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NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS FOR A CLASS OF BILINEAR SYSTEMS IN INFINITE-DIMENSIONAL SPACES

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Abstract. We investigate optimal control problems for abstract bilinear evolution systems in infinite-dimensional Hilbert spaces. Working within a Gelfand triple structure, we analyze the minimization of a tracking-type cost functional subject to a bilinear state equation. We establish well-posedness results for weak solutions under minimal regularity assumptions and prove existence of optimal controls via direct methods. We then derive first-order necessary conditions through differentiation of the control-to-state map, and develop a comprehensive second-order analysis yielding sufficient conditions for local optimality: we prove that controls satisfying first-order conditions and positive definiteness of the cost functional Hessian are strict local minimizers. Our analysis accounts for the two-norm discrepancy between cost functional differentiability and coercivity spaces. The theoretical results provide a rigorous foundation for numerical implementation, while their practical relevance is illustrated through an application to a coupled reaction-diffusion system modeling biological populations.

Keywords. Bilinear control; Optimality conditions; Second-order analysis.
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1. INTRODUCTION

Optimal control of distributed parameter systems has long attracted attention due to its mathematical richness and broad range of applications. Within this framework, bilinear control systems, where the control acts multiplicatively on the state, are of particular practical relevance. Unlike linear control models, where the control enters as an external additive input such as a force or heat source, bilinear models capture mechanisms that directly modulate the internal parameters or structure of a system—such as reaction rates, conductances, or transition probabilities—making them particularly suited to applications where the control action is intrinsically multiplicative. As observed in the foundational works [28, 31], such structures arise naturally across engineering, physics, and biology. In neurobiology, the Hodgkin–Huxley model demonstrates that nerve excitation is governed by voltage-dependent modifications of membrane conductances, a fundamentally multiplicative mechanism [22, 23]. In thermodynamics and chemical engineering, bilinear controls appear when manipulating heat transfer coefficients or catalytic reaction rates. Reaction–diffusion systems in population dynamics and mathematical biology provide a further important class of bilinear models [16, 20, 21]. In each of these settings, the control acts not as an additive external input but as a modulator of the

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system's internal dynamics, motivating the abstract operator-theoretic framework developed in this paper. Although finite-dimensional bilinear systems are comparatively well understood, modeling such continuous-space processes requires infinite-dimensional formulations, which introduce substantial analytical difficulties through the interaction of unbounded differential operators with functional analytic structures.

This work addresses these difficulties within an abstract operator-theoretic framework. Bilinear evolution systems are considered in real separable Hilbert spaces. Let $V \hookrightarrow Y \hookrightarrow V^*$ be a Gelfand triple with dense and compact embeddings, where V, Y, and V^* denote the state, pivot, and dual spaces, respectively. The dynamics are governed by the Cauchy problem

$$\begin{cases} \dot{y}(t) = Ay(t) + B(u(t), y(t)) + f(t), \quad t \in (0,T), \\ y(0) = y_0 \in Y, \end{cases} \tag{1.1}$$

where $y(t)$ denotes the state of the system at time t, $\dot{y}(t)$ denotes the derivative of the state with respect to time, $u(t)$ is the control variable, $A : V \to V^*$ is a bounded linear operator such that $-A$ is coercive, $B : U \times Y \to Y$ is a bounded and bilinear control operator, U is a Banach control space assumed to be the dual of a separable Banach space with the Radon–Nikodym property and continuously embedded into a Hilbert space H, and $f \in L^2(0,T;V^*)$ represents an external forcing term.

The control objective is to minimize the quadratic cost functional

$$J(u) = \tfrac{a}{2}\|y_u(T) - y_d\|_Y^2 + \tfrac{b}{2}\int_0^T \|y_u(t) - y_d\|_Y^2 \,\mathrm{d}t + \tfrac{r}{2}\int_0^T \|u(t)\|_H^2 \,\mathrm{d}t, \tag{1.2}$$

over admissible controls $U_{\mathrm{ad}} \subset L^\infty(0,T;U)$, where y_u denotes the weak solution associated with the control u, $y_d \in Y$ being a desired state, and the parameters $a, b \geq 0$ and $r > 0$ satisfy $a + b > 0$. To ensure the existence of minimizers, the admissible set U_{ad} is assumed to be nonempty, convex, bounded, and closed in $L^\infty(0,T;U)$.

The quadratic structure of (1.2) is adopted for two complementary reasons. From a practical perspective, the tracking terms quantify the mean-square deviation of the system trajectory from a desired profile, while the control penalty $\frac{r}{2}\int_0^T \|u(t)\|_H^2 \,\mathrm{d}t$ corresponds to the energy expenditure of the actuator and constitutes the natural L^2-cost of the control effort in distributed-parameter systems; together, these three terms form the standard formulation encountered in engineering applications of optimal control. From a mathematical viewpoint, the quadratic structure is fundamental for the validity of the second-order sufficient conditions established later in the paper, since for non-quadratic cost functionals the positivity condition considered here is generally insufficient for optimality, as shown in [10, 24].

The main contributions of this paper are the following. First, the well-posedness of bilinear state equation (1.1) is established by proving existence, uniqueness, and energy estimates for weak solutions. Based on these results, the existence of optimal controls is obtained through direct variational arguments, and first-order necessary conditions are derived via differentiability properties of the control-to-state mapping. Owing to the bilinear coupling between the control and the state, the optimization problem is inherently nonconvex, so that a control satisfying the first-order condition may correspond to a saddle point rather than a local minimizer; consequently, second-order conditions become indispensable for characterizing local optimality. A complete second-order analysis is therefore developed, yielding both necessary and sufficient conditions. In particular, if an admissible control $\bar{u}$ satisfies the first-order condition together

with $J''(\bar{u})(v,v) > 0$, for all nonzero critical directions v, then $\bar{u}$ is a strict local minimizer in $L^2(0,T;H)$. A central analytical difficulty is the *two-norm discrepancy*: the objective functional J is differentiable with respect to controls in $L^2(0,T;U)$, whereas the positivity of the Hessian $J''(\bar{u})$ is established only in the weaker topology of $L^2(0,T;H)$. This structural mismatch, first identified in [7] for elliptic systems, complicates the derivation of quadratic growth conditions and remains a fundamental challenge in PDE-constrained optimization. Early pioneering works showed that the positive definiteness of the Hessian on an appropriate critical cone yields local optimality in stronger topologies, a principle subsequently extended in [36] to the Navier–Stokes equations and in [9] to state-constrained problems. An essential theoretical basis for the present analysis is the abstract framework developed in [10]. Working in general Banach spaces U_2 and U_∞ satisfying the continuous embedding $U_\infty \hookrightarrow U_2$, the authors resolved the two-norm discrepancy by proving that, under suitable continuity assumptions, standard second-order conditions imply a quadratic growth condition in the weaker norm. This framework has subsequently been extended in several directions, including nonsmooth semilinear parabolic equations [5], sparsity-promoting L^1-regularized control problems [8, 38], nonlocal equations with integral constraints [26], and weighted spaces involving point sources [19]. Additional applications include infinite-horizon problems [12], singular Allen–Cahn systems [34], and problems with mixed pointwise constraints [33].

In contrast, second-order analyses for bilinear control systems remain comparatively limited despite their strong practical relevance. In [3], first- and second-order optimality conditions were established for Fokker–Planck equations with scalar controls, while [29] extended these results to ensemble control problems with space–time dependent controls, and [17] investigated infinite-horizon bilinear systems with spatially localized controls. Complete second-order conditions for a bilinear wave equation posed on an infinite time horizon were derived in [15]. The Goh transform was employed in [2] for heat and wave equations to compensate for the effects of unbounded operators. More recent developments concern degenerate parabolic equations with boundary degeneracy [27], bilinear systems involving fractional Laplacian operators [4, 32], and discretized elliptic systems [13].

Despite these advances, abstract bilinear evolution systems in Gelfand triple settings remain comparatively underexplored. Most existing studies exploit PDE-specific structures that do not generalize to the operator-theoretic setting. The most closely related work to the present paper is [18], which derived second-order optimality conditions for an abstract bilinear system in a Gelfand triple over an infinite time horizon with vector-valued controls. However, the sufficient condition established therein is of coercivity type, requiring the existence of $\delta > 0$ such that $J''(\bar{u})(v,v) \geq \delta \|v\|^2$, for all nonzero critical directions v, which imposes a quantitative lower bound on the Hessian. This is a strictly stronger requirement than mere positivity, and its verification in practice is considerably more demanding. Bridging the gap between these two conditions requires handling the two-norm discrepancy between the space in which J is differentiable and the space in which the Hessian acts–a structural difficulty sidestepped there precisely by imposing coercivity, and which the present work resolves through Assumption 5.1 and Proposition 5.1. Beyond their theoretical role, second-order sufficient conditions provide essential tools for sensitivity and stability analysis [6, 14, 18, 25] and underpin convergence

theories in infinite-horizon settings [12]. Thus, they simultaneously establish rigorous optimality and ensure robustness of optimal controls under perturbations, offering both analytic and computational benefits.

The paper is structured as follows. Section 2 establishes well-posedness of bilinear evolution systems. Section 3 proves existence of optimal controls. Section 4 develops first- and second-order derivative representations. Section 5 derives necessary and sufficient optimality conditions. Section 6 applies the framework to a reaction–diffusion population model. Section 7, the last section, concludes with the main findings.

2. Well-posedness of the State Equation

This section establishes the mathematical framework and conditions guaranteeing existence and uniqueness of solutions to state equation (1.1). We begin by precisely specifying the structural assumptions on the operators and spaces involved.

We work within the Gelfand triple $V \hookrightarrow Y \hookrightarrow V^*$, where the embeddings are dense and compact. We denote by $\langle \cdot, \cdot \rangle_*$ the duality pairing between V^* and V, and by $\langle \cdot, \cdot \rangle$ the scalar product in the pivot space Y. The operator $A\colon V \to V^*$ satisfies two fundamental properties. First, the negative of A is coercive, meaning there exist constants $\alpha > 0$ and $\lambda \geq 0$ such that

$$\langle -Ay, y \rangle_* \geq \alpha \|y\|_V^2 - \lambda \|y\|_Y^2 \quad \forall y \in V. \tag{2.1}$$

Second, A is bounded, i.e., there exists $C_A > 0$ such that $|\langle Ay, z \rangle_*| \leq C_A \|y\|_V \|z\|_V$ for all $y, z \in V$. For the bilinear control operator $B\colon U \times Y \to Y$, we assume the following boundedness condition: there exists a constant $C_B > 0$ such that

$$\|B(u,y)\|_Y \leq C_B \|u\|_U \|y\|_Y \quad \forall u \in U, y \in Y. \tag{2.2}$$

We define the solution space $W(0,T)$ as $W(0,T) = \left\{ y \in L^2(0,T;V) \mid \dot{y} \in L^2(0,T;V^*) \right\}$, equipped with the norm $\|y\|_{W(0,T)} = \left(\|y\|^2_{L^2(0,T;V)} + \|\dot{y}\|^2_{L^2(0,T;V^*)} \right)^{1/2}$. By the Aubin-Lions lemma, $W(0,T)$ is compactly embedded in $L^2(0,T;Y)$ (see, e.g., [30, p. 58]). Furthermore, the continuous embedding $W(0,T) \hookrightarrow C([0,T];Y)$ ensures that the initial condition in our problem is well-defined. Within this framework, a function $y_u \in W(0,T)$ is called a *weak solution* to the state equation if it satisfies

$$\begin{cases} \dot{y}(t) = Ay(t) + B(u(t), y(t)) + f(t) & \text{in } L^2(0,T;V^*), \\ y(0) = y_0 & \text{in } Y. \end{cases}$$

Our primary well-posedness result is given by the following proposition, which establishes existence, uniqueness, and stability of solutions.

Proposition 2.1. *Let $y_0 \in Y$, $f \in L^2(0,T;V^*)$, and $u \in L^2(0,T;U)$. The state equation admits a unique weak solution $y_u \in W(0,T) \cap L^\infty(0,T;Y)$ satisfying*

$$\|y_u\|_{W(0,T)} + \|y_u\|_{L^\infty(0,T;Y)} \leq C_u \left(\|y_0\|_Y + \|f\|_{L^2(0,T;V^*)} \right), \tag{2.3}$$

where C_u is a non-decreasing function of $\|u\|_{L^2(0,T;U)}$ and is independent of f and y_0. Moreover, given two solutions y_1, y_2 corresponding to controls $u_1, u_2 \in L^2(0,T;U)$ and data $f_1, f_2 \in L^2(0,T;V^*)$, the stability estimate

$$\|y_1 - y_2\|_{W(0,T)} + \|y_1 - y_2\|_{L^\infty(0,T;Y)} \leq C_{1,2} \left(\|u_1 - u_2\|_{L^2(0,T;U)} + \|f_1 - f_2\|_{L^2(0,T;V^*)} \right)$$

holds, where $C_{1,2}$ depends on $\|u_1\|_{L^2(0,T;U)}, \|u_2\|_{L^2(0,T;U)}, \|f_1\|_{L^2(0,T;V^*)}, \|f_2\|_{L^2(0,T;V^*)}$.

