,\tau}\big)_{\mathscr{H}} - \left(\tau\sum_{j=1}^{k}\mathscr{C}(t_k - t_{j-1}, \varepsilon(u_{j-1}^{h,\tau}), \zeta_{j-1}^{h,\tau}), \varepsilon(v^h) - \varepsilon(\delta u_k^{h,\tau})\right)_{\mathscr{H}},
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
&\big(\mathbf{B}\nabla\varphi_k^{h,\tau}, \nabla\psi^h\big)_{\mathscr{H}} - \big(\mathscr{P}\varepsilon(u_k^{h,\tau}), \nabla\psi^h\big)_{\mathscr{H}} + \big(\mathscr{N}\theta_k^{h,\tau}, \nabla\psi^h\big)_{\mathscr{H}} + \big(\ell(u_k^{h,\tau}, \varphi_k^{h,\tau}) \\
= & \big(q_{e_k}, \psi^h\big)_{\Psi}, \psi^h\big)_{\Psi},
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
&\big(\delta\theta_k^{h,\tau}, \xi^h\big)_Y + \big(\mathscr{K}\nabla\theta_k^{h,\tau}, \nabla\xi^h\big)_H + \big(\mathscr{R}\varepsilon(\dot{u}_k^{h,\tau}), \xi^h\big)_H - \big(\mathscr{N}\nabla\varphi_k^{h,\tau}, \xi^h\big)_H \\
= & \big(h_k, \xi^h\big)_{\Theta} - \big(\vartheta(u_k^{h,\tau}, \theta_k^{h,\tau}), \xi^h\big)_{\Theta},
\end{aligned} \tag{5.3}$$

$$\big(\delta\zeta_k^{h,\tau}, \rho^h - \zeta_k^{h,\tau}\big)_Y + a(\zeta_k^{h,\tau}, \rho^h - \zeta_k^{h,\tau}) \geq \big(\mathscr{S}(\varepsilon(u_{n-1}^{hk}), \zeta_{k-1}^{h,\tau}), \rho^h - \zeta_k^{h,\tau}\big)_Y, \tag{5.4}$$

$$u_0^{hk} = u_0^h, \quad \theta_0^{hk} = \theta_0^h, \quad \zeta_0^{hk} = \zeta_0^h, \quad \varphi_0^{hk} = \varphi_0^h.$$

for all $v^h \in V^h$, $\psi^h \in \Psi^h$, $\xi^h \in \Theta^h$, $\forall\rho^h \in \Xi^h$, and $k = 1, ..., K$.

Theorem 5.1. *Under the hypotheses of Theorem 3.1, there exists* $c > 0$ *independent of* h *and* τ *such that:*

$$\begin{aligned}
&\max_{k=1,\dots,K}\Big\{\|u_k - u_k^{h,\tau}\|_{\mathbb{V}} + \|\zeta_k - \zeta_k^{h,\tau}\|_Y + \|\theta_k - \theta_k^{h,\tau}\|_{\Theta} + \|\varphi_k - \varphi_k^{h,\tau}\|_{\Psi}\Big\} \\
&+ \tau\sum_{i=1}^{K}\|\nabla\zeta_i - \nabla\zeta_i^{h,\tau}\|_Y \\
\leq & c\tau\Bigg\{\sum_{j=1}^{K-1}\Big(\|u_j - u_j^{h,\tau}\|_{\mathbb{V}} + \|u_j - u_{j-1}^{h,\tau}\|_{\mathbb{V}} + \|\zeta_j - \zeta_j^{h,\tau}\|_Y + \|\zeta_j - \zeta_{j-1}^{h,\tau}\|_Y \\
&\qquad + \|\varphi_j - \varphi_j^{h,\tau}\|_{\Psi} + \|\theta_j - \theta_j^{h,\tau}\|_{\Theta}\Big) \\
&+ \sum_{i=1}^{K}\Big(\|\rho_j^h - \zeta_j\|_Y + \|\dot{\zeta}_j - \delta\zeta_j\|_Y + \|\theta_j - \delta\theta_j^{h,\tau}\|_{\Theta} + \|\dot{\theta}_j - \delta\theta_j\|_{\Theta} + \|\theta_j - \xi_j^h\|_{\Theta}\Big)\Bigg\} \\
&+ c\max_{k=1,\dots,K}\Bigg\{\Big\|\int_0^{t_k}\mathscr{C}(t_k - s, \varepsilon(u(s)), \zeta(s))ds - \tau\sum_{j=1}^{k}\mathscr{C}(t_k - t_{j-1}, \varepsilon(u_{j-1}), \zeta_{j-1})\Big\|_{\mathscr{H}} \\
&\qquad + \|\varphi_k - \psi_k^h\|_{\Psi} + \|\theta_k - \xi_k^h\|_{\Theta} + \|\dot{u}_k - \delta u_k\|_{\mathbb{V}}\Bigg\} \\
&+ c\Big\{\|\zeta_0 - \zeta_0^{h,\tau}\|_Y + \|u_0 - u_0^{h,\tau}\|_{\mathbb{V}} + \|\varphi_0 - \varphi_0^{h,\tau}\|_{\Psi} + \|\theta_0 - \theta_0^{h,\tau}\|_{\Theta}\Big\} \\
&+ c\Big\{\mathscr{I}_\tau(\dot{u}) + \mathscr{I}_\tau(\dot{\theta}) + \mathscr{I}_\tau(\dot{\zeta})\Big\} \\
&+ c\tau\Big\{\|u\|_{W^{1,\infty}(\mathbb{I};\mathbb{V})} + \|\zeta\|_{W^{1,\infty}(\mathbb{I};Y)} + \|\varphi\|_{C(\mathbb{I};\Psi)} + \|\theta\|_{W^{1,\infty}(\mathbb{I};\Theta)}\Big\},
\end{aligned} \tag{5.5}$$

