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A NEW APPROXIMATION APPROACH TO VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we develop a new approximation approach for solving variational inequalities in real Hilbert spaces. The main novelty lies in the construction of two solution mappings, for which strong quasi-nonexpansiveness is established. These mappings are then incorporated into a relaxed projection scheme for solving a class of bilevel variational inequalities. Under pseudomonotonicity assumptions on the associated cost mappings, we establish the strong convergence of the proposed iterative algorithms to a solution of the bilevel variational inequalities. Furthermore, several numerical experiments are conducted to illustrate the computational performance and practical effectiveness of the proposed approach.

Keywords. Inexact projection; Pseudomonotone mapping; Quasi-nonexpansiveness; Variational inequalities.

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1. Introduction

Let $\mathscr{H}$ be a real Hilbert space which associated with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $\mathbb{Q}\subset\mathscr{H}$ be nonempty, closed, and convex. We consider the variational inequalities, shortly VI($\mathbb{Q},\mathscr{A}$), as follows:

$$\text{Find } \hat{x}\in\mathbb{Q} \text{ such that } \langle\mathscr{A}(\hat{x}), x-\hat{x}\rangle\geq 0, \quad \forall x\in\mathbb{Q}.$$

The mapping $\mathscr{A}:\mathscr{H}\to\mathscr{H}$ is usually called a *cost mapping* and $c_i:\mathscr{H}\to\mathscr{R}$, for each $i\in I:=\{1,2,\cdots,m\}$, is lower semicontinuous, convex and *Gâteaux* differentiable, and $\mathbb{Q}=\cap_{i\in I}\{x\in\mathscr{H}: c_i(x)\leq 0\}$ satisfies the Slater condition. As usual, the solution set of Problem VI($\mathbb{Q},\mathscr{A}$) is denoted by $\Theta(\mathbb{Q},\mathscr{A})$. Note that the $\Theta(\mathbb{Q},\mathscr{A})$ is convex, when $\mathscr{A}$ is monotone [23, Propostion 1.1.2].

Problem VI($\mathbb{Q},\mathscr{A}$) addresses a wide range of real-world problems such as the Nash cooperation game theory, network analysis, image restoration, signal processing [16, 20, 25]. It is worth

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to mention that, during the recent years, there are numerous effective methods for solving Problem VI($\mathbb{Q},\mathscr{A}$) when cost mapping $\mathscr{A}$ is monotone. Let us briefly recall some excellent solution methods such as PPM (proximal point method) of Rockafellar [27], PM (projection methods) of Solodov et al. [29], IPPM (inexact proximal point methods) of Burachik et al. [8, 9], SEM (subgradient extragradient methods) given by Censor et al. [12, 13], OAM (outer approximation methods) of Gibali et al. [17], FPIM (fixed point iteration methods) of Cegielski and Zalas [10, 11], IEM (inertial extragradient methods) of Shehu and Iyiola [28], MSEM (modified subgradient extragradient methods) of Yang et al. [31], and so on; see, e.g., [1, 2, 3, 14, 15, 22, 28] and the references cited therein. For most of these solution methods, one projection or two projections are used at each iterate. Note that, in some cases of $\mathbb{Q}$, the $\mathscr{P}_{\mathbb{Q}}:\mathscr{H}\to\mathbb{Q}$ is computed in the simple form. Examples as, when set $\mathbb{Q}$ is the nonnegative orthant, a box, a half space or the intersection of two half spaces. Then $\mathscr{P}_{\mathbb{Q}}(x)$ is provided in an explicit form for all $x\in\mathscr{H}$. However, in some general cases of $\mathbb{Q}$, the projection $\mathscr{P}_{\mathbb{Q}}$ is too expensive to compute.

It is known that a popular approach to solve Problem VI($\mathbb{Q},\mathscr{A}$) is the solution mapping methods via the metric projection $\mathscr{P}_{\mathbb{Q}}$. In fact, a point $q^*\in\mathbb{Q}$ belongs to $\Theta(\mathbb{Q},\mathscr{A})$ if and only if $q^*\in Fix(S_\lambda)$, where

$$S_\lambda(x)=\mathscr{P}_{\mathbb{Q}}[x-\lambda\mathscr{A}(x)],\quad \forall x\in\mathscr{H},$$

where $\lambda>0$. If $\mathscr{A}$ is L-Lipschitz continuous and θ-strongly monotone, in [11, Theorem 5], then S_λ is *contractive* under the restrictions $0<\lambda<\frac{2\theta}{L^2}$. Then, solving Problem VI($\mathbb{Q},\mathscr{A}$) is evaluated by Banach's contraction mapping principle to find a unique point of the solution mapping S_λ. When $\mathscr{A}$ is montone and Lipschitz continuous, in [21, Lemma 3.1], Kraikaew and Saejung, based on the method SEM, proposed the mapping $G_\tau:\mathscr{H}\to\mathscr{H}$ as follows, for each $x\in\mathscr{H}$,

$$\begin{aligned}&G_\tau(x)=\mathscr{P}_{K_x}[x-\tau\mathscr{A}(\mathscr{P}_{\mathbb{Q}}(x-\tau\mathscr{A}(x)))],\\ &K_x:=\{w\in\mathscr{H}:\langle x-\tau\mathscr{F}(x)-\mathscr{P}_{\mathbb{Q}}(x-\tau\mathscr{A}(x)),w-\mathscr{P}_{\mathbb{Q}}(x-\tau\mathscr{A}(x))\rangle\le 0\}.\end{aligned}$$

Then, G_τ is *quasi-nonexpansive* under the condition $0<\tau<\frac{1}{L}$. Moreover,

$$\begin{aligned}\|G_\tau(x)-q^*\|^2\le&\|x-q^*\|^2-(1-\tau L)\|x-\mathscr{P}_{\mathbb{Q}}(x-\tau\mathscr{A}(x))\|^2\\ &-(1-\tau L)\|G_\tau(x)-\mathscr{P}_{\mathbb{Q}}(x-\tau\mathscr{A}(x))\|^2,\quad\forall q^*\in\Theta(\mathbb{Q},\mathscr{A}).\end{aligned}$$

The authors demonstrated that $\Theta(\mathbb{Q},\mathscr{A})\subset Fix(G_\tau)$ and solving Problem VI($\mathbb{Q},\mathscr{A}$) is equivalent to finding a fixed point of the G_τ. However, a difficult problem in computing of the solution mapping G_τ that is the projection $\mathscr{P}_{\mathbb{Q}}$. In this paper, we focus on the following new two outer approximation mappings related to Problem VI($\mathbb{Q},\mathscr{A}$):

- For all $\chi>0$, we propose $\Psi^\chi:\mathscr{H}\to\mathscr{H}$ given in the form
$$\Psi^\chi(x)=\mathscr{P}_{C_x}[x-\chi\mathscr{A}(x)],\quad\forall x\in\mathscr{H},\tag{1.1}$$
where $\nabla c_i(x)$ is the gradient vector of the function c_i at x, $C_x:=\cap_{i\in I}\{y\in\mathscr{H}:c_i(x)+\langle\nabla c_i(x),y-x\rangle\le 0\}$;
- For each $\gamma>0,\chi>0$ and $\zeta>0$, the mapping $T^{\chi,\zeta}:\mathscr{H}\to\mathscr{H}$ is given as follows:
$$T^{\chi,\zeta}(x)=\mathscr{P}_{W_x}\left[x-\chi\zeta\mathscr{A}\left(\Psi^\chi(x)\right)\right],\quad\forall x\in\mathscr{H},\tag{1.2}$$
where $\Psi^\chi(x)$ is defined in (1.1), for each $x\in\mathscr{H}$, W_x is a half space formulated by $W_x=\{t\in\mathscr{H}:\langle x-\chi\mathscr{A}(x)-\Psi^\chi(x),t-\Psi^\chi(x)\rangle\le\gamma\|x-\Psi^\chi(x)\|^2\}$.

Inspired and motivated by the ongoing research, by using the mappings Ψ^{χ} defined in (1.1) and $T^{\chi,\zeta}$ defined in (1.2), we have the following three aspects: *First*, we show that the fixed point sets of two solution mappings Ψ^{χ} and $T^{\chi,\tau}$ coincide with the solution set $\Theta(\mathbb{Q},\mathscr{A})$. *Second*, we prove their strong quasi-nonexpansivexness of $T^{\chi,\zeta}$ and firmly contractiveness of Ψ^{χ} under partially monotone and Lipschitz continuous assumptions of $\mathscr{A}$. *Third*, by using the mappings, we design new algorithms for solving Problem VI($\mathbb{Q},\mathscr{A}$). Then, weak and strong convergence of proposed algorithms are established. Our strong and weak convergence theorems are quite different from the schemes proposed by current results. In fact, we do not need to calculate the projections onto the set $\mathbb{Q}$ at each iterate.

The remainder of this paper is organized as follows. Section 2 introduces the notation, basic concepts, and auxiliary results that are used throughout the paper, and establishes the relationship between the solution mappings and the solution set $\Theta(\mathbb{Q},\mathscr{A})$. In Section 3, we establish weak and strong convergence results for the proposed iterative schemes. Finally, Section 4 is devoted to numerical experiments and computational discussions, which are presented to assess the performance and effectiveness of the proposed algorithms.

2. New Solution Mappings

Throughout this paper, we use the following notation. The weak convergence of a sequence $x^k \subset \mathscr{H}$ to $x \in \mathscr{H}$ is denoted by $x^k \rightharpoonup x$, while its strong convergence to x is denoted by $x^k \to x$. For a nonempty, closed, and convex set $\mathbb{Q} \subset \mathscr{H}$, we denote by $\mathscr{P}_{\mathbb{Q}}(x)$ the metric projection of $x \in \mathscr{H}$ onto $\mathbb{Q}$. By the definition, we have $\|\mathscr{P}_{\mathbb{Q}}(x)-x\| \le \|y-x\|$ for all y in $\mathbb{Q}$. Then, for each $x \in \mathscr{H}, v \in \mathbb{Q}$,

$$\|\mathscr{P}_{\mathbb{Q}}(x)-v\|^2 \le \|x-v\|^2 - \|x-\mathscr{P}_{\mathbb{Q}}(x)\|^2. \tag{2.1}$$

We recall the explicit formula for the metric projection onto the half space $D := \{y \in \mathscr{H} : \langle a,y\rangle \le b\}$, where $0 \ne a \in \mathscr{H}$ and $b \in \mathscr{R}$. Then, the following relation holds:

$$\mathscr{P}_D(x) = \begin{cases} x - \frac{\langle a,x\rangle - b}{\|a\|^2} a & \text{if } x \notin D, \\ x & \text{if otherwise.} \end{cases}$$

For each $i \in I$, the function $c_i : \mathscr{H} \to \mathscr{R}$ is said to be *Gâteaux differentiable* at $x \in \mathscr{H}$ if there exists an element in $\mathscr{H}$, shortly $\nabla c_i(x)$, such that

$$\lim_{\lambda \to 0} \frac{c_i(x+\lambda v) - c_i(x)}{\lambda} = \langle \nabla c_i(x), v\rangle, \quad \forall v \in \mathscr{H}.$$

The function c_i is said to be *Gâteaux differentiable* on $\mathscr{H}$ if it is *Gâteaux differentiable* at very point $x \in \mathscr{H}$. Vector $w_x \in \mathscr{H}$ is called a *subgradient* of c_i at $x \in \mathscr{H}$ if

$$c_i(y) - c_i(x) \ge \langle w_x, y-x\rangle, \quad \forall y \in \mathscr{H}.$$

The *subdifferential* of c_i at x is the set of its subgradients and denoted by $\partial c_i(x)$. When c_i is *Gâteaux differentiable* at x, $\partial c_i(x) = \{\nabla c_i(x)\}$.

