Appl. Set-Valued Anal. Optim. 8 (2026), No. 3, pp. 311-335
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.8.2026.3.03

EVALUATING CONTAGION SYSTEMIC RISK WITH REGULATORY INTERVENTIONS BY ADAPTIVE SEIR MODELLING

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Abstract. In this paper, we highlight the important role of dynamic systems in modelling contagion risks within financial networks. We mainly focus on how systemic risk spreads under regulatory interventions, using the adaptive SEIR model to show stability locally and globally. Through our simulations, we aim to investigate more about these processes, which help guide decisions made by regulators and strengthen the financial system. We also explore, through optimal control strategies, the best ways to manage these risks. The results of this paper are designed to contribute to discussions about financial stability, stressing the need for rules that can be adjusted to effectively handle potential risks.

Keywords. Contagion systemic risk; Dynamic systems; Financial modelling; Financial networks; SEIR Model.

2020 Mathematics Subject Classification. 92D30, 34D20, 49K15, 49J15.

1. Introduction

Systemic risk in banking arises when one or more major financial institutions fail, causing problems in the entire financial network, similar to when one virulent disease finds its way amongst the population. The potential for this sort of risk is concerning as the stability of economies is threatened, which has widespread effects on a large number of individuals and businesses. The effects can be hard, including the increased unemployment, the loss of savings, and decreased economic activity more generally. Mathematical modelling has been very helpful to understand systemic risk, as modelling makes it possible to study the interactions and dependencies between several different financial institutions. On the other hand, the SEIR (susceptible, exposed, infected, recovered) model is very important to track the infectious disease spread among populations in epidemiology. It defines a metric, $\mathscr{R}_0$ such that if $\mathscr{R}_0 < 1$ which means that the transmission of the disease is limited, we are in a stable situation. However, in the opposite, it leads to rapid growth in the epidemic [1, 2, 3, 4]. By the projection of this model in the financial contagion propagation of the systemic risk, researchers and policymakers can locate and measure critical contagion dynamics and assess the potential effectiveness of different intervention strategies to reduce systemic risk. To obtain an overview of how distress moves

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Received 27 May 2025; Accepted 6 July 2026; Published online 15 September 2026.

through the bank network, researchers and policymakers can reinforce financial stability and reduce the probability of crises. For example, some researchers demonstrated how connectedness causes systemic risk, which indicates why good regulatory frameworks are necessary [5, 6]. Some research focused on measuring systemic risk by using network analysis and the implications of such risks for financial stability [7, 8]. The basic model for analysing systemic risk in the bank ecosystem was developed from a system of ordinary differential equations that specify the connections among the different types of banks: undistressed banks (U), exposed banks (E), distressed banks (D), recovered banks (R), and liquidated banks (L). This model represents a mathematical way to demonstrate how these categories change over time by different forces, such as contagion risk, recovery rates, and exposure rates [9, 10, 11, 12, 13]. The model captures the dynamics of these states, and shows how banks can transition between categories given their interactions and provides a complete representation of systemic risk [14, 15, 16]. Our baseline model, studied in [9], is given by:

$$\begin{cases} \dfrac{dU}{dt} = -\beta UD + \mu E + \theta R, \\ \dfrac{dE}{dt} = \beta UD - (\mu + \gamma)E, \\ \dfrac{dD}{dt} = \gamma E - (\gamma_1 + \gamma_2)D, \\ \dfrac{dR}{dt} = \gamma_1 D - \theta R, \\ \dfrac{dL}{dt} = \gamma_2 D. \end{cases}$$

where

- U: Undistressed banks. They are currently healthy, and not distressed but remain at some risk.
- E: Exposed banks. They had exposure to distressed institutions, and began to be concerned about their performance but have not yet experienced significant loss.
- D: Distressed banks. They are in distress now, and possibly with a threat of failure.
- R: Recovered banks. They were distressed but have recovered, they are not considered to be at risk.
- L: Liquidated banks. They are distressed and have been liquidated.

These categories are important in examining the nature of systemic risk in the banking sector because they demonstrate how different banks behave depending on the level of distress and recovery that they experience. This problem can be modelled in the SEIR model framework. As a financial interpretation of the parameters, β is the rate of transition of default, or the speed of propagation of financial distress across banks in a system, μ is the variance of the first transition speed, which measures how fast a bank goes from being a stable bank to that of distressed. θ, the recovery rate is how fast a distressed bank can return to stability, and γ the observed response of banks, or the sensitivity of the banks to shock, is how much exposure the bank will have to default. γ_1, the recovery rate of distressed banks is how quickly these banks can return to stability, while, γ_2, the speed that banks leave the system after the crisis, such as liquidation. In conclusion, these parameters can be used to model the dynamics of systemic risk of the transition and recovery processes observed in the banking system in times of crisis.

Lower operation extremes capture the relationship between these measurements, which produce stabilising turbulence on the system level for the entire banking system. Let our dynamic system be given as:

$$\begin{cases} \frac{dU}{dt} = -\beta UD + \mu E + \theta R, \\ \frac{dE}{dt} = \beta UD - (\mu + \gamma)E, \\ \frac{dD}{dt} = \gamma E - (\gamma_1 + \gamma_2)D - \frac{cD}{b+D}, \\ \frac{dR}{dt} = \gamma_1 D - \theta R + \frac{cD}{b+D}, \\ \frac{dL}{dt} = \gamma_2 D, \end{cases} \tag{1.1}$$

where the initial conditions are satisfied:

$$U(0) \geq 0, \quad E(0) \geq 0, \quad D(0) \geq 0, \quad R(0) \geq 0, \quad L(0) \geq 0. \tag{1.2}$$

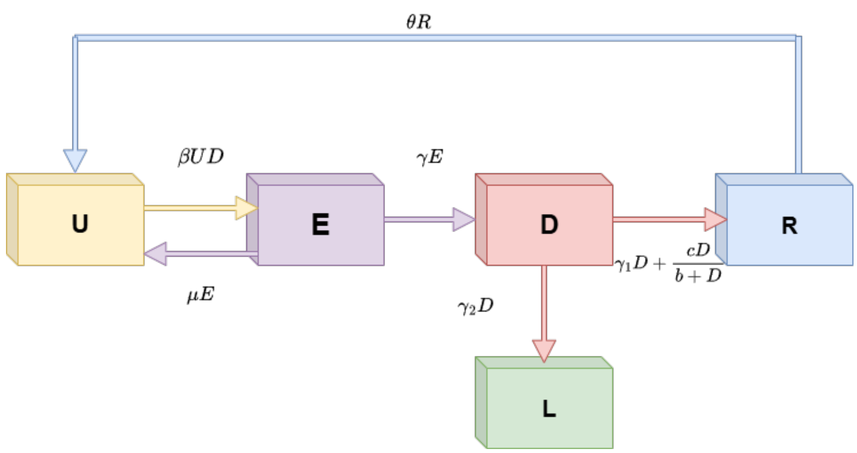


FIGURE 1. Flow Chart of Regulatory Interventions on Systemic Risk Transmission

In this paper, we express $\frac{cD}{b+D}$, where c is the contagion coefficient, which measures how easy it is for one bank's failure to cause the failure of another bank, and b the resistance coefficient that expresses the financial system's ability to absorb shocks from bank failures without the system collapsing. Thus, this term is an indication to measure systemic risk and the probability of contagion in banks. We propose to subtract this term from the distressed banks compartment D and add it to the recovered banks compartment R so that we could model the recovery process for distressed banks. This term expresses the flow rate of distressed banks to recovered banks, which indicates that recovery is based on the current state of distressed banks and exogenous shocks existing in the model of systemic risk [17]. In explicitly modelling, this transition captures the dynamic interactions between compartments and how the distressed state of banks influences their chances of recovery, which is key to identifying systemic risk [18]. Additionally, this adjustment ensures stability in the system and incorporates sensitivities to current situations, as not all banks that are distressed will recover due to systemic pressure, as was done in studies focusing on recovery behaviour of financial systems [19, 20]. The main benefit of this approach is to improve the realism of the model and provide a better sense of systemic risk and recovery dynamics; the latter is essential for risk management and regulatory policies. It is reasonable to question the integrated value of the transition term $\frac{cD}{b+D}$ in the model when the optimal control strategies are applied. This term is important because it describes the nonlinear dynamics of bank recovery processes and represents the real-world interactions between distressed and recovering banks. Realistic modelling of these dynamics yields meaningful results from which we can deduce recovery rates and system behaviour characteristics, which are important for the development of effective control strategies [21]. Understanding how this term affects the decisions made in the model, with respect to a sensitivity analysis, allows updating the control parameters so that actions depend on various levels of systemic risk [22]. In conclusion, incorporating this transition term strengthens the structure of optimal control models, then improves the financial risk management efforts and establishes them in real-world activities [23, 24, 25]. After incorporating the transition term and adding the recovery dynamics of distressed banks into the model, it is essential to employ optimal control methods. Financial systems exhibit nonlinear interactions and time-varying parameters, requiring dynamic risk management. Optimal control generates strategies that can dynamically respond to variations in system parameters and prevent crises in a timely manner. Dynamic strategies that employ robust control methods have increased the resilience of financial systems through the best possible responses under uncertainty [26, 27]. Moreover, research demonstrated that optimal control can help improve key performance indicators for the entire system, allowing policymakers to deal with complex interrelationships of risk within financial networks [28]. Therefore, optimal control is necessary for future systemic risk management.

The model is based on several important assumptions from the theoretical basis of the standard SEIR model.

1) The total value of the banking sector assumes that it is one closed system, and any global value changes will not be considered. So, the total capacity of the banking network N is a constant value over time.

2) At that time $t = 0$, every bank is considered undistressed and has a default exposure, except the first bank, which is considered distressed since it is in a state of default.

3) We take the model to be homogeneous. The totals are homogeneous banks of different types of financial institutions.

4) Any recovered bank can lose immunity and become undistressed again.

Our paper is divided into four sections. Section 2 focuses on the analysis of the model, where we study both local, global stability and sensitivity analysis. Section 3 is devoted to finding a solution by using optimal control strategies. In Section 4, we present the numerical method to obtain a numerical solution. Finally, Section 5 verifies the theoretical analysis.

2. Analysis of the Model

2.1. **Positivity and boundedness of solutions.**

Theorem 2.1. *Let system (1.1) be given, and let initial conditions $U(0)$, $E(0)$, $D(0)$, $R(0)$, and $L(0)$ satisfy (1.2). Then, this system admits a unique solution. Moreover, this solution is bounded and positive for all $t \geq 0$.*

Proof. Define the state vector $X(t) = \big(U(t), E(t), D(t), R(t), L(t)\big)^T$, and consider the function $F : \mathbb{R}^5 \to \mathbb{R}^5$, defined as

$$F(X) = \begin{pmatrix} -\beta UD + \mu E + \theta R \\ \beta UD - (\mu+\gamma)E \\ \gamma E - (\gamma_1+\gamma_2)D - \frac{cD}{b+D} \\ \gamma_1 D - \theta R + \frac{cD}{b+D} \\ \gamma_2 D \end{pmatrix}.$$

So, our dynamic system can be written as

$$\begin{cases} \dot{X}(t) = F(X), \\ X(t_0) = X(0). \end{cases} \tag{2.1}$$

This is an autonomous differential equation.

