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OPTIMALITY CONDITIONS FOR MULTIOBJECTIVE FRACTIONAL INTERVAL-VALUED OPTIMIZATION PROBLEMS ON HADAMARD MANIFOLDS

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Abstract. This paper considers a multiobjective fractional interval-valued optimization problem on Hadamard manifolds. By utilizing Clarke directional derivatives and Clarke subdifferentials, we derive the necessary and sufficient optimality conditions for weak *LU*-Pareto optimal solutions to the problem. In addition, some Mond-Weir type duality theorems are also obtained.

Keywords. Hadamard manifold; Interval-valued optimization problem; Mond-Weir type duality theorems; Optimality conditions.

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1. INTRODUCTION

As the complexity of the environment increases and the inherent subjectivity and ambiguity of human thought come into play, decision-making information is often uncertain in numerical cases. This imprecision leads to various data and economic problems in real life. To address this, the concept of interval-valued optimization problems was introduced in [1]. Methods for these problems in Euclidean space have been studied extensively in [2, 3, 4]. In [5], Wu explored KKT type necessary optimality conditions for scalar differentiable interval-valued optimization problems. In [6, 7], Wu examined Wolfe type and Lagrange duality theorems in the context of nonlinear interval-valued scalar optimization problems. In addition, Wu investigated the application of interval-valued objective functions in multiobjective programming and derived sufficient optimality conditions of KKT type for multiobjective interval-valued optimization problems under convexity assumptions in [8]. In [9], Steuer proposed three algorithms to tackle challenges arising from interval-valued objective function coefficients in linear programming, aiming to address the uncertainties associated with these coefficients. In [10], Urli and Nadeau employed an interactive method to solve multiobjective stochastic linear programming under conditions of partial uncertainty.

Recently, the geometric structure of manifolds offers a new perspective for tackling complex optimization problems. By leveraging manifold structures, many difficult, constrained,

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and nonconvex optimization problems originally framed in Euclidean spaces can be reformulated as simpler equivalent unconstrained convex problems within the framework of Hadamard manifolds; see, e.g., [11]. Researchers have successfully adapted a wide range of Euclidean optimization methods on manifolds. For instance, under the assumption of geodesic convexity, the sufficient optimality conditions for multiobjective mathematical programming problems with equilibrium constraints on Hadamard manifolds were derived (see [12]). Furthermore, KKT type optimality conditions were derived for differentiable interval-valued optimization problems defined on Hadamard manifolds [12]. Furthermore, Chen [13] explored KKT type optimality conditions for optimization problems involving interval-valued objective functions that are directionally differentiable on Hadamard manifolds. Upadhyay et.al [14, 15, 16] extended the concept of convexification from Euclidean spaces to Hadamard manifolds and studied the optimality conditions, duality theorems, and constraint rules for various types of nonconvex optimization problems. This work provides novel perspectives and powerful tools for studying nonsmooth optimization problems on manifolds. It is noteworthy that Hosseini and Pouryayevali [17] introduced the Clarke directional derivative and Clarke subdifferential for locally Lipschitz functions on Hadamard manifolds, and also established their fundamental properties. In [18], Antczak investigated nonsmooth vector optimization problems with multiobjective interval-valued functions in Euclidean spaces, and established KKT type optimality conditions and Mond-Weir type duality theory for such problems, thereby extending classical vector optimization results to the interval-valued framework.

Inspired by the results mentioned above, we extend the corresponding results from Euclidean space in [18] to the framework of Hadamard manifolds. Unlike the model in [18], we adopted in this work is a multiobjective fractional interval-valued model. The rest of the paper is organized as follows: Section 2 presents the fundamental notations and concepts used throughout the paper. Section 3 discusses the optimality conditions for multiobjective fractional interval-valued optimization problems on Hadamard manifolds. Section 4 addresses the dual problems of these optimization issues and presents the corresponding duality theorems. Finally, Section 5 provides a concluding summary to the paper.

2. Preliminaries

In this section, We first present some known fundamental concepts and mathematical notation used throughout this work, see [19].