Proof. Testing the weak formulation with $y_u(t)$ and using (2.1) and (2.2), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_u(t)\|_Y^2 + \alpha\|y_u(t)\|_V^2 \le \lambda\|y_u(t)\|_Y^2 + C_B\|u(t)\|_U\|y_u(t)\|_Y^2 + \|f(t)\|_{V^*}\|y_u(t)\|_V. \tag{2.4}$$

Using Young's inequality on the last term $\|f(t)\|_{V^*}\|y_u(t)\|_V \le \frac{\alpha}{2}\|y_u(t)\|_V^2 + \frac{1}{2\alpha}\|f(t)\|_{V^*}^2$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_u(t)\|_Y^2 + \frac{\alpha}{2}\|y_u(t)\|_V^2 \le \lambda\|y_u(t)\|_Y^2 + C_B\|u(t)\|_U\|y_u(t)\|_Y^2 + \frac{1}{2\alpha}\|f(t)\|_{V^*}^2.$$

Dropping the $\frac{\alpha}{2}\|y_u(t)\|_V^2$ term and applying Grönwall's inequality yields

$$\|y_u(t)\|_Y^2 \le \left(\|y_0\|_Y^2 + \frac{1}{\alpha}\|f\|_{L^2(0,T;V^*)}^2\right) e^{2\lambda T + 2C_B\|u\|_{L^1(0,T;U)}}.$$

Since $L^2(0,T;U) \hookrightarrow L^1(0,T;U)$, we have

$$\|y_u\|_{L^\infty(0,T;Y)} \le C_1(\|u\|_{L^2(0,T;U)})\left(\|y_0\|_Y + \|f\|_{L^2(0,T;V^*)}\right),$$

where $C_1(\rho) := \sqrt{1+\frac{1}{\alpha}}\, e^{\lambda T + C_B\sqrt{T}\rho}$ is non-decreasing in ρ. For the $L^2(0,T;V)$ estimate, integrating inequality (2.4), we obtain

$$\frac{\alpha}{2}\|y_u\|_{L^2(0,T;V)}^2 \le \|y_0\|_Y^2 + (\lambda T + C_B\sqrt{T}\|u\|_{L^2(0,T;U)})\|y_u\|_{L^\infty(0,T;Y)}^2 + \frac{1}{2\alpha}\|f\|_{L^2(0,T;V^*)}^2.$$

Substituting the $L^\infty(0,T;Y)$-bound yields

$$\|y_u\|_{L^2(0,T;V)} \le C_2(\|u\|_{L^2(0,T;U)})\left(\|y_0\|_Y + \|f\|_{L^2(0,T;V^*)}\right),$$

where $C_2(\rho) := \sqrt{\frac{2}{\alpha}\left(1 + (\lambda T + C_B\sqrt{T}\rho)C_1(\rho)^2 + \frac{1}{\alpha}\right)}$. For the time derivative, from $\dot{y}_u = Ay_u + B(u,y_u) + f$, we arrive at

$$\|\dot{y}_u\|_{L^2(0,T;V^*)} \le C_A\|y_u\|_{L^2(0,T;V)} + C_E C_B\|u\|_{L^2(0,T;U)}\|y_u\|_{L^\infty(0,T;Y)} + \|f\|_{L^2(0,T;V^*)},$$

where $C_E > 0$ is the embedding constant for $Y \hookrightarrow V^*$. Combining previous estimates gives

$$\|\dot{y}_u\|_{L^2(0,T;V^*)} \le C_3(\|u\|_{L^2(0,T;U)})\left(\|y_0\|_Y + \|f\|_{L^2(0,T;V^*)}\right),$$

where $C_3(\rho) := C_A C_2(\rho) + C_E C_B \rho C_1(\rho) + 1$.

Combining these estimates gives the norm bound in $W(0,T)$. Note that the existence and uniqueness follow from the a priori estimates and Faedo–Galerkin method. To prove the stability estimate, let y_1, y_2 solve (1.1) with controls u_1, u_2 and data f_1, f_2. The difference $w := y_1 - y_2$ satisfies $\dot{w} = Aw + B(u_1, w) + B(u_1 - u_2, y_2) + (f_1 - f_2)$. Testing with $w(t)$ and using Young's inequality yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t)\|_Y^2 + \frac{\alpha}{2}\|w(t)\|_V^2 &\le (\lambda + \frac{1}{2} + C_B\|u_1(t)\|_U)\|w(t)\|_Y^2 \\ &\quad + \frac{C_B^2}{2}\|u_1(t) - u_2(t)\|_U^2\|y_2(t)\|_Y^2 + \frac{1}{2\alpha}\|f_1(t) - f_2(t)\|_{V^*}^2.\end{aligned}$$

Applying Grönwall's inequality and using the $L^\infty(0,T;Y)$-bound for y_2, we see that

$$\|w\|_{W(0,T)} + \|w\|_{L^\infty(0,T;Y)} \le C_{1,2}\left(\|u_1 - u_2\|_{L^2(0,T;U)} + \|f_1 - f_2\|_{L^2(0,T;V^*)}\right),$$

where $C_{1,2}$ depends on $\|u_1\|_{L^2(0,T;U)}$, $\|u_2\|_{L^2(0,T;U)}$, $\|f_1\|_{L^2(0,T;V^*)}$, and $\|f_2\|_{L^2(0,T;V^*)}$ through the exponential terms. □

3. Existence of Optimal Controls

In this section, we establish the existence of at least one solution to the optimal control problem, which we formulate as

$$\min_{u\in U_{\text{ad}}} J(u). \tag{P}$$

We recall that the admissible control set U_{ad} is assumed to be a bounded, closed, and convex subset of $L^\infty(0,T;U)$.

The existence of optimal controls is ensured by the following theorem.

Theorem 3.1. *The optimal control problem* (P) *admits at least one solution.*

Proof. Because the set of admissible controls U_{ad} is non-empty and the cost functional $J(u)$ is bounded from below by zero, its infimum $j^* = \inf_{u\in U_{ad}} J(u)$ is finite. Consequently, we can find a minimizing sequence $(u_n) \subset U_{ad}$ such that $\lim_{n\to\infty} J(u_n) = j^*$. By assumption, U_{ad} is a bounded subset of $L^\infty(0,T;U)$, which means our minimizing sequence (u_n) is uniformly bounded. Recall that $L^\infty(0,T;U)$ is the dual of the separable space $L^1(0,T;U')$, where U' is the predual of U. Thanks to the Banach-Alaoglu theorem, we can extract a subsequence (which we still denote by (u_n)) that converges weakly-star to some limit $\overline{u} \in L^\infty(0,T;U)$. Furthermore, the closedness and convexity of U_{ad} guarantee that it is weakly-star closed, ensuring $\overline{u} \in U_{ad}$. For each n, let $y_n \in W(0,T)$ denote the solution to the state equation associated with the control u_n. As established in our earlier proposition, the bounds on y_n depend on a constant C_{u_n} which is non-decreasing with respect to $\|u_n\|_{L^2(0,T;U)}$. Since (u_n) is uniformly bounded, these constants C_{u_n} remain bounded independently of n. This naturally yields the uniform boundedness of the sequence of states (y_n) in $W(0,T) \cap L^\infty(0,T;Y)$. Because $W(0,T)$ is a Hilbert space, this boundedness allows us to extract a weakly converging subsequence, again denoted by (y_n), with a limit $\overline{y} \in W(0,T)$. Moreover, leveraging the continuous embedding $W(0,T) \hookrightarrow C([0,T];Y)$ alongside the compact embedding $W(0,T) \hookrightarrow L^2(0,T;Y)$, we can extract a further subsequence satisfying the following convergence:

- $y_n \rightharpoonup \overline{y}$ weakly in $W(0,T)$,
- $y_n \to \overline{y}$ strongly in $L^2(0,T;Y)$,
- $y_n(T) \rightharpoonup \overline{y}(T)$ weakly in Y.

Our next step is to verify that the limit $\overline{y}$ is indeed the state solution corresponding to the control $\overline{u}$. Knowing that y_n solves the state equation for u_n, we begin by testing its weak formulation against a sufficiently regular function $w \in W(0,T)$:

$$\int_0^T \langle \dot{y}_n, w\rangle_* dt = \int_0^T \langle Ay_n + B(u_n, y_n) + f, w\rangle_* dt.$$

As $n \to \infty$, the linear terms behave nicely. Specifically, $\dot{y}_n \rightharpoonup \dot{\overline{y}}$ weakly in $L^2(0,T;V^*)$, and the linearity and boundedness of A imply that $Ay_n \rightharpoonup A\overline{y}$ weakly in $L^2(0,T;V^*)$. Therefore, we easily obtain

$$\int_0^T \langle \dot{y}_n - Ay_n, w\rangle_* dt \longrightarrow \int_0^T \langle \dot{\overline{y}} - A\overline{y}, w\rangle_* dt.$$

Handling the bilinear term requires a bit more care. We split the difference into two parts

$$\int_0^T \langle B(u_n,y_n)-B(\overline{u},\overline{y}),w\rangle_* dt = \int_0^T \langle B(u_n,y_n-\overline{y}),w\rangle_* dt + \int_0^T \langle B(u_n-\overline{u},\overline{y}),w\rangle_* dt.$$

For the first term on the right-hand side, we can estimate it by

$$\begin{aligned}\int_0^T |\langle B(u_n,y_n-\overline{y}),w\rangle_*| dt &= \int_0^T |\langle B(u_n,y_n-\overline{y}),w\rangle| dt \\ &\leq |B|\|u_n\|_{L^2(0,T;U)}\|y_n-\overline{y}\|_{L^2(0,T;Y)}\|w\|_{L^\infty(0,T;Y)} \\ &\leq C\|y_n-\overline{y}\|_{L^2(0,T;Y)} \longrightarrow 0.\end{aligned}$$

Here, the constant C absorbs the uniform bound of the sequence (u_n) and the norm of the test function w. The entire expression vanishes as $n\to\infty$ thanks to the strong convergence of y_n to $\overline{y}$ in $L^2(0,T;Y)$. Regarding the second term, observe that, for the given $\overline{y}\in W(0,T)$ and $w\in W(0,T)$, the bilinear and bounded nature of B ensures that $u\mapsto \int_0^T \langle B(u,\overline{y}),w\rangle_* dt$ is a continuous and linear functional on $L^2(0,T;U)$. Thus, the weak-star convergence $u_n \rightharpoonup^* \overline{u}$ in $L^2(0,T;U)$ naturally implies that $\int_0^T \langle B(u_n-\overline{u},\overline{y}),w\rangle_* dt \longrightarrow 0$ as $n\to\infty$. Gathering all these limits, we successfully pass to the limit as $n\to\infty$ in the weak formulation, obtaining for every test function $w\in W(0,T)$ $\int_0^T \langle \dot{\overline{y}},w\rangle_* dt = \int_0^T \langle A\overline{y}+B(\overline{u},\overline{y})+f,w\rangle_* dt$. However, the standard weak formulation requires this equality to hold for any test function $v\in L^2(0,T;V)$, not just those in $W(0,T)$. We bridge this gap by using a standard density argument. Since $W(0,T)$ is dense in $L^2(0,T;V)$, any $v\in L^2(0,T;V)$ can be approximated by a sequence of functions $(w_k)\subset W(0,T)$. Passing to the limit as $k\to\infty$ confirms that the equality holds universally for all $v\in L^2(0,T;V)$. This rigorously establishes that $\overline{y}$ is indeed the weak solution corresponding to $\overline{u}$.

To conclude the proof, we evaluate the cost functional at the limit by considering the specific convergence of each term. The weak convergence $y_n(T)\rightharpoonup \overline{y}(T)$ in Y and the weak convergence $u_n\rightharpoonup^*\overline{u}$ in $L^2(0,T;H)$ allow us to invoke sequential lower semicontinuity for their respective norms. Meanwhile, the strong convergence $y_n\to\overline{y}$ in $L^2(0,T;Y)$ provides actual continuity for the tracking term. Combining these facts, we deduce

$$\begin{aligned}\|\overline{y}(T)-y_d\|_Y^2 &\leq \liminf_{n\to\infty}\|y_n(T)-y_d\|_Y^2, \\ \int_0^T \|\overline{y}-y_d\|_Y^2 dt &= \lim_{n\to\infty}\int_0^T \|y_n-y_d\|_Y^2 dt, \\ \int_0^T \|\overline{u}\|_H^2 dt &\leq \liminf_{n\to\infty}\int_0^T \|u_n\|_H^2 dt.\end{aligned}$$

Summing these relations yields $J(\overline{u})\leq \liminf_{n\to\infty} J(u_n)=j^*$. In view of $\overline{u}\in U_{ad}$, we see that $\overline{u}$ achieves the minimum, proving that it is an optimal control.

□

4. Differentiability Properties of the Cost Functional

In this section, we study the differentiability of the cost functional, characterizing its first- and second-order derivatives with explicit expressions involving adjoint variables and linearized states.