where $\mathscr{I}_\tau(\dot{u})$, $\mathscr{I}_\tau(\dot{u})$, and $\mathscr{I}_\tau(\dot{u})$, are given as follows

$$\mathscr{I}_\tau(\dot{u}) = \sum_{j=1}^{K} \int_{\iota_{j-1}}^{\iota_j} \|\dot{u}(s) - \delta u_j\| ds$$
$$\mathscr{I}_\tau(\dot{\theta}) = \sum_{j=1}^{K} \int_{\iota_{j-1}}^{\iota_j} \|\dot{\theta}(s) - \delta \theta_j\| ds$$
$$\mathscr{I}_\tau(\dot{\varphi}) = \sum_{j=1}^{K} \int_{\iota_{j-1}}^{\iota_j} \|\dot{\varphi}(s) - \delta \varphi_j\| ds.$$

Proof. From (5.4) and (2.31), we can see that

$$\begin{aligned}&\frac{1}{2\tau}\left(\|\zeta_k - \zeta_k^{h,\tau}\|_Y^2 - \|\zeta_{k-1} - \zeta_{k-1}^{h,\tau}\|_Y^2\right) + \kappa\|\nabla(\zeta_k - \zeta_k^{h,\tau})\|_Y^2 \\ \leq & c\Big(\|\rho^h - \zeta_k\|_Y + \|\zeta_k - \zeta_{k-1}^{h,\tau}\|_Y^2 + \|u_k - u_{k-1}^{h,\tau}\|_V^2 + \|\dot{\zeta}_k - \delta\zeta_k\|_Y^2\Big).\end{aligned}$$

Summing from $i = 1$ to k, we find

$$\begin{aligned}&\|\zeta_k - \zeta_k^{h,\tau}\|_Y^2 + \tau\sum_{i=1}^{k}\|\nabla\zeta_i - \zeta_i^{h,\tau})\|_Y^2 \\ \leq & c\tau\sum_{i=1}^{k}\left(\|\zeta_i - \zeta_{i-1}^{h,\tau}\|_Y^2 + \|u_i - u_{i-1}^{h,\tau}\|_V^2\right) + c\|\zeta_0 - \zeta_0^{hk}\|_Y^2 \\ &+ c\tau\sum_{i=1}^{n}\left(\|\rho^h - \zeta_i\|_{L^2} + \|\dot{\zeta}_i - \delta\zeta_i\|_Y^2\right).\end{aligned} \tag{5.6}$$