Let $\mathscr{M}$ be a connected m-dimensional manifold, with $T_x\mathscr{M}$ representing the tangent space of $\mathscr{M}$ at $x \in \mathscr{M}$, and $T\mathscr{M} = \bigcup_{x\in\mathscr{M}} T_x\mathscr{M}$ denoting the tangent bundle of $\mathscr{M}$. If there is a Riemannian metric $\langle .,. \rangle_x$ defined on $T_x\mathscr{M}$, then $\mathscr{M}$ is referred to as a Riemannian manifold. As demonstrated in [19], for all $x \in \mathscr{M}$ and all $u,v,w \in T_x\mathscr{M}$,

(1) $\langle u,v \rangle_x = \langle v,u \rangle_x$;

(2) $\langle u,av+bw \rangle_x = a\langle u,v \rangle_x + b\langle u,w \rangle_x, a,b \in R$;

(3) If $u = 0$, then $\langle u,v \rangle_x = 0$.

For any $x \in \mathscr{M}$, the exponential map is defined as $\exp_x : T_x\mathscr{M} \to \mathscr{M}$. Given any two points x and y, the unique geodesic segment joining x to y is defined by $\gamma(t) = \exp_x(t\exp_x^{-1}(y))$ ($t \in [0,1]$) , where $\gamma(t)$ is the geodesic curve passing through x. Moreover, according to the inverse theorem, $\exp_x$ is a local diffeomorphism. The inverse of the exponential map is denoted as $\exp_x^{-1}$.

A complete, connected Riemannian manifold with non-positive curvature is referred to as a Hadamard manifold. If $\mathscr{M}$ is a Hadamard manifold, then, for every pair of points $x, y \in \mathscr{M}$, there exists a unique minimum geodesic connecting x to y. Unless otherwise specified, $\mathscr{M}$ refers to a Hadamard manifold in the remainder of this paper.

Definition 2.1. [17] A function $f : E \subseteq \mathscr{M} \to R$ is said to be locally Lipschitz at $\bar{x} \in E$ if there exists constant $L > 0$ such that, for any x and y located within a certain open neighborhood around $\bar{x}$, $|f(x) - f(y)| \leq Ld(x,y)$, where $d(x,y)$ denotes the Riemannian distance between points x and y. If it is locally Lipschitz for all $x \in E$, then f is said to be locally Lipschitz function on E.

Definition 2.2. [20] A set $E \subseteq \mathscr{M}$ is geodesic convex set if, for any geodesic $\gamma : [0,1] \to \mathscr{M}$ with $\gamma(0) = x$ and $\gamma(1) = y$, $\gamma(t) \in E$, for all $t \in [0,1]$.

Definition 2.3. [20] Let $E \subseteq \mathscr{M}$ be non-empty open geodesic convex set. A function $f : E \to R$ is called geodesic convexity at $x \in E$ if, for any $y \in E$, $f(\gamma(t)) \leq tf(y) + (1-t)f(x)$ for all t in $[0,1]$, where $\gamma : [0,1] \to \mathscr{M}$ is a geodesic satisfying $\gamma(0) = x$ and $\gamma(1) = y$. If the above inequality is strict when $x \neq y$, then f is called strictly geodesic convexity at x. Furthermore, if the inequality holds for all $x \in E$, then f is referred to as geodesic convex function on E.

Definition 2.4. [17] Let f be a locally Lipschitz function on E. Then the generalized directional derivative of f at $\bar{x} \in E$ in the direction $v \in T_{\bar{x}}\mathscr{M}$ is denoted by

$$f^{\circ}(\bar{x}, v) = \limsup_{y \to \bar{x}, t \downarrow 0} \frac{f(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1} y} v) - f(\bar{x})}{t},$$

where $(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1} y}$ denotes the differential of $\exp_{\bar{x}}$ at $\exp_{\bar{x}}^{-1}$.

Definition 2.5. [17] Let f be a locally Lipschitz function on E. Then the Clarke subdifferential of f at a point $\bar{x} \in E$ is defined as $\partial f(\bar{x}) = \{\zeta \in T_{\bar{x}}\mathscr{M} \mid f^{\circ}(\bar{x}; v) \geq \langle \zeta, v \rangle_{\bar{x}}$ for all $v \in T_{\bar{x}}\mathscr{M}\}$.