We begin by establishing the Fréchet differentiability of the control-to-state mapping.

Lemma 4.1. *The control-to-state mapping $G : L^2(0,T;U) \to W(0,T)$, $G(u) = y_u$ is twice Fréchet differentiable. For any $u, v, v_1, v_2 \in L^2(0,T;U)$, the derivatives $z_v = G'(u)v$ and $z_{v_1,v_2} = G''(u)(v_1,v_2)$ satisfy:*

$$\begin{cases} \dot{z}_v = Az_v + B(u,z_v) + B(v,y_u) & \text{in } L^2(0,T;V^*), \\ z_v(0) = 0 & \text{in } Y, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \dot{z}_{v_1,v_2} = Az_{v_1,v_2} + B(u,z_{v_1,v_2}) + B(v_1,z_{v_2}) + B(v_2,z_{v_1}) & \text{in } L^2(0,T;V^*), \\ z_{v_1,v_2}(0) = 0 & \text{in } Y, \end{cases} \tag{4.2}$$

Proof. Consider the nonlinear operator $F : W(0,T) \times L^2(0,T;U) \to L^2(0,T;V^*) \times Y$, defined by

$$F(y,u) = \left(\frac{\partial y}{\partial t} - Ay - B(u,y) - f,\ y(0) - y_0 \right).$$

The bilinear structure of B ensures that F is of class C^2. To see this, observe that the Fréchet derivative of F with respect to y at a point (y_u,u) is given by the linear operator

$$\frac{\partial F}{\partial y}(y_u,u)z = \left(\frac{\partial z}{\partial t} - Az - B(u,z),\ z(0) \right),$$

which acts continuously on $W(0,T)$. Proposition 2.1 guarantees that this derivative constitutes an isomorphism between $W(0,T)$ and $L^2(0,T;V^*) \times Y$. The implicit function theorem can now be applied to $F(y,u) = (0,0)$ at the point (y_u,u). This yields a neighborhood of u in $L^2(0,T;U)$, where the control-to-state mapping G is well-defined and of class C^2. To compute the first Fréchet derivative $z_v = G'(u)v$, differentiate the identity $F(G(u),u) = (0,0)$ with respect to u in the direction v. This results in $\frac{\partial F}{\partial y}(y_u,u)z_v + \frac{\partial F}{\partial u}(y_u,u)v = 0$. Substituting the expressions for the partial derivatives of F, we obtain the linear evolution equation (4.1) for z_v. For the second Fréchet derivative $z_{v_1,v_2} = G''(u)(v_1,v_2)$, differentiate the equation for $G'(u)v_1$ with respect to u in the direction v_2. This yields

$$\frac{\partial F}{\partial y}(y_u,u)z_{v_1,v_2} + \frac{\partial^2 F}{\partial y \partial u}(y_u,u)(z_{v_1},v_2) + \frac{\partial^2 F}{\partial u \partial y}(y_u,u)(v_1,z_{v_2}) = 0.$$

The bilinear terms $B(v_1,z_{v_2})$ and $B(v_2,z_{v_1})$ emerge from the cross-derivatives, leading to the equation (4.2) for z_{v_1,v_2}. The initial conditions $z_v(0) = 0$ and $z_{v_1,v_2}(0) = 0$ follow from the compatibility conditions in the definition of F. □

Following [1], we define the adjoint operator $B_U^* : U \times Y \to Y$ by $\langle x, B_U^*(u,w)\rangle = \langle B(u,x), w\rangle$ for all $(u,w,x) \in U \times Y \times Y$.

We now examine the existence and regularity of the adjoint state.

Lemma 4.2. *Let $u \in L^2(0,T;U)$ be a control and let $y_u \in W(0,T) \cap L^\infty(0,T;Y)$ denote the corresponding state solution to (1.1). Then, the adjoint equation*

$$\begin{cases} -\dot{\phi}(t) = A^*\phi(t) + B_U^*(u(t),\phi(t)) + b(y_u(t) - y_d) & \text{in } L^2(0,T;V^*), \\ \phi(T) = a(y_u(T) - y_d) & \text{in } Y, \end{cases} \tag{4.3}$$

admits a unique weak solution $\phi \in W(0,T) \cap L^\infty(0,T;Y)$. *Moreover, the solution satisfies the stability estimate*

$$\|\phi\|_{W(0,T)} + \|\phi\|_{L^\infty(0,T;Y)} \leq C_u^2 \left(a + C_E b\sqrt{T}\right) \left(\|y_d\|_Y + \|y_0\|_Y + \|f\|_{L^2(0,T;V^*)}\right), \tag{4.4}$$

where C_u *is the constant from Proposition 2.1, and* C_E *is the embedding constant of* $Y \hookrightarrow V^*$. *Furthermore, if* $u_1, u_2 \in L^2(0,T;U)$ *are two controls with corresponding adjoint states* ϕ_1, ϕ_2, *then the following estimate holds:*

$$\|\phi_1 - \phi_2\|_{W(0,T)} + \|\phi_1 - \phi_2\|_{L^\infty(0,T;Y)} \leq C'_{1,2} \|u_1 - u_2\|_{L^2(0,T;U)}, \tag{4.5}$$

where $C'_{1,2}$ *depends on* a, b, T, C_E, *and the norms* $\|u_1\|_{L^2(0,T;U)}$, $\|u_2\|_{L^2(0,T;U)}$, $\|f\|_{L^2(0,T;V^*)}$, *and* $\|y_d\|_Y$.

Proof. Let $\psi(t) = \phi(T-t)$. This transforms the backward adjoint equation into a forward equation

$$\begin{cases} \dot{\psi} = A^*\psi + B_U^*(u(T-t), \psi) + b(y_u(T-t) - y_d), \\ \psi(0) = a(y_u(T) - y_d). \end{cases}$$

The operator $A^* : V \to V^*$ inherits boundedness and coercivity from A. By definition of B_U^* and boundedness of B, one sees that $B_U^* : U \times Y \to Y$ is bounded and bilinear. Since $y_u \in L^\infty(0,T;Y) \hookrightarrow L^2(0,T;Y)$, using the continuous embedding $Y \hookrightarrow V^*$ with embedding constant C_E, one sees that $b(y_u - y_d)$ belongs to $L^2(0,T;V^*)$. The initial condition $\psi(0) = a(y_u(T) - y_d)$ lies in Y by the embedding $W(0,T) \hookrightarrow C([0,T];Y)$. Proposition 2.1 ensures the existence of a unique solution $\psi \in W(0,T) \cap L^\infty(0,T;Y)$ with

$$\|\psi\|_{W(0,T)} + \|\psi\|_{L^\infty(0,T;Y)} \leq C_u \left(\|\psi(0)\|_Y + \|b(y_u - y_d)\|_{L^2(0,T;V^*)}\right).$$

Using the estimate (2.3), we bound

$$\|\psi(0)\|_Y \leq a\left(\|y_u(T)\|_Y + \|y_d\|_Y\right) \leq a\left(C_u(\|y_0\|_Y + \|f\|_{L^2(0,T;V^*)}) + \|y_d\|_Y\right),$$

and, using the continuous embedding $Y \hookrightarrow V^*$, we have

$$\begin{aligned} \|b(y_u - y_d)\|_{L^2(0,T;V^*)} &\leq C_E b\sqrt{T}\left(\|y_u\|_{L^\infty(0,T;Y)} + \|y_d\|_Y\right) \\ &\leq C_E b\sqrt{T}\left(C_u(\|y_0\|_Y + \|f\|_{L^2(0,T;V^*)}) + \|y_d\|_Y\right). \end{aligned}$$

Combining these estimates yields

$$\begin{aligned} \|\psi\|_{W(0,T)} + \|\psi\|_{L^\infty(0,T;Y)} &\leq C_u \left(a + C_E b\sqrt{T}\right) \left(C_u(\|y_0\|_Y + \|f\|_{L^2(0,T;V^*)}) + \|y_d\|_Y\right) \\ &\leq C_u^2 \left(a + C_E b\sqrt{T}\right) \left(\|y_0\|_Y + \|f\|_{L^2(0,T;V^*)} + \|y_d\|_Y\right), \end{aligned}$$

where we used the fact that $1 \leq C_u$. Transforming back via $\phi(t) = \psi(T-t)$, we obtain a unique solution $\phi \in W(0,T) \cap L^\infty(0,T;Y)$ satisfying the adjoint equation and the energy estimate (4.4).

Finally, the Lipschitz continuity estimate (4.5) follows by repeating the same arguments as in the proof of Proposition 2.1, applied to the difference of two adjoint equations. □

Building upon the previous results, we now proceed to establish the second-order differentiability of the cost functional.

Corollary 4.1. *The cost functional J is twice Fréchet differentiable over $L^2(0,T;U)$. For controls $u,v,v_1,v_2 \in L^2(0,T;U)$, its derivatives are given by*

$$J'(u)v = a\langle z_v(T), y_u(T)-y_d\rangle + b\int_0^T \langle z_v(t), y_u(t)-y_d\rangle \,\mathrm{d}t + r\int_0^T \langle u(t),v(t)\rangle_H \,\mathrm{d}t, \tag{4.6}$$

and

$$\begin{aligned} J''(u)(v_1,v_2) &= a\langle z_{v_1}(T), z_{v_2}(T)\rangle + a\langle z_{v_1,v_2}(T), y_u(T)-y_d\rangle \\ &\quad + b\int_0^T \left(\langle z_{v_1}(t), z_{v_2}(t)\rangle + \langle z_{v_1,v_2}(t), y_u(t)-y_d\rangle\right)\mathrm{d}t + r\int_0^T \langle v_1(t),v_2(t)\rangle_H \,\mathrm{d}t, \end{aligned} \tag{4.7}$$

where $z_v = G'(u)v$, $z_{v_i} = G'(u)v_i$, and $z_{v_1,v_2} = G''(u)(v_1,v_2)$ solve the linearized and second-order linearized equations in (4.1) and (4.2), respectively. Furthermore, the derivatives admit equivalent expressions in terms of the adjoint variable ϕ

$$J'(u)v = \int_0^T \langle B(v(t), y_u(t)), \phi(t)\rangle \,\mathrm{d}t + r\int_0^T \langle u(t),v(t)\rangle_H \,\mathrm{d}t, \tag{4.8}$$

and

$$\begin{aligned} J''(u)(v_1,v_2) &= a\langle z_{v_1}(T), z_{v_2}(T)\rangle + b\int_0^T \langle z_{v_1}(t), z_{v_2}(t)\rangle \,\mathrm{d}t \\ &\quad + \int_0^T \langle B(v_1(t), z_{v_2}(t)) + B(v_2(t), z_{v_1}(t)), \phi(t)\rangle \,\mathrm{d}t + r\int_0^T \langle v_1(t),v_2(t)\rangle_H \,\mathrm{d}t, \end{aligned} \tag{4.9}$$

where $\phi \in W(0,T)\cap L^\infty(0,T;Y)$ uniquely solves adjoint equation (4.3).

Proof. The twice Fréchet differentiability of J follows directly from Lemma 4.1, which establishes the C^2-regularity of the control-to-state map G, and the smoothness of the quadratic terms in the cost functional. The expressions (4.6) and (4.7) are immediate consequences of the chain rule applied to the composition of G with the tracking terms and control penalty.

To derive the adjoint formulations (4.8)–(4.9), we consider the unique solution ϕ to (4.3) guaranteed by Lemma 4.2. Pair the adjoint equation with z_v over $[0,T]$

$$-\int_0^T \left\langle z_v(t), \dot\phi(t)\right\rangle_* \mathrm{d}t = \int_0^T \left\langle z_v(t), A^*\phi(t) + B_U^*(u(t),\phi(t)) + b(y_u(t)-y_d)\right\rangle_* \mathrm{d}t.$$

Integrating the left-hand side by parts and utilizing the initial condition $z_v(0)=0$ from (4.1), we have

$$-\int_0^T \left\langle z_v(t), \dot\phi(t)\right\rangle_* \mathrm{d}t = -\langle z_v(T),\phi(T)\rangle + \int_0^T \langle \dot z_v(t),\phi(t)\rangle_* \,\mathrm{d}t.$$

Substitute $\dot z_v$ from the linearized equation (4.1) and the terminal condition $\phi(T) = a(y_u(T)-y_d)$

$$\begin{aligned} &\int_0^T \langle Az_v(t) + B(u(t),z_v(t)) + B(v(t),y_u(t)),\phi(t)\rangle_* \,\mathrm{d}t - \langle z_v(T),\phi(T)\rangle \\ &\quad = \int_0^T \langle z_v(t), A^*\phi(t) + B_U^*(u(t),\phi(t))\rangle_* \,\mathrm{d}t + b\int_0^T \langle z_v(t), y_u(t)-y_d\rangle \,\mathrm{d}t. \end{aligned}$$

Invoking the adjoint relationships $\langle Az_v,\phi\rangle_* = \langle z_v, A^*\phi\rangle_*$ and $\langle B(u,z_v),\phi\rangle_* = \langle z_v, B_U^*(u,\phi)\rangle_*$, we cancel matching terms on both sides, yielding

$$\int_0^T \langle B(v(t),y_u(t)),\phi(t)\rangle \,\mathrm{d}t = a\langle z_v(T), y_u(T)-y_d\rangle + b\int_0^T \langle z_v(t), y_u(t)-y_d\rangle \,\mathrm{d}t.$$

Substituting this identity into the original expression (4.6) for $J'(u)v$ eliminates the state tracking terms and introduces the adjoint variable

$$J'(u)v = \int_0^T \langle B(v(t), y_u(t)), \phi(t)\rangle \, \mathrm{d}t + r\int_0^T \langle u(t), v(t)\rangle_H \, \mathrm{d}t,$$

which establishes (4.8).