On the other hand, function F is continuous and belongs to C^1 on $\mathbb{R}^5$, ensuring that F is locally Lipschitz. Note that system (1.1) admits a unique solution defined on the interval $[0, t_{\max})$. Now, let $F = (f_1, f_2, f_3, f_4, f_5)^T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$, where

$$f_1(X) = -\beta UD + \mu E + \theta R,\ f_2(X) = \beta UD - (\mu+\gamma)E,$$
$$f_3(X) = \gamma E - (\gamma_1+\gamma_2)D - \frac{cD}{b+D},\ f_4(X) = \gamma_1 D - \theta R + \frac{cD}{b+D},\ f_5(X) = \gamma_2 D.$$

We recall that $\mathbb{R}^5$ is an open set of $\mathbb{R}^5$ that contains $\overline{\mathbb{R}}^5_+$. On the other hand, for all $X = (U, E, D, R, L)^T \in \overline{\mathbb{R}}^5_+$, we have

$$\begin{aligned}
f_1(X) &= \mu E + \theta R \geq 0, && \text{for } U = 0,\\
f_2(X) &= \beta UD \geq 0, && \text{for } E = 0,\\
f_3(X) &= \gamma E \geq 0, && \text{for } D = 0,\\
f_4(X) &= \gamma_1 D + \frac{cD}{b+D} \geq 0, && \text{for } R = 0,\\
f_5(X) &= \gamma_2 D \geq 0, && \text{for } L = 0.
\end{aligned}$$

From [29, Definition 4.1], F is essentially nonnegative. By applying [29, Proposition 4.1], we conclude that $\overline{\mathbb{R}}^5_+$ is an invariant set with respect to (2.1). It follows from (1.2) that

$$U(t) \geq 0, \quad E(t) \geq 0, \quad D(t) \geq 0, \quad R(t) \geq 0, \quad L(t) \geq 0, \quad \forall t \in [0, t_{max}).$$

Thus $||X(t)|| = U(t) + E(t) + D(t) + R(t) + L(t)$.

Next, let us consider the total number of banks $N(t)$, defined as

$$N(t) = U(t) + E(t) + D(t) + R(t) + L(t).$$

Taking the derivative of both sides of the equation, we find

$$\frac{dN(t)}{dt} = \frac{dU(t)}{dt} + \frac{dE(t)}{dt} + \frac{dD(t)}{dt} + \frac{dR(t)}{dt} + \frac{dL(t)}{dt}.$$

From system (1.1), we see that $\frac{dN(t)}{dt} = 0$, which implies that

$$N(t) = N, \quad \forall t \in [0, t_{max}), \tag{2.2}$$

where N represents the total initial number of banks. Thus $||X(t)|| \leq N$ for all $t \in [0, t_{max})$. By using this boundedness and invoking the blow-up theorem, we deduce that the maximal time of existence is $t_{\max} = +\infty$, Which concludes the proof immediately. $\square$

Remark 2.1. The feasible region for the banking population is defined as

$$\mathscr{J} = \left\{(U_0, E_0, D_0, R_0, L_0) \in \mathbb{R}_+^5 : ||X(t)|| = U(t) + E(t) + D(t) + R(t) + L(t) = N(t) = N, \forall t \geq 0\right\},$$

where $N > 0$ represents the total population capacity, which is positively invariant.

Indeed, $X(t, X_0) = X_0 + \int_0^t F(\eta, X(\eta))d\eta$. Let $X_0 \in \mathbb{R}^5$ and the map be define by $\varphi(t) : X_0 \longrightarrow X(t, X_0)$. Then $\varphi(0)(X_0) = X(0, X_0) = X_0$. Let $t, s \geq 0$ and $Z(t) = \varphi(t+s)(X_0) = X(t+s, X_0)$. Then $Z'(t) = F(Z(t))$ and $Z(0) = X(s, X_0)$, which yield $Z(t) = X(t, X(s, X_0))$. Finally, $X(t+s)(X_0) = X(t, X(s, X_0))$, which concludes $\varphi(t+s)(X_0) = \varphi(t)(\varphi(s))(X_0)$. On the other hand, we have $\varphi(t)(X_0 + Y_0) \neq \varphi(t)(X_0) + \varphi(t)(Y_0)$, which concludes that (1.1) generates a nonlinear semigroup on $\mathbb{R}^5$. By (2.2) and the invariance of $\overline{\mathbb{R}}_+^5$, for all $X_0 \in \mathscr{J}$, $\varphi(t)(X_0) \in \mathscr{J}$, system (1.1) is well posed in $\mathscr{J}$.

2.1.1. *Equilibrium state without systemic risk.* The equilibrium state without systemic risk refers to the condition within a financial system where all financial institutions operate without any distress or risk contagion. In this equilibrium state, the distressed banks satisfy $(D = E = 0)$, which means that there are no financial instabilities or vulnerabilities present. Mathematical model (1.1) admits a unique Risk-Free Equilibrium (RFE), denoted by $\mathscr{P}_0$. This equilibrium satisfies the following system of equations:

$$\begin{cases} -\beta UD + \mu E + \theta(N - U - E - D) = 0, \\ \beta UD - (\mu + \gamma)E = 0, \\ \gamma E - (\gamma_1 + \gamma_2)D - \dfrac{cD}{b+D} = 0, \\ \gamma_1 D - \theta R + \dfrac{cD}{b+D} = 0. \end{cases} \tag{2.3}$$

By the conditions of the risk-free equilibrium $E = D = 0$, we deduce $U = N$. Consequently, the risk-free equilibrium is given by $\mathscr{P}_0 = (N, 0, 0, 0, 0)$.

2.1.2. *Basic reproduction number of the UEDR framework for systemic risk* ($\mathscr{R}_0$).

Remark 2.2. The basic reproduction number, denoted by $\mathscr{R}_0$ [9, 30], is a critical metric adapted to assess systemic risk within financial networks. Quantifies the average number of secondary distressed banks that can appear from a single distressed bank interacting with a population of undistressed banks.

Let us calculate $\mathscr{R}_0$. In our dynamic system, we have two compartments of distress E and D. Thus

$$\mathscr{F} = \begin{pmatrix} \beta UD \\ 0 \end{pmatrix}, \mathscr{V} = \begin{pmatrix} (\mu + \gamma)E \\ (\gamma_1 + \gamma_2)D + \frac{cD}{b+D} - \gamma E \end{pmatrix}.$$

It follows that

$$\mathbf{F} = \begin{pmatrix} 0 & \beta\Lambda \\ 0 & 0 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} \mu + \gamma & 0 \\ -\gamma & \gamma_1 + \gamma_2 + \frac{b}{c} \end{pmatrix},$$

where Λ the number of undistressed banks at the risk- free equilibrium point, $\mathbf{F}$ represents the matrix of new contagion between the financial banks and $\mathbf{V}$ the transition matrix representing the transition terms related to other dynamics in the financial model. Then

$$\mathbf{F}\mathbf{V}^{-1} = \begin{pmatrix} \frac{\gamma\beta\Lambda}{(\mu+\gamma)(\gamma_1+\gamma_2+\frac{c}{b})} & \beta\Lambda \\ 0 & 0 \end{pmatrix}.$$

Finally,

$$\mathscr{R}_0 = \rho(\mathbf{F}\mathbf{V}^{-1}) = \max\left\{|\lambda| \mid \lambda \in \mathrm{Sp}(\mathbf{F}\mathbf{V}^{-1})\right\} = \frac{\gamma\beta\Lambda}{(\mu+\gamma)(\gamma_1+\gamma_2+\frac{c}{b})},$$

where $\mathrm{Sp}(\mathbf{F}\mathbf{V}^{-1})$ is the set of eigenvalues of the matrix $\mathbf{F}\mathbf{V}^{-1}$.

2.1.3. *Persistence equilibrium of distress.* Refers to a state in which the network of U, E, D, R classes are greater than zero. This equilibrium indicates that the risk continues to persist within the bank network, with banks present in each category. It is a critical point for analysing the dynamics of distress transmission, as it reflects ongoing interactions and recovery within the bank's ecosystem.

Theorem 2.2. *Let U,E,D and $R \neq 0$. The system (2.3) admits a unique solution which is the persistent equilibrium of distress, denoted by $\mathscr{P}_* = (U^*,E^*,D^*,R^*)$.*

Proof. From the system (2.3), we have $\beta U^*D^* = \mu E^* + \theta(N-U^*-E^*-D^*)$. We substitute this value into the second equation and we can see that $\mu E^* + \theta(N-U^*-E^*-D^*) = (\mu+\gamma)E^*$, that is, $E^* = \frac{\theta(N-U^*)-\theta D^*}{\theta+\gamma}$.

On the other hand, we have the third equation $\gamma E^* - (\gamma_1+\gamma_2)D^* - \frac{cD^*}{b+D^*} = 0$. Thus,

$$(\gamma_1+\gamma_2)D^{*2} + [(\gamma_1+\gamma_2)b + c - \gamma E^*]\,D^* - \gamma E^* b = 0.$$

Substituting the value of E^* in this equation, we have

$$(\gamma+\theta)(\gamma_1+\gamma_2)D^{*2} + [(\theta+\gamma)\left((\gamma_1+\gamma_2)b+c\right) - (\theta+\gamma\theta(N-U^*))]\,D^* - b\gamma\theta(N-U^*) = 0.$$

The discriminant of this equation is given by

$$\Delta = [(\theta+\gamma)\left((\gamma_1+\gamma_2)b+c\right) - (\theta+\gamma\theta(N-U^*))]^2 + 4(\gamma+\theta)(\gamma_1+\gamma_2)b\gamma\theta(N-U^*),$$

which is positive. Then this equation has two solutions:

$$D_1^* = \frac{-[(\theta+\gamma)\left((\gamma_1+\gamma_2)b+c\right) - (\theta+\gamma\theta(N-U^*))] - \sqrt{\Delta}}{2(\gamma+\theta)(\gamma_1+\gamma_2)},$$

and

$$D_2^* = \frac{-[(\theta+\gamma)\left((\gamma_1+\gamma_2)b+c\right) - (\theta+\gamma\theta(N-U^*))] + \sqrt{\Delta}}{2(\gamma+\theta)(\gamma_1+\gamma_2)}.$$

Note that $D_1^*D_2^* = \frac{-b\gamma\theta(N-U^*)}{(\gamma+\theta)(\gamma_1+\gamma_2)} < 0$. $D_2^* > 0$ yields

$$D^* = \frac{-[(\theta+\gamma)\left((\gamma_1+\gamma_2)b+c\right) - (\theta+\gamma\theta(N-U^*))] + \sqrt{\Delta}}{2(\gamma+\theta)(\gamma_1+\gamma_2)}.$$

We further have

$$D^* = \frac{2b\theta\gamma\Lambda(\mathscr{R}_0-1)}{[(\theta+\gamma)\left((\gamma_1+\gamma_2)b+c\right) - (\theta+\gamma\theta(N-U^*))] + \sqrt{\Delta}}.$$