Definition 2.1. Denote the fixed point set of a mapping $\mathscr{A} : \mathscr{H} \to \mathscr{H}$ by $Fix(\mathscr{A}) := \{x \in \mathscr{H} : \mathscr{A}(x) = x\}$. Then, $\mathscr{A}$ is called to be

- $\beta-$*strongly monotone* on $\mathbb{Q}$ if, for every $u,y \in \mathbb{Q}$, $\langle \mathscr{A}(u)-\mathscr{A}(y), u-y\rangle \ge \beta\|u-y\|^2$;
- *monotone* on $\mathbb{Q}$ if, for every $u,y \in \mathbb{Q}$, $\langle \mathscr{A}(u)-\mathscr{A}(y), u-y\rangle \ge 0$;

- *$L-$Lipschitz continuous* on $\mathbb{Q}$ where $L>0$ if, for every $u,y\in\mathbb{Q}$, $\|\mathscr{A}(u)-\mathscr{A}(y)\|\le L\|u-y\|$;
- *$\eta-$partially monotone* where $\eta>0$ if, for every $u\in\mathbb{Q}$ and $v\in Fix(\mathscr{A})$, $\langle\mathscr{A}(u)-\mathscr{A}(v),u-v\rangle\ge\eta\|u-v\|^2$;
- *partially monotone* if, for every $u\in\mathbb{Q}$ and $v\in Fix(\mathscr{A})$, $\langle\mathscr{A}(u)-\mathscr{A}(v),u-v\rangle\ge 0$;
- *$\eta-$strongly quasi-nonexpansive* where $\eta>0$ if, for every $u\in\mathscr{H}$ and $z\in Fix(\mathscr{A})$,
$$\|\mathscr{A}(u)-z\|^2\le\|u-z\|^2-\eta\|\mathscr{A}(u)-u\|^2;$$
- *quasi-nonexpansive* if, for every $u\in\mathscr{H}$ and $z\in Fix(\mathscr{A})\neq\emptyset$, $\|\mathscr{A}(u)-z\|\le\|u-z\|$;
- *partially contractive* with constant $\eta\in[0,1)$ if, for every $u\in\mathscr{H}$ and $z\in Fix(\mathscr{A})$, $\|\mathscr{A}(u)-z\|\le\eta\|u-z\|$;
- *$\eta-$contractive* where $\eta\in[0,1)$ if, for every $u,y\in\mathscr{H}$, $\|S(u)-S(y)\|\le\eta\|u-y\|$.

To study properties of solution mappings Ψ^{χ} and $T^{\chi,\zeta}$, we assume that $\mathscr{A}$ and $c_i (i\in I)$ (of Problem VI($\mathbb{Q},\mathscr{A}$)) satisfy the following restrictions

(C_1) $\Theta(\mathbb{Q},\mathscr{A})\neq\emptyset$;
(C_2) $\mathscr{A}:\mathscr{H}\to\mathscr{H}$ is monotone and $L-$Lipschitz continuous;
(C_3) for each $i\in I$, c_i is lower semicontinuous, convex, and differentiable. Gradient vector ∇c_i is L_i-Lipschitz continuous;
(C_4) for each $\hat{x}\in\mathscr{H}$, there exists $M>0$ such that $\|\mathscr{A}(\hat{x})\|\le M\inf\left\{\|v\|:v\in conv\{\nabla c_i(\hat{x}):i\in I_{\hat{x}}\}\right\}$, where $I_{\hat{x}}:=\{i\in I:c_i(\hat{x})=0\}$ and $conv\{\nabla c_i(\hat{x}):i\in I_{\hat{x}}\}$ is the convex hull of the set $\{\nabla c_i(\hat{x}):i\in I_{\hat{x}}\}$.

Lemma 2.1. *Let $g:\mathscr{H}\to\mathscr{R}$ be continuous and Gâteaux differentiable function with L_g-Lipschitz continuous gradient ∇g, i.e., there exists a constant $L_g>0$ such that $\|\nabla g(x)-\nabla g(y)\|\le L_g\|x-y\|$ for all $x,y\in\mathscr{H}$. Then,*
$$g(y)\le g(x)+\langle\nabla g(x),y-x\rangle+\frac{L}{2}\|y-x\|^2,\quad\forall x,y\in\mathscr{H},L\ge L_g.$$

Proof. See [6, 26]. □

Lemma 2.2. [4, Lemma 2.39] *Assume that the $\mathbb{Q}\subset\mathscr{H}$ is nonempty, and $\{x^k\}\subset\mathscr{H}$ satisfies the following:*

(i) *For every $x\in\mathbb{C}$, $\lim\limits_{k\to\infty}\|x^k-x\|$ exists;*
(ii) *If $\bar{x}$ is a sequential weak cluster point of the sequence $\{x^k\}$, then $\bar{x}\in\mathbb{Q}$.*

Thus $\{x^k\}$ converges weakly to a point in $\mathbb{Q}$.

Lemma 2.3. [19, 24] *Let the mapping $\mathscr{A}$ and $\mathbb{Q}$ be defined in Problem VI($\mathbb{Q},\mathscr{A}$), and the solution set $\Theta(\mathbb{Q},\mathscr{A})\neq\emptyset$. Then, $\hat{x}\in\mathbb{Q}$ belongs to $\Theta(\mathbb{Q},\mathscr{A})$ if and only if one of the following assertions holds:*

(i) *Vector $\hat{x}$ belongs to the interior of the set $\mathbb{Q}$. Then, $\mathscr{A}(\hat{x})=0$;*
(ii) *Vector $\hat{x}$ belongs to the boundary of the set $\mathbb{Q}$ and there exist some constant $\beta_{\hat{x}}>0$ (depending on $\hat{x}$) and $u\in conv\{\nabla c_i(\hat{x}):i\in I_{\hat{x}}\}$ such that $\mathscr{A}(\hat{x})=-\beta_{\hat{x}}u$.*

The following Fejér monotonicity was first introduced by Bauschke and Borwein in [5, Theorem 2.16(v)].

Lemma 2.4. *Assume that the set $D \subset \mathscr{H}$ is nonempty, closed and convex. Let the sequence $\{x^k\} \subset \mathscr{H}$ be Fejér monotone with respect to D, i.e., for each $q^* \in D$, $\|x^{k+1} - q^*\| \leq \|x^k - q^*\|, \quad \forall k \geq 0$. Then, $\{\mathscr{P}_D(x^k)\}$ converges strongly to some $\bar{x} \in D$.*

Lemma 2.5. *Let three sequences $\{a_k\} \subset [0,\infty)$, $\{b_k\} \subset [0,\infty)$ and $\{\alpha_k\} \subset (0,1)$ be such that $\lim_{k\to\infty} \alpha_k = \infty$, $\sum_{k=0}^{\infty} b_k < +\infty$, and $\limsup_{k\to\infty} \gamma_k \leq 0$. If $a_{k+1} \leq (1-\alpha_k)a_k + \alpha_k\gamma_k + b_k$, for all $k \geq 0$. Then, $\lim_{k\to\infty} a_k = 0$.*

For each a point $x \in \mathscr{H}$ and a constant $\chi > 0$, we introduce a new mapping Ψ^χ of Problem VI$(\mathbb{Q},\mathscr{A})$ given by (1.1). The following lemma shows the relation between the fixed point set $Fix(\Psi^\chi)$ and the solution set $\Theta(\mathbb{Q},\mathscr{A})$.

Theorem 2.1. *Under assumption (C_1), a point $\hat{x} \in \Theta(\mathbb{Q},\mathscr{A})$ if and only if $\hat{x} \in Fix(\Psi^\chi)$.*

Proof. Let $\chi > 0$. Since $\Psi^\chi(x) = \mathscr{P}_{C_x}[x - \chi\mathscr{A}(x)]$ and the definition of $\mathscr{P}_{C_x}$, we have

$$\left\langle \mathscr{A}(x) + \frac{1}{\chi}\left[\Psi^\chi(x) - x\right], y - \Psi^\chi(x) \right\rangle \geq 0, \quad \forall y \in C_x. \tag{2.2}$$

Assume that $\hat{x} \in Fix(\Psi^\chi)$, i.e., $\hat{x} = \Psi^\chi(\hat{x})$. From (2.2), it follows that $\langle \mathscr{A}(\hat{x}), y - \hat{x}\rangle \geq 0$ for all y in $C_{\hat{x}}$. Note that $\mathbb{Q} \subset C_{\hat{x}}$ for all $\hat{x} \in \mathscr{H}$. Consequently, $\langle \mathscr{A}(\hat{x}), y - \hat{x}\rangle \geq 0$ for all y in C. From $\hat{x} = \Psi^\chi(\hat{x}) \in C_{\hat{x}}$, it implies that

$$c_i(\hat{x}) + \left\langle \nabla c_i(\hat{x}), \Psi^\chi(\hat{x}) - \hat{x} \right\rangle \leq 0 \Rightarrow c_i(\hat{x}) \leq 0, \quad \forall i \in I.$$

Thus, the point $\hat{x} \in C$ belongs to $\Theta(\mathbb{Q},\mathscr{A})$.