For the second derivative $J''(u)(v_1, v_2)$, we proceed similarly. Pair the adjoint equation with z_{v_1,v_2} over $[0,T]$

$$-\int_0^T \left\langle z_{v_1,v_2}(t), \dot{\phi}(t)\right\rangle_* \mathrm{d}t = \int_0^T \left\langle z_{v_1,v_2}(t), A^*\phi(t) + B_U^*(u(t), \phi(t)) + b(y_u(t) - y_d)\right\rangle_* \mathrm{d}t.$$

Integrating the left-hand side by parts and using $z_{v_1,v_2}(0) = 0$ and $\phi(T) = a(y_u(T) - y_d)$, we have

$$-\int_0^T \left\langle z_{v_1,v_2}(t), \dot{\phi}(t)\right\rangle_* \mathrm{d}t = -\langle z_{v_1,v_2}(T), \phi(T)\rangle + \int_0^T \left\langle \dot{z}_{v_1,v_2}(t), \phi(t)\right\rangle_* \mathrm{d}t.$$

Substitute $\dot{z}_{v_1,v_2}$ in the second-order linearized equation (4.2), we have

$$\begin{aligned}
&\langle z_{v_1,v_2}(T), \phi(T)\rangle - \int_0^T \left\langle z_{v_1,v_2}(t), \dot{\phi}(t)\right\rangle_* \mathrm{d}t \\
&\quad = \int_0^T \langle Az_{v_1,v_2}(t) + B(u(t), z_{v_1,v_2}(t)) + B(v_1(t), z_{v_2}(t)) + B(v_2(t), z_{v_1}(t)), \phi(t)\rangle_* \, \mathrm{d}t.
\end{aligned}$$

Using the adjoint relationships and canceling terms, we obtain

$$\begin{aligned}
&\int_0^T \langle B(v_1(t), z_{v_2}(t)) + B(v_2(t), z_{v_1}(t)), \phi(t)\rangle \, \mathrm{d}t \\
&\qquad = a\langle z_{v_1,v_2}(T), y_u(T) - y_d\rangle + b\int_0^T \langle z_{v_1,v_2}(t), y_u(t) - y_d\rangle \, \mathrm{d}t.
\end{aligned}$$

Substituting this into the original expression (4.7) for $J''(u)(v_1, v_2)$ eliminates the state tracking terms and introduces the adjoint variable

$$\begin{aligned}
J''(u)(v_1, v_2) &= a\langle z_{v_1}(T), z_{v_2}(T)\rangle + b\int_0^T \langle z_{v_1}(t), z_{v_2}(t)\rangle \, \mathrm{d}t \\
&\quad + \int_0^T \langle B(v_1(t), z_{v_2}(t)) + B(v_2(t), z_{v_1}(t)), \phi(t)\rangle \, \mathrm{d}t + r\int_0^T \langle v_1(t), v_2(t)\rangle_H \, \mathrm{d}t,
\end{aligned}$$

which establishes (4.9). □

Next, we derive bounds on the first and second derivatives of the cost functional, which are essential for the second-order optimality analysis.

Lemma 4.3. *There exists a positive constant C such that, for all $u_1, u_2 \in U_{\mathrm{ad}}$ and $v_1, v_2 \in L^2(0,T;U)$, the following bounds hold:*

$$|J'(u_1)v_1| \le C\|v_1\|_{L^2(0,T;U)}, \tag{4.10}$$

$$|J''(u_1)(v_1, v_2)| \le C\|v_1\|_{L^2(0,T;U)}\|v_2\|_{L^2(0,T;U)}, \tag{4.11}$$

$$|J'(u_1)v_1 - J'(u_2)v_1| \le C\|u_1 - u_2\|_{L^2(0,T;U)}\|v_1\|_{L^2(0,T;U)}, \tag{4.12}$$

$$|J''(u_1)(v_1, v_1) - J''(u_2)(v_1, v_1)| \le C\|u_1 - u_2\|_{L^2(0,T;U)}\|v_1\|^2_{L^2(0,T;U)}. \tag{4.13}$$

Proof. Let $u_1, u_2 \in U_{\text{ad}}$ and $v_1, v_2 \in L^2(0,T;U)$. Let (y_i, ϕ_i) represent the state and adjoint state corresponding to the control u_i for $i = 1,2$. Additionally, let z_j^i denote the linearized state associated with control u_i in the direction of v_j.

First, since U_{ad} is bounded in $L^2(0,T;U)$, we can bound u_1, u_2, and their corresponding state/adjoint state by a constant independent of u_1 and u_2. For the linearized state, we have

$$\|z_j^i\|_{W(0,T)} + \|z_j^i\|_{L^\infty(0,T;Y)} \leq C\|v_j\|_{L^2(0,T;U)}.$$

From Corollary 4.1, we have the expressions for $J'(u)$ and $J''(u)$. We now establish the required bounds. For (4.10), using (4.8), the Cauchy-Schwarz inequality gives

$$\begin{aligned}|J'(u_1)v_1| &= \left| \int_0^T \langle B(v_1(t), y_1(t)), \phi_1(t)\rangle \, \mathrm{d}t + r\int_0^T \langle u_1(t), v_1(t)\rangle_H \, \mathrm{d}t \right| \\ &\leq \int_0^T \|B(v_1(t), y_1(t))\|_Y \|\phi_1(t)\|_Y \mathrm{d}t + r\|u_1\|_{L^2(0,T;H)}\|v_1\|_{L^2(0,T;H)}.\end{aligned}$$

Using the boundedness of B, y_1, and ϕ_1, we obtain $|J'(u_1)v_1| \leq C\|v_1\|_{L^2(0,T;U)}$. For (4.11), using (4.9), we apply the Cauchy-Schwarz inequality:

$$\begin{aligned}|J''(u_1)(v_1,v_2)| \leq\ & a\|z_1^1(T)\|_Y\|z_2^1(T)\|_Y + b\int_0^T \|z_1^1(t)\|_Y\|z_2^1(t)\|_Y \mathrm{d}t \\ &+ \int_0^T \Big(\|B(v_1(t), z_2^1(t))\|_Y\|\phi_1(t)\|_Y + \|B(v_2(t), z_1^1(t))\|_Y\|\phi_1(t)\|_Y\Big)\mathrm{d}t \\ &+ r\|v_1\|_{L^2(0,T;H)}\|v_2\|_{L^2(0,T;H)}.\end{aligned}$$

Using the boundedness of B, ϕ_1, and the estimates on z_j^1, we obtain

$$|J''(u_1)(v_1,v_2)| \leq C\|v_1\|_{L^2(0,T;U)}\|v_2\|_{L^2(0,T;U)}.$$

For (4.12), using the expression for $J'(u)$, we estimate

$$\begin{aligned}|J'(u_1)v_1 - J'(u_2)v_1| &= \left| \int_0^T \langle B(v_1(t), y_1(t)), \phi_1(t)\rangle \, \mathrm{d}t - \int_0^T \langle B(v_1(t), y_2(t)), \phi_2(t)\rangle \, \mathrm{d}t \right. \\ &\quad \left. + r\int_0^T \langle u_1(t) - u_2(t), v_1(t)\rangle_H \, \mathrm{d}t \right| \\ &\leq \left| \int_0^T \langle B(v_1(t), y_1(t) - y_2(t)), \phi_1(t)\rangle \, \mathrm{d}t \right| + \left| \int_0^T \langle B(v_1(t), y_2(t)), \phi_1(t) - \phi_2(t)\rangle \, \mathrm{d}t \right| \\ &\quad + r\|u_1 - u_2\|_{L^2(0,T;H)}\|v_1\|_{L^2(0,T;H)} \\ &\leq C\|y_1 - y_2\|_{L^2(0,T;Y)}\|v_1\|_{L^2(0,T;U)} + C\|\phi_1 - \phi_2\|_{L^2(0,T;Y)}\|v_1\|_{L^2(0,T;U)} \\ &\quad + r\|u_1 - u_2\|_{L^2(0,T;H)}\|v_1\|_{L^2(0,T;H)}.\end{aligned}$$

Since y_i and ϕ_i depend Lipschitz-continuously on u_i, we obtain

$$|J'(u_1)v_1 - J'(u_2)v_1| \leq C\|u_1 - u_2\|_{L^2(0,T;U)}\|v_1\|_{L^2(0,T;U)}.$$

For (4.13), we use the definition of $J''(u)$ from (4.9) and estimate

$$\begin{aligned}|J''(u_1)(v_1,v_1)-J''(u_2)(v_1,v_1)| \le a\left|\langle z_1^1(T),z_1^1(T)\rangle - \langle z_1^2(T),z_1^2(T)\rangle\right| \\ + b\left|\int_0^T \left(\langle z_1^1(t),z_1^1(t)\rangle - \langle z_1^2(t),z_1^2(t)\rangle\right)\mathrm{d}t\right| \\ + \left|\int_0^T \left\langle 2B(v_1(t),z_1^1(t)),\phi_1(t)\right\rangle - \left\langle 2B(v_1(t),z_1^2(t)),\phi_2(t)\right\rangle \mathrm{d}t\right|.\end{aligned}$$

For the first term, using the polarization identity:

$$\langle z_1^1(T),z_1^1(T)\rangle - \langle z_1^2(T),z_1^2(T)\rangle = \langle z_1^1(T)-z_1^2(T),z_1^1(T)+z_1^2(T)\rangle,$$

we bound it by $a\|z_1^1(T)-z_1^2(T)\|_Y\|z_1^1(T)+z_1^2(T)\|_Y \le C\|u_1-u_2\|_{L^2(0,T;U)}\|v_1\|^2_{L^2(0,T;U)}$. Similarly, the second term becomes

$$b\int_0^T \langle z_1^1(t)-z_1^2(t),z_1^1(t)+z_1^2(t)\rangle \mathrm{d}t \le C\|u_1-u_2\|_{L^2(0,T;U)}\|v_1\|^2_{L^2(0,T;U)}.$$

For the third term, we split it as

$$\begin{aligned}&2\int_0^T \left(\langle B(v_1(t),z_1^1(t)-z_1^2(t)),\phi_1(t)\rangle + \langle B(v_1(t),z_1^2(t)),\phi_1(t)-\phi_2(t)\rangle\right)\mathrm{d}t \\ &\le C\left(\|z_1^1-z_1^2\|_{L^2(0,T;Y)}\|\phi_1\|_{L^2(0,T;Y)} + \|z_1^2\|_{L^2(0,T;Y)}\|\phi_1-\phi_2\|_{L^2(0,T;Y)}\right) \\ &\le C\|u_1-u_2\|_{L^2(0,T;U)}\|v_1\|^2_{L^2(0,T;U)}.\end{aligned}$$

Combining these estimates yields

$$|J''(u_1)(v_1,v_1)-J''(u_2)(v_1,v_1)| \le C\|u_1-u_2\|_{L^2(0,T;U)}\|v_1\|^2_{L^2(0,T;U)}.$$

Thus, all the required bounds hold, completing the proof. □

5. Optimality Conditions

This section establishes the mathematical characterization of optimal controls through necessary and sufficient conditions. We first derive first-order necessary conditions that any optimal control must satisfy, then develop a second-order analysis that provide deeper insights characterization and identification of potential solutions.

5.1. **First-order necessary conditions.**

The first-order optimality condition is expressed as a variational inequality involving the derivative of the cost functional. It characterizes optimal controls in terms of the state and adjoint variables, which are linked through a coupled system.

Theorem 5.1. *Let $\bar{u}$ be an optimal control for Problem (P). Then, there exist state and adjoint variables $\bar{y},\bar{\phi} \in W(0,T)$ satisfying:*

$$\begin{cases}\dot{\bar{y}}(t) = A\bar{y}(t) + B(\bar{u}(t),\bar{y}(t)) + f(t) \quad \text{in } L^2(0,T;V^*),\\ \bar{y}(0) = y_0 \quad \text{in } Y,\end{cases}$$

and

$$\begin{cases}-\dot{\bar{\phi}}(t) = A^*\bar{\phi}(t) + B_U^*(\bar{u}(t),\bar{\phi}(t)) + b(\bar{y}(t)-y_d) \quad \text{in } L^2(0,T;V^*),\\ \bar{\phi}(T) = a(\bar{y}(T)-y_d) \quad \text{in } Y,\end{cases}$$

along with the variational inequality: $J'(\bar{u})(u-\bar{u}) \ge 0$ for all $u \in U_{\text{ad}}$.