Finally,

$$E^* = \frac{\theta\Lambda(\mathscr{R}_0 - 1)\left[[(\theta+\gamma)((\gamma_1+\gamma_2)b+c) - (\theta+\gamma\theta(N-U^*))] + \sqrt{\Delta} - 2b\theta\right]}{(\theta+\gamma)\left[[(\theta+\gamma)((\gamma_1+\gamma_2)b+c) - (\theta+\gamma\theta(N-U^*))] + \sqrt{\Delta}\right]},$$

$$R^* = \frac{2b\theta\gamma\Lambda(\mathscr{R}_0-1)}{\theta\left[\big((\theta+\gamma)\big((\gamma_1+\gamma_2)b+c\big) - \big(\theta+\gamma\theta(N-U^*)\big)\big) + \sqrt{\Delta}\right]}$$
$$\times\left[\gamma_1 + \frac{c\left[\big((\theta+\gamma)\big((\gamma_1+\gamma_2)b+c\big) - \big(\theta+\gamma\theta(N-U^*)\big)\big) + \sqrt{\Delta}\right]}{b\left[\big((\theta+\gamma)\big((\gamma_1+\gamma_2)b+c\big) - \big(\theta+\gamma\theta(N-U^*)\big)\big) + \sqrt{\Delta}\right] + 2b\theta\gamma\Lambda(\mathscr{R}_0-1)}\right],$$

and

$$U^* = \frac{(\mu+\gamma)E^*}{\beta D^*}.$$

□

2.2. **Stability Analysis of the Model.**

2.2.1. *Local stability.*

Theorem 2.3. *If $\mathscr{R}_0 < 1$, the risk-free equilibrium of distress $\mathscr{P}_0$ in dynamic system (1.1) is locally asymptotically stable for all $t \geq 0$, and this equilibrium is unstable in the other case.*

Proof. Calculating the Jacobian matrix based on system (2.3), we have

$$J = \begin{pmatrix} -\beta D - \theta & \mu - \theta & -\beta U - \theta & 0 \\ \beta D & -(\mu+\gamma) & \beta U & 0 \\ 0 & \gamma & -(\gamma_1+\gamma_2) - \frac{cb}{(b+D)^2} & 0 \\ 0 & 0 & \gamma_1 + \frac{cb}{(b+D)^2} & -\theta \end{pmatrix}.$$

The characteristic polynomial associated with the risk-free equilibrium is given by

$$\chi_J(X) = \begin{vmatrix} X+\beta D+\theta & \theta-\mu & \beta U+\theta & 0 \\ -\beta D & X+(\mu+\gamma) & -\beta U & 0 \\ 0 & -\gamma & X+(\gamma_1+\gamma_2)+\frac{cb}{(b+D)^2} & 0 \\ 0 & 0 & -\gamma_1 - \frac{cb}{(b+D)^2} & X+\theta \end{vmatrix}.$$

By replacing $D = 0$ and $U = N$, we find

$$\chi_{\mathscr{P}_0}(X) = \begin{vmatrix} X+\theta & \theta-\mu & \beta N+\theta & 0 \\ 0 & X+(\mu+\gamma) & -\beta N & 0 \\ 0 & -\gamma & X+(\gamma_1+\gamma_2)+\frac{c}{b} & 0 \\ 0 & 0 & -\gamma_1-\frac{c}{b} & X+\theta \end{vmatrix},$$

where

$$\chi_{\mathscr{P}_0}(X) = (X+\theta)^2\left[X^2 + (\mu+\gamma+\gamma_1+\gamma_2+\frac{c}{b})X + (\mu+\gamma)(\gamma_1+\gamma_2+\frac{c}{b}) - \gamma\beta N\right].$$

Note that $X_0 = -\theta$ is a root of the polynomial $(X+\theta)^2$ which is negative, and the polynomial $X^2 + (\mu+\gamma+\gamma_1+\gamma_2+\frac{c}{b})X + (\mu+\gamma)(\gamma_1+\gamma_2+\frac{c}{b}) - \gamma\beta N$ admits two roots, X_1 and X_2 that satisfy

$$\begin{cases} X_1 + X_2 = -(\mu+\gamma+\gamma_1+\gamma_2+\frac{c}{b}), \\ X_1 X_2 = (\mu+\gamma)(\gamma_1+\gamma_2+\frac{c}{b}) - \gamma\beta N. \end{cases}$$

In view of $(\mu+\gamma+\gamma_1+\gamma_2+\frac{c}{b}) > 0$, for the polynomial $\chi_{\mathscr{P}_0}$ to be the Routh-Hurwitz (see [31]) should be two negative roots, $(\mu+\gamma)(\gamma_1+\gamma_2+\frac{c}{b}) - \gamma\beta N > 0$. Therefore $\mathscr{P}_0$ is locally asymptotically stable if $\mathscr{R}_0 < 1$. □

Theorem 2.4. *If $\mathscr{R}_0 > 1$, then the equilibrium of persistence of distress $\mathscr{P}_*$ in dynamic system (1.1) is locally asymptotically stable for all $t \geq 0$, and it is unstable in the other case.*

Proof. The characteristic polynomial associated with the risk-Persistence Equilibrium $\mathscr{P}_*$ is

$$\begin{aligned}
\chi_{\mathscr{P}_*}(X) &= \begin{vmatrix} X+\beta D^*+\theta & \theta-\mu & \beta U^*+\theta & 0 \\ -\beta D^* & X+(\mu+\gamma) & -\beta U^* & 0 \\ 0 & -\gamma & X+(\gamma_1+\gamma_2)+\frac{cb}{(b+D^*)^2} & 0 \\ 0 & 0 & -\gamma_1-\frac{cb}{(b+D^*)^2} & X+\theta \end{vmatrix} \\
&= (X+\theta)\begin{vmatrix} X+\beta D^*+\theta & \theta-\mu & \beta U^*+\theta \\ -\beta D^* & X+(\mu+\gamma) & -\beta U^* \\ 0 & -\gamma & X+(\gamma_1+\gamma_2)+\frac{cb}{(b+D^*)^2} \end{vmatrix} \\
&= (X+\theta)\chi_{\mathscr{A}}(X),
\end{aligned}$$

where

$$\mathscr{A} = \begin{pmatrix} -\beta D^*-\theta & \mu-\theta & -\beta U^*-\theta \\ \beta D^* & -(\gamma+\mu) & \beta U^* \\ 0 & \gamma & -(\gamma_1+\gamma_2)-\frac{cb}{(b+D^*)^2} \end{pmatrix}.$$

Calculating $\chi_{\mathscr{A}}$ (the characteristic polynomial of matrix $\mathscr{A}$), we have $\chi_{\mathscr{A}}(X) = X^3 + a_1X^2 + a_2X + a_3$, where

$$a_1 = -\mathrm{Tr}(\mathscr{A}), \quad a_2 = \frac{1}{2}\left(\mathrm{Tr}(\mathscr{A})^2 - \mathrm{Tr}(\mathscr{A}^2)\right), \quad a_3 = -\det(\mathscr{A}) \text{ (see [32]).}$$

By calculating these values, we find

$$\begin{aligned}
a_1 &= \beta D^* + \theta + \gamma + \mu + \gamma_1 + \gamma_2 + \frac{cb}{(b+D^*)^2}, \\
a_2 &= (\beta D^*+\theta)(\gamma+\mu) + (\beta D^*+\theta+\gamma+\mu)\left(\gamma_1+\gamma_2+\frac{cb}{(b+D^*)^2}\right) \\
&\quad + \beta D^*(\mu-\theta) + \gamma\beta U^*, \\
a_3 &= \left[\frac{(\beta D^*+\theta)(\mu+\gamma)+(\mu-\theta)\beta D^*}{(\beta D^*)}\right]^2 \frac{(\beta D^*+\theta)}{\gamma}(\gamma_1+\gamma_2+\frac{cb}{(b+D^*)^2}).
\end{aligned}$$

Then $a_1 > 0$ and $a_3 > 0$, by the Routh-Hurwitz criteria, $a_1a_2 > a_3$ verified if $\mathscr{R}_0 \geq 1$. Finally, it follows that $\chi_{\mathscr{P}_*}$ admits 4 roots X_i such that $\mathrm{Re}(X_i) < 0$ for all $i = 1,2,3,4$, which completes the proof. □

$\mathrm{Tr}(\mathscr{A})$: trace of matrix $\mathscr{A}$, defined as the sum of the elements on the main diagonal.

2.2.2. *Global stability.* In this section, we study the global stability of the two equilibria. Let us first recall the following lemma.

Lemma 2.1. [33] *Let $D \subset \mathbb{R}^n$ be a compact set containing an equilibrium point X^*, and consider the dynamical system $\dot{X} = f(X)$, defined on D. Suppose that there exists a Lyapunov function $V : D \to \mathbb{R}$ such that*

(1) $V(X) > 0 \quad \forall X \in D \setminus \{X^*\}$, $V(X^*) = 0$,
(2) $\dot{V}(X) = \langle \nabla V(X), f(X) \rangle < 0$, $\forall X \in D \setminus \{X^*\}$,
(3) Any trajectory starting in D remains in D. (D is invariant).

Then the equilibrium point X^* is globally asymptotically stable on D.

Theorem 2.5. *If the basic reproduction number is $\mathscr{R}_0 < 1$, then the risk-free equilibrium of distress $\mathscr{P}_0$ in dynamic system (1.1) is globally asymptotically stable for all $t \geq 0$, and this equilibrium is unstable in the other case.*

Proof. Define a function $V : \mathscr{J} \longrightarrow \mathbb{R}$ by $V(X) = a_1 E + a_2 D + a_3 \int_0^D \frac{cA}{b+A}\, dA$, where $X = (U,E,D,R,L)^T$ and $\mathscr{J}$ the region of the solution is defined in 2.1. We have $V(\mathscr{P}_*) = 0$ and

$$\begin{aligned}\dot{V}(X) &= \left\langle \begin{pmatrix} 0 \\ a_1 \\ a_2 + a_3 \frac{cD}{b+D} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\beta U D + \mu E + \theta R \\ \beta U D - (\mu+\gamma) E \\ \gamma E - (\gamma_1+\gamma_2) D - \frac{cD}{b+D} \\ \gamma_1 D - \theta R + \frac{cD}{b+D} \\ \gamma_2 D \end{pmatrix} \right\rangle \\ &\leq a_1 \beta D \Lambda - a_1(\mu+\gamma)E + a_2 \gamma E - a_2 \left((\gamma_1+\gamma_2) D + \frac{cD}{b+D} \right) \\ &\quad + a_3 \frac{cD}{b+D} \left(\gamma E - (\gamma_1+\gamma_2) D - \frac{cD}{b+D} \right)\end{aligned}$$

By posing $a_2 = \frac{(\mu+\gamma)a_1}{\gamma}$, and $a_3 = \frac{a_2}{bc}$, with $a_1 \geq 0$, $(V(X) > 0, \forall X \in \mathscr{J} \setminus \{\mathscr{P}_0\})$, we find

$$V(\dot{X}) \leq a_1 D \left(\beta \Lambda - \frac{(\gamma+\mu)(\gamma_1+\gamma_2+\frac{c}{b})}{\gamma} \right) + \frac{a_1(\mu+\gamma)}{\gamma b c} \left(\gamma E - (\gamma_1+\gamma_2) D \right).$$

That is true for all $X \in \mathscr{J} \setminus \{\mathscr{P}_0\}$. Taking $E = 0$, one has $\dot{V}(X) < 0$ if $\mathscr{R}_0 < 1$. By Lemma 2.1, we conclude the result immediately. ☐

Theorem 2.6. *The persistence equilibrium of distress $\mathscr{P}_*$ of system (1.1) is globally asymptotically stable for all $t \geq 0$.*