Now, we suppose $q^* \in \Theta(\mathbb{Q},\mathscr{A})$. By [18, Proposition 2.2], we find that there exist Lagrange multipliers $\lambda_i^*(i \in I)$ such that

$$\begin{cases} \mathscr{A}(q^*) + \sum\limits_{i\in I} \lambda_i^* \nabla c_i(q^*) = 0, \\ c_i(q^*) \leq 0, \lambda_i^* \geq 0, \lambda_i^* c_i(q^*) = 0, \quad i \in I. \end{cases}$$

By using [7, Theorem 6.1.6] and Slater's constraint qualification, we find that q^* is also a solution to the variational inequalities: $\langle \mathscr{A}(q^*), y - q^*\rangle \geq 0$ for all y in C_{q^*}. From $\Psi^\chi(q^*) \in C_{q^*}$, it yields $\left\langle \mathscr{A}(q^*), \Psi^\chi(q^*) - q^* \right\rangle \geq 0$. Substitute $x := q^* \in C$ and $y = q^*$ into (2.2), we obtain

$$\left\langle \mathscr{A}(q^*) + \frac{1}{\chi}\left[\Psi^\chi(q^*) - q^*\right], q^* - \Psi^\chi(q^*) \right\rangle \geq 0.$$

Thus $\Psi^\chi(q^*) = q^*$, where $\chi > 0$. This means that $q^* \in Fix(\Psi^\chi)$ for all $\chi > 0$. □

Next, we present some concepts of fixed points of the solution mapping $T^{\chi,\tau}$ defined in (1.2).

Theorem 2.2. *Let the $L-$Lipschitz continuous mapping $\mathscr{A}$ and*

$$\zeta \in (0,1), \chi \in \left(0, \min\left\{\frac{2}{\zeta L}, \frac{1}{L}\right\}\right), \gamma \in \left(0, \frac{1-\chi L}{2}\right). \tag{2.3}$$

Under (C_1), a point $\hat{x}$ belongs to the fixed point set of $T^{\chi,\zeta}$ if and only if it is a solution point to Problem VI$(\mathbb{Q},\mathscr{A})$.

Proof. For each $x \in \mathscr{H}$, using $T^{\chi,\tau}(x) \in W_x$, we deduce

$$\langle x-\Psi^{\chi}(x), T^{\chi,\tau}(x)-\Psi^{\chi}(x)\rangle - \gamma\|x-\Psi^{\chi}(x)\|^2 \leq \chi\langle \mathscr{A}(x), T^{\chi,\tau}(x)-\Psi^{\chi}(x)\rangle. \tag{2.4}$$

Combining the definition of $\mathscr{P}_{W_x}$ and $T^{\chi,\tau}(x)$ in (1.2), we have

$$\left\langle \zeta\chi\mathscr{A}(\Psi^{\chi}(x)) + T^{\chi,\zeta}(x) - x, t - T^{\chi,\zeta}(x) \right\rangle \geq 0, \quad \forall t \in W_x.$$

Consequently,

$$\zeta\chi\left\langle \mathscr{A}(\Psi^{\chi}(x)), t - T^{\chi,\zeta}(x)\right\rangle \geq \left\langle x - T^{\chi,\zeta}(x), t - T^{\chi,\zeta}(x)\right\rangle, \quad \forall t \in W_x. \tag{2.5}$$

From the $L-$Lipschitz continuous assumption of $\mathscr{A}$, (2.4), (2.5), and the inequality $2\langle a,b\rangle \leq \|a\|\|b\|$ for all $a,b \in \mathscr{H}$, we find

$$\begin{aligned}
&\langle x - T^{\chi,\zeta}(x), t - T^{\chi,\zeta}(x)\rangle - \zeta\chi\langle \mathscr{A}(\Psi^{\chi}(x)), t - \Psi^{\chi}(x)\rangle \\
\leq& -\zeta\chi\langle \mathscr{A}(\Psi^{\chi}(x)), T^{\chi,\zeta}(x) - \Psi^{\chi}(x)\rangle \\
=& -\zeta\chi\langle \mathscr{A}(\Psi^{\chi}(x)) - F(x), T^{\chi,\zeta}(x) - \Psi^{\chi}(x)\rangle - \zeta\chi\langle \mathscr{A}(x), T^{\chi,\zeta}(x) - \Psi^{\chi}(x)\rangle \\
\leq& \zeta\chi L\|\Psi^{\chi}(x) - x\|\|T^{\chi,\zeta}(x) - \Psi^{\chi}(x)\| - \zeta(\langle x - \Psi^{\chi}(x), T^{\chi,\zeta}(x) - \Psi^{\chi}(x)\rangle \\
&-\gamma\|x - \Psi^{\chi}(x)\|^2) \\
\leq& \zeta\langle \Psi^{\chi}(x) - x, T^{\chi,\zeta}(x) - \Psi^{\chi}(x)\rangle + \zeta\gamma\|x - \Psi^{\chi}(x)\|^2 + \frac{1}{2}L\zeta\chi\|x - \Psi^{\chi}(x)\|^2 \\
&+\frac{1}{2}L\zeta\chi\|T^{\chi,\zeta}(x) - \Psi^{\chi}(x)\|^2,
\end{aligned} \tag{2.6}$$

where $t \in \mathscr{W}_x$. Applying the relations $2\langle u,v\rangle = \|u\|^2 + \|v\| - \|u-v\|^2$ and $2\langle u,v\rangle = \|u+v\|^2 - \|u\| - \|v\|^2$ for all $u,v \in \mathscr{H}$, we deduce

$$2\langle x - T^{\chi,\zeta}(x), t - T^{\chi,\zeta}(x)\rangle = \|x - T^{\chi,\zeta}(x)\|^2 + \|t - T^{\chi,\zeta}(x)\|^2 - \|t - x\|^2$$

and

$$2\langle \Psi^{\chi}(x) - x, T^{\chi,\zeta}(x) - \Psi^{\chi}(x)\rangle = \|x - T^{\chi,\zeta}(x)\|^2 - \|\Psi^{\chi}(x) - x\|^2 - \|T^{\chi,\zeta}(x) - \Psi^{\chi}(x)\|^2.$$

Combining the above equalities and (2.6), it implies that, for all $w \in W_x$,

$$\begin{aligned}
\|T^{\chi,\zeta}(x) - w\|^2 \leq& \|x - w\|^2 - (1-\zeta)\|x - T^{\chi,\zeta}(x)\|^2 - \zeta(1 - 2\gamma - \chi L)\|x - \Psi^{\chi}(x)\|^2 \\
&- \zeta(1 - \chi L)\|T^{\chi,\zeta}(x) - \Psi^{\chi}(x)\|^2 + 2\zeta\chi\langle \mathscr{A}(\Psi^{\chi}(x)), w - \Psi^{\chi}(x)\rangle.
\end{aligned} \tag{2.7}$$

Inversely, let $T^{\chi,\tau}(\bar{x}) = \bar{x}$. Replacing $x := \bar{x}$ and $w := \Psi^{\chi}(\bar{x}) \in W_{\bar{x}}$ into (2.7), we see that

$$\begin{aligned}
\|T^{\chi,\zeta}(\bar{x}) - \Psi^{\chi}(\bar{x})\|^2 =& \|\bar{x} - \Psi^{\chi}(\bar{x})\|^2 \\
\leq& \|\Psi^{\chi}(\bar{x}) - \bar{x}\|^2 - (1-\zeta)\|\bar{x} - T^{\chi,\zeta}(\bar{x})\|^2 - \zeta(1 - 2\gamma - \chi L)\|\bar{x} - \Psi^{\chi}(\bar{x})\|^2 \\
&- \zeta(1 - \chi L)\|T^{\chi,\zeta}(\bar{x}) - \Psi^{\chi}(\bar{x})\|^2 + 2\zeta\chi\langle \mathscr{A}(\Psi^{\chi}(\bar{x})), \Psi^{\chi}(\bar{x}) - \Psi^{\chi}(\bar{x})\rangle.
\end{aligned}$$

Then, using (2.3), we have $\Psi^{\chi}(\bar{x}) = \bar{x}$. Therefore, $\bar{x} \in Fix(\Psi^{\chi})$. By Theorem 2.1, we see that $\bar{x}$ also belongs to $\Theta(\mathbb{Q}, \mathscr{A})$.

Next, we assume $q^* \in \Theta(\mathbb{Q}, \mathscr{A})$. By Theorem 2.1, we have $\Psi^{\chi}(q^*) = q^*$ for all $\chi > 0$. Replacing w by q^*, x by q^* in (2.7), we obtain $(2 - \zeta\chi L)\|T^{\chi,\zeta}(q^*) - q^*\|^2 \leq 0$. Using assumption

(2.3), we see $T^{\chi,\zeta}(q^*) = q^*$. We conclude that $q^* \in \Theta(\mathbb{Q},\mathscr{A})$ belongs to the $Fix(T^{\chi,\zeta})$. The proof is completed. $\square$

Theorem 2.3. *Under the assumptions $(C_1)-(C_4)$ and the following restrictions:*

$$\chi \in \left(0,\frac{1}{L+2M\mathscr{L}}\right), \gamma \in \left(0,\frac{1-\chi(L+2M\mathscr{L})}{2}\right), \zeta \in (0,1), \tag{2.8}$$

where $\mathscr{L} := \max\left\{\frac{L_i}{2} : i \in I\right\}$, and $T^{\chi,\zeta}$ is strongly quasi-nonexpansive. Moreover,

$$\begin{aligned}
&\|T^{\chi,\zeta}(x)-q^*\|^2\\
\le&\|x-q^*\|^2-(1-\zeta)\left\|x-T^{\chi,\zeta}(x)\right\|^2-\zeta(1-2\gamma-\chi L-2\chi M\mathscr{L})\left\|x-\Psi^\chi(x)\right\|^2\\
&-\zeta(1-\chi L)\left\|T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\|^2, \quad \forall x \in \mathscr{H}, q^* \in \Theta(\mathbb{Q},\mathscr{A}).
\end{aligned}$$

Proof. Let $q^* \in \Theta(\mathbb{Q},\mathscr{A})$. From the definition of the projection $\mathscr{P}_{C_x}$ in (1.1), it follows

$$\left\langle x-\chi\mathscr{A}(x)-\Psi^\chi(x), y-\Psi^\chi(x)\right\rangle \le 0, \quad \forall y \in C_x, x \in \mathscr{H}.$$

Then, the assumption $\gamma > 0$ deduces

$$\left\langle x-\chi\mathscr{A}(x)-\Psi^\chi(x), y-\Psi^\chi(x)\right\rangle \le \gamma\|x-\Psi^\chi(x)\|^2, \quad \forall y \in C_x, x \in \mathscr{H}, \gamma > 0.$$