Building upon the results established in [17], we obtain two key properties of subdifferentials for locally Lipschitz functions.

(a) Given that f is locally Lipschitz at $\bar{x} \in E$, it holds for any scalar λ that $\partial(\lambda f)(\bar{x}) = \lambda \partial f(\bar{x})$.

(b) Suppose each f_i, $i = 1, 2, \ldots, n$, is locally Lipschitz at $\bar{x} \in E$, and $\lambda_i \in R$, $i = 1, 2, \ldots, n$, then the function $f := \sum_{i=1}^{n} \lambda_i f_i$ is also locally Lipschitz function at $\bar{x} \in E$. Moreover, the following inclusion holds: $\partial \left(\sum_{i=1}^{n} \lambda_i f_i\right)(\bar{x}) \subset \sum_{i=1}^{n} \lambda_i \partial f_i(\bar{x})$.

Lemma 2.1. [17] *Let f be locally Lipschitz function on E, $\bar{x} \in E$. For any $v \in T_{\bar{x}}\mathscr{M}$, it holds that $f^{\circ}(\bar{x}; v) = \max\{\langle \zeta, v \rangle_{\bar{x}} : \zeta \in \partial f(\bar{x})\}$.*

Lemma 2.2. [21] *Let f be locally Lipschitz function on E. If f is geodesic convexity at $\bar{x} \in E$, then, for any $y \in E$ and $\zeta \in \partial f(\bar{x})$, $f(y) - f(\bar{x}) \geq \langle \zeta, exp_{\bar{x}}^{-1} y \rangle_{\bar{x}}$.*

Lemma 2.3. [21] *Let f_k be locally Lipschitz at $\bar{x} \in E$ for $k \in K = \{1, \ldots, m\}$. Then $f(x) = \max_{k \in K} f_k(x)$ is also locally Lipschitz at $\bar{x}$. Furthermore $\partial f(\bar{x}) \subseteq conv\{\partial f_k(\bar{x}) : k \in K\}$.*

Lemma 2.4. *If f and g are locally Lipschitz at $\bar{x} \in E$ and $g(\bar{x}) \neq 0$, then*

(a) $\partial(f \cdot g)(\bar{x}) \subseteq g(\bar{x}) \cdot \partial f(\bar{x}) + f(\bar{x}) \cdot \partial g(\bar{x})$;

(b) $\partial \left(\frac{f}{g}\right)(\bar{x}) \subseteq \frac{g(\bar{x}) \cdot \partial f(\bar{x}) - f(\bar{x}) \cdot \partial g(\bar{x})}{g^2(\bar{x})}$.

Proof.

$$(a)\quad (f\cdot g)^\circ(\bar{x},v) = \limsup_{y\to\bar{x},\, t\downarrow 0} \frac{f(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v))\cdot g(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - f(\bar{x})g(\bar{x})}{t}$$

$$- \limsup_{y\to\bar{x},\, t\downarrow 0} \frac{f(\bar{x})g(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - f(\bar{x})g(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v))}{t}$$

$$= \limsup_{y\to\bar{x},\, t\downarrow 0}\left[g(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v))\cdot \frac{f(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - f(\bar{x})}{t}\right.$$

$$\left. + f(\bar{x})\cdot\frac{g(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - g(\bar{x})}{t}\right]$$

$$\leq g(\bar{x})\limsup_{y\to\bar{x},\, t\downarrow 0}\frac{f(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - f(\bar{x})}{t}$$

$$+ f(\bar{x})\limsup_{y\to\bar{x},\, t\downarrow 0}\frac{g(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - g(\bar{x})}{t}$$

$$= g(\bar{x})f^\circ(\bar{x},v) + f(\bar{x})g^\circ(\bar{x},v).$$

For any $\zeta\in\partial(f\cdot g)(\bar{x})$, there exist $\xi\in\partial f(\bar{x})$ and $\eta\in\partial g(\bar{x})$ such that

$$\langle\zeta,v\rangle_{\bar{x}} \leq (f\cdot g)^\circ(\bar{x},v) \leq g(\bar{x})f^\circ(\bar{x},v) + f(\bar{x})g^\circ(\bar{x},v) = \langle g(\bar{x})\xi + f(\bar{x})\eta, v\rangle_{\bar{x}}.$$