Proof. Let $W : \mathscr{J} \longrightarrow \mathbb{R}$ be a function defined as

$$W(X) = \left(U - U^* - U^* \ln\left(\frac{U}{U^*}\right) \right) + \left(E - E^* - E^* \ln\left(\frac{E}{E^*}\right) \right) + \left(D - D^* - D^* \ln\left(\frac{D}{D^*}\right) \right),$$

where $X = (U,E,D,R,L)^T$. First, We have $W(U^*,E^*,D^*,R^*,L^*) = W(\mathscr{P}_*) = 0$. Let us now verify the positivity of W. Note that, fora ll $x > 0$, $x \geq 1 + \ln(x)$. By replacing x with $\frac{U}{U^*}$, we find $U - U^* - U^* \ln\left(\frac{U}{U^*}\right) \geq 0$. Finally, by the same argument, we show that $W(X) > 0$, for all

$X \in \mathscr{J} \setminus \{\mathscr{P}_*\}$. Calculate $\dot{W}$ by

$$\dot{W}(X) = \left\langle \begin{pmatrix} 1-\frac{U^*}{U} \\ 1-\frac{E^*}{E} \\ 1-\frac{D^*}{D} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\beta UD+\mu E+\theta R \\ \beta UD-(\mu+\gamma)E \\ \gamma E-(\gamma_1+\gamma_2)D-\frac{cD}{b+D} \\ \gamma_1 D-\theta R+\frac{cD}{b+D} \\ \gamma_2 D \end{pmatrix} \right\rangle$$
$$= (-\beta UD+\mu E+\theta R) - \frac{U^*}{U}(-\beta UD+\mu E+\theta R)$$
$$+ (\beta UD-(\mu+\gamma)E) - \frac{E^*}{E}(\beta UD-(\mu+\gamma)E)$$
$$+ \left(\gamma E-(\gamma_1+\gamma_2)D-\frac{cD}{b+D}\right) - \frac{D^*}{D}\left(\gamma E-(\gamma_1+\gamma_2)D-\frac{cD}{b+D}\right).$$

Summing the term of system (2.3) with D (no D^*), we have $\beta U^* D = (\gamma_1+\gamma_2)D + \frac{cD}{b+D}$, which implies $\beta U^* = (\gamma_1+\gamma_2) + \frac{c}{b+D}$. Thus

$$\dot{W}(X) = \theta R - \frac{U^*}{U}(\mu E+\theta R) - \frac{E^*}{E}(\beta UD-(\mu+\gamma)) - \frac{D^*}{D}\left(\gamma E-(\gamma_1+\gamma_2)D-\frac{cD}{b+D}\right).$$

On the other hand,

$$\dot{W}(X) = \theta R - \frac{U^*}{U}\left(\frac{(\beta U^*D^*-\theta R^*)}{E^*}E+\theta R\right) - \frac{E^*}{E}\beta UD + \frac{E^*}{E}(\mu+\gamma) - \gamma E\frac{D^*}{D}$$
$$+ (\gamma_1+\gamma_2+\frac{c}{b+D})D^*$$
$$= \theta R - \frac{\beta U^{*2}D^*E}{UE^*} + \frac{\theta R^*U^*E}{UE^*} - \frac{\theta RU^*}{U} - \frac{\beta UDE^*}{E} + \beta U^*D^* - \gamma E\frac{D^*}{D} + \beta U^*D^*$$
$$= \beta U^*D^*\left(2-\frac{U^*E}{UE^*}-\frac{UDE^*}{U^*D^*E}\right) + \theta R + \frac{\theta R^*U^*E}{UE^*} - \frac{\theta U^*R}{U} - \gamma E\frac{D^*}{D}$$
$$= \beta U^*D^*\left(2-\frac{U^*E}{UE^*}-\frac{UDE^*}{U^*D^*E}\right) + \theta R\left(\frac{U-U^*}{U}\right) + \gamma E\left(\frac{U^*D-UD^*}{UD}\right).$$

Let us denote

$$b_1 = \frac{U^*E}{UE^*} \quad \text{and} \quad b_2 = \frac{UDE^*}{U^*D^*E}.$$

If $D = D^*$, then, $b_1 b_2 = 1$. By using the fact that $\frac{b_1+b_2}{2} \geq \sqrt{b_1 b_2}$, we find $b_1+b_2 \geq 2$. Finally,

$$\dot{W}(X) = \beta U^*D^*(2-(b_1+b_2)) + \theta R\left(\frac{U-U^*}{U}\right) + \gamma E\left(\frac{U^*D-UD^*}{UD}\right).$$

For $U = U^*, \dot{W} < 0$ and for $\begin{cases} U = U^*, \\ D = D^*, \\ E = E^*, \end{cases}$ $\dot{W} = 0$, which completes the proof. $\square$

2.3. **Sensitivity analysis.** In the following analysis of the sensitivity index, we examine the impact of parameters b and c, as well as the effects of interventions or measures on our model. Additionally, this analysis provides important information on how these parameters can influence $\mathscr{R}_0$, allowing us to understand how to use them effectively for optimal results.

Definition 2.1. ([34]) The normalised sensitivity index is calculated for the variable $\mathscr{R}_0$ with respect to the parameter ρ. This index is given by $\mathscr{C}_\rho^{\mathscr{R}_0} = \frac{\partial \mathscr{R}_0}{\partial \rho} \times \frac{\rho}{\mathscr{R}_0}$, where $\mathscr{R}_0$ is the variable analysed and ρ is the parameter.

Parameter	**Sensitivity Index**
γ	$\frac{\mu}{\mu+\gamma}$
μ	$\frac{-\mu}{\mu+\gamma}$
γ_1	$\frac{-\gamma_1}{\gamma_1+\gamma_2+\frac{c}{b}}$
γ_2	$\frac{-\gamma_2}{\gamma_1+\gamma_2+\frac{c}{b}}$
c	$\frac{-c}{b(\gamma_1+\gamma_2+\frac{c}{b})}$
b	$\frac{c}{b(\gamma_1+\gamma_2+\frac{c}{b})}$
β	1

TABLE 1. Sensitivity index of $\mathscr{R}_0$.

3. OPTIMAL CONTROL STRATEGY

In Section 2 (and Section 5.1), we describe the evaluation and measurement process for risk transmission, identifying key stages which, in crisis situations, can act as alarms. In this section, we aim to address our problem with real recovery possibilities based on mathematics and, in particular, optimal control theory. This is about intervention to stop the systemic risk from continuing with preventive actions that are effective and at a lower cost. These approaches are especially suited for disease modelling, which is illustrated by the SEIR model. The basic idea is to recast the model to include a strong control pair to move our system from an initial state to a defined final state, where we can succeed to minimise this risk. Our dynamic system (1.1) after amelioration is given as

$$\begin{cases} \frac{dU}{dt} = -\beta(1-u_1(t))UD + \mu E + \theta R, \\ \frac{dE}{dt} = \beta(1-u_1(t))UD - (\mu+\gamma)E, \\ \frac{dD}{dt} = \gamma E - (\gamma_1+\gamma_2+u_2(t))D - \frac{cD}{b+D}, \\ \frac{dR}{dt} = (\gamma_1+u_2(t))D - \theta R + \frac{cD}{b+D}, \\ \frac{dL}{dt} = \gamma_2 D. \end{cases} \tag{3.1}$$

with the initial conditions (1.2) satisfied . So our admissible control set is

$$\mathbb{U}=\left\{(u_1(.),u_2(.))\in L^2(0,T)\times L^2(0,T)\mid 0\le u_1,u_2\le 1,\quad t\in[0,T]\right\},$$

where T is the period designated for the control operation. Now let us define the objective function as

$$J(u_1,u_2)=\int_0^T W_1E(t)+W_2D(t)+W_3\frac{u_1(t)^2}{2}+W_4\frac{u_2(t)^2}{2}\,dt.$$

where
-W_1,W_2 weight costs associated with controlling financial systemic network connection ,
-W_3,W_4 Weight costs related to establishing measures aimed at treating or recovering distressed banks, which can include intervention or financial support. Then we have to determine the optimal controls (u_1^*,u_2^*) such that

$$\begin{cases}J(u_1^*,u_2^*)=\min_{(u_1,u_2)\in\mathbb{U}}J(u_1,u_2),\\ \text{subject to (3.1)}.\end{cases}\tag{3.2}$$

Theorem 3.1. *Optimal control problem* (3.2) *admits a solution.*

Proof. To prove this, we have to demonstrate
$\mathscr{P}_1$: The set of admissible controls and their corresponding state solutions must be nonempty.
$\mathscr{P}_2$: The admissible control set U is convex and closed in $L^2(0,T)\times L^2(0,T)$.
$\mathscr{P}_3$: The state system's vector field is bounded by a linear control function.
$\mathscr{P}_4$: The integral in the objective function denoted by $\mathscr{L}(X,u_1,u_2)$ is required to be convex on $\mathbb{U}$. Verify the existence of constants $\xi_1>0$, $\xi_2>0$, $j>1$ such that $\mathscr{L}(X,u_1,u_2)>\xi_1+\xi_2(|u_1|^2+|u_2|^2)^{\frac{j}{2}}$. Now, taking two functions f,g such that $f(t)=g(t)=\frac{1}{2}$ for all t in $[0,T]$, one sees that (f,g) is in the admissible set $\mathbb{U}$ which ensures that $\mathbb{U}$ is not empty. Let $X(t)=(U(t),E(t),D(t),R(t),L(t))^T$. Our system with control (3.1) can be written as

$$\begin{cases}\dot{X}(t)=F(X(t)),t\ge 0,\\ X(0)=X_0,\end{cases}$$

where $F(X(t))=F(X(t),Z(t))$ for $Z(t)=(u_1(t),u_2(t))^T$, and $F(X(t))=AX(t)+K(X(t))$, with

$$A=\begin{pmatrix}0 & \mu & 0 & \theta & 0\\ 0 & -(\mu+\gamma) & 0 & 0 & 0\\ 0 & \gamma & -(\gamma_1+\gamma_2+u_2(t)) & 0 & 0\\ 0 & 0 & (\gamma_1+u_2(t)) & -\theta & 0\\ 0 & 0 & \gamma_2 & 0 & 0\end{pmatrix}$$

and

$$K(X(t))=\begin{pmatrix}-\beta(1-u_1(t))U(t)D(t)\\ \beta(1-u_1(t))U(t)D(t)\\ -\frac{cD(t)}{b+D(t)}\\ \frac{cD(t)}{b+D(t)}\\ 0\end{pmatrix}.$$

Let $X_1, X_2 \in \mathbb{R}^5_+$. Then

$$\begin{aligned}
&|F(X_1)-F(X_2)| \\
&\leq \|A\|\,|X_1-X_2| + 2\left(\beta\,|U_1(t)D_1(t)-U_2(t)D_2(t)| + \left|\frac{cD_1(t)}{b+D_1(t)} - \frac{cD_2(t)}{b+D_2(t)}\right|\right) \\
&\leq \|A\|\,|X_1-X_2| + 2\left(\beta\left(|D_1(t)|\,|U_1(t)-U_2(t)| + |U_2(t)|\,|D_1(t)-D_2(t)|\right)\right. \\
&\quad \left. + \frac{bc\,|D_1(t)-D_2(t)|}{(b+D_1(t))(b+D_2(t))}\right) \\
&\leq \|A\|\,|X_1-X_2| + 2\left(\beta \max\left(|D_1(t)|,|U_2(t)|\right)\left(|U_1(t)-U_2(t)| + |D_1(t)-D_2(t)|\right)\right. \\
&\quad \left. + \frac{c}{b}\,|D_1(t)-D_2(t)|\right).
\end{aligned}$$

Let $M = 2\beta \max\left(|D_1(t)|,|U_2(t)|\right)$ and $M' = \max\left(M + \frac{2c}{b}, \|A\|\right)$. Finally, by simple verification, we obtain $|F(X_1)-F(X_2)| \leq M'\,|X_1-X_2|$. Hence, F is continuous and uniformly Lipschitz on $\mathbb{R}^5_+$. Thus this system admits a unique solution. This completes the demonstration of property $(\mathscr{P}_1)$. Let $(u,v),(u',v') \in \mathbb{U}$ and $\lambda \in [0,1]$. Then $\lambda(u,v)+(1-\lambda)(u',v') = (\lambda u + (1-\lambda)u',\ \lambda v + (1-\lambda)v')$. By using the fact that $L^2(0,T)$ is a vector space, we have $\lambda u + (1-\lambda)u' \in L^2(0,T)$.