Thus $\Theta(\mathbb{Q},\mathscr{A}) \subset C \subset C_x \subset W_x$ for all x in $\mathscr{H}$. (2.1) yields

$$\begin{aligned}
\|T^{\chi,\zeta}(x)-q^*\|^2 \le& \left\|x-\chi\zeta\mathscr{A}\left(\Psi^\chi(x)\right)-q^*\right\|^2-\left\|x-\chi\zeta\mathscr{A}\left(\Psi^\chi(x)\right)-T^{\chi,\zeta}(x)\right\|^2\\
=&\|x-q^*\|^2-2\chi\zeta\langle\mathscr{A}(\Psi^\chi(x)), x-q^*\rangle-\left\|x-T^{\chi,\zeta}(x)\right\|^2\\
&+2\chi\zeta\left\langle\mathscr{A}\left(\Psi^\chi(x)\right), x-T^{\chi,\zeta}(x)\right\rangle\\
=&\|x-q^*\|^2-2\chi\zeta\langle\mathscr{A}(\Psi^\chi(x)), T^{\chi,\zeta}(x)-q^*\rangle-\left\|x-T^{\chi,\zeta}(x)\right\|^2\\
=&\|x-q^*\|^2-2\chi\zeta\langle\mathscr{A}(\Psi^\chi(x))-F(q^*), \Psi^\chi(x)-q^*\rangle\\
&-2\chi\zeta\left\langle\mathscr{A}(\Psi^\chi(x)), T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\rangle-\left\|x-T^{\chi,\zeta}(x)\right\|^2\\
&-2\chi\zeta\left\langle\mathscr{A}(q^*), \Psi^\chi(x)-q^*\right\rangle\\
\le&\|x-q^*\|^2-2\chi\zeta\left\langle F(\Psi^\chi(x)), T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\rangle\\
&-2\chi\zeta\left\langle\mathscr{A}(q^*), \Psi^\chi(x)-q^*\right\rangle-\left\|x-T^{\chi,\zeta}(x)\right\|^2,
\end{aligned} \tag{2.9}$$

where the last inequality is deduced from the monotonicity of $\mathscr{A}$. Since $\mathscr{A}$ is $L-$Lipschitz continuous on $\mathscr{H}$ and using the Cauchy-Schwarz inequality and $T^{\chi,\zeta}(x) \in W_x$ for all $x \in \mathscr{H}$, we have

$$
\begin{aligned}
& -2\chi\left\langle \mathscr{A}(\Psi^\chi(x)), T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\rangle \\
=& 2\chi\left\langle \mathscr{A}(x)-F(\Psi^\chi(x)), T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\rangle - 2\chi\left\langle \mathscr{A}(x), T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\rangle \\
\leq& 2\chi\left\|\mathscr{A}(x)-F(\Psi^\chi(x))\right\|\left\|T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\| - 2\chi\left\langle \mathscr{A}(x), T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\rangle \\
\leq& 2\chi L\left\|x-\Psi^\chi(x)\right\|\left\|T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\| - 2\chi\left\langle \mathscr{A}(x), T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\rangle \\
\leq& \chi L\left\|x-\Psi^\chi(x)\right\|^2 + \chi L\left\|T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\|^2 - 2\chi\left\langle \mathscr{A}(x), T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\rangle \\
=& \chi L\left\|x-\Psi^\chi(x)\right\|^2 + \chi L\left\|T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\|^2 \\
& + 2\left\langle x-\chi\mathscr{A}(x)-\Psi^\chi(x), T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\rangle - 2\left\langle x-\Psi^\chi(x), T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\rangle \\
\leq& \chi L\left\|T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\|^2 - (1-2\gamma-\chi L)\left\|x-\Psi^\chi(x)\right\|^2 + \left\|T^{\chi,\zeta}(x)-x\right\|^2 \\
& - \left\|T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\|^2. \qquad (2.10)
\end{aligned}
$$

Combining (2.9) and (2.10), we obtain

$$
\begin{aligned}
& \|T^{\chi,\zeta}(x)-q^*\|^2 \\
\leq& \|x-q^*\|^2 - 2\chi\zeta\left\langle \mathscr{A}(q^*), \Psi^\chi(x)-q^*\right\rangle - \left\|x-T^{\chi,\zeta}(x)\right\|^2 \\
& - 2\chi\zeta\left\langle \mathscr{A}(\Psi^\chi(x)), T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\rangle \\
\leq& \|x-q^*\|^2 - 2\chi\zeta\left\langle \mathscr{A}(q^*), \Psi^\chi(x)-q^*\right\rangle - \left\|x-T^{\chi,\zeta}(x)\right\|^2 \\
& + \chi\zeta L\left\|T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\|^2 - \zeta(1-2\gamma-\chi L)\left\|x-\Psi^\chi(x)\right\|^2 + \zeta\left\|T^{\chi,\zeta}(x)-x\right\|^2 \\
& - \zeta\left\|T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\|^2 \\
=& \|x-q^*\|^2 - (1-\zeta)\left\|x-T^{\chi,\zeta}(x)\right\|^2 - \zeta(1-2\gamma-\chi L)\left\|x-\Psi^\chi(x)\right\|^2 \\
& - \zeta(1-\chi L)\left\|T^{\chi,\zeta}(x)-\Psi^\chi(x)\right\|^2 - 2\chi\zeta\left\langle \mathscr{A}(q^*), \Psi^\chi(x)-q^*\right\rangle. \qquad (2.11)
\end{aligned}
$$

Let us consider the following cases.

Case 1: $\mathscr{A}(q^*) \neq 0$.

By using Lemma 2.3, there exist $\alpha_i \geq 0, \beta_{q^*} > 0$ such that

$$\mathscr{A}(q^*) = -\beta_{q^*} \sum_{i\in I_{q^*}} \alpha_i \nabla c_i(q^*), \qquad (2.12)$$

where $I_{q^*} := \{i \in I : c_i(q^*) = 0\}$ and $\sum_{i\in I_{q^*}} \alpha_i = 1$. Since c_i is convex and differentiable for all $i \in I$, we have $c_i(\Psi^\chi(x)) \geq c_i(q^*) + \langle \nabla c_i(q^*), \Psi^\chi(x) - q^*\rangle$ for all i in I. Consequently,

$$c_i(\Psi^\chi(x)) \geq \left\langle \nabla c_i(q^*), \Psi^\chi(x) - q^*\right\rangle, \quad \forall i \in I_{q^*}. \qquad (2.13)$$

From (2.12) and (2.13), it follows that

$$\left\langle \mathscr{A}(q^*), q^* - \Psi^\chi(x) \right\rangle = \beta_{q^*} \sum_{i \in I_{q^*}} \alpha_i \left\langle \nabla c_i(q^*), \Psi^\chi(x) - q^* \right\rangle \leq \beta_{q^*} \sum_{i \in I_{q^*}} \alpha_i c_i(\Psi^\chi(x)).$$

Applying the L_i−Lipschitz continuous assumption in (C_3) of $\nabla c_i (i \in I)$ and Lemma 2.1, we conclude

$$c_i\left(\Psi^\chi(x)\right) \leq c_i(x) + \langle \nabla c_i(x), \Psi^\chi(x) - x \rangle + \frac{L_i}{2} \|\Psi^\chi(x) - x\|^2 \leq \mathscr{L} \|\Psi^\chi(x) - x\|^2,$$

where $\mathscr{L} := \max\{\frac{L_i}{2} : i \in I\}$ and the last inequality is deduced from $\Psi^\chi(x) \in C_x$, i.e., $c_i(x) + \langle \nabla c_i(x), \Psi^\chi(x) - x \rangle \leq 0$ for all $i \in I$. By using Lemma 2.3(ii), there exist a positive number β_{q^*} and $u \in conv\{\nabla c_i(q^*) : i \in I_{q^*}\}$ such that $\mathscr{A}(q^*) = -\beta_{q^*} u$. Combining this and the assumption (C_4), we have

$$\|\mathscr{A}(q^*)\| = \beta_{q^*} \|u\| \leq M \inf\left\{ \|v\| : v \in conv\{\nabla c_i(q^*) : i \in I_{q^*}\} \right\} \leq M\|u\|.$$

In view of $u \neq 0$, one has $\beta_{q^*} \leq M$. Note that

$$\left\langle \mathscr{A}(q^*), q^* - \Psi^\chi(x) \right\rangle \leq \beta_{q^*} \sum_{i \in I_{q^*}} \alpha_i c_i(\Psi^\chi(x)) \leq \beta_{q^*} \mathscr{L} \|\Psi^\chi(x) - x\|^2 \sum_{i \in I_{q^*}} \alpha_i \leq M\mathscr{L} \|\Psi^\chi(x) - x\|^2. \tag{2.14}$$

Since (2.11) and (2.14), we deduce

$$\begin{aligned}
&\|T^{\chi,\zeta}(x) - q^*\|^2 \\
\leq &\|x - q^*\|^2 - (1-\zeta)\left\|x - T^{\chi,\zeta}(x)\right\|^2 - \zeta(1 - 2\gamma - \chi L)\left\|x - \Psi^\chi(x)\right\|^2 \\
&- \zeta(1-\chi L)\left\|T^{\chi,\zeta}(x) - \Psi^\chi(x)\right\|^2 - 2\chi\zeta\left\langle \mathscr{A}(q^*), \Psi^\chi(x) - q^* \right\rangle \\
\leq &\|x - q^*\|^2 - (1-\zeta)\left\|x - T^{\chi,\zeta}(x)\right\|^2 - \zeta(1 - 2\gamma - \chi L - 2\chi M\mathscr{L})\left\|x - \Psi^\chi(x)\right\|^2 \\
&- \zeta(1-\chi L)\left\|T^{\chi,\zeta}(x) - \Psi^\chi(x)\right\|^2.
\end{aligned}$$

Case 2: $\mathscr{A}(q^*) = 0$.