For the temperature, we use (2.30) and (5.3) to get

$$\begin{aligned}&\frac{1}{2\tau}\left(\|\theta_k - \theta_k^{h,\tau}\|_\Theta^2 - \|\theta_{k-1} - \theta_{k-1}^{h,\tau}\|_\Theta^2\right) + m_{\mathscr{K}}\|\theta_k - \theta_k^{h,\tau}\|_\Theta^2 \\ \leq & c\Big(\|\theta_k - \delta\theta_k^{h,\tau}\|_\Theta^2 + \|\dot{\theta}_k - \delta\theta_k\|_\Theta^2 + \|\theta_k - \xi^h\|_\Theta^2 + \|\varphi_k - \varphi_k^{h,\tau}\|_\Psi^2 + \|u_k - u_k^{h,\tau}\|_{\mathbb{V}}^2\Big).\end{aligned}$$

Substituting k by i and summing from $i = 1$ to k, we obtain

$$\begin{aligned}&\|\theta_k - \theta_k^{h,\tau}\|_\Theta^2 \\ \leq & c\tau\sum_{i=1}^{k}\left[\|\theta_i - \delta\theta_i^{h,\tau}\|_\Theta^2 + \|\dot{\theta}_i - \delta\theta_i\|_\Theta^2 + \|\theta_i - \xi^h\|_\Theta^2 + \|\varphi_i - \varphi_i^{h,\tau}\|_\Psi^2 + \|u_i - u_i^{h,\tau}\|_{\mathbb{V}}^2\right] \\ &+ \|\theta_0 - \theta_0^{h,\tau}\|_\Theta^2.\end{aligned} \tag{5.7}$$

Next, it follows from (2.29) and (5.2) that

$$\|\varphi_k - \varphi_k^{h,\tau}\|_\Psi^2 \leq c\left(\|u_k - u_n^{h,\tau}\|_{\mathbb{V}}^2 + \|\theta_k - \theta_k^{h,\tau}\|_\Theta^2 + \|\varphi_k - \psi^h\|_\Psi^2\right). \tag{5.8}$$

To derive the displacement field error, we combine (2.28) and (5.1) to have

$$\begin{aligned}\|u_k-u_k^{h,\tau}\|_{\mathbb{V}}^2 \leq & c\tau\sum_{j=1}^{k}\left(\|u_{j-1}-u_{j-1}^{h,\tau}\|_{\mathbb{V}}^2+\|\zeta_{j-1}-\zeta_{j-1}^{h,\tau}\|_Y^2\right)+c\|\varphi_k-\varphi_{k-1}^{h,\tau}\|_{\Psi}^2\\ &+\left\|\int_0^{t_k}\mathscr{C}(t_k-s,\varepsilon(u(s)),\zeta(s))ds-\tau\sum_{j=1}^{k}\mathscr{C}(t_k-t_{j-1},\varepsilon(u_{j-1}),\zeta_{j-1})\right\|_{\mathscr{H}}^2\\ &+\|\theta_k-\theta_{k-1}^{h,\tau}\|_{\Theta}^2+\|\dot{u}_k-\delta u_k\|_{\mathbb{V}}^2.\end{aligned} \tag{5.9}$$

Finally, combining (5.6)-(5.9) and using the following estimates yield

$$\|\theta_k-\theta_{k-1}^{hk}\|_{\Theta}\leq c\left(\mathscr{I}_\tau(\dot{\theta})+\tau\|\theta\|_{W^{1,\infty}(\Theta)}\right),$$
$$\|\varphi_k-\varphi_{k-1}^{hk}\|_{\Psi}\leq c\left(\mathscr{I}_\tau(\dot{\varphi})+\tau\|\varphi\|_{C(\mathbb{I};\Psi)}\right),$$
$$\|u_k-u_{k-1}^{h,\tau}\|_{\mathbb{V}}\leq c\left(\mathscr{I}_\tau(\dot{u})+\tau\|u\|_{W^{1,\infty}(\mathbb{V})}\right),$$

with discrete Grönwall's lemma yields the error estimate (5.5). □

Proposition 5.1. *Under Theorem 5.1 hypotheses and solution regularity:*

$$u\in W^{1,\infty}(\mathbb{I};\mathbb{V})\cap C(\mathbb{I};H^2(\Omega;\mathbb{R}^d)),$$
$$\varphi\in C(\mathbb{I};H^2(\Omega)),$$
$$\theta\in W^{1,\infty}(\mathbb{I};\Theta)\cap C(\mathbb{I};H^2(\Omega)),$$
$$\zeta\in W^{1,\infty}(\mathbb{I};L^2(\Omega))\cap L^\infty(\mathbb{I};H^2(\Omega))\cap H^1(\mathbb{I};H^1(\Omega)),$$