Proof. For any $u \in U_{\text{ad}}$ and $\theta \in (0,1)$, the convex combination $\bar{u} + \theta(u - \bar{u})$ remains in U_{ad}. The optimality of $\bar{u}$ implies

$$0 \leq \lim_{\theta \to 0^+} \frac{J(\bar{u} + \theta(u - \bar{u})) - J(\bar{u})}{\theta} = J'(\bar{u})(u - \bar{u}).$$

This concludes the proof. □

5.2. **Second-order optimality conditions.** We now derive both necessary and sufficient second-order optimality conditions, based on the positive definiteness of the second derivative of the cost functional over appropriate critical cones. These conditions require additional regularity of the state and adjoint variables, which is ensured by the following assumption on the spaces and operators involved.

Assumption 5.1. There exists a Banach space X, densely and continuously embedded in Y, such that

(1) For every control $u \in L^2(0,T;U)$, initial and target states $y_0, y_d \in X$, and forcing term $f \in L^\infty(0,T;X)$, the corresponding state y_u satisfies $y_u \in L^\infty(0,T;X)$, and the corresponding adjoint state ϕ_u also belongs to $L^\infty(0,T;X)$.
(2) The bilinear operator $B : U \times Y \to Y$ admits continuous extensions

$$B : H \times X \to Y, \qquad B : H \times Y \to X^*, \qquad B : H \times Y \to V^*,$$

Remark 5.1.
(1) Assumption 5.1 is trivially satisfied when $U = H$, which includes the important case of scalar controls $u = u(t) \in \mathbb{R}$. In this setting, one can take $X = Y$, and no extension of the bilinear operator is needed.
(2) When $U \neq H$, the assumption requires identifying a suitable state space X that provides the additional regularity needed. Specifically, the state variable must have enough smoothness or integrability so that the bilinear operator B can be continuously extended to act on $H \times X$ with values in Y, and on $H \times Y$ with values in the dual space X^*.
(3) In the case of parabolic PDEs, such regularity can often be ensured through an appropriate choice of the system data.
(4) When controls valued only in H, Assumption 5.1 (2) indeed requires both extensions $B : H \times Y \to X^*$ and $B : H \times Y \to V^*$; since X^* and V^* are both superspaces of Y that are not comparable by inclusion, neither extension can be derived from the other, and each fulfills a genuinely distinct role: the V^*-valued extension underlies the continuity estimates in the proof of Lemma 5.1, whereas the X^*-valued extension is precisely what drives the convergence arguments in Proposition 5.1.

For the subsequent analysis, we define the following control spaces:

$$U_\infty := L^\infty(0,T;U), \quad \text{with norm } \|u\|_\infty := \operatorname*{ess\,sup}_{t \in (0,T)} \|u(t)\|_U,$$

$$U_2 := L^2(0,T;H), \quad \text{with norm } \|u\|_2 := \left(\int_0^T \|u(t)\|_H^2 \, dt \right)^{1/2}.$$

We first establish continuity estimates for the state, adjoint, and linearized equations.

Lemma 5.1. *Let $y_0, y_d \in X$ and $f \in L^\infty(0,T;X)$, and assume that Assumption 5.1 holds. For $u_1, u_2 \in U_\infty$ with corresponding states y_1, y_2 and adjoint states ϕ_1, ϕ_2, there exists a constant $C > 0$, depending non-decreasingly on $\|u_1\|_\infty$, $\|u_2\|_\infty$, $\|y_d\|_X$, $\|y_1\|_{L^\infty(0,T;X)}$, $\|y_2\|_{L^\infty(0,T;X)}$, such that*

$$\|y_1 - y_2\|_{W(0,T)} + \|y_1 - y_2\|_{L^\infty(0,T;Y)} \le C\,\|u_1 - u_2\|_2,$$
$$\|\phi_1 - \phi_2\|_{W(0,T)} + \|\phi_1 - \phi_2\|_{L^\infty(0,T;Y)} \le C\,\|u_1 - u_2\|_2.$$

Moreover, for $u \in U_\infty$ and $v \in U_2$, linearized system (4.1) admits a unique solution z_v satisfying $\|z_v\|_{W(0,T)} + \|z_v\|_{L^\infty(0,T;Y)} \le C\,\|v\|_2$, and for $u_1, u_2 \in U_\infty$ with corresponding linearized states z_v^1, z_v^2, it holds $\|z_v^1 - z_v^2\|_{W(0,T)} + \|z_v^1 - z_v^2\|_{L^\infty(0,T;Y)} \le C\,\|u_1 - u_2\|_2\,\|v\|_2$.

Proof. **Step 1. State difference estimates.** Let $u_1, u_2 \in U_\infty$ be given and let $y_1, y_2 \in W(0,T) \cap L^\infty(0,T;X)$ denote the corresponding state solutions. Set $y := y_1 - y_2$. Subtracting the two state equations yields

$$\begin{cases} \dot y(t) = Ay(t) + B(u_1(t), y(t)) + B\big(u_1(t) - u_2(t), y_2(t)\big), & t \in (0,T), \\ y(0) = 0. \end{cases}$$

Applying Proposition 2.1 to this problem with forcing term $B(u_1 - u_2, y_2)$ gives

$$\|y\|_{W(0,T)} + \|y\|_{L^\infty(0,T;Y)} \;\le\; C\,\|B(u_1 - u_2, y_2)\|_{L^2(0,T;V^*)}, \tag{5.1}$$

where C depends on $\|u_1\|_\infty$ but not on y_0 or f. By Assumption 5.1 (2), $B : H \times X \to Y$ is continuous. Since $Y \hookrightarrow V^*$ continuously, there exists $C > 0$ such that

$$\|B(u_1(t) - u_2(t), y_2(t))\|_{V^*} \;\le\; C\,\|u_1(t) - u_2(t)\|_H\,\|y_2(t)\|_X \quad \text{for a.e. } t \in (0,T).$$

Taking $L^2(0,T;\mathbb{R})$-norm to both side of the pointwise inequality

$$\|B(u_1 - u_2, y_2)\|_{L^2(0,T;V^*)} \;\le\; C\,\|u_1 - u_2\|_{L^2(0,T;H)}\,\|y_2\|_{L^\infty(0,T;X)}.$$

Substituting into (5.1) yields $\|y_1 - y_2\|_{W(0,T)} + \|y_1 - y_2\|_{L^\infty(0,T;Y)} \;\le\; C\,\|u_1 - u_2\|_2$ with C depending on $\|u_1\|_\infty, \|u_2\|_\infty, \|y_2\|_{L^\infty(0,T;X)}$.

Step 2. Adjoint difference estimates. Let $\phi_1, \phi_2 \in W(0,T) \cap L^\infty(0,T;X)$ be the adjoint states corresponding to u_1, u_2 and y_1, y_2, satisfying (4.3). Define $\phi := \phi_1 - \phi_2$. Then the adjoint system for ϕ can be written as

$$\begin{cases} -\dot\phi(t) = A^*\phi(t) + B_U^*(u_1(t), \phi(t)) + B_U^*(u_1(t) - u_2(t), \phi_2(t)) + b(y_1(t) - y_2(t)), & t \in (0,T), \\ \phi(T) = a(y_1(T) - y_2(T)). \end{cases}$$

Applying Lemma 4.2 to this system and using the continuity of $B_U^* : H \times X \to Y$ yields

$$\|\phi\|_{W(0,T)} + \|\phi\|_{L^\infty(0,T;Y)} \le C\Big(\|u_1 - u_2\|_2 + \|y_1 - y_2\|_{L^2(0,T;V^*)} + \|y_1(T) - y_2(T)\|_Y\Big).$$

Using the estimate from Step 1, we conclude $\|\phi_1 - \phi_2\|_{W(0,T)} + \|\phi_1 - \phi_2\|_{L^\infty(0,T;Y)} \le C\,\|u_1 - u_2\|_2$, with C depending non-decreasingly on $\|u_1\|_\infty$,$\|u_2\|_\infty$, $\|y_1\|_{L^\infty(0,T;X)}$,$\|y_2\|_{L^\infty(0,T;X)}$, $\|y_d\|_X$, a, b, T.

Step 3. Linearized system estimates. For $u \in U_\infty$ and $v \in U_2$, consider the linearized system

$$\begin{cases} \dot{z}_v(t) = Az_v(t) + B(u(t), z_v(t)) + B(v(t), y_u(t)), \\ z_v(0) = 0. \end{cases}$$

We first verify that this system admits a unique solution $z_v \in W(0,T) \cap L^\infty(0,T;Y)$. By Assumption 5.1, the operator $B : H \times X \to Y$ is continuous, so that, for a.e. $t \in (0,T)$,

$$\|B(v(t), y_u(t))\|_Y \leq C_B \|v(t)\|_H \|y_u(t)\|_X.$$

Since $v \in U_2 = L^2(0,T;H)$ and $y_u \in L^\infty(0,T;X)$, we deduce that $B(v, y_u) \in L^2(0,T;Y) \hookrightarrow L^2(0,T;V^*)$, which serves as a source term with sufficient regularity. Moreover, $B(u(\cdot), z_v(\cdot))$ is well-defined and $z \mapsto B(u,z)$ is bounded from $L^2(0,T;Y)$ to $L^2(0,T;Y)$. Since $u \in U_\infty$ and $-A$ is coercive, Proposition 2.1 applies directly and guarantees the existence of a unique solution $z_v \in W(0,T) \cap L^\infty(0,T;Y)$, satisfying

$$\|z_v\|_{W(0,T)} + \|z_v\|_{L^\infty(0,T;Y)} \leq C \|B(v, y_u)\|_{L^2(0,T;V^*)} \leq C \|v\|_2 \|y_u\|_{L^\infty(0,T;X)}.$$

This gives $\|z_v\|_{W(0,T)} + \|z_v\|_{L^\infty(0,T;Y)} \leq C\|v\|_2$.

Step 4. Lipschitz estimate for linearized states. Let $u_1, u_2 \in U_\infty$ and $v \in U_2$, and denote z_v^1, z_v^2 the corresponding solutions of (4.1). Set $z := z_v^1 - z_v^2$. Then z satisfies

$$\begin{cases} \dot{z}(t) = Az(t) + B(u_1(t), z(t)) + B(u_1(t) - u_2(t), z_v^2(t)) + B(v(t), y_1(t) - y_2(t)), \\ z(0) = 0. \end{cases}$$

Using Proposition 2.1, Steps 1 and 3, and the continuity of B from Assumption 5.1, we obtain

$$\|z_v^1 - z_v^2\|_{W(0,T)} + \|z_v^1 - z_v^2\|_{L^\infty(0,T;Y)} \leq C \|u_1 - u_2\|_2 \|v\|_2,$$

where C depends on $\|u_1\|_\infty, \|u_2\|_\infty$, and $\|y_1\|_{L^\infty(0,T;X)}, \|y_2\|_{L^\infty(0,T;X)}$. This completes the proof. □

Remark 5.2. Let Assumption 5.1 hold, and let $y_0, y_d \in X$ and $f \in L^\infty(0,T;X)$. For a given $u \in U_\infty$, the derivatives $J'(u)$ and $J''(u)$ are respectively a linear and a bilinear form originally defined on U_∞ and $U_\infty \times U_\infty$. A key question is whether these derivatives admit well-defined and continuous extensions to U_2 and $U_2 \times U_2$, respectively. We first address the well-definedness here, and the continuity of these extensions is then established in Lemma 5.2 and Proposition 5.1 below. By Lemma 5.1, the state and adjoint satisfy $y_u, \phi_u \in L^\infty(0,T;X) \cap W(0,T)$, and the linearized state z_v is well-posed with $z_v \in W(0,T) \cap L^\infty(0,T;Y)$ for every $v \in U_2$.

First derivative. By Assumption 5.1 (2), the operator $B : H \times X \to Y$ is continuous. Since $v(t) \in H$ and $y_u(t) \in X$ for almost every $t \in (0,T)$, we have $B(v(t), y_u(t)) \in Y$ for a.e. t, with $\|B(v(t), y_u(t))\|_Y \leq C\|v(t)\|_H \|y_u(t)\|_X$. Since $v \in L^2(0,T;H)$ and $y_u \in L^\infty(0,T;X)$, we see that $t \mapsto B(v(t), y_u(t))$ belongs to $L^2(0,T;Y)$. Together with $\phi_u \in L^2(0,T;Y)$, the integral

$$\int_0^T \langle B(v(t), y_u(t)), \phi_u(t) \rangle \, \mathrm{d}t$$

appearing in the first derivative (4.8) is therefore well-defined for every $v \in U_2$.

Second derivative. For the second derivative, the bilinear terms $B(v_i(t), z_{v_j}(t))$ involve $v_i(t) \in H$ and $z_{v_j}(t) \in Y$, so the extension $B : H \times Y \to X^*$ from Assumption 5.1 (2) is required. This extension ensures $B(v_i(t), z_{v_j}(t)) \in X^*$ for a.e. t, with $\|B(v_i(t), z_{v_j}(t))\|_{X^*} \leq C\|v_i(t)\|_H \|z_{v_j}(t)\|_Y$.