Substituting this into (2.11), we have

$$\begin{aligned}
&\|T^{\chi,\zeta}(x) - q^*\|^2 \\
\leq &\|x - q^*\|^2 - (1-\zeta)\left\|x - T^{\chi,\zeta}(x)\right\|^2 - \zeta(1 - 2\gamma - \chi L)\left\|x - \Psi^\chi(x)\right\|^2 \\
&- \zeta(1-\chi L)\left\|T^{\chi,\zeta}(x) - \Psi^\chi(x)\right\|^2 - 2\chi\zeta\left\langle \mathscr{A}(q^*), \Psi^\chi(x) - q^* \right\rangle \\
= &\|x - q^*\|^2 - (1-\zeta)\left\|x - T^{\chi,\zeta}(x)\right\|^2 - \zeta(1 - 2\gamma - \chi L)\left\|x - \Psi^\chi(x)\right\|^2 \\
&- \zeta(1-\chi L)\left\|T^{\chi,\zeta}(x) - \Psi^\chi(x)\right\|^2 \\
\leq &\|x - q^*\|^2 - (1-\zeta)\left\|x - T^{\chi,\zeta}(x)\right\|^2 - \zeta(1 - 2\gamma - \chi L - 2\chi M\mathscr{L})\left\|x - \Psi^\chi(x)\right\|^2 \\
&- \zeta(1-\chi L)\left\|T^{\chi,\zeta}(x) - \Psi^\chi(x)\right\|^2.
\end{aligned}$$

The proof is complete. $\square$

Theorem 2.4. *Assume that $\mathscr{A}:\mathscr{H}\to\mathscr{H}$ is $\eta-$strongly monotone and the assumptions $(C_1)-(C_4)$ hold. Under the conditions:*

$$\tau\in\left(0,\frac{\eta}{L}\right),\gamma\in\left(0,\frac{1}{2}\right),\chi\in\left(0,\min\left\{\frac{1}{2(\eta-\tau L)},\frac{2\tau(1-2\gamma)}{L+4\tau M\mathscr{L}}\right\}\right), \tag{2.15}$$

Ψ^χ is firmly contractive, where $\delta:=1-2\chi(\eta-\tau L)\in(0,1)$. Moreover, for all $x\in\mathscr{H}$ and $\{\bar{x}\}=\Theta(\mathbb{Q},\mathscr{A})$,

$$\|\Psi^\chi(x)-\bar{x}\|^2\le[1-2\chi(\eta-\tau L)]\|x-\bar{x}\|^2-\left[1-\chi\left(\frac{L}{2\tau}+2M\mathscr{L}\right)-2\gamma\right]\|x-\Psi^\chi(x)\|^2.$$

Proof. Let $\bar{x}\in\Theta(\mathbb{Q},\mathscr{A})$. Since $\mathbb{Q}\subset C_x\subset W_x$ for all $x\in\mathscr{H}$, for each $y\in\mathbb{Q}$, we have

$$\left\langle x-\chi\mathscr{A}(x)-\Psi^\chi(x),y-\Psi^\chi(x)\right\rangle\le\gamma\|\Psi^\chi(x)-x\|^2.$$

Substituting $y=\bar{x}\in\mathbb{Q}$ into the last inequality, we deduce that

$$\left\langle x-\chi\mathscr{A}(x)-\Psi^\chi(x),\bar{x}-\Psi^\chi(x)\right\rangle\le\gamma\|\Psi^\chi(x)-x\|^2,\quad\forall x\in\mathbb{Q}.$$

Consequently,

$$\gamma\|\Psi^\chi(x)-x\|^2+\chi\langle\mathscr{A}(x),\bar{x}-\Psi^\chi(x)\rangle\ge\left\langle x-\Psi^\chi(x),\bar{x}-\Psi^\chi(x)\right\rangle. \tag{2.16}$$

Since $\mathscr{A}$ is $\eta-$strongly monotone and $L-$Lipschitz continuous, we get

$$\begin{aligned}
&\chi\langle\mathscr{A}(x),\bar{x}-\Psi^\chi(x)\rangle\\
=&\chi\langle\mathscr{A}(x)-\mathscr{A}(\bar{x}),\bar{x}-\Psi^\chi(x)\rangle+\chi\langle\mathscr{A}(\bar{x}),\bar{x}-\Psi^\chi(x)\rangle\\
=&\chi\langle\mathscr{A}(\bar{x}),\bar{x}-\Psi^\chi(x)\rangle+\chi\langle\mathscr{A}(x)-\mathscr{A}(\bar{x}),\bar{x}-x\rangle\\
&+\chi\langle\mathscr{A}(x)-\mathscr{A}(\bar{x}),x-\Psi^\chi(x)\rangle\\
\le&-\chi\eta\|x-\bar{x}\|^2+\chi L\|x-\bar{x}\|\|x-\Psi^\chi(x)\|+\chi\langle\mathscr{A}(\bar{x}),\bar{x}-\Psi^\chi(x)\rangle\\
\le&\chi\left(\tau L-\eta\right)\|x-\bar{x}\|^2+\frac{\chi L}{4\tau}\|x-\Psi^\chi(x)\|^2+\chi\left\langle\mathscr{A}(\bar{x}),\bar{x}-\Psi^\chi(x)\right\rangle.
\end{aligned} \tag{2.17}$$

Note that $\langle u,v\rangle=\frac{1}{2}(\|u\|^2+\|v\|^2-\|u-v\|^2)$ for all $u,v\in\mathscr{H}$. For $u=x-\Psi^\chi(x)$ and $v=\bar{x}-\Psi^\chi(x)$, we obtain

$$\|x-\Psi^\chi(x)\|^2+\|\bar{x}-\Psi^\chi(x)\|^2-\|x-\bar{x}\|^2=2\langle x-\Psi^\chi(x),\bar{x}-\Psi^\chi(x)\rangle. \tag{2.18}$$

Combining (2.16), (2.17), and (2.18), we obtain

$$\begin{aligned}
&\|x-\Psi^\chi(x)\|^2+\|\bar{x}-\Psi^\chi(x)\|^2-\|x-\bar{x}\|^2\\
=&2\langle x-\Psi^\chi(x),\bar{x}-\Psi^\chi(x)\rangle\\
\le&2\gamma\|\Psi^\chi(x)-x\|^2+2\chi\langle\mathscr{A}(x),\bar{x}-\Psi^\chi(x)\rangle\\
\le&2\chi\left(\tau L-\eta\right)\|x-\bar{x}\|^2+\left(2\gamma+\frac{\chi L}{2\tau}\right)\|x-\Psi^\chi(x)\|^2+2\chi\left\langle\mathscr{A}(\bar{x}),\bar{x}-\Psi^\chi(x)\right\rangle,
\end{aligned}$$

which yields

$$\begin{aligned}\|\Psi^{\chi}(x)-\bar{x}\|^2 \le&[1-2\chi(\eta-\tau L)]\|x-\bar{x}\|^2-\left(1-\frac{\chi L}{2\tau}-2\gamma\right)\|x-\Psi^{\chi}(x)\|^2\\&+2\chi\left\langle \mathscr{A}(\bar{x}),\bar{x}-\Psi^{\chi}(x)\right\rangle.\end{aligned}\tag{2.19}$$

Using a similar way in the proof of Theorem 2.2, we also consider two following cases.

Case 1: $\mathscr{A}(\bar{x})\neq 0$.

It follows from(2.14) that $\left\langle \mathscr{A}(\bar{x}),\bar{x}-\Psi^{\chi}(x)\right\rangle \le M\mathscr{L}\|\Psi^{\chi}(x)-x\|^2$ for all x in $\mathscr{H}$. Combining this and (2.19), we find

$$\|\Psi^{\chi}(x)-\bar{x}\|^2\le[1-2\chi(\eta-\tau L)]\|x-\bar{x}\|^2-\left[1-\chi\left(\frac{L}{2\tau}+2M\mathscr{L}\right)-2\gamma\right]\|x-\Psi^{\chi}(x)\|^2.$$

Case 2: $\mathscr{A}(\bar{x})=0$.

Substituting $\mathscr{A}(\bar{x})=0$ into (2.19), we deduce

$$\begin{aligned}&\|\Psi^{\chi}(x)-\bar{x}\|^2\\\le&[1-2\chi(\eta-\tau L)]\|x-\bar{x}\|^2-\left(1-\frac{\chi L}{2\tau}-2\gamma\right)\|x-\Psi^{\chi}(x)\|^2\\\le&[1-2\chi(\eta-\tau L)]\|x-\bar{x}\|^2-\left[1-\chi\left(\frac{L}{2\tau}+2M\mathscr{L}\right)-2\gamma\right]\|x-\Psi^{\chi}(x)\|^2.\end{aligned}$$

This implies the proof. $\square$

Remark 2.1. In [30, Lemma 2.1], Yamada showed that if $\mathscr{A}:\mathscr{H}\to\mathscr{H}$ is $\beta-$strongly monotone and $L-$Lipschitz continuous, then the mapping $S=\mathscr{P}_{\mathbb{Q}}(I-\zeta\mathscr{A})$ is contractive for $\zeta\in(0,\frac{2\beta}{L^2})$, where I is the identity mapping.

3. Outer Approximation Schemes

In this section, we apply the properties of the solution mappings Ψ^{χ} and $T^{\chi,\zeta}$ in the above to propose new outer approximation schemes for solving Problem VI($\mathbb{Q},\mathscr{A}$).

Theorem 3.1. *Let the mapping $\mathscr{A}$ be $\eta-$strongly monotone, and the functions $c_i(i\in I)$ satisfy assumptions $(C_1)-(C_4)$. Let $\{x^k\}$ be a sequence defined as follows:*

$$x^k\in\mathscr{H}, x^{k+1}=\mathscr{P}_{C_{x^k}}[x^k-\chi\mathscr{A}(x^k)],\quad \forall k\ge 0,$$

where $C_{x^k}:=\cap_{i\in I}\{y\in\mathscr{H}: c_i(x^k)+\langle\nabla c_i(x^k),y-x^k\rangle\le 0\}$ and χ satisfies restriction (2.15). Then, $\{x^k\}$ converges strongly to $q^*\in\Theta(\mathbb{Q},\mathscr{A})$. Moreover, $\{x^k\}$ converges linearly to q^*, and, for all $k\ge 0$, the priori and posteriori error estimates as follows:

$$\|x^{k+1}-q^*\|\le\frac{\delta^k}{1-\delta}\|x^1-x^0\|,$$

and

$$\|x^{k+1}-q^*\|\le\frac{\delta}{1-\delta}\|x^{k+1}-x^k\|,$$

where $\delta=1-2\chi(\eta-\tau L)\in(0,1)$.

Proof. It follows from condition (2.15) that

$$0 < \delta < 1, 1 - \chi\left(\frac{L}{2\tau} + 2M\mathscr{L}\right) - 2\gamma > 0.$$

By using Theorem 2.4 and the mapping Ψ^χ in (1.1), we obtain

$$\|x^{k+1} - q^*\| = \left\|\mathscr{P}_{C_{x^k}}[x^k - \chi\mathscr{A}(x^k)] - q^*\right\| = \left\|\Psi^\chi(x^k) - q^*\right\| \leq \delta\|x^k - q^*\|, \quad \forall k \geq 0, \quad (3.1)$$

which implies $\|x^k - q^*\| \leq \|x^{k+1} - x^k\| + \|x^{k+1} - q^*\| \leq \|x^{k+1} - x^k\| + \delta\|x^k - q^*\|$. Thus

$$\|x^k - q^*\| \leq \frac{1}{1-\delta}\|x^{k+1} - x^k\| \Rightarrow \|x^0 - q^*\| \leq \frac{1}{1-\delta}\|x^1 - x^0\|.$$

By using induction for (3.1), we find

$$\|x^{k+1} - q^*\| \leq \delta\|x^k - q^*\| \leq \cdots \leq \delta^{k+1}\|x^0 - q^*\| \leq \frac{\delta^{k+1}}{1-\delta}\|x^1 - x^0\|,$$

and $\|x^{k+1} - q^*\| \leq \delta\|x^k - q^*\| \leq \frac{\delta}{1-\delta}\|x^{k+1} - x^k\|$. The proof is complete. □

Now, we are in a position to apply the strong quasi-nonexpansiveness of the solution mapping $T^{\chi,\zeta}$ defined in (1.2) for solving Problem VI($\mathbb{Q}, \mathscr{A}$). We add the following assumption on $\mathscr{A}$

(C_5) Assume that $a^k \rightharpoonup \hat{a} \in \mathscr{H}$ as $k \to \infty$. Then, $\lim_{k\to\infty}\langle\mathscr{A}(a^k), a^k - t\rangle \geq \langle\mathscr{A}(\hat{a}), \hat{a} - t\rangle$ for all $t \in \mathscr{H}$.