Therefore, $\partial(f\cdot g)(\bar{x}) \subseteq g(\bar{x})\partial f(\bar{x}) + f(\bar{x})\partial g(\bar{x})$.

$$(b)\quad \left(\frac{f}{g}\right)^\circ(\bar{x},v) = \limsup_{y\to\bar{x},\, t\downarrow 0}\frac{g(\bar{x})f(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - f(\bar{x})g(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v))}{g(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v))g(\bar{x})\cdot t}$$

$$= \frac{1}{g^2(\bar{x})}\limsup_{y\to\bar{x},\, t\downarrow 0}\frac{g(\bar{x})f(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - g(\bar{x})f(\bar{x})}{t}$$

$$- \frac{1}{g^2(\bar{x})}\limsup_{y\to\bar{x},\, t\downarrow 0}\frac{f(\bar{x})g(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - f(\bar{x})g(\bar{x})}{t}$$

$$= \frac{1}{g^2(\bar{x})}\limsup_{y\to\bar{x},\, t\downarrow 0}\left[g(\bar{x})\cdot\frac{f(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - f(\bar{x})}{t}\right.$$

$$\left. - f(\bar{x})\cdot\frac{g(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - g(\bar{x})}{t}\right]$$

$$\leq \frac{1}{g^2(\bar{x})}\left[g(\bar{x})\limsup_{y\to\bar{x},\, t\downarrow 0}\frac{f(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - f(\bar{x})}{t}\right.$$

$$\left. - f(\bar{x})\limsup_{y\to\bar{x},\, t\downarrow 0}\frac{g(\exp_y(t(D\exp_{\bar{x}})_{\exp_{\bar{x}}^{-1}y}v)) - g(\bar{x})}{t}\right]$$

$$= \frac{1}{g^2(\bar{x})}\left[g(\bar{x})f^\circ(\bar{x},v) - f(\bar{x})g^\circ(\bar{x},v)\right].$$

For any $\zeta \in \partial(f\cdot g)(\bar{x})$, there exist $\xi \in \partial f(\bar{x})$ and $\eta \in \partial g(\bar{x})$ such that

$$\langle \zeta, v\rangle_{\bar{x}} \leq \left(\frac{f}{g}\right)^{\circ}(\bar{x},v) \leq \frac{1}{g^2(\bar{x})}\left[g(\bar{x})f^{\circ}(\bar{x},v) - f(\bar{x})g^{\circ}(\bar{x},v)\right] = \left\langle \frac{g(\bar{x})\xi - f(\bar{x})\eta}{g^2(\bar{x})}, v\right\rangle_{\bar{x}}.$$

Therefore, we obtain $\partial\left(\frac{f}{g}\right)(\bar{x}) \subseteq \frac{g(\bar{x})\partial f(\bar{x}) - f(\bar{x})\partial g(\bar{x})}{g^2(\bar{x})}$.

Next, we introduce operations related on intervals. Denote by M_I the collection of all bounded closed intervals in R, $M_I = \left\{\left[x^L, x^U\right] : x^L, x^U \in R,\ \ x^L \leq x^U\right\}$. Given $X = \left[x^L, x^U\right]$, $Y = \left[y^L, y^U\right] \in M_I$, and a scalar $\lambda \in R$, we define the following operations:

$$X + Y = \left[x^L + y^L, x^U + y^U\right],\ \lambda X = \begin{cases}\left[\lambda x^L, \lambda x^U\right], & \lambda \geq 0,\\ \left[\lambda x^U, \lambda x^L\right], & \lambda < 0.\end{cases}$$

$$X \times Y = \left[\min(x^L y^L, x^L y^U, x^U y^L, x^U y^U), \max(x^L y^L, x^L y^U, x^U y^L, x^U y^U)\right],$$

$$\frac{X}{Y} = \left[x^L, x^U\right] \times \left[\frac{1}{y^U}, \frac{1}{y^L}\right],\ \text{where}\quad y^L \neq 0, y^U \neq 0.$$