On the other hand, $0 \leq \lambda u + (1-\lambda)u' \leq 1$. Then, $\lambda(u,v)+(1-\lambda)(u',v') \in \mathbb{U}$, which ensures the convexity of $\mathbb{U}$. Let us now demonstrate that $\mathbb{U}$ is closed. Let $((u_1)_n,(u_2)_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathbb{U}$ such that $((u_1)_n,(u_2)_n) \to (u_1,u_2)$ in $L^2(0,T)\times L^2(0,T)$. Then there exists a subsequence $(((u_1)_{n_k},(u_2)_{n_k}))_{k\in\mathbb{N}}$ of $((u_1)_n,(u_2)_n)_{n\in\mathbb{N}}$ such that $(((u_1(t))_{n_k},(u_2(t))_{n_k})) \to (u_1(t),u_2(t))$ almost everywhere. Note that, for all $n \in \mathbb{N}$, $0 \leq (u_1)_n,(u_2)_n \leq 1$ almost everywhere. There exists $\omega \subset [0,T]$, $\mu(\omega)=0$, where μ represents the Borel measure (see [35]), such that, for all $t \in [0,T]\setminus\omega$, $0 \leq u_{1,n_k}(t),\ u_{2,n_k}(t) \leq 1$. Passing to the limit gives $0 \leq u_1,u_2 \leq 1$ almost everywhere. Since $L^2(0,T)\times L^2(0,T)$ is closed, $(u_1,u_2) \in \mathbb{U}$. Hence $\mathbb{U}$ is closed, which completes the property $(\mathscr{P}_2)$. Using the same approach, one sees the boundlessness. Thus, the vector of the state system is still bounded by the sum of bounded state variables and input controls. Hence, $(\mathscr{P}_3)$. By the expression of the integral function,

$$\mathscr{L}(X,u_1,u_2) = W_1E(t) + W_2D(t) + W_3\frac{u_1(t)^2}{2} + W_4\frac{u_2(t)^2}{2},$$

we have $\mathscr{H}_{u_1,u_2} = \begin{pmatrix} W_3 & 0 \\ 0 & W_4 \end{pmatrix}$, where $\mathscr{H}_{u_1,u_2}$ is the Hessian matrix corresponds to $\mathscr{L}$ on $\mathbb{U}$, that is, positive definite. Hence $\mathscr{L}$ is strictly convex on $\mathbb{U}$. On the other hand, we have

$$\mathscr{L}(X,u_1,u_2) \geq \inf_{t\in[0,T]}\left(W_1E(t)+W_2D(t)\right) + \frac{1}{2}\min(W_3,W_4)\left(u_1(t)^2+u_2(t)^2\right).$$

Hence the desired inequality with $\xi_1 = \inf_{t\in[0,T]}\left(W_1E(t)+W_2D(t)\right)$ and $\xi_2 = \frac{1}{2}\min(W_3,W_4)$, $j = 2$ completes the demonstration of $(\mathscr{P}_4)$. According to the Filippov-Cesari theorem [36], optimal control problem (3.2) admits a solution. □

Now, we apply a principle known as Pontryagin's Maximum Principle [37] by defining the following function:

$$\mathscr{H}(U,E,D,R,L,\lambda_i,u)_{i=1\dots5} = \mathscr{L} + \lambda_1\frac{dU}{dt} + \lambda_2\frac{dE}{dt} + \lambda_3\frac{dD}{dt} + \lambda_4\frac{dR}{dt} + \lambda_5\frac{dL}{dt}.$$

This function is called the Hamiltonian function, where $\mathscr{L}$ is the Lagrangian, defined as follows:

$$\mathscr{L}(E,D,u_1,u_2) = W_1E(t) + W_2D(t) + W_3\frac{u_1(t)^2}{2} + W_4\frac{u_2(t)^2}{2},$$

where $\lambda(\cdot)_i, \quad i = 1,\dots,5$ are the adjoint variables verifying:

$$\frac{d\lambda_1(t)}{dt} = -\frac{\partial\mathscr{H}}{\partial U},\ \frac{d\lambda_2(t)}{dt} = -\frac{\partial\mathscr{H}}{\partial E},\ \frac{d\lambda_3(t)}{dt} = -\frac{\partial\mathscr{H}}{\partial D},\ \frac{d\lambda_4(t)}{dt} = -\frac{\partial\mathscr{H}}{\partial R},\ \frac{d\lambda_5(t)}{dt} = -\frac{\partial\mathscr{H}}{\partial L}.$$

Thus

$$\begin{cases} \frac{d\lambda_1(t)}{dt} = D(\beta - u_1(t))(\lambda_1(t) - \lambda_2(t)), \\ \frac{d\lambda_2(t)}{dt} = \gamma(\lambda_2(t) - \lambda_3(t)) + \mu(\lambda_2(t) - \lambda_1(t)) - W_1, \\ \frac{d\lambda_3(t)}{dt} = (\beta - u_1(t))U(\lambda_1(t) - \lambda_2(t)) + \gamma_2(\lambda_3(t) - \lambda_5(t)) \\ \qquad\qquad +(\lambda_3(t) - \lambda_4(t))\left(\gamma_1 + u_2(t) + \frac{cb}{(b+D)^2}\right) - W_2, \\ \frac{d\lambda_4(t)}{dt} = \theta(\lambda_4(t) - \lambda_1(t)), \\ \frac{d\lambda_5(t)}{dt} = 0. \end{cases} \tag{3.3}$$

with the transversality conditions $\lambda(T)_i = 0, \quad i = 1,\dots,5$. Regarding the control equations, their expressions are satisfying:

$$\frac{dU(t)}{dt} = \frac{\partial\mathscr{H}}{\partial\lambda_1},\ \frac{dE(t)}{dt} = \frac{\partial\mathscr{H}}{\partial\lambda_2},\ \frac{dD(t)}{dt} = \frac{\partial\mathscr{H}}{\partial\lambda_3},\ \frac{dR(t)}{dt} = \frac{\partial\mathscr{H}}{\partial\lambda_4},\ \frac{dL(t)}{dt} = \frac{\partial\mathscr{H}}{\partial\lambda_5}.$$

Theorem 3.2. *([37]) Consider control problems (3.1)- (3.2). Let $(U^*_{op}, E^*_{op}, D^*_{op}, R^*_{op}, L^*_{op})$ be an optimal solution, and $\lambda^*_{i\ op}, i = 1,\dots,5$ be the solution of (3.3). Then the optimal control that solves our problem is given by*

$$\begin{cases} u^*_{1op} = \max\left(0, \min\left(u^*_1, 1\right)\right), \\ u^*_{2op} = \max\left(0, \min\left(u^*_2, 1\right)\right), \end{cases}$$

where

$$\begin{cases} u^*_1 = \frac{U^*_{op}D^*_{op}(\lambda_2-\lambda_1)}{W_3}, \\ u^*_2 = \frac{D^*_{op}(\lambda_3-\lambda_4)}{W_4}. \end{cases}$$

Proof. Note that

$$\begin{cases} \left.\frac{\partial\mathscr{H}}{\partial u_1}\right|_{u=u^*_1} = u^*_1W_3 + \beta U^*_{op}D^*_{op}(\lambda_1 - \lambda_2) = 0, \\ \left.\frac{\partial\mathscr{H}}{\partial u_2}\right|_{u=u^*_2} = u^*_2W_4 + D^*_{op}(\lambda_4 - \lambda_3) = 0. \end{cases}$$

By solving this system, the control solutions u_1^* and u_2^* can be written as

$$\begin{cases} u_{1op}^* = \begin{cases} 0, u_1^* < 0, \\ \frac{U_{op}^* D_{op}^* (\lambda_2 - \lambda_1)}{W_3}, 0 \leq u_1^* \leq 1, \\ 1, u_1^* > 1, \end{cases} \\ u_{2op}^* = \begin{cases} 0, u_2^* < 0, \\ \frac{D_{op}^* (\lambda_3 - \lambda_4)}{W_4}, 0 \leq u_2^* \leq 1, \\ 1, u_2^* > 1. \end{cases} \end{cases}$$

By considering the boundedness of the control variable set $\mathbb{U}$, we conclude the proof immediate. □

4. The Numerical Method

In this section, we use a method called Euler explicit to present a numerical solution to our control problem. The state system associated with controlled model (3.1) is solved forward in time by using this method on a uniform temporal grid. Starting from the initial conditions, the state variables are updated at each time step by a forward finite-difference approximation. Once the state trajectories are obtained, the adjoint system is solved backward in time from the prescribed terminal conditions, also using the explicit Euler scheme with a reversed time step. The control is then updated by using the optimality condition derived from the Pontryagin principle. This forward–backward iterative procedure is repeated until convergence of the state, adjoint, and control variables is achieved. Let N be the number of time steps, and consider a discretisation of time as $t_n = nh$, for $n = 0, \ldots, N$. Define the state approximations

$$U(t_n) \approx U_n, \quad E(t_n) \approx E_n, \quad D(t_n) \approx D_n, \quad R(t_n) \approx R_n, \quad L(t_n) \approx L_n,$$

the adjoints approximations

$$\lambda_1(t_n) \approx \lambda_{1,n}, \quad \lambda_2(t_n) \approx \lambda_{2,n}, \quad \lambda_3(t_n) \approx \lambda_{n,3}, \quad \lambda_4(t_n) \approx \lambda_{n,4}, \quad \lambda_5(t_n) \approx \lambda_{n,5},$$

and the controls approximations $u_1(t_n) \approx u_{1,n}$ and $u_2(t_n) \approx u_{2,n}$. The forwards state equations based on Euler explicit is given by

$$\begin{cases} U_{n+1} = U_n + h\big(-\beta(1-u_{1,n})U_n D_n + \mu E_n + \theta R_n\big), \\ E_{n+1} = E_n + h\big(\beta(1-u_{1,n})U_n D_n - (\mu+\gamma)E_n\big), \\ D_{n+1} = D_n + h\Big(\gamma E_n - (\gamma_1 + \gamma_2 + u_{2,n})D_n - \dfrac{cD_n}{b+D_n}\Big), \qquad n = 0 \ldots N-1, \\ R_{n+1} = R_n + h\big((\gamma_1 + u_{2,n})D_n - \theta R_n + \dfrac{cD_n}{b+D_n}\big), \\ L_{n+1} = L_n + h\gamma_2 D_n. \end{cases}$$