Theorem 3.2. *Let mapping $\mathscr{A}$ and functions $c_i (i \in I)$ satisfy assumptions $(C_1) - (C_5)$. Let two sequences $\{x^k\}$ and $\{y^k\}$ be defined by*

$$\begin{cases} y^k = \mathscr{P}_{C_{x^k}}[x^k - \chi\mathscr{A}(x^k)], \\ x^{k+1} = \mathscr{P}_{W_{x^k}}\left[x^k - \chi\zeta\mathscr{A}\left(y^k\right)\right], \quad \forall k \geq 0, \end{cases} \quad (3.2)$$

where $x^0 \in \mathscr{H}$, C_{x^k} is defined by Theorem 3.1, and $W_{x^k} = \{w \in \mathscr{H} : \langle x^k - \chi\mathscr{A}(x^k) - y^k, w - y^k\rangle \leq \gamma\|x^k - y^k\|^2\}$. Under condition (2.8), it holds that $x^k \rightharpoonup \bar{x}$ and $y^k \rightharpoonup \bar{x} \in \Theta(\mathbb{Q}, \mathscr{A})$. Moreover, $\bar{x} = \lim\limits_{k\to\infty}\mathscr{P}_{\Theta(\mathbb{Q},\mathscr{A})}(x^k)$.

Proof. Let $\bar{x} \in \Theta(\mathbb{Q}, \mathscr{A})$. It is clear that $y^k = \Psi^\chi(x^k)$ and $x^{k+1} = T^{\chi,\zeta}(x^k)$. By Theorem 2.3, we have

$$\begin{aligned} &\|x^{k+1} - \bar{x}\|^2 \\ \leq&\|x^k - \bar{x}\|^2 - (1-\zeta)\left\|x^k - T^{\chi,\zeta}(x^k)\right\|^2 \\ &- \zeta(1 - 2\gamma - \chi L - 2\chi M\mathscr{L})\left\|x^k - \Psi^\chi(x^k)\right\|^2 - \zeta(1-\chi L)\left\|T^{\chi,\zeta}(x^k) - \Psi^\chi(x^k)\right\|^2 \\ =&\|x^k - \bar{x}\|^2 - (1-\zeta)\left\|x^k - x^{k+1}\right\|^2 - \zeta(1 - 2\gamma - \chi L - 2\chi M\mathscr{L})\left\|x^k - y^k\right\|^2 \\ &- \zeta(1-\chi L)\left\|x^{k+1} - y^k\right\|^2. \end{aligned} \quad (3.3)$$

It follows from condition (2.8) that $1 - \zeta > 0$, $1 - 2\gamma - \chi L - 2\chi M\mathscr{L} > 0$, $1 - \chi L > 0$, and

$$\alpha\|x^k - y^k\|^2 \leq \|x^k - \bar{x}\|^2, \|x^{k+1} - \bar{x}\|^2 \leq \|x^k - \bar{x}\|^2 - \alpha\|x^k - y^k\|^2, \quad \forall k \geq 0,$$

where $\alpha := 1-2\gamma-\chi L-2\chi M\mathscr{L} > 0$. Consequently,

$$\alpha\|x^{k+1}-y^{k+1}\|^2 \le \|x^{k+1}-\bar{x}\|^2 \le \|x^k-\bar{x}\|^2-\alpha\|x^k-y^k\|^2,$$

and hence

$$\alpha(\|x^{k+1}-y^{k+1}\|^2+\|x^k-y^k\|^2) \le \|x^k-\bar{x}\|^2 \le \|x^{k-1}-\bar{x}\|^2-\alpha\|x^{k-1}-y^{k-1}\|^2,\ \forall k\ge 1.$$

By using induction with k, we have $\alpha\sum_{k=0}^{n}\|x^k-y^k\|^2 \le \|x^0-\bar{x}\|^2$ for all $n\ge 0$. Consequently, the $\sum_{k=0}^{+\infty}\|x^k-y^k\|^2$ is convergent and hence $\lim_{k\to\infty}\|x^k-y^k\|^2=0$. From $\|x^{k+1}-\bar{x}\|\le\|x^k-\bar{x}\|$ for every $k\ge 0$, we see that $A=\lim\limits_{k\to\infty}\|x^k-\bar{x}\|<+\infty$, and $\{x^k\}$ and $\{y^k\}$ also are bounded. Thus there exists at least one weak accumulation point and we assume that $\{x^{k_j}\}\subset\{x^k\}$ such that $x^{k_j}\rightharpoonup\bar{x}$. From $x^k-y^k\to 0$, it follows $y^{k_j}\rightharpoonup\bar{x}$ as $j\to\infty$. Since $\mathscr{A}$ is $L-$Lipschitz continuous and $\{x^k\}$ is bounded, then $\{F(x^k)\}$ is also bounded. By the definition of $y^{k_j}=\mathscr{P}_{C_{x^{k_j}}}[x^{k_j}-\chi\mathscr{A}(x^{k_j})]$, for each $y\in C_{x^{k_j}}$, we have

$$\begin{aligned}0\le&\langle x^{k_j}-\chi\mathscr{A}(x^{k_j})-y^{k_j},y^{k_j}-y\rangle\\ =&\langle x^{k_j}-y^{k_j},y^{k_j}-y\rangle-\chi\langle\mathscr{A}(x^{k_j}),y^{k_j}-y\rangle\\ =&\langle x^{k_j}-y^{k_j},y^{k_j}-y\rangle-\chi\langle\mathscr{A}(x^{k_j}),y^{k_j}-x^{k_j}\rangle-\chi\langle\mathscr{A}(x^{k_j}),x^{k_j}-y\rangle.\end{aligned}$$

By the boundedness of $\{x^k\}$ and $\{y^k\}$, $\lim_{k\to\infty}\|x^k-y^k\|=0$, (C_5) and $C_{x^{k_j}}\to C_{\bar{x}}$ as $j\to\infty$, we have

$$\begin{aligned}\chi\langle\mathscr{A}(\bar{x}),\bar{x}-y\rangle\le&\chi\lim_{j\to\infty}\langle\mathscr{A}(x^{k_j}),x^{k_j}-y\rangle\\ \le&\lim_{j\to\infty}\left[\langle x^{k_j}-y^{k_j},y^{k_j}-y\rangle-\chi\langle\mathscr{A}(x^{k_j}),y^{k_j}-x^{k_j}\rangle\right]\\ \le&\lim_{j\to\infty}\left[\|x^{k_j}-y^{k_j}\|\|y^{k_j}-y\|+\chi\|\mathscr{A}(x^{k_j})\|\|y^{k_j}-x^{k_j}\|\right]=0,\quad\forall y\in C_{\bar{x}}.\end{aligned}$$

Note that the multivalued mapping C_x is weakly upper semicontinuous, so $y^{k_j}\rightharpoonup\bar{x}\in C_{\bar{x}}$. By Theorem 2.1, the weakly cluster point $\bar{x}$ of $\{x^k\}$ belongs to $\Theta(\mathbb{Q},\mathscr{A})$. Then, by Lemma 2.2, it implies that $\{x^k\}$ converges weakly to an element belonging to $\Theta(\mathbb{Q},\mathscr{A})$. In view of the Fejér-monotone assumption of $\{x^k\}$ and Lemma 2.4, we have that $\{\mathscr{P}_{\Theta(\mathbb{Q},\mathscr{A})}(x^k)\}$ converges strongly to some $\hat{x}\in\Theta(C,F)$. We also have

$$\left\langle x^k-\mathscr{P}_{\Theta(\mathbb{Q},\mathscr{A})}(x^k),y-\mathscr{P}_{\Theta(\mathbb{Q},\mathscr{A})}(x^k)\right\rangle\le 0,\quad\forall y\in\Theta(\mathbb{Q},\mathscr{A}).$$

Taking the limit and $y:=\bar{x}$, we obtain $\langle\bar{x}-\hat{x},\bar{x}-\hat{x}\rangle\le 0$, and hence $\{x^k\}$ converges weakly to $\hat{x}=\bar{x}$. The proof is completed. □

By combining the Halpern iteration technique and the scheme in Theorem 3.2, we next propose a new theorem and prove its strong convergence of sequences.

Theorem 3.3. *Let the mapping $\mathscr{A}$ and the functions $c_i (i \in I)$ satisfy assumptions $(C_1)-(C_5)$. Let two sequences $\{x^k\}$ and $\{y^k\}$ are defined as follows:*

$$\begin{cases} x^0 \in \mathscr{H}, \\ y^k = \mathscr{P}_{C_{x^k}}[x^k - \chi\mathscr{A}(x^k)], \\ x^{k+1} = \alpha_k x^0 + (1-\alpha_k)\mathscr{P}_{W_{x^k}}\left[x^k - \chi\zeta\mathscr{A}\left(y^k\right)\right], \end{cases}$$

where the polyhedron C_{x^k} and the half space W_{x^k} are given in Theorem 3.2. Under the condition (2.8) on parameters and $\{\alpha_k\} \subset (0,1), \lim_{k\to\infty}\alpha_k = 0, \sum_{k=0}^{\infty}\alpha_k = \infty$, $\{x^k\}$ and $\{y^k\}$ converge strongly to q^* belonging to $\Theta(\mathbb{Q},\mathscr{A})$, where $q^* = \mathscr{P}_{\Theta(\mathbb{Q},\mathscr{A})}(x^0)$.

Proof. Assume that $q^* = \mathscr{P}_{\Theta(\mathbb{Q},\mathscr{A})}(x^0)$. As (3.3), under condition (2.8), we have

$$\left\|\mathscr{P}_{W_{x^k}}\left[x^k - \chi\zeta\mathscr{A}\left(y^k\right)\right] - q^*\right\| \le \|x^k - q^*\|, \quad \forall k \ge 0.$$

In view of $\|a+b\|^2 \le \|a\|^2 + 2\langle b, a+b\rangle$ for all $a, b \in \mathscr{H}$, we find

$$\begin{aligned} \|x^{k+1} - q^*\|^2 &= \left\|\alpha_k x^0 + (1-\alpha_k)\mathscr{P}_{W_{x^k}}\left[x^k - \chi\zeta\mathscr{A}\left(y^k\right)\right] - q^*\right\|^2 \\ &\le (1-\alpha_k)^2\left\|\mathscr{P}_{W_{x^k}}\left[x^k - \chi\zeta\mathscr{A}\left(y^k\right)\right] - q^*\right\|^2 \\ &\quad + 2\alpha_k\langle x^0 - q^*, x^{k+1} - q^*\rangle \\ &\le (1-\alpha_k)^2\left\|x^k - q^*\right\|^2 + 2\alpha_k\langle x^0 - q^*, x^{k+1} - q^*\rangle \\ &\quad - \zeta(1 - 2\gamma - \chi L - 2\chi M\mathscr{L})\left\|x^k - y^k\right\|^2. \end{aligned} \tag{3.4}$$

We consider the following cases.