Since $v_i \in L^2(0,T;H)$ and $z_{v_j} \in L^\infty(0,T;Y)$, the map $t \mapsto B(v_i(t), z_{v_j}(t))$ belongs to $L^2(0,T;X^*)$. Together with $\phi_u \in L^\infty(0,T;X)$, the duality pairing

$$\int_0^T \langle B(v_1(t), z_{v_2}(t)) + B(v_2(t), z_{v_1}(t)),\, \phi_u(t)\rangle_{X^*,X}\, \mathrm{d}t$$

appearing in the second derivative (4.9) is well-defined for all $v_1, v_2 \in U_2$.

The next result establishes that the well-defined expressions above are in fact bounded, and hence extend continuously from U_∞ to the larger spaces U_2 and $U_2 \times U_2$.

Lemma 5.2. *Let Assumption 5.1 hold, and let $y_0, y_d \in X$ and $f \in L^\infty(0,T;X)$. There exists a constant $C > 0$ such that, for all $u_1, u_2 \in U_{\mathrm{ad}}$ and $v_1, v_2 \in U_2$, the following estimates hold:*

$$|J'(u_1)v_1| \le C\|v_1\|_2,$$

$$|J''(u_1)(v_1,v_2)| \le C\|v_1\|_2\|v_2\|_2,$$

$$|J'(u_1)v_1 - J'(u_2)v_1| \le C\|u_1 - u_2\|_2\|v_1\|_2, \tag{5.2}$$

$$|J''(u_1)(v_1,v_1) - J''(u_2)(v_1,v_1)| \le C\|u_1 - u_2\|_2\|v_1\|_2^2. \tag{5.3}$$

Proof. Let $u_1, u_2 \in U_{\mathrm{ad}} \subset U_\infty$, and let y_i, ϕ_i be the state and adjoint associated with u_i $(i = 1,2)$. For $u \in U_\infty$ and $v \in U_2$, denote by z_v the solution of (4.1). By Lemma 5.1, one has

$$\|y_u\|_{L^\infty(0,T;X)} + \|\phi_u\|_{L^\infty(0,T;X)} \le C, \quad \|z_v\|_{L^\infty(0,T;Y)} + \|z_v\|_{W(0,T)} \le C\|v\|_2, \tag{5.4}$$

and, for $u_1, u_2 \in U_\infty$, $v \in U_2$,

$$\|z_v^1 - z_v^2\|_{L^\infty(0,T;Y)} + \|z_v^1 - z_v^2\|_{W(0,T)} \le C\|u_1 - u_2\|_2\,\|v\|_2, \tag{5.5}$$

as well as

$$\begin{aligned}\|y_1 - y_2\|_{W(0,T)} + \|y_1 - y_2\|_{L^\infty(0,T;Y)} \\ + \|\phi_1 - \phi_2\|_{W(0,T)} + \|\phi_1 - \phi_2\|_{L^\infty(0,T;Y)} \le C\|u_1 - u_2\|_2.\end{aligned} \tag{5.6}$$

(i) Bound for $J'(u_1)v_1$.

Using (4.8) with the extension $B : H \times X \to Y$ from Assumption 5.1 (2), and Hölder's inequality, we find by (5.4) that

$$|J'(u_1)v_1| \le \int_0^T \|B(v_1(t), y_1(t))\|_Y\, \|\phi_1(t)\|_Y\, \mathrm{d}t + r\|u_1\|_2\|v_1\|_2 \le C\|v_1\|_2.$$

(ii) Bound for $J''(u_1)(v_1,v_2)$.

Using (4.9) and the extension $B : H \times Y \to X^*$ from Assumption 5.1 (2), and applying Cauchy–Schwarz, one has

$$\begin{aligned}|J''(u_1)(v_1,v_2)| &\le a\|z_{v_1}(T)\|_Y\, \|z_{v_2}(T)\|_Y + b\int_0^T \|z_{v_1}(t)\|_Y\, \|z_{v_2}(t)\|_Y\, \mathrm{d}t \\ &\quad + \int_0^T \|B(v_1(t), z_{v_2}(t)) + B(v_2(t), z_{v_1}(t))\|_{X^*}\|\phi_1(t)\|_X\, \mathrm{d}t + r\|v_1\|_2\|v_2\|_2 \\ &\le C\|v_1\|_2\|v_2\|_2,\end{aligned}$$

where the last inequality follows from (5.4), the bound $\|\phi_1\|_{L^\infty(0,T;X)} \le C$, and the continuity of $B : H \times Y \to X^*$.

(iii) Lipschitz bound for J'.

From (4.8) and the estimates (5.6), it follows that

$$\begin{aligned}|J'(u_1)v_1 - J'(u_2)v_1| &\le \left|\int_0^T \langle B(v_1(t), y_1(t) - y_2(t)), \phi_1(t)\rangle \, \mathrm{d}t\right| \\ &\quad + \left|\int_0^T \langle B(v_1(t), y_2(t)), \phi_1(t) - \phi_2(t)\rangle \, \mathrm{d}t\right| + r\|u_1 - u_2\|_2 \, \|v_1\|_2 \\ &\le C\Big(\|y_1 - y_2\|_{L^2(0,T;Y)} + \|\phi_1 - \phi_2\|_{L^2(0,T;Y)} + \|u_1 - u_2\|_2\Big)\|v_1\|_2 \\ &\le C\|u_1 - u_2\|_2 \, \|v_1\|_2.\end{aligned}$$

(iv) Lipschitz bound for J''.

Let $z_i := z^i_{v_1}$ for $i = 1, 2$. Using (4.9) and the extension $B : H \times Y \to X^*$, one has

$$\begin{aligned}|J''(u_1)(v_1, v_1) - J''(u_2)(v_1, v_1)| &\le a\,|\langle z_1(T), z_1(T)\rangle - \langle z_2(T), z_2(T)\rangle| \\ &\quad + b\int_0^T |\langle z_1(t), z_1(t)\rangle - \langle z_2(t), z_2(t)\rangle| \, \mathrm{d}t \\ &\quad + 2\left|\int_0^T \langle B(v_1(t), z_1(t)) - B(v_1(t), z_2(t)), \phi_1(t)\rangle_{X^*,X} \, \mathrm{d}t\right| \\ &\quad + 2\left|\int_0^T \langle B(v_1(t), z_2(t)), \phi_1(t) - \phi_2(t)\rangle_* \, \mathrm{d}t\right|.\end{aligned}$$

Using the polarization identity, $\langle z_1, z_1\rangle - \langle z_2, z_2\rangle = \langle z_1 - z_2, z_1 + z_2\rangle$, Cauchy–Schwarz, the continuity of $B : H \times Y \to X^*$, the bound $\|\phi_i\|_{L^\infty(0,T;X)} \le C$, and the estimates (5.4)–(5.5), we obtain

$$\begin{aligned}&|J''(u_1)(v_1, v_1) - J''(u_2)(v_1, v_1)| \\ &\quad \le C\,\|z_1 - z_2\|_{W(0,T)}\left(\|z_1\|_{W(0,T)} + \|z_2\|_{W(0,T)}\right) + C\|\phi_1 - \phi_2\|_{L^2(0,T;V)}\,\|z_2\|_{L^\infty(0,T;Y)}\|v_1\|_2.\end{aligned}$$

Applying (5.5), (5.4), and the adjoint Lipschitz estimate from Lemma 5.1 yields

$$|J''(u_1)(v_1, v_1) - J''(u_2)(v_1, v_1)| \le C\|u_1 - u_2\|_2 \, \|v_1\|_2^2.$$

The four estimates are established, which proves the lemma. □

Building on the above continuity properties, we now examine the behavior of the derivatives of the cost functional under appropriate convergence conditions, and the following proposition plays a crucial role in deriving the second-order optimality conditions, where $\mathscr{L}(U_2, \mathbb{R})$ denotes the space of bounded linear functionals on U_2 and $\mathscr{B}(U_2 \times U_2, \mathbb{R})$ denotes the space of bounded bilinear forms on $U_2 \times U_2$, i.e., bilinear maps $\beta : U_2 \times U_2 \to \mathbb{R}$ satisfying $|\beta(v_1, v_2)| \le C\|v_1\|_2\|v_2\|_2$ for some $C > 0$.

Proposition 5.1. *Let Assumption 5.1 hold, and let $y_0, y_d \in X$ and $f \in L^\infty(0,T;X)$. The first and second derivatives of J extend uniquely to continuous operators in the following spaces:*

$$J'(u) \in \mathscr{L}\big(U_2, \mathbb{R}\big), \quad J''(u) \in \mathscr{B}\big(U_2 \times U_2, \mathbb{R}\big), \quad \forall u \in U_{\mathrm{ad}}. \tag{5.7}$$

For any sequence $(u_n, v_n) \subset U_{\text{ad}} \times U_2$ satisfying $u_n \to \bar{u}$ strongly in U_2 and $v_n \rightharpoonup v$ weakly in U_2, the following convergence properties hold:

$$J'(\bar{u})v = \lim_{n\to\infty} J'(u_n)v_n, \tag{5.8}$$

$$J''(\bar{u})(v,v) \le \liminf_{n\to\infty} J''(u_n)(v_n,v_n). \tag{5.9}$$

Moreover, if $v = 0$, then there exists a constant $\Lambda > 0$, independent of the sequence (v_n), such that

$$\Lambda \liminf_{n\to\infty} \|v_n\|_2^2 \le \liminf_{n\to\infty} J''(u_n)(v_n,v_n). \tag{5.10}$$

Proof. The twice continuous Fréchet differentiability of J was established in Corollary 4.1. The continuous extensions in (5.7) follow directly from the boundedness of the derivatives on U_2, as shown in Lemma 5.2. For property (5.8), we decompose the difference as follows

$$|J'(u_n)v_n - J'(\bar{u})v| \le |J'(u_n)v_n - J'(\bar{u})v_n| + |J'(\bar{u})v_n - J'(\bar{u})v|.$$

For the first term, we apply the estimate (5.2) from Lemma 5.2

$$|J'(u_n)v_n - J'(\bar{u})v_n| \le C\|u_n - \bar{u}\|_2 \|v_n\|_2.$$

Since $u_n \to \bar{u}$ strongly in U_2 and (v_n) is bounded in U_2 (as a weakly convergent sequence), this term converges to zero.

For the second term, since $J'(\bar{u}) \in \mathscr{L}(U_2, \mathbb{R})$ is a bounded and linear functional on U_2, it is weakly continuous. Therefore, the weak convergence $v_n \rightharpoonup v$ in U_2 directly implies $J'(\bar{u})v_n \to J'(\bar{u})v$ as $n \to \infty$. Combining both estimates, we conclude that $J'(u_n)v_n \to J'(\bar{u})v$ as $n \to \infty$, which establishes (5.8).

To prove (5.9) and (5.10), we first show that the problem can be reduced to the case where $u_n = \bar{u}$ for all $n \in \mathbb{N}$. From inequality (5.3) in Lemma 5.2, we have

$$|(J''(u_n) - J''(\bar{u}))(v_n,v_n)| \le C\|u_n - \bar{u}\|_2 \|v_n\|_2^2.$$

Since $u_n \to \bar{u}$ strongly in U_2, and (v_n) is bounded in U_2 by the uniform boundedness principle applied to the weakly convergent sequence (v_n) in U_2, so that $\sup_n \|v_n\|_2^2 < \infty$, it follows that

$$\begin{aligned}
\liminf_{n\to\infty} J''(u_n)(v_n,v_n) &= \liminf_{n\to\infty} \left[J''(\bar{u})(v_n,v_n) + (J''(u_n) - J''(\bar{u}))(v_n,v_n)\right] \\
&= \liminf_{n\to\infty} J''(\bar{u})(v_n,v_n) + \lim_{n\to\infty} (J''(u_n) - J''(\bar{u}))(v_n,v_n) \\
&= \liminf_{n\to\infty} J''(\bar{u})(v_n,v_n),
\end{aligned}$$

where the last equality due to $\lim_{n\to\infty}(J''(u_n) - J''(\bar{u}))(v_n,v_n) = 0$, as $C\|u_n - \bar{u}\|_2 \sup_n \|v_n\|_2^2 \to 0$. Therefore, it suffices to verify conditions (5.9) and (5.10) for the fixed control $\bar{u}$ instead of the sequence (u_n).

For each n, let $z_n = G'(\bar{u})v_n$ be the linearized state and ϕ be the adjoint state associated with the control $\bar{u}$ in the direction v_n. The linearized states are defined by the linear and continuous operator $G'(\bar{u}) : U_2 \to W(0,T)$. Due to this continuity, (z_n) converges weakly in $W(0,T)$ to $z = G'(\bar{u})v$. Thus z is the linearized state associated with $\bar{u}$ in the direction of the weak limit v. By the definition of liminf, up to a subsequence, one has $\liminf_{n\to\infty} J''(\bar{u})(v_n,v_n) = \lim_{n\to\infty} J''(\bar{u})(v_n,v_n)$. The corresponding sequence (z_n) then converges weakly to z in $W(0,T)$.