Next, we write the backwards adjoint equations

$$\begin{cases} \lambda_{1,n} &= \lambda_{1,n+1} - h\big(D_{n+1}(\beta - u_{1,n+1})(\lambda_{1,n+1} - \lambda_{2,n+1})\big), \\ \lambda_{2,n} &= \lambda_{2,n+1} - h\big(\gamma(\lambda_{2,n+1} - \lambda_{3,n+1}) + \mu(\lambda_{2,n+1} - \lambda_{1,n+1}) - W_1\big), \\ \lambda_{3,n} &= \lambda_{3,n+1} - h\big((\beta - u_{1,n+1})U_{n+1}(\lambda_{1,n+1} - \lambda_{2,n+1}) + \gamma_2(\lambda_{3,n+1} - \lambda_{5,n+1}) \\ & \quad + (\lambda_{3,n+1} - \lambda_{4,n+1})\left(\gamma_1 + u_{2,n+1} + \frac{cb}{(b+D_{n+1})^2}\right) - W_2\big), \quad n = N-1\ldots 0, \\ \lambda_{4,n} &= \lambda_{4,n+1} - h\big(\theta(\lambda_{4,n+1} - \lambda_{1,n+1})\big), \\ \lambda_{5,n} &= \lambda_{1,n+5}. \end{cases}$$

Finally, we obtain the controls updates

$$u_{1,n+1} = \max\left(0, \min\left(\frac{U_n D_n(\lambda_{2,n} - \lambda_{1,n})}{W_3}, 1\right)\right),\ u_{2,n+1} = \max\left(0, \min\left(\frac{D_n(\lambda_{3,n} - \lambda_{4,n})}{W_4}, 1\right)\right).$$

5. Numerical Simulation

5.1. **Asymptotic behaviour.** In this section, we choose a significant number in a banking network, set N=140, with initial conditions given by $U(0) = 50, E(0) = 40, D(0) = 30$ and $R(0) = 20$, and consider the cases described in the following table

Parameter	$\mathscr{R}_0 < 1$		$\mathscr{R}_0 > 1$	
	Case 1	Case 2	Case 1	Case 2
μ	0.2	0.3	0.025	0.025
β	0.002	0.002	0.06	0.05
γ	0.03	0.03	0.02	0.03
γ_1	0.05	0.05	0.02	0.04
γ_2	0.001	0.001	0.0001	0.0001
θ	0.3233	0.0233	0.05	0.06
b	0.17	0.05	0.05	0.09
c	0.004	0.004	0.08	0.06

TABLE 2. Parameter values for various scenarios of $\mathscr{R}_0$

In Figure 2, the bar chart of the parameter indices shows the relative influence of each model parameter on the dynamical system. However, in Figure 3, we outline the substantial roles played by the parameters c and b in this study and their impact on $\mathscr{R}_0$. Moreover, Figure 4 presents a contour plot illustrating how the basic reproduction number varies with respect to γ_1 and γ_2. In addition, Figure 5 provides further insight into the sensitivity of the model by varying the parameters γ and μ. Finally, we examine the role of b and c by combining them with other parameters, as shown in Figure 6 and Figure 7, which also offers insight into their functionality under continuous parameter variations within the model.

The sensitivity indices show that these parameters are some of the most influential parameters in $\mathscr{R}_0$, and the analysis demonstrates how the modification of these parameters can significantly vary the behaviour of the system. This suggests that it is important to monitor c and b closely in order to appreciate our understanding of how they each contribute to the model as well as to maintain control of the dynamics of the system. We will conduct simulations using Python 3.12 on a Windows system AMD Ryzen 7 5700U processor with 12GB of RAM and an integrated AMD Radeon$^{\text{TM}}$ Graphics GPU. The aim is to measure the dynamic risk by varying the parameters c and b.

The following simulations allow us to analyse how changes in these parameters influence the dynamics of the banking system and the propagation of systemic risk within the network. By observing the results, we can gain a better understanding of the interactions between different categories of banks and assess the potential impact of various scenarios on overall financial stability. Figure 8 described at two different values of $\mathscr{R}_0 < 1$; $\mathscr{R}_0 = 0.5$ and $\mathscr{R}_0 = 0.2$, the risk-free equilibrium $\mathscr{P}_0 = (140, 0, 0, 0)$ is globally asymptotically stable, and the solution tends to this equilibrium, as shown in the illustration. In the other case $\mathscr{R}_0 > 1$, Figure 9(a) shows that the risk persistence equilibrium point $\mathscr{P}_* = (0.7, 55.1, 59.7, 22.5)$ in the first case is globally asymptotically stable. For the second case, Figure 9(b) where $\mathscr{R}_0 = 5.4$, the solution converges and $\mathscr{P}_*$ is in this situation $\mathscr{P}_* = (0.67, 41.3, 67.2, 25.2)$. In this step, as illustrated in Figure 10, even when varying the values of b, which means increasing resilience to risk to a high value, the term $\frac{cD}{b+D}$ takes on small values. This suggests that, even as the level of risk increases, the effect of its transmission remains limited. On the other hand, if the values of c are decreased, it indicates that even if defaults transpire, their propagation is constrained. This reinforces the stability of the system, as banks are less likely to be affected by the defaults of others.

Finally, Figure 11 shows that a low c and a high b in the term make the banking system more susceptible to crises.

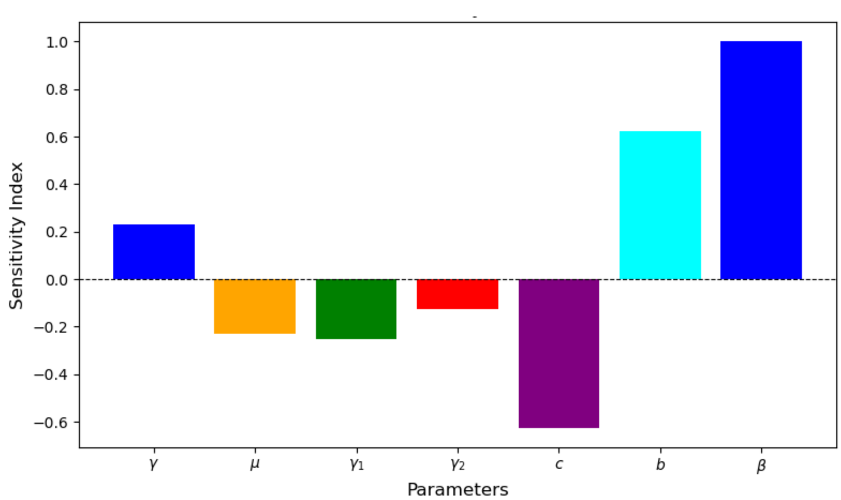


FIGURE 2. Illustration of parameters sensitivity indices associated with $\mathscr{R}_0$.

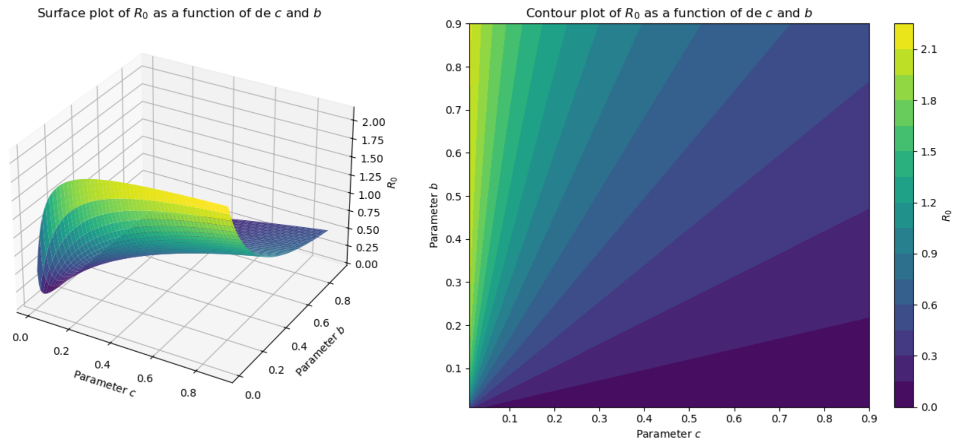


FIGURE 3. Visualization of the basic reproduction number $\mathscr{R}_0$ with respect to b and c.

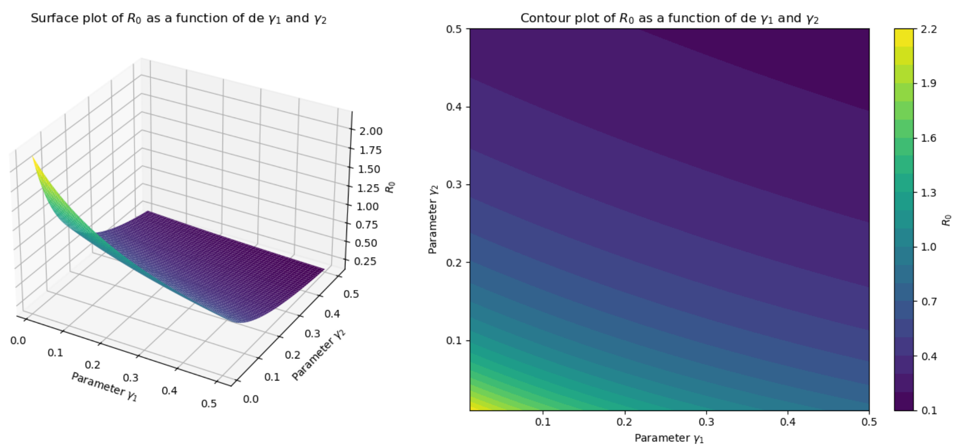


FIGURE 4. Visualization of the basic reproduction number $\mathscr{R}_0$ with respect to γ_1 and γ_2.

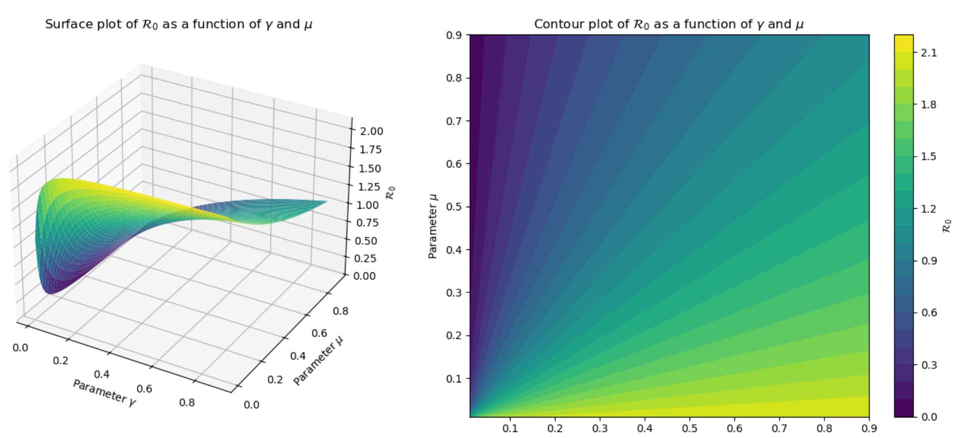


FIGURE 5. Visualization of the basic reproduction number $\mathscr{R}_0$ with respect to γ and μ.