with initial approximations:

$$\zeta_0^{h,\tau}=\Pi_{\Xi^h}\zeta_0,\quad u_0^{h,\tau}=\Pi_{V^h}u_0,\quad \theta_0^{h,\tau}=\Pi_{\Theta^h}\theta_0,\quad \varphi_0^{h,\tau}=\Pi_{\Psi^h}\varphi_0,$$

where Π_{V^h}, Π_{Θ^h}, Π_{Ψ^h}, *and* Π_{Ξ^h}, *are interpolation operators,*

$$\max_{k=1,\dots,K}\left\{\|\zeta_k-\zeta_k^{h,\tau}\|_Y+\|\theta_k-\theta_k^{h,\tau}\|_{\Theta}+\|\varphi_k-\varphi_k^{h,\tau}\|_{\Psi}+\|u_k-u_k^{h,\tau}\|_V\right\}\leq c(h+\tau). \tag{5.10}$$

Proof. By using the interpolation estimates see [3, Theorem 4.4.20], we obtain

$$\begin{aligned}\max_{k=1,\dots,K}\inf_{\rho^h\in\Xi^h}\|\zeta_k-\rho^h\|_{H^1(\Omega)}&\leq ch\|\zeta\|_{C(\mathbb{I};H^2(\Omega))},\\ \max_{k=1,\dots,K}\inf_{v^h\in\mathbb{V}^h}\|u_n-v^h\|_{\mathbb{V}}&\leq ch\|u\|_{C(\mathbb{I};H^2(\Omega))},\\ \max_{k=1,\dots,K}\inf_{\xi^h\in\Theta^h}\|\theta_k-\xi^h\|_{\Theta}&\leq ch\|\theta\|_{C(\mathbb{I};H^2(\Omega))},\\ \max_{k=1,\dots,K}\inf_{\psi^h\in\Psi^h}\|\varphi_k-\psi^h\|_{\Psi}&\leq ch\|\varphi\|_{C(\mathbb{I};H^2(\Omega))},\end{aligned} \tag{5.11}$$

and initial error estimates:

$$\begin{aligned}\|\zeta_0-\zeta_0^{h,\tau}\|_Y&\leq ch\|\zeta_0\|_{H^2(\Omega)},\\ \|u_0-u_0^{h,\tau}\|_V&\leq ch\|u_0\|_{H^2(\Omega)},\\ \|\theta_0-\theta_0^{h,\tau}\|_{\Theta}&\leq ch\|\theta_0\|_{H^2(\Omega)},\\ \|\varphi_0-\varphi_0^{h,\tau}\|_{\Psi}&\leq ch\|\varphi_0\|_{H^2(\Omega)}.\end{aligned} \tag{5.12}$$

Moreover, it easy to see that the quadrature error satisfies

$$\max_{k=1,\ldots,K}\left\| \int_0^{\iota_k} \mathscr{C}(\iota_k - s, \varepsilon(u(s)), \zeta(s))ds - \tau\sum_{i=1}^{k}\mathscr{C}(\iota_k - \iota_{i-1}, \varepsilon(u_{i-1}), \zeta_{i-1})\right\|_{\mathscr{H}} \\ \leq c\tau\|\mathscr{C}\|_{W^{1,\infty}(\mathbb{I};\mathscr{H})}. \tag{5.13}$$

Therefore, the time discretization errors are bounded by

$$\begin{aligned} &\mathscr{I}_\tau(\dot{u}) \leq c\tau\|\ddot{u}\|_{L^1(\mathbb{I};\mathbb{V})}, \quad \sum_{i=1}^{K}\|\dot{u}_i - \delta u_i\|_V \leq c\tau\|\ddot{u}\|_{L^2(\mathbb{I};\mathbb{V})}, \\ &\mathscr{I}_\tau(\dot{\theta}) \leq c\tau\|\ddot{\theta}\|_{L^1(\mathbb{I};\Theta)}, \quad \sum_{i=1}^{K}\|\dot{\theta}_i - \delta\theta_i\|_\Theta \leq c\tau\|\ddot{\theta}\|_{L^2(\mathbb{I};\mathbb{V})}, \\ &\mathscr{I}_\tau(\dot{\varphi}) \leq c\tau\|\ddot{\varphi}\|_{L^1(\mathbb{I};\Psi)}, \quad \sum_{i=1}^{K}\|\dot{\varphi}_i - \delta\varphi_i\|_\Psi \leq c\tau\|\ddot{\varphi}\|_{L^2(\mathbb{I};\Psi)}. \end{aligned} \tag{5.14}$$

Finally, we substitute estimates (5.11)-(5.14) into Theorem 5.1 to obtain (5.10). □

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REFERENCES

[1] M. Shillor, M. Sofonea, J.J. Telega, Models and Analysis of Quasistatic Contact: Variational Methods, Vol. 655, Springer, 2004.