By the compact embedding of $W(0,T)$ into $L^2(0,T;Y)$, a further subsequence (still denoted by z_n) converges strongly to z:

$$z_n \to z \text{ in } L^2(0,T;Y). \tag{5.11}$$

From the expression of J'', we have

$$J''(\bar{u})(v_n,v_n) = a\|z_n(T)\|_Y^2 + b\|z_n\|_{L^2(0,T;Y)}^2 + 2\int_0^T \langle B(v_n,z_n),\phi\rangle_{X^*,X}\mathrm{d}t + r\|v_n\|_2^2.$$

We now verify (5.9) term by term. Given that $z_n \to z$ strongly in $L^2(0,T;Y)$ by (5.11) and $v_n \rightharpoonup v$ weakly in U_2, we have that $B(v_n,z_n)$ converges weakly in $L^1(0,T;X^*)$ to $B(v,z)$ by the properties of bilinear forms. Since $\phi \in L^\infty(0,T;X)$, we obtain

$$2\int_0^T \langle B(v_n,z_n),\phi\rangle_{X^*,X}\mathrm{d}t \to 2\int_0^T \langle B(v,z),\phi\rangle_{X^*,X}\mathrm{d}t \quad \text{as } n\to\infty. \tag{5.12}$$

For the term involving $\|v_n\|_2^2$, the weak convergence $v_n \rightharpoonup v$ in U_2 and the weak lower semicontinuity of the norm yield

$$r\|v\|_2^2 \le \liminf_{n\to\infty} r\|v_n\|_2^2. \tag{5.13}$$

Since the mapping $z \mapsto z(T)$ is linear and continuous from $W(0,T)$ to Y, it is also weakly continuous. However, for the quadratic term $\|z(T)\|_Y^2$, we need weak lower semicontinuity, which gives

$$a\|z(T)\|_Y^2 \le \liminf_{n\to\infty} a\|z_n(T)\|_Y^2. \tag{5.14}$$

Additionally, from the strong convergence in (5.11), we have $b\|z_n\|_{L^2(0,T;Y)}^2 \to b\|z\|_{L^2(0,T;Y)}^2$. Combining this with (5.12), (5.13), and (5.14), we obtain

$$\begin{aligned}
J''(\bar{u})(v,v) &= a\|z(T)\|_Y^2 + b\|z\|_{L^2(0,T;Y)}^2 + 2\int_0^T \langle B(v,z),\phi\rangle_{X^*,X}\mathrm{d}t + r\|v\|_2^2 \\
&\le \liminf_{n\to\infty} a\|z_n(T)\|_Y^2 + \lim_{n\to\infty} b\|z_n\|_{L^2(0,T;Y)}^2 \\
&\quad + \lim_{n\to\infty} 2\int_0^T \langle B(v_n,z_n),\phi\rangle_{X^*,X}\mathrm{d}t + \liminf_{n\to\infty} r\|v_n\|_2^2 \\
&\le \liminf_{n\to\infty} \left(a\|z_n(T)\|_Y^2 + b\|z_n\|_{L^2(0,T;Y)}^2 + 2\int_0^T \langle B(v_n,z_n),\phi\rangle_{X^*,X}\mathrm{d}t + r\|v_n\|_2^2 \right) \\
&= \liminf_{n\to\infty} J''(\bar{u})(v_n,v_n),
\end{aligned}$$

which establishes (5.9).

Finally, we prove (5.10). Assume $v = 0$. Then, $v_n \rightharpoonup 0$ weakly in U_2. By the linearity and continuity of $G'(\bar{u})$, this implies that the corresponding limit $z = G'(\bar{u})v$ of the sequence (z_n) is also zero. Consequently, all terms in $J''(\bar{u})(v,v)$ involving z vanish. As before, for the stationary sequence $u_n = \bar{u}$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\liminf_{n\to\infty} J''(\bar{u})(v_n,v_n) = \\
&\liminf_{n\to\infty} \left(a\|z_n(T)\|_Y^2 + b\|z_n\|_{L^2(0,T;Y)}^2 + 2\int_0^T \langle B(v_n,z_n),\phi\rangle_{X^*,X}\mathrm{d}t + r\|v_n\|_2^2 \right).
\end{aligned}$$

Since $v_n \rightharpoonup 0$ and $z_n \to 0$, from (5.12) we know that $2\int_0^T \langle B(v_n, z_n), \phi \rangle_{X^*,X} \mathrm{d}t \to 0$ as $n \to \infty$. Furthermore, the terms $a\|z_n(T)\|_Y^2$ and $b\|z_n\|^2_{L^2(0,T;Y)}$ are non-negative. Therefore,

$$\liminf_{n\to\infty} J''(\bar{u})(v_n, v_n) \geq \liminf_{n\to\infty} r\|v_n\|_2^2,$$

which establishes (5.10) with $\Lambda := r > 0$, thereby concluding the proof. □

To formulate the second-order optimality conditions, we need to define appropriate cones that characterize the feasible directions and critical directions at a given control. These sets will determine the space over which the positivity of the Hessian must be verified. Given a fixed control $\bar{u} \in U_{\mathrm{ad}}$, we define

$$\begin{aligned} S_{\bar{u}} &= \{v \in U_\infty : v = \lambda(u - \bar{u}) \text{ for some } \lambda > 0 \text{ and } u \in U_{\mathrm{ad}}\}, \\ C_{\bar{u}} &= \mathrm{cl}_2(S_{\bar{u}}) \cap \{v \in U_2 : J'(\bar{u})v = 0\}, \\ D_{\bar{u}} &= \{v \in S_{\bar{u}} : J'(\bar{u})v = 0\}, \end{aligned}$$

where $\mathrm{cl}_2(K)$ denotes the closure of a subset $K \subset U_2$ in U_2. Here, $S_{\bar{u}}$ represents the cone of feasible directions, and $\mathscr{C}_{\bar{u}}$ is the cone of critical directions.

Having established the necessary technical framework, we can now state the second-order necessary condition for optimality. This result characterizes local minimizers by requiring non-negative curvature of the cost functional along all critical directions.

Theorem 5.2. *Let $\bar{u}$ be a local solution of the optimal control problem (P), with given initial state $y_0 \in X$, desired state $y_d \in X$, and source term $f \in L^\infty(0,T;X)$. Assume that Assumption 5.1 holds, and that the regularity condition $C_{\bar{u}} = \mathrm{cl}_2(D_{\bar{u}})$ is satisfied. Then, the second-order necessary condition for optimality is $J''(\bar{u})(v,v) \geq 0$ for all $v \in C_{\bar{u}}$.*

The previous result is a direct consequence of [10, Theorem 2.2] and Proposition 5.1.

To complement the necessary condition, we now present a sufficient condition that guarantees the local optimality of a control. This result provides a quadratic growth condition that ensures strict local optimality in the space U_2 when the Hessian is strictly positive definite on the cone of critical directions.

Theorem 5.3. *Let Assumption 5.1 hold, and let $y_0, y_d \in X$ and $f \in L^\infty(0,T;X)$ be given. Suppose that $\bar{u} \in U_{\mathrm{ad}}$ satisfies the first-order necessary condition. Furthermore, let $J''(\bar{u})(v,v) > 0$ for all $v \in C_{\bar{u}} \setminus \{0\}$. Then, there exist $\delta > 0$ and $\rho > 0$ such that $J(\bar{u}) + \delta\|u - \bar{u}\|_2^2 \leq J(u)$, for all $u \in U_{\mathrm{ad}}$ with $\|\bar{u} - u\|_2 \leq \rho$. In particular, $\bar{u}$ is a strict local solution to (P).*

This result is an immediate consequence of [10, Theorem 2.3].

6. Application to a Coupled System

We demonstrate the applicability of our abstract results through a coupled reaction-diffusion system with bilinear control. Consider the evolution of interacting populations y_1 and y_2 in a

bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, with boundary $\partial\Omega$. The system is governed by:

$$\begin{cases} \dot{y}_1 - \Delta y_1 = y_2 & \text{in } Q := (0,T)\times\Omega, \\ \dot{y}_2 - \Delta y_2 = y_1 + u y_2 & \text{in } Q, \\ \partial_\nu y_1 = \partial_\nu y_2 = 0 & \text{on } \Gamma := (0,T)\times\partial\Omega, \\ y_1(0,x) = y_{1,0}(x),\ y_2(0,x) = y_{2,0}(x) & \text{in } \Omega, \end{cases} \tag{6.1}$$

where $y_{1,0}$ and $y_{2,0}$ are the initial states, and $u(t,x)$ serves as a bilinear control influencing the growth rate of y_2. The boundary condition $\partial_\nu y_1 = 0$ and $\partial_\nu y_2 = 0$ specifies the vanishing normal derivative, where ν denotes the outward unit normal vector on the boundary $\partial\Omega$, reflecting the absence of flux across the boundary.

The control objective is to steer the system state (y_1, y_2) towards a desired trajectory $(y_{1,d}, y_{2,d})$ over the time interval $(0,T)$ while minimizing the control effort. This is formalized through the following cost functional:

$$\mathscr{J}_c(u) = \frac{1}{2}\int_Q |y_1 - y_{1,d}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_Q |y_2 - y_{2,d}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{r}{2}\int_Q u^2 \,\mathrm{d}x\,\mathrm{d}t.$$

The first two terms quantify the deviation of each population from its desired distribution, while the last term penalizes the control effort with regularization parameter $r > 0$. The admissible control set is defined as $\mathscr{U}_{\text{ad}} = \{u \in L^\infty(Q) \mid m \le u(t,x) \le M \ \text{ a.e. in } Q\}$, where $m < M$ are constants representing the lower and upper bounds on the control intensity, respectively.

The optimal control problem can thus be formulated as:

$$\min_{u\in\mathscr{U}_{\text{ad}}} \mathscr{J}_c(u). \tag{$\mathscr{P}_c$}$$

We assume the following regularity on the initial and desired data: $y_{1,0}, y_{1,d} \in L^2(\Omega)$ and $y_{2,0}, y_{2,d} \in C(\overline{\Omega})$. The higher regularity of $y_{2,0}$ and $y_{2,d}$ will be used in the technical analysis.

6.1. **Abstract formulation of the problem.** We define the state variable $y = (y_1, y_2)$ and take $V = W_0 \times W_0$, where $W_0 = \{w \in H^1(\Omega) \mid \partial_\nu w = 0 \text{ on } \partial\Omega\}$. The pivot space is $Y = L^2(\Omega) \times L^2(\Omega)$, the control space is $U = L^\infty(\Omega)$, and we set $H = L^2(\Omega)$. In this setting, the system reads

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = Ay(t) + B(u(t), y(t)),$$

where $A : V \to V^*$ is $A(y_1,y_2) = \big(\Delta y_1 + y_2, \Delta y_2 + y_1\big)$ and $B : U \times Y \to Y$ is $B(u,(y_1,y_2)) = (0, u y_2)$. The abstract state equation is

$$\begin{cases} \dot{y}(t) = Ay(t) + B(u(t), y(t)), & \text{in } L^2(0,T;V^*), \\ y(0) = y_0, & \text{in } Y. \end{cases} \tag{6.2}$$

We fix the notation for norms in the various spaces. For $y = (y_1, y_2)$ in a product space $X = X_1 \times X_2$, the norm is defined by $\|y\|_X = \left(\|y_1\|_{X_1}^2 + \|y_2\|_{X_2}^2\right)^{1/2}$. In particular, for $Y = L^2(\Omega)\times L^2(\Omega)$ and $V = W_0 \times W_0$,

$$\|y\|_Y = \left(\|y_1\|^2 + \|y_2\|^2\right)^{1/2}, \quad \|y\|_V = \left(\|y_1\|^2 + \|\nabla y_1\|^2 + \|y_2\|^2 + \|\nabla y_2\|^2\right)^{1/2},$$

with $\|\cdot\|$ denoting the natural $L^2(\Omega)$-norm.

Applying the theoretical results from the previous sections requires verifying that the operators in the abstract formulation meet the necessary assumptions.