The gh-difference of X ang Y is defined as

$$\left[x^L, x^U\right] \ominus_{gh} \left[y^L, y^U\right] = \left[\min\left\{x^L - y^L, x^U - y^U\right\}, \max\left\{x^L - y^L, x^U - y^U\right\}\right].$$

Definition 2.6. ([7]) For closed intervals $X = \left[x^L, x^U\right]$ and $Y = \left[y^L, y^U\right]$,

(a) $X \leq_{LU} Y$ if and only if $x^L \leq y^L$ and $x^U \leq y^U$;

(b) $X <_{LU} Y$ if and only if $X \leq_{LU} Y$ and $X \neq Y$, which can be expressed in one of the following ways:

$$\begin{cases} a^L < b^L \\ a^U \leq b^U \end{cases} \quad or \quad \begin{cases} a^L \leq b^L \\ a^U < b^U \end{cases} \quad or \quad \begin{cases} a^L < b^L \\ a^U < b^U. \end{cases}$$

A mapping $f : E \subset \mathscr{M} \to M_I$ is called an interval-valued function if, for every $x \in E$, $f(x)$ is a closed interval in R. This can be expressed as $f(x) = [f^L(x), f^U(x)]$, where f^L and f^U are real-valued functions satisfying $f^L(x) \leq f^U(x)$ for every $x \in E$. The functions f^L and f^U are referred to as the endpoint functions of f. By using interval operations, $\frac{f}{g}(x)$ can be expressed as (see [22])

$$\frac{f(x)}{g(x)} = \left[f^L(x), f^U(x)\right] \times \left[\frac{1}{g^U(x)}, \frac{1}{g^L(x)}\right] = \left[\frac{f^L(x)}{g^U(x)}, \frac{f^U(x)}{g^L(x)}\right],$$

where $g^L(x) > 0$ for any $x \in E$.

3. Optimality Conditions

Throughout this section, we consider the multiobjective fractional interval-valued optimization problem on Hadamard manifolds (MFIOP) outlined in [22]

$$\begin{aligned} \min \qquad & \frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)}\right) \\ \text{s.t.} \qquad & h(x) = (h_1(x), \ldots, h_p(x)) \leq_{LU} [0,0], \end{aligned}$$

where f_k, g_k, h_s are interval-valued functions and $g_k^L > 0$, $(k \in K = \{1, \ldots, m\}; s \in S = \{1, \ldots, p\})$. Let $f_k^L, f_k^U, g_k^L, g_k^U, h_s^L, h_s^U : E \subset \mathscr{M} \to R$ be locally Lipschitz functions, with E being a geodesic convex set. Furthermore, we define D as the set of feasible solutions:

$$D = \{x \in E : h_s(x) \leq_{LU} [0,0], s \in S\}.$$

Definition 3.1. ([21]) A feasible point $\bar{x}$ is defined as a weak LU-efficient (weak LU-Pareto) solution to the problem (MFIOP) if no other feasible point $x \in D$ satisfies, for every $k \in K$, $\frac{f_k(x)}{g_k(x)} <_{LU} \frac{f_k(\bar{x})}{g_k(\bar{x})}$.

Definition 3.2. ([21]) A feasible point $\bar{x}$ is called an LU-efficient (LU-Pareto) solution to the problem (MFIOP) if no other feasible point $x \in D$ satisfies, for every $k \in K$, $\frac{f_k(x)}{g_k(x)} \leq_{LU} \frac{f_k(\bar{x})}{g_k(\bar{x})}$, and there exists at least one $k \in K$ for which $\frac{f_k(x)}{g_k(x)} <_{LU} \frac{f_k(\bar{x})}{g_k(\bar{x})}$.