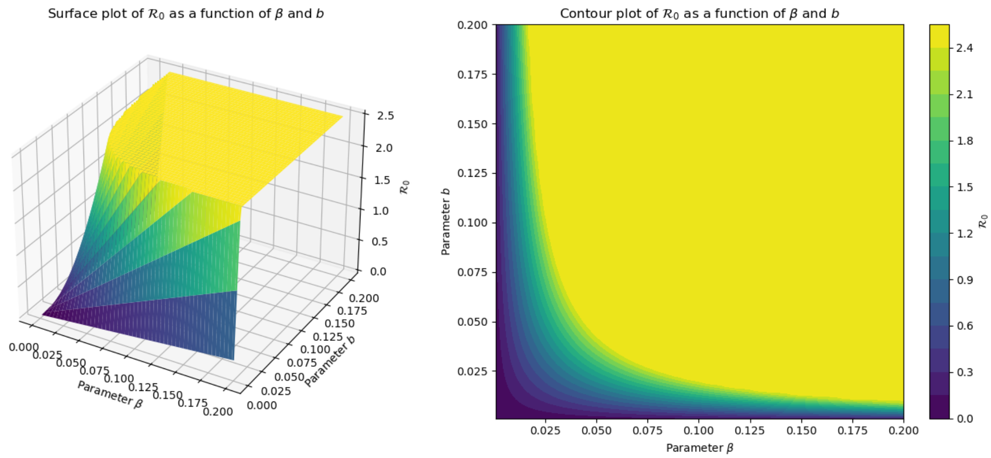


FIGURE 6. Visualization of the basic reproduction number $\mathscr{R}_0$ with respect to β and b.

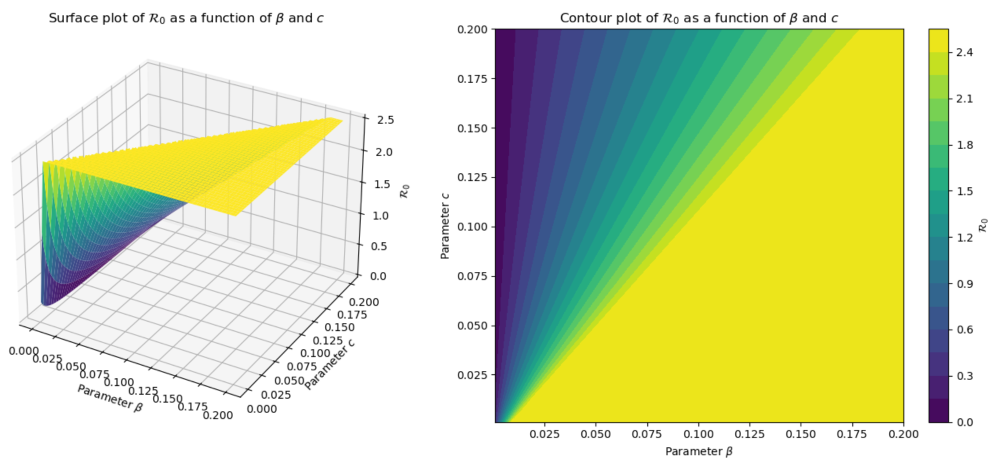
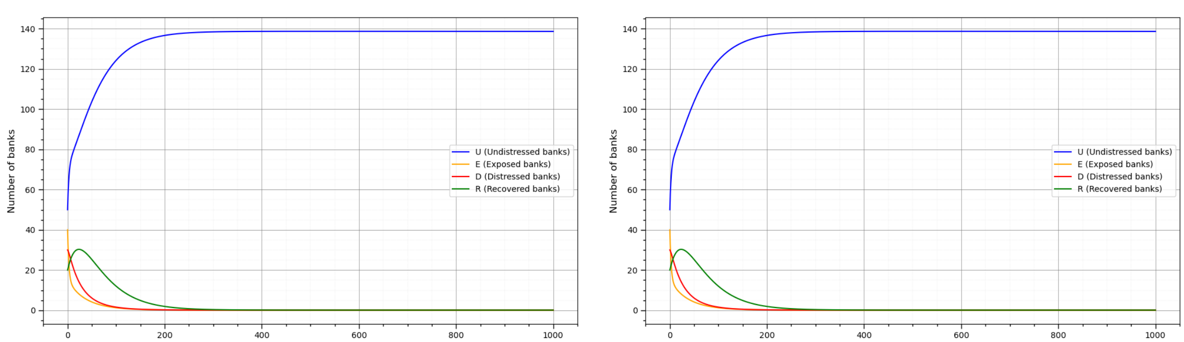
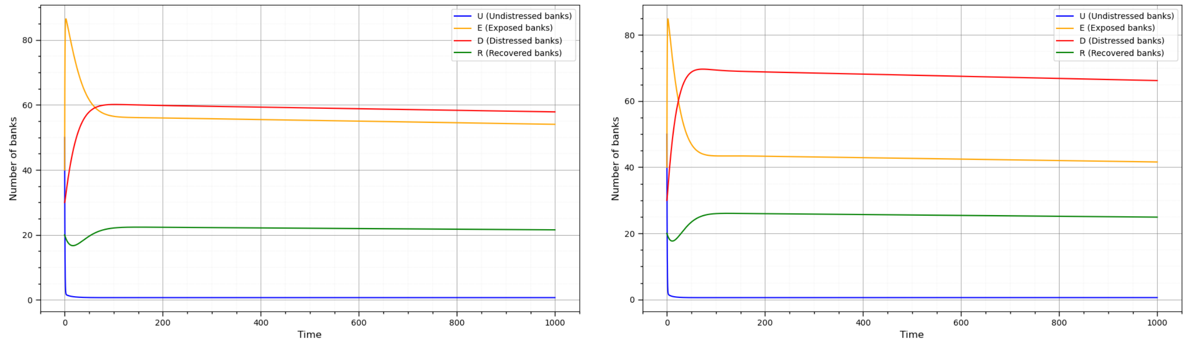


FIGURE 7. Visualization of the basic reproduction number $\mathscr{R}_0$ with respect to β and c.



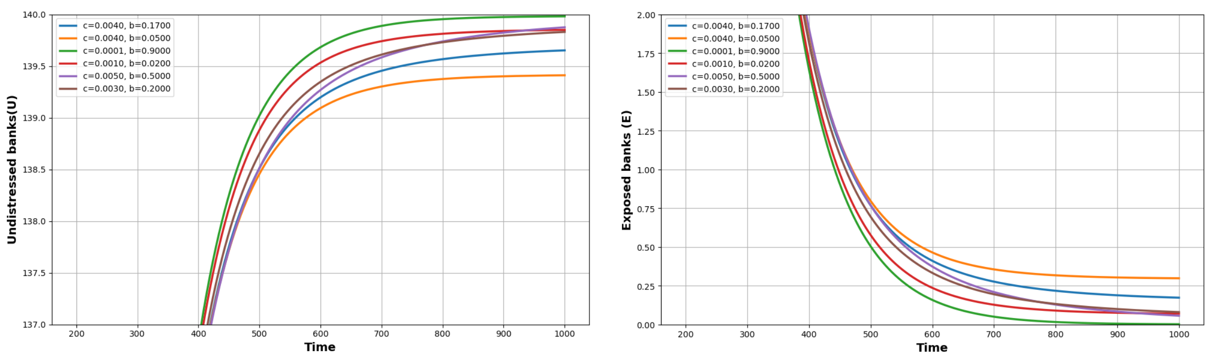
(A) $\mathscr{R}_0 = 0.5$ (B) $\mathscr{R}_0 = 0.2$

FIGURE 8. Illustration of bank network system for $\mathscr{R}_0 < 1$.

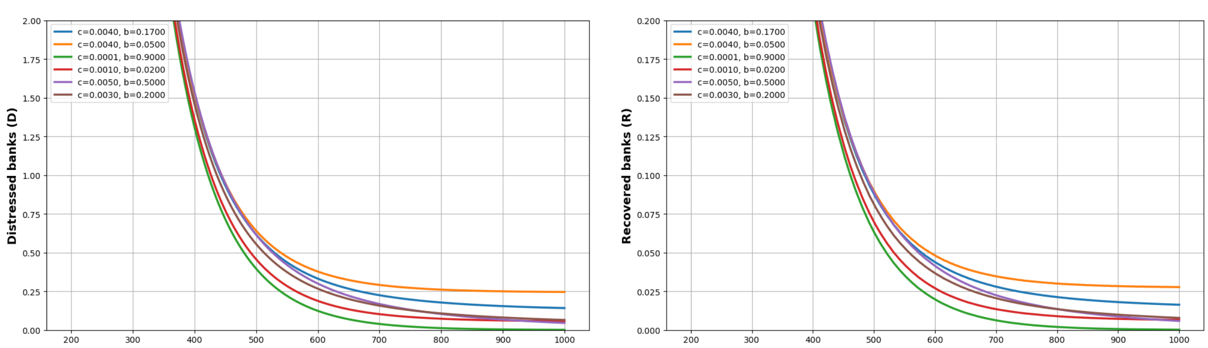


(A) $\mathscr{R}_0 = 2.3$ (B) $\mathscr{R}_0 = 5.4$

FIGURE 9. Illustration of bank network system for $\mathscr{R}_0 > 1$.

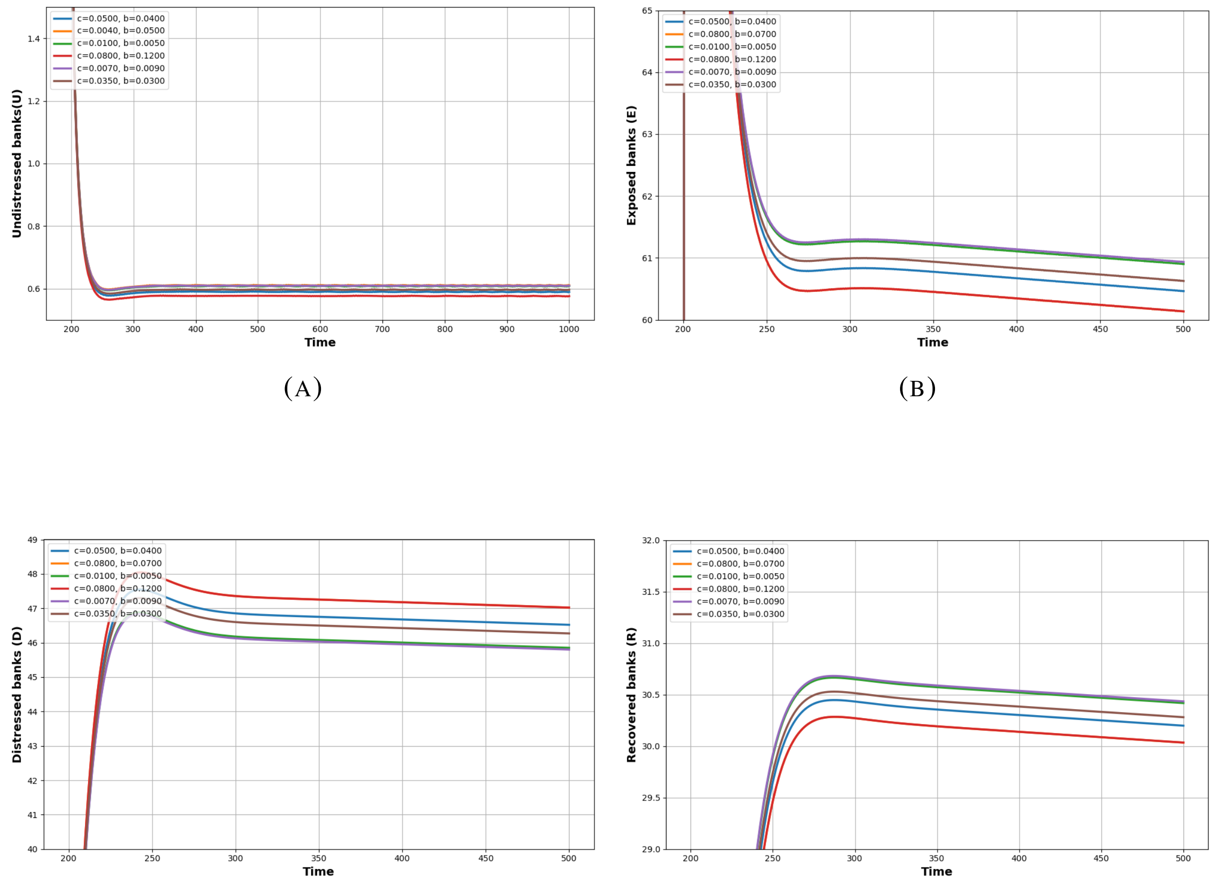


(A) (B)