- Gelfand triple: The embeddings $V \hookrightarrow Y \hookrightarrow V^*$ are dense and continuous. The embedding $V \hookrightarrow Y$ is compact by the Rellich–Kondrachov theorem [35, p. 356] for $n = 2,3$.
- Boundedness of B: For any $u \in U$ and $y = (y_1, y_2) \in Y$:

$$\|B(u,y)\|_Y = \|(0,uy_2)\|_Y = \|uy_2\| \leq \|u\|_{L^\infty(\Omega)}\|y_2\| \leq \|u\|_U\|y\|_Y.$$

- Boundedness of A: Let $y = (y_1, y_2)$ and $w = (w_1, w_2)$ in V:

$$\begin{aligned}\langle Ay, w\rangle_* &= \int_\Omega (-\nabla y_1 \cdot \nabla w_1 + y_2 w_1 - \nabla y_2 \cdot \nabla w_2 + y_1 w_2)\,\mathrm{d}x \\ &\leq \|\nabla y_1\|\|\nabla w_1\| + \|y_2\|\|w_1\| + \|\nabla y_2\|\|\nabla w_2\| + \|y_1\|\|w_2\| \\ &\leq (\|\nabla y_1\| + \|y_1\|)\,(\|\nabla w_1\| + \|w_1\|) \\ &\quad + (\|\nabla y_2\| + \|y_2\|)\,(\|\nabla w_2\| + \|w_2\|) \\ &\leq 2\|y\|_V\|w\|_V.\end{aligned}$$

- Coercivity of $-A$: For $y = (y_1, y_2) \in V$:

$$\begin{aligned}\langle -Ay, y\rangle_* &= -\int_\Omega (\Delta y_1 y_1 + \Delta y_2 y_2 + y_2 y_1 + y_1 y_2)\,\mathrm{d}x \\ &= \int_\Omega (|\nabla y_1|^2 + |\nabla y_2|^2)dx - 2\int_\Omega y_1 y_2\,\mathrm{d}x \\ &\geq \|\nabla y_1\|^2 + \|\nabla y_2\|^2 - (\|y_1\|^2 + \|y_2\|^2) \\ &\geq \|y\|_V^2 - 2\|y\|_Y^2.\end{aligned}$$

6.2. **Well-posedness and optimal control existence.** Having established the abstract framework and verified the required assumptions, we present the well-posedness of the coupled reaction-diffusion system together with the existence of optimal controls.

Lemma 6.1. *Let $u \in L^\infty(Q)$. Then, coupled system (6.1) possesses a unique weak solution $y_u = (y_1, y_2)$ satisfying*

$$\begin{aligned}y_1 &\in L^2(0,T;W_0) \cap H^1(0,T;W_0^*) \cap L^\infty(0,T;L^2(\Omega)), \\ y_2 &\in L^2(0,T;W_0) \cap H^1(0,T;W_0^*) \cap C(\overline{Q}).\end{aligned}$$

Moreover, the solution depends continuously on the initial data and the control function.

Proof. The existence of a solution $y_u = (y_1, y_2) \in W(0,T) \cap L^\infty(0,T;Y)$ follows directly from Proposition 2.1 applied to equation (6.2), as all the required assumptions have been verified in the previous subsection. From this abstract result, we deduce that each component belongs to the space $L^2(0,T;W_0) \cap H^1(0,T;W_0^*) \cap L^\infty(0,T;L^2(\Omega))$.

We now establish the enhanced regularity of y_2, namely $y_2 \in C(\overline{Q})$, under the assumption that $y_{2,0} \in C(\overline{\Omega})$. Consider the linear parabolic equation satisfied by y_2:

$$\begin{cases} \dot{y}_2 - \Delta y_2 = y_1 + uy_2 & \text{in } Q, \\ \partial_\nu y_2 = 0 & \text{on } \Gamma := (0,T) \times \partial\Omega, \\ y_2(0,x) = y_{2,0}(x) & \text{in } \Omega. \end{cases}$$

Since $y_1 \in L^2(0,T;W_0) \cap H^1(0,T;W_0^*)$, we can invoke the embedding result from [29, Corollary 2.5] to conclude that $y_1 \in L^3(Q)$. Furthermore, $u \in L^\infty(Q)$ and $y_2 \in L^2(0,T;W_0) \cap H^1(0,T;W_0^*) \subset$

$L^3(Q)$, so the right-hand side of the equation for y_2 has sufficient regularity. Applying the parabolic regularity result from [37, Theorem 5.5], and noting that $y_{2,0} \in C(\overline{\Omega})$, we obtain $y_2 \in C(\overline{Q})$, which completes the proof. The continuous dependence on initial data and control follows from the abstract theory as well, specifically from the energy estimates established in Proposition 2.1. □

Remark 6.1. The enhanced regularity $y_2 \in C(\overline{Q})$ is crucial for our subsequent analysis, particularly for the second-order conditions. It ensures that the bilinear term uy_2 is well-defined pointwise, which simplifies the derivation of optimality conditions. From an applied perspective, this regularity implies that the second population distribution remains continuous in both space and time, which is often a realistic assumption in biological models.

The well-posedness of the state equation established above allows us to immediately deduce the existence of optimal controls as a direct consequence of our theoretical framework.

Theorem 6.1. *Under the standing assumptions, Problem* ($\mathscr{P}_c$) *admits at least one solution* $\bar{u} \in \mathscr{U}_{\text{ad}}$.

Proof. The existence follows directly from Theorem 3.1. □

6.3. **Optimality conditions.** We now specialize the abstract optimality conditions to the coupled reaction-diffusion system. First, we state the first-order necessary conditions via the adjoint system, followed by the second-order sufficient conditions.

6.3.1. *First-order necessary conditions.* The adjoint system associated with $\mathscr{J}_c$ reads

$$\begin{cases} -\dot{\phi}_1 - \Delta\phi_1 = \phi_2 + (y_1 - y_{1,d}) & \text{in } Q, \\ -\dot{\phi}_2 - \Delta\phi_2 = \phi_1 + u\phi_2 + (y_2 - y_{2,d}) & \text{in } Q, \\ \partial_\nu \phi_1 = \partial_\nu \phi_2 = 0 & \text{on } \Gamma, \\ \phi_1(T) = 0,\ \phi_2(T) = 0 & \text{in } \Omega, \end{cases} \tag{6.3}$$

where $\phi_u = (\phi_1, \phi_2)$ is the adjoint state corresponding to the control u and associated state $y_u = (y_1, y_2)$.

Lemma 6.2. *Let $u \in L^\infty(Q)$ and assume that the associated state y_u satisfies the regularity of Lemma 6.1. Then backward system (6.3) admits a unique weak solution $\phi_u = (\phi_1, \phi_2)$ satisfying*

$$\phi_1 \in L^2(0,T;W_0) \cap H^1(0,T;W_0^*) \cap L^\infty(0,T;L^2(\Omega)),$$
$$\phi_2 \in L^2(0,T;W_0) \cap H^1(0,T;W_0^*) \cap C(\overline{Q}).$$

In particular,

$$\phi_u \in L^\infty(0,T;X) \cap W(0,T), \quad X := L^2(\Omega) \times L^\infty(\Omega).$$

Proof. The result follows along the same lines as Lemma 6.1, noting that the source terms in (6.3), namely $y_1 - y_{1,d}$ and $y_2 - y_{2,d}$, inherit the regularity from y_u and y_d. Standard energy estimates for linear backward parabolic systems then yield the stated regularity of ϕ_u. □

The first derivative of $\mathscr{J}_c$ at $\bar{u}$ in direction $v \in L^2(Q)$ is:

$$\mathscr{J}_c'(\bar{u})v = \int_Q (\phi_2 \bar{y}_2 + r\bar{u})\, v \, \mathrm{d}x\, \mathrm{d}t.$$

Using this derivative, we can now state the first-order necessary conditions:

Theorem 6.2. *Let $\bar{u} \in \mathscr{U}_{ad}$ be a local minimizer of Problem ($\mathscr{P}_c$). Then, the following variational inequality holds:*

$$\mathscr{J}_c'(\bar{u})(v-\bar{u}) \geq 0, \quad \forall v \in \mathscr{U}_{ad}. \tag{6.4}$$

Moreover, the optimal control can be characterized pointwise almost everywhere in Q by:

$$\bar{u}(t,x) = \begin{cases} m & \text{if } \phi_2(t,x)\bar{y}_2(t,x) + r\bar{u}(t,x) > 0, \\ M & \text{if } \phi_2(t,x)\bar{y}_2(t,x) + r\bar{u}(t,x) < 0. \end{cases}$$

Equivalently, the optimal control satisfies the projection formula:

$$\bar{u}(t,x) = \mathbb{P}_{[m,M]}\left(-\frac{1}{r}\phi_2(t,x)\bar{y}_2(t,x)\right),$$

where $\mathbb{P}_{[m,M]}$ denotes the pointwise projection onto the interval $[m,M]$.

Proof. The variational inequality (6.4) follows directly from the abstract optimality conditions established in the theoretical sections. The pointwise characterization and projection formula are standard results, as seen in Chapter 2, Section 8 of [37]. □

6.3.2. *Second-order analysis.* We verify Assumption 5.1 for coupled system (6.1). Recall the notation

$$X = L^2(\Omega) \times L^\infty(\Omega), \quad Y = L^2(\Omega) \times L^2(\Omega), \quad H = L^2(\Omega), \quad U = L^\infty(\Omega).$$

By Lemma 6.1, the state satisfies $y_u = (y_1, y_2) \in L^\infty(0,T;X)$ for any $u \in L^\infty(Q)$, thereby verifying condition Assumption 5.1 (1). Similarly, the adjoint state satisfies $\phi_u \in L^\infty(0,T;X)$. The bilinear operator $B(u,y) = (0, uy_2)$ admits two continuous extensions. First, for $B : H \times X \to Y$, we have, for all $u \in H$ and $y \in X$:

$$\|B(u,y)\|_Y = \|uy_2\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}\|y_2\|_{L^\infty(\Omega)} \leq \|u\|_H\|y\|_X.$$

Second, for $B : H \times Y \to X^* \supset L^2(\Omega) \times L^1(\Omega)$, it holds that, for all $u \in H$ and $y \in Y$:

$$\|B(u,y)\|_{X^*} \leq \|uy_2\|_{L^1(\Omega)} \leq \|u\|_{L^2(\Omega)}\|y_2\|_{L^2(\Omega)} \leq \|u\|_H\|y\|_Y.$$

Hence, both extensions are continuous, and Assumption 5.1 is satisfied.

We now introduce the *cone of critical directions* at $\bar{u} \in \mathscr{U}_{\text{ad}}$ as follows

$$C_{\bar{u}} = \left\{ v \in L^2(Q) \,\middle|\, v \geq 0 \text{ if } \bar{u} = m,\ v \leq 0 \text{ if } \bar{u} = M, \text{ and } \mathscr{J}_c'(\bar{u})v = 0 \right\}.$$

This definition coincides with the cone introduced in the previous theoretical section, where $C_{\bar{u}}$ is given by $C_{\bar{u}} = \text{cl}_2(S_{\bar{u}}) \cap \{v \in U_2 : \mathscr{J}_c'(\bar{u})v = 0\}$. Moreover, this cone satisfies $C_{\bar{u}} = \text{cl}_2(D_{\bar{u}})$. The proof of this statement is standard in the literature; see [3] or [10] for detailed arguments.

With these preliminaries in place, we can now state the second-order conditions.

Theorem 6.3 (Second-order necessary condition)**.** *Let $\bar{u}$ be a local solution of the optimal control problem ($\mathscr{P}_c$). Then the second-order necessary condition for optimality is $\mathscr{J}_c''(\bar{u})(v,v) \geq 0$ for all $v \in C_{\bar{u}}$.*

Theorem 6.4 (Second-order sufficient condition)**.** *For $\bar{u} \in \mathscr{U}_{ad}$, assume that the first-order necessary condition (6.4) holds. If $\mathscr{J}_c''(\bar{u})(v,v) > 0$ for all $v \in C_{\bar{u}} \setminus \{0\}$, then $\bar{u}$ is a strict local minimizer of Problem ($\mathscr{P}_c$). Moreover, there exist constants $\delta > 0$ and $\rho > 0$ such that $\mathscr{J}_c(\bar{u}) + \delta\|u - \bar{u}\|_2^2 \leq \mathscr{J}_c(u)$ for all $u \in \mathscr{U}_{ad}$ with $\|\bar{u} - u\|_2 \leq \rho$.*

7. Conclusion

This study establishes a rigorous framework for the optimal control of abstract bilinear evolution systems within Gelfand triple structures. Using weak solution theory and direct methods in the calculus of variations, the well-posedness of the state equation and the existence of optimal controls were proved. Differentiating the control-to-state map allowed the derivation of first-order necessary conditions, and second-order sufficient conditions were extended to operator-theoretic dynamics, ensuring local optimality. A key aspect of this work lies in the formulation of Assumption 5.1, which required careful functional-analytic reasoning to reconcile the bilinear operator's mapping properties with the regularity needed for the second-order analysis. Identifying such an assumption in the abstract setting demanded substantial analytical effort but also led to a formulation that applies to a wide range of problems, beyond the specific PDE example addressed here. Looking ahead, several directions appear natural. One is the extension of positivity sufficient condition from finite-horizon, bounded settings to infinite-horizon problems or unbounded operator frameworks. Another is the treatment of nonlinear bilinear systems or systems with time delays, which introduce additional analytical challenges and nonlocal dynamics. The framework could also be adapted to systems with second-order time derivatives, such as wave equations, necessitating refined regularity and adjoint analyses. Finally, stochastic perturbations or non-smooth control objectives present opportunities to broaden the applicability of these methods to more complex and practical control scenarios. These perspectives provide a roadmap for further theoretical and numerical developments in infinite-dimensional control systems.

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