Theorem 3.1. *If $\bar{x} \in D$ is a weak LU-efficient solution to problem* (MFIOP) *and the Slater condition is satisfied at $\bar{x}$. Then, there exist $\lambda_k^L, \lambda_k^U > 0$, and $\mu_s^L, \mu_s^U \geq 0$, for $k \in K$ and $s \in S$ such that*

$$0 \in \sum_{k=1}^{m} \lambda_k^L(\partial f_k^L(\bar{x}) - z_k \partial g_k^U(\bar{x})) + \sum_{k=1}^{m} \lambda_k^U(\partial f_k^U(\bar{x}) - c_k \partial g_k^L(\bar{x})) + \sum_{s=1}^{p} \mu_s^L \partial h_s^L(\bar{x}) + \sum_{s=1}^{p} \mu_s^U \partial h_s^U(\bar{x}),$$

$$\mu_s h_s(\bar{x}) = [0,0],$$

where $z_k = \frac{f_k^L}{g_k^U}(\bar{x})$ and $c_k = \frac{f_k^U}{g_k^L}(\bar{x})$.

Proof. Let $\bar{x} \in D$ be a weak *LU*-efficient solution to the (MFIOP). We define the function $F : E \subset \mathscr{M} \to R$ as follows:

$$F(x) = \max\left\{[f_k^L(x) - z_k g_k^U(x)] - [f_k^L(\bar{x}) - z_k g_k^U(\bar{x})], [f_k^U(x) - c_k g_k^L(x)] - [f_k^U(\bar{x}) - c_k g_k^L(\bar{x})]\right.,$$
$$\left. h_s^L(x), h_s^U(x) \mid k \in K, s \in S\right\}.$$

It is obvious that $F(x) \geq 0$. Due to the Slater condition being satisfied at $\bar{x}$, we observe that $h_s(\bar{x}) <_{LU} [0,0]$, $s \in S$. Thus $F(\bar{x}) = 0$. Hence, $F(x)$ attain its minimum at $\bar{x}$. Since $f_k^L(x)$, $f_k^U(x)$, $g_k^L(x)$, $g_k^U(x)$, $h_s^L(x)$, $h_s^U(x) : E \subset \mathscr{M} \to R$ are all locally Lipschitz functions, we also see that $F(x)$ is locally Lipschitz function. Thus $0 \in \partial F(\bar{x})$. Since $\bar{x}$ is a weak *LU*-efficient solution to the problem (MFIOP), we obtain

$$\partial[(f_k^L(x) - z_k g_k^U(x)) - (f_k^L(\bar{x}) - z_k g_k^U(\bar{x}))] = \partial f_k^L(x) - z_k \partial g_k^U(x), \quad k \in K,$$
$$\partial[(f_k^U(x) - c_k g_k^L(x)) - (f_k^U(\bar{x}) - c_k g_k^L(\bar{x}))] = \partial f_k^U(x) - c_k \partial g_k^L(x), \quad k \in K.$$

Hence $0 \in \operatorname{conv}\left\{\partial f_k^L(\bar{x}) - z_k \partial g_k^U(\bar{x}), \partial f_k^U(\bar{x}) - c_k \partial g_k^L(\bar{x}), \partial h_s^L(\bar{x}), \partial h_s^U(\bar{x}), k \in K, s \in S\right\}$. Then there exist $\lambda_k^L, \lambda_k^U > 0, \mu_s^L, \mu_s^U \geq 0$, $k \in K, s \in S$ such that

$$0 \in \sum_{k=1}^{m} \lambda_k^L(\partial f_k^L(\bar{x}) - z_k \partial g_k^U(\bar{x})) + \sum_{k=1}^{m} \lambda_k^U(\partial f_k^U(\bar{x}) - c_k \partial g_k^L(\bar{x})) + \sum_{s=1}^{p} \mu_s^L \partial h_s^L(\bar{x}) + \sum_{s=1}^{p} \mu_s^U \partial h_s^U(\bar{x}),$$

with $\sum_{k=1}^m \lambda_k^L + \sum_{k=1}^m \lambda_k^U + \sum_{s=1}^p \mu_s^L + \sum_{s=1}^p \mu_s^U = 1$. Furthermore, when $h_s(\bar{x}) <_{LU} [0,0]$, we have $\mu_s^L = \mu_s^U = 0$ for those $s \in S$.