(C) (D)

FIGURE 10. Capture of each category in the network under evaluation by varying b and c in the case of $\mathscr{R}_0 < 1$



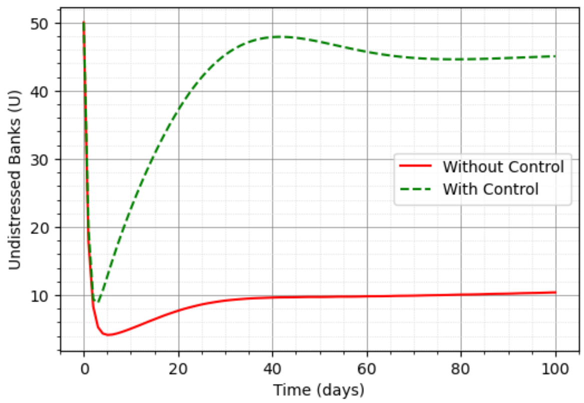
(A) (B)

(C) (D)

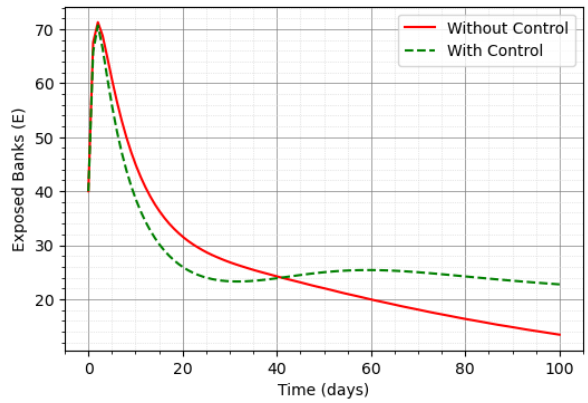
FIGURE 11. Capture of each category in the network under evaluation by varying b and c in the case of $\mathscr{R}_0 > 1$

5.2. **Optimal control simulations.** In this step, we choose the instability-critical measured situation given in the instability case. Using our numerical Euler explicit method, we develop the real-world problem solved in the financial network as obtained in the simulation. Figure 12a shows that our control strategy can optimally curb the distress by stopping the contagion effect between banks and applying the liquidity principle. We succeeded in diminishing the number of distressed banks quickly during the first days of the distress outbreak, as shown in Figure 12c.

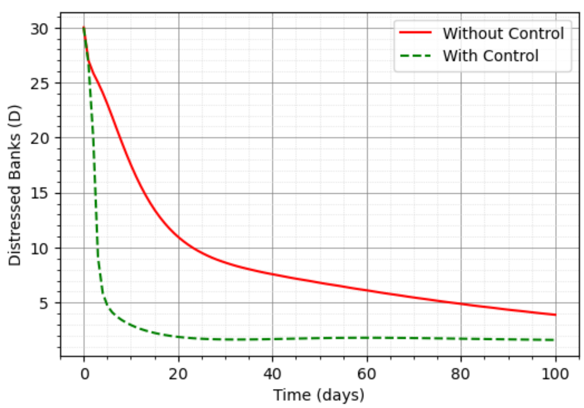
On the other hand, our intervention strategy gives good results in recovering the banks that are in distress, as illustrated in Figure 12d. In Figure 12, we show through these simulations how this strategy can quickly and optimally stop the propagation of systemic risk between the categories of the UEDRL dynamic system.



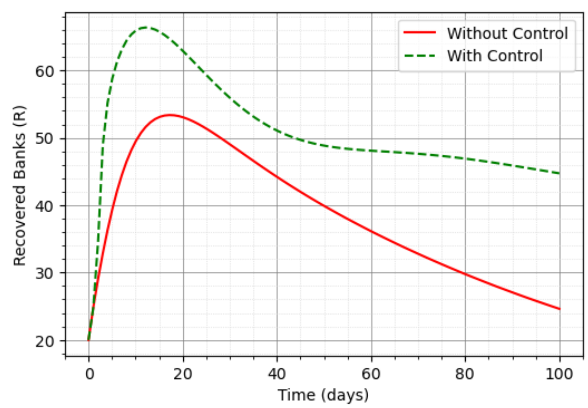
(A)



(B)



(C)



(D)

FIGURE 12. Impact of control measures on different categories of banks: a comparative simulation with and without control

REFERENCES

[1] H. W. Hethcote, The mathematics of infectious diseases, SIAM Rev. 42 (2000), 599–653.

[2] O. Diekmann, J. A. P. Heesterbeek, M. G. Roberts, The construction of next-generation matrices in epidemiology: A review, J. Royal Soc. Interface 7 (2010), 873–885.

[3] R. M. Anderson, R. M. May, Infectious Diseases of Humans: Dynamics and Control, Oxford University Press, 1991.

[4] P. Van den Driessche, J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180 (2002), 29–48.

[5] F. Caccioli, M. Shrestha, C. Moore, J. Doyne Farmer, Stability analysis of financial contagion due to overlapping portfolios, J. Banking & Finance, 46 (2014), 233–245.

[6] P. Gai, A. Haldane, S. Kapadia, Complexity, concentration and contagion, J. Monetary Economics, 58 (2011), 453–470.

[7] S. Markose, C. Bisiere, S. Minocha, Financial network instability: an application to the uk banking system, J. Economic Dyn. Control 36 (2012), 1039–1055.

[8] X. Zhou, Qian Li, A new measure of systemic risk: Asset contagion, J. Banking & Finance, 73 (2016), 164–177.
[9] I. Irakoze, F. Nahayo, D. Ikpe, S.A. Gyamerah, F. Vien, Mathematical modeling and stability analysis of systemic risk in the banking ecosystem, J. Appl. Math. 2023 (2023), 5628621.
[10] V.V. Acharya, L.H. Pedersen, G.M. Phillips, M. Richardson, Measuring systemic risk, Rev. Financial Stud, 30 (2017), 1–47.
[11] R. Cifuentes, G. Ferrucci, H.S. Shin, Liquidity risk and contagion, J. Eur. Economic Ass. 3 (2005), 556–566.
[12] F. Allen, D. Gale, Financial contagion, J Political Economy, 108 (2000), 1–33.
[13] L. Laeven, F. Valencia, Systemic banking crises database, IMF Economic Rev. 61 (2013), 225–270.
[14] V. Fanelli, L. Maddalena, A nonlinear dynamic model for credit risk contagionm, Math. Comput. Simul. 174 (2020), 45–58.
[15] P. Gai, S. Kapadia, Contagion in financial networks, Technical Report Working Paper No. 383, Bank of England, 2010.
[16] D. Bisias, M. Flood, A.W. Lo, S. Valavanis, A survey of systemic risk analytics, Ann. Rev. Financial Economics, 4 (2012), 255–296.
[17] V.V. Acharya, M. Richardson, Causes of the financial crisis, Critical Review: A Journal of Politics and Society, 21 (2009), 195–210.
[18] B.I. Carlin, S. Gervais, Choosing the wrong competitor: The role of benchmarking in the competition between financial institutions, J. Financial Economics, 94 (2009), 194–210.
[19] D.W. Diamond. P.H. Dybvig, Bank runs, deposit insurance, and liquidity, J. Political Economy, 91 (1983), 401–419.
[20] P. Gai, S. Kapadia, Contagion in financial networks, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 466 (2010), 2401–2423.
[21] S. Battiston, G. Caldarelli, M. D'Errico, S. Gurciullo, Leveraging the network: A stress-test framework based on debtrank, Statistics & Risk Modeling, 33 (2016), 117–138.
[22] P. Glasserman, H. Peyton Young, Contagion in financial networks, J. Economic Literature, 54 (2016), 779–831.
[23] S. Battiston, M. Puliga, R. Kaushik, P. Tasca, G. Caldarelli, Debtrank: Too central to fail? financial networks, the fed and systemic risk, Scientific Reports, 2 (2012), 541.
[24] N. Chen, H. Fan, Credit risk contagion and optimal dual control—an sis/r model, Math. Comput. Simul. 210 (2023), 448–472.
[25] M.E. Dehshalie, M. Kabiri, M.E. Dehshali, Stability analysis and fixed-time control of credit risk contagion, Math. Comput. Simul. 190 (2021), 131–139.
[26] P. Glasserman, H.P. Young, How likely is contagion in financial networks? J. Banking & Finance, 50 (2015), 383–399.
[27] D. Bentaleb, S. Harroudi, S. Amine, K. Allali, Analysis and optimal control of a multistrain seir epidemic model with saturated incidence rate and treatment, Differential Equ. Dyn. Sys. 31 (2023), 907–923.
[28] A. Capponi, P.C. Chen, Systemic risk mitigation in financial networks, J. Economic Dyn. Control, 58 (2015), 152–166.
[29] W.M. Haddad, V. Chellaboina, Stability and dissipativity theory for nonnegative dynamical systems: A unified analysis framework for biological and physiological systems, Nonlinear Anal. 6 (2005), 35–65.
[30] M. Fang, Mathematical modelling, observation and identification of epidemiological models with reinfection, PhD thesis, Sorbonne Universite, 2024.
[31] R. Mahardika, Widowati, Y. D. Sumanto, Routh-hurwitz criterion and bifurcation method for stability analysis of tuberculosis transmission model, J. Physics: Conference Series 1217 (2019), 012056.
[32] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 2012.
[33] J.P. LaSalle, An invariance principle in the theory of stability, Technical Report Technical Report 66-1, Center for Dynamical Systems, Brown University, 1966.
[34] S. Annas, M.I. Pratama, M. Rifandi, W. Sanusi, S. Side, Stability analysis and numerical simulation of seir model for pandemic covid-19 spread in indonesia, Chaos, Solitons & Fractals, 139 (2020), 11007.
[35] E. Kowalski, Measure and Integral, 2011.

[36] L. Cesari, Optimization Theory and Applications: Problems with Ordinary Differential Equations, Springer-Verlag, New York, 1983.
[37] L. S. Pontryagin, Mathematical Theory of Optimal Processes, Routledge, London, 2018.