[2] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, Springer, New York, 2008.

[3] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, SIAM, Philadelphia, 2002.

[4] A. Oultou, O. Baiz, H. Benaissa, Analysis of a system consisting of a fractional history-dependent hemivariational inequality and a nonlocal delay fractional differential equation, J. Math. Sci. 290 (2024), 745–764.

[5] R.C. Batra, J.S. Yang, Saint-Venant's principle in linear piezoelectricity, J. Elasticity 38 (1995), 209–218.

[6] J. Yang, An Introduction to the Theory of Piezoelectricity, Springer, Boston, 2005.

[7] M. Campo, J.R. Fernández, A. Rodriguez-Aros, A quasistatic contact problem with normal compliance and damage involving viscoelastic materials with long memory, Appl. Numer. Math. 58 (2008), 1274–1290.

[8] M. Rochdi, M. Shillor, M. Sofonea, Quasistatic viscoelastic contact with normal compliance and friction, J. Elasticity 51 (1998), 105–126.

[9] A. Oultou, O. Baiz, H. Benaissa, A doubly history-dependent quasivariational inequality arising in viscoelastic frictional contact problems with wear, J. Appl. Math. Mech. 104 (2024), e202400012.

[10] M. Frémond, B. Nedjar, Damage, gradient of damage and principle of virtual power, Int. J. Solids Structures 33 (1996), 1083–1103.

[11] B. Nedjar, Elastoplastic-damage modelling including the gradient of damage: formulation and computational aspects, Int. J. Solids Structures 38 (2001), 5421–5451.

[12] S. Mohamed, A dynamic problem with adhesion and damage in electro-viscoelasticity with long-term memory, J. Inequal. Pure Appl. Math. 10 (2009), 6.

[13] T.A. Angelov, On a rolling problem with damage and wear, Mech. Res. Commun. 26 (1999), 281–286.

[14] M. Selmani, A frictional contact problem involving piezoelectric materials with long memory, Mediterranean J. Math. 12 (2015), 1177–1197.

[15] C. Miehe, Discontinuous and continuous damage evolution in Ogden-type large-strain elastic materials, Eur. J. Mech. A. Solids 14 (1995), 697–720.

[16] E. Bonetti, M. Frémond, Damage Theory: Microscopic effects of vanishing macroscopic motions, Comput. Appl. Math. 22 (2003), 313–333.

[17] M.S. Mesai Aoun, M. Selmani, A.A. Ahmed, Variational analysis of a frictional contact problem with wear and damage, Math. Model. Anal. 26 (2021), 170–187.

[18] O. Baiz, H. Benaissa, D. El Moutawakil, R. Fakhar, Variational and numerical analysis of a quasistatic thermo-electro-visco-elastic frictional contact problem, ZAMM Z. Angew. Math. Mech. 99 (2019), e201800138.

[19] O. Baiz, H. Benaissa, Optimal control for a dynamic thermo-electro-viscoelastic contact problem with frictional heating, Acta Appl. Math. 171 (2021), 22.

[20] M. Bouallala, E.-H. Essoufi, M. Alaoui, Variational and numerical analysis for frictional contact problem with normal compliance in thermo-electro-viscoelasticity, Int. J. Differ. Equ. 2019 (2019), 6972742.

[21] A. Oultou, O. Baiz, H. Benaissa, Numerical analysis of a piezoelectric contact problem with locking material and thermal effects, Discrete Contin. Dyn. Syst. Ser. S18 (2025), 218–237.

[22] A. Oultou, O. Baiz, H. Benaissa, Thermo-electro-elastic contact problem with temperature dependent friction law, Nonlinear Dyn. Sys. Theory 24 (2024), 80–98.

[23] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland Pub. Co, Amsterdam, 1973.

[24] L, Byszewski, Nonlinear second-order delay differential equation, Technical Transactions, 3 (2019) 141-148.