Case 1: There exists a positive integer k_0 such that $\|x^{k+1} - q^*\| \le \|x^k - q^*\|$ for all $k \ge k_0$ and hence $\lim_{k\to\infty}\|x^k - q^*\|^2 = A < \infty$. From (3.4) and condition $\lim_{k\to\infty}\alpha_k = 0$, we see that

$$\lim_{k\to\infty}\left\|\mathscr{P}_{W_{x^k}}\left[x^k - \chi\zeta\mathscr{A}\left(y^k\right)\right] - q^*\right\|^2 = A \text{ and } \lim_{k\to\infty}\left\|x^k - y^k\right\| = 0.$$

Following the proof in Theorem 3.2, we also obtain that all weak cluster points of the sequence $\{x^k\}$ belongs to $\Theta(\mathbb{Q},\mathscr{A})$. There exists a subsequence $\{x^{k_j}\} \subset \{x^k\}$ such that $x^{k_j+1} \rightharpoonup \bar{x} \in \Theta(\mathbb{Q},\mathscr{A})$ and

$$\limsup_{k\to\infty}\langle x^0 - q^*, x^{k+1} - q^*\rangle = \lim_{j\to\infty}\langle x^0 - q^*, x^{k_j+1} - q^*\rangle = \langle x^0 - q^*, \bar{x} - q^*\rangle \le 0, \tag{3.5}$$

where the last inequality is deduced from $q^* = \mathscr{P}_{\Theta(\mathbb{Q},\mathscr{A})}(x^0)$. Applying Lemma 2.5 for

$$a_k := \|x^k - q^*\|^2, \gamma_k := 2\langle x^0 - q^*, x^{k+1} - q^*\rangle$$

and

$$b_k := -\zeta(1 - 2\gamma - \chi L - 2\chi M\mathscr{L})\left\|x^k - y^k\right\|^2,$$

we have $\lim_{k\to\infty} a_k = 0$. Thus both $\{x^k\}$ and $\{y^k\}$ converge strongly to the same point $q^* = \mathscr{P}_{\Theta(\mathbb{Q},\mathscr{A})}(x^0)$.

Case 2: There does not exist a positive integer k_0 such that $\|x^{k+1}-q^*\| \leq \|x^k-q^*\|$ for all $k \geq k_0$. Then, we have a nondecreasing sequence $\{k_j\} \subset \mathscr{N}$ such that $\lim_{j\to\infty} k_j = \infty$ and the following inequalities hold

$$\|x^{k_j}-q^*\| \leq \|x^{k_j+1}-q^*\| \text{ and } \|x^{k_j}-q^*\| \leq \|x^{k_j+1}-q^*\|, \quad \forall j \geq 0. \tag{3.6}$$

Combining (3.3), (3.6), and Theorem 2.4, we have

$$\begin{aligned}
\|x^{k_j}-q^*\|^2 \leq & \left\| \alpha_{k_j} x^0 + (1-\alpha_{k_j}) \mathscr{P}_{W_{x^{k_j}}} \left[x^{k_j} - \chi\zeta\mathscr{A}\left(y^{k_j}\right)\right] - q^* \right\|^2 \\
\leq & \alpha_{k_j}\|x^0-q^*\|^2 + (1-\alpha_{k_j}) \left\| \mathscr{P}_{W_{x^{k_j}}} \left[x^{k_j} - \chi\zeta\mathscr{A}\left(y^{k_j}\right)\right] - q^* \right\|^2 \\
\leq & \alpha_{k_j}\|x^0-q^*\|^2 - (1-\alpha_{k_j})\zeta(1-2\gamma-\chi L - 2\chi M\mathscr{L})\|x^{k_j}-y^{k_j}\|^2 \\
& -(1-\alpha_{k_j})(1-\zeta) \left\| x^{k_j} - \mathscr{P}_{W_{x^{k_j}}} \left[x^{k_j} - \chi\zeta\mathscr{A}\left(y^{k_j}\right)\right] \right\|^2 \\
& + (1-\alpha_{k_j})\|x^{k_j}-q^*\|^2.
\end{aligned} \tag{3.7}$$

From (3.7) and condition $\alpha_{k_j} > 0$ for all $j \geq 0$, it follows that $\{x^{k_j}\}$ is bounded and

$$\lim_{j\to\infty} \left\| x^{k_j} - \mathscr{P}_{W_{x^{k_j}}} \left[x^{k_j} - \chi\zeta\mathscr{A}\left(y^{k_j}\right)\right] \right\| = 0.$$

Consequently,

$$\begin{aligned}
\|x^{k_j+1}-x^{k_j}\| = & \left\| \alpha_k x^0 + (1-\alpha_k)\mathscr{P}_{W_{x^k}} \left[x^k - \chi\zeta\mathscr{A}\left(y^k\right)\right] - x^{k_j} \right\| \\
\leq & \alpha_k\|x^0-x^{k_j}\| + (1-\alpha_k) \left\| \mathscr{P}_{W_{x^k}} \left[x^k - \chi\zeta\mathscr{A}\left(y^k\right)\right] - x^{k_j} \right\| \\
\to & 0 \text{ as } j\to\infty.
\end{aligned}$$

Therefore, there exists a subsequence of $\{x^{k_j}\}$ such that it converges weakly to $\bar{x}$. Without loss of generality, we can assume that $x^{k_j} \rightharpoonup \bar{x}$ as $j\to\infty$. By the similar works as in the proof of Theorem 3.2, we also claim $\bar{x} \in \Theta(\mathbb{Q},\mathscr{A})$. Using (3.4) and (3.6), we have

$$\|x^{k_j}-q^*\|^2 \leq \|x^{k_j+1}-q^*\|^2 \leq (1-\alpha_{k_j})^2 \left\| x^{k_j}-q^* \right\|^2 + 2\alpha_{k_j}\langle x^0-q^*, x^{k_j+1}-q^*\rangle, \tag{3.8}$$

and hence $\limsup_{j\to\infty}\|x^{k_j}-q^*\|^2 \leq \limsup_{j\to\infty}\langle x^0-q^*, x^{k_j+1}-q^*\rangle$. Similarly, we have

$$\limsup_{j\to\infty}\langle x^0-q^*, x^{k_j+1}-q^*\rangle \leq 0.$$

Therefore, $x^k \to q^* = \mathscr{P}_{\Theta(\mathbb{Q},\mathscr{A})}(x^0)$. This implies the proof. □

4. COMPUTATIONAL EXPERIMENTS

At each iterate k, it is easy to see that the projection $\mathscr{P}_{W_{x^k}}$ in Theorem 3.2 and Theorem 3.3 are computed in the expilcit form:

$$\mathscr{P}_{W_{x^k}}[x^k-\chi\zeta\mathscr{A}(y^k)] = x^k - \chi\zeta\mathscr{A}(y^k) - \frac{\langle v^k, w-y^k\rangle - \gamma\|x^k-y^k\|^2}{\|v^k\|^2}v^k,$$

where $v^k = x^k - \chi\mathscr{A}(x^k) - y^k$.

Now, we discuss computation of the projection $\mathscr{P}_{C_{x^k}}\left[x^k - \chi\mathscr{A}(x^k)\right]$, at each iterate k, in Theorems 3.1-3.3. We consider the following cases of the set $\mathbb{Q}$:

Case 1: With $m = 1$, $\mathbb{Q} := \{x \in \mathscr{H} : c(x) \leq 0\}$, where $c : \mathscr{H} \to \mathscr{R}$ is lower semicontinuous, convex and *Gâteaux* differentiable. It is clear that $\mathscr{P}_{C_{x^k}}\left[x^k - \chi\mathscr{A}(x^k)\right]$ is computed in the explicit form:

$$\mathscr{P}_{C_{x^k}}\left[x^k - \chi\mathscr{A}(x^k)\right] = \begin{cases} x^k - \chi\mathscr{A}(x^k) - \frac{t^k}{\|\nabla c(x^k)\|^2}\nabla c(x^k) & \text{if } t^k \leq 0, \\ x^k - \chi\mathscr{A}(x^k) & \text{if } t^k > 0, \end{cases}$$

where $t^k = c(x^k) - \chi\langle\nabla c(x^k), F(x^k)\rangle$.

Case 2: With $m = 2$, $\mathbb{Q} := \{x \in \mathscr{H} : c_1(x) \leq 0, c_2(x) \leq 0\}$, where $c_i : \mathscr{H} \to \mathscr{R}$ is lower semicontinuous, convex and *Gâteaux* differentiable for each $i \in \{1,2\}$. By using [4, Proposition 28.19], we have

$$\mathscr{P}_{C_{x^k}}\left[x^k - \chi\mathscr{A}(x^k)\right] = x^k - \chi\mathscr{A}(x^k) + \max\{0,\lambda_1\}\nabla c_1(x^k) + \max\{0,\lambda_2\}\nabla c_2(x^k),$$

where (λ_1, λ_2) is a unique solution to the linear equation system:

$$\begin{cases} \lambda_1\|\nabla c_1(x^k)\|^2 - \chi\lambda_2\langle\nabla c_1(x^k), \mathscr{A}(x^k)\rangle = \chi\langle\nabla c_1(x^k), \mathscr{A}(x^k)\rangle - c_1(x^k) \\ \lambda_1\|\nabla c_2(x^k)\|^2 - \chi\lambda_2\langle\nabla c_2(x^k), \mathscr{A}(x^k)\rangle = \chi\langle\nabla c_2(x^k), \mathscr{A}(x^k)\rangle - c_2(x^k). \end{cases}$$

Case 3: With $m \geq 3$, the set $\mathbb{Q} := \{x \in \mathscr{H} : c_i(x) \leq 0 \quad \forall i \in I\}$, $I = \{1, 2, \ldots, m\}$. Then, C_{x^k} is a polyhedron with m edges and computing $\mathscr{P}_{C_{x^k}}\left[x^k - \chi\mathscr{A}(x^k)\right]$ can be evaluated by solving the following strongly convex quadratical programming:

$$\mathscr{P}_{C_{x^k}}\left[x^k - \chi\mathscr{A}(x^k)\right] = \text{argmin}\{\|x^k - \chi\mathscr{A}(x^k) - y\|^2 : y \in C_{x^k}\}.$$

In Matlab software, this quadratical programming is very effective by using the quadratic - program solvers from the Matlab Optimization Toolbox.

Example 4.1. Consider the problem VI($\mathbb{Q}, \mathscr{A}$) with the feasible set $\mathbb{Q} \subset \mathscr{R}^n$

$$\mathbb{Q} := \left\{x = (x_1, x_2, ..., x_n)^\top \in \mathscr{R}^n : c_1(x) := \|x\|^2 - 4 \leq 0, c_2(x) := x_1^2 - x_2 \leq 0\right\},$$

and the cost mapping $\mathscr{A} : \mathscr{R}^n \to \mathscr{R}^n$ is defined by $\mathscr{A}(x) = (3h(x_1), 2x_1 + x_2, x_3, ..., x_n)^\top$, where

$$h(t) = \begin{cases} e(t-1) + e, & \text{if } t > 1, \\ e^t, & \text{if } t \in [-1,1], \\ e^{-1}(t+1) + e^{-1}, & \text{otherwise.} \end{cases}$$

One can easily evaluate that the solution set $\Theta(\mathbb{Q}, \mathscr{A})$ of the problem VI($\mathbb{Q}, \mathscr{A}$) is nonempty and $(-1,1,0,...,0)^\top \in \Theta(\mathbb{Q}, \mathscr{A})$, the cost mapping $\mathscr{A}$ is Lipschitz continuous and monotone. For the implementation of the schemes, we first need to estimate data of $\mathscr{A}$ and c_i as follows:

- The $\mathscr{A}$ is monotone;
- The Lipschitz constant of $\mathscr{A}$ is $L = \sqrt{9e^2 + 5}$;
- The Lipschitz constant $\nabla c_i (i = 1,2)$ is $L_i = 2$ and hence $\mathscr{L} = 1$;
- Bounded parameter in (C_4): $M = 3\sqrt{e^2 + 1}$.

Then, assumptions $(C_1)-(C_4)$ are satisfied. The outer approximation sets are written in the following explicit forms:

$$C_{x^k} = \{x \in \mathscr{R}^n : \|x^k\|^2 - 4 + \langle 2x^k, x - x^k\rangle \leq 0, (x_1^k)^2 - x_2^k + \langle t^k, x - x^k\rangle \leq 0\},$$

where $t^k := (2x_1^k, -1, 0, ..., 0)^\top, x^k = (x_1^k, ..., x_n^k)^\top$, and

$$W_{x^k} = \{w \in \mathscr{R}^n : \langle x^k - \chi\mathscr{A}(x^k) - y^k, w - y^k\rangle \leq \gamma\|x^k - y^k\|^2\}.$$

All the following programming is implemented in Matlab 2025a running on a PC with Intel(R) Core(TM) $i9-9900KS\ CPU$ @ $4.00GHz$ 32.0 GB Ram.

Test 1. By using Theorem 2.1, iteration x^k belongs to $\Theta(\mathbb{Q}, \mathscr{A})$ if and only if $y^k = x^k$. Therefore, in this test, we evaluate the scheme (3.2) with $\Gamma_k := \|y^k - x^k\|$ and different starting points x^0 in the space $\mathscr{R}^5$. By Condition (2.8), the parameters are chosen randomly as follows:

$$\chi = \frac{1.5}{2(L+2M\mathscr{L})}, \zeta = \frac{1}{2}, \gamma = \frac{1-\chi(L+2M\mathscr{L})}{5}.$$

The numerical results of the algorithm are displayed in Figures 3-4 and Table 1.

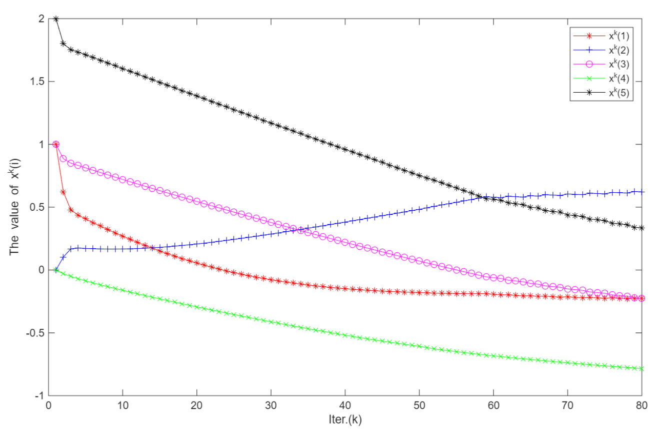

FIGURE 1. Performance of the sequence $\{x^k\}$ in the scheme (26) in the space $\mathscr{R}^5$ with $x^0 = (1,0,1,0,2)^\top$ and the stopping criterion $\|y^k - x^k\| \leq 10^{-5}$.

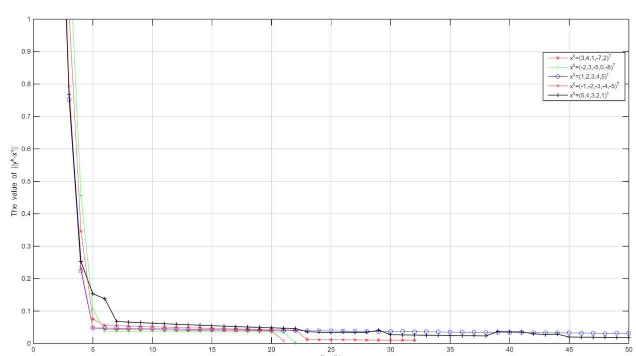

FIGURE 2. Performance of the sequence $\{\Gamma_k := \|y^k - x^k\|\}$ of Scheme (26) with five different points x^0.

Problem	Starting point x^0	Iteration	CPU time
1	$(3,4,1,-7,2)^\top$	32	0.35000
2	$(-2,3,-5,0,-8)^\top$	22	0.15156
3	$(1,2,3,4,5)^\top$	103	0.48750
4	$(-1,-2,-3,-4,-5)^\top$	21	0.10005
5	$(5,4,3,2,1)^\top$	82	0.23438
6	$(-5,-4,-3,-2,-1)^\top$	25	0.13438
7	$(0.1,0.2,0.3,0.4,0.5)^\top$	53	0.27500
8	$(-0.1,-0.2,-0.3,-0.4,-0.5)^\top$	28	0.10156
9	$(0.1,-0.2,0.3,-0.4,0.5)^\top$	35	0.27188
10	$(0,-2,4,-6,8)^\top$	56	0.35469

TABLE 1. Iterations (Iter.) and CPU times (Sec.) with randomly different points x^0 (Start.).

Test 2. Compare the performances of Scheme (3.2) with $TRSSEM-$ Totally Relaxed and Subgradient Extragradient Method of He et al. in [19, Algorithm 3.1] using approximation projections and $SEA-$ Subgradient Extragradient Algorithm proposed by Censor et al. in [12, Algorithm 2.1] using two approximation and metric projections. As usual, we use the stopping criterion $\Gamma_k := \|y^k - x^k\| \leq \varepsilon$ with the tolerance error $\varepsilon > 0$.
Parameters in each algorithm are chosen as follows:

- Scheme (3.2): Parameters are chosen as in **Test 1**;
- TRSSEM: $L = \max L_1, L_2 = 2, M' = ML = 6\sqrt{e^2+1}, \nu = 0.99, \sigma = 1, \rho = 0.04$;
- SEA: $\tau = 0.04$.

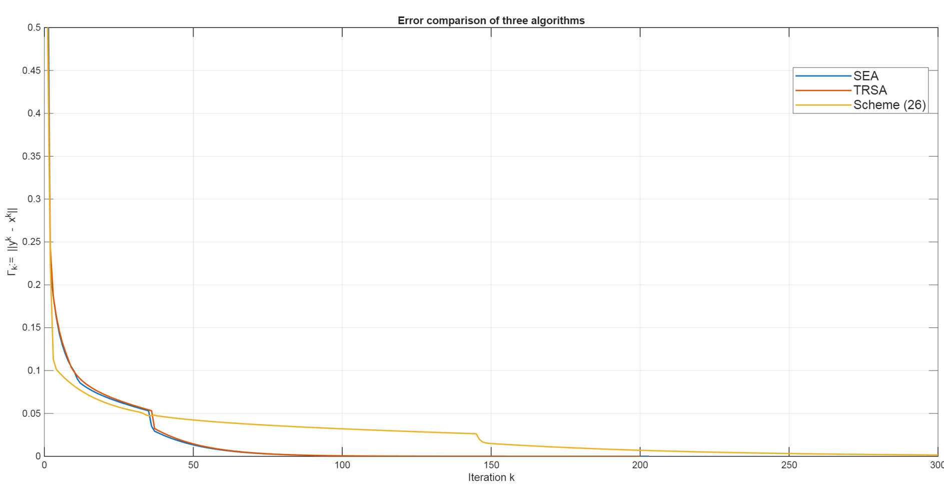

FIGURE 3. Performance of the sequence $\{\Gamma_k := \|y^k - x^k\|\}$ of Algorithms $TRSSEM, SEA$ and Scheme (26) with $x^0 = (3,4,1,-7,2)^\top \in \mathscr{R}^5$.

The comparative results are shown in Table 2 and Figure 5 with different dimensions n. Since

Dimension n	TRSSEM		SEA		Scheme (26)	
	CPU time	Iteration	CPU time	Iteration	CPU time	Iteration
5	2.070000	434	0.110000	83	0.10800	34
10	1.810000	454	0.080000	86	0.07660	51
20	1.890000	467	0.220000	89	0.32000	95
30	1.890000	468	0.480000	89	0.43200	146
50	1.960000	468	0.580000	89	0.94200	281

TABLE 2. Comparison of CPU time and number of iterations for three algorithms with $x^0 = (1, 0, 0.5, \ldots, 0.5)^T$.

the preliminary numerical results reported in Figures 3-4-5 and Tables 1-2 of three the above algorithms, we can see that the convergence speed of all the algorithms is very sensitive to the different starting points x^0. In some cases, the number of iterations of Scheme (26) are larger. However, the CPU time (second) of Scheme (26) are quite effective, less than two algorithms: *TRSSEM* using approximation projections and *SEA* using metric projections, when solving Problem VI($\mathbb{Q}, \mathscr{A}$).

Finally, we give a concluding remark to end this paper. This work presents two new solution mappings to the variational inequalities in Hilbert spaces. We claim that if $\mathscr{A}$ is partially monotone and Lipschitz continuous, then it has strong quasi-nonexpansiveness or firmly contractiveness. Then, the outer approximation schemes are constructed by the solution mappings and metric projection technique onto popyhedra. It is interesting here that we do not need to calculate the projections onto set $\mathbb{Q}$ at each iterate.

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