Example 3.1. Let $\mathscr{M} \subset R^2$ be defined as $\mathscr{M} := \{x = (x_1, x_2) \in R^2, x > 0\}$. Thus $\mathscr{M}$ is a Riemannian manifold equipped with a certain metric. A bilinear form is defined as

$$\langle v_1, v_2 \rangle_x = \langle \rho(x) v_1, v_2 \rangle_x, v_1, v_2 \in T_x\mathscr{M}, x \in \mathscr{M}.$$

where $\rho(x) = \begin{pmatrix} \frac{1}{x_1} & 0 \\ 0 & \frac{1}{x_2} \end{pmatrix}$. Thus $\mathscr{M}$ be termed a Hadamard manifold. For any $x \in \mathscr{M}, v \in T_x\mathscr{M}$, we define the exponential map $\exp_x : T_x\mathscr{M} \to \mathscr{M}$ as $\exp_x(v) = (x_1 e^{\frac{v_1}{x_1}}, x_2 e^{\frac{v_2}{x_2}})$ for all

$v = (v_1, v_2) \in \mathscr{M}$. For any given $x, y \subset \mathscr{M}$, we have $\exp_x^{-1} y = \left(x_1 \ln \frac{y_1}{x_1}, x_2 \ln \frac{y_2}{x_2}\right)$. We consider the function $f(x), g(x), h(x) : \mathscr{M} \to M_I$ defined as follows for optimization over the Hadamard manifold (P1):

$$\begin{aligned} \min \quad & \frac{f(x)}{g(x)} = \frac{[f^L(x), 2f^L(x)]}{[x_1+1, x_1+2]} \\ \text{s.t.} \quad & h(x) = [x_2-2, x_2-3] \leq_{LU} [0,0], \end{aligned}$$

where

$$f^L(x) = \begin{cases} x_1^2 + x_2^2, & x_1 + x_2 \leq 1, \\ 3, & x_1 + x_2 = 1. \end{cases}$$

Given that $g^L(x) > 0$, we can express $\frac{f(x)}{g(x)} = \begin{cases} \left[\frac{x_1^2+x_2^2}{x_1+2}, \frac{2(x_1^2+x_2^2)}{x_1+1}\right], & x_1+x_2 \leq 1, \\ \left[\frac{3}{x_1+2}, \frac{6}{x_1+1}\right], & x_1+x_2 = 1. \end{cases}$ The function is locally Lipschitz, and $f(x)$ is defined by $x_1 + x_2 = 1$, which is non-volatile. We have

$$\frac{f(0)}{g(0)} \ominus_{gh} \frac{f(x)}{g(x)} = [0,0] \ominus_{gh} \frac{f(x)}{g(x)} = \begin{cases} \left[-\frac{2(x_1^2+x_2^2)}{x_1+1}, -\frac{x_1^2+x_2^2}{x_1+2}\right], & x_1+x_2 \leq 1 \\ \left[-\frac{6}{x_1+1}, -\frac{3}{x_1+2}\right], & x_1+x_2 = 1 \end{cases} \leq_{LU} [0,0].$$

Thus $\frac{f(0)}{g(0)} \leq_{LU} \frac{f(x)}{g(x)}$, and $(0,0)$ is feasible for problem (P1) as a weak LU-efficient solution. Moreover, we have $h(0) <_{LU} [0,0]$. Furthermore, due to Slater's conditions holding at $(0,0)$, We derive

$$\partial f^L(0,0) - z\partial g^U(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \partial f^U(0,0) - c\partial g^L(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\partial h^L(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \partial h^U(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By setting $\lambda^L = \frac{1}{2}, \lambda^U = \frac{1}{2}, \mu^L = 0, \mu^U = 0$, We conclude that

$$0 \in \lambda^L(\partial f^L(0,0) - z\partial g^U(0,0)) + \lambda^U(\partial f^U(0,0) - c\partial g^L(0,0)) + \mu^L \partial h^L(0,0) + \mu^U \partial h^U(0,0),$$

$$\mu h(0,0) = [0,0].$$

The necessary optimality conditions describe the fundamental requirements that weak LU-efficient solution must fulfill. However, solutions satisfying these conditions may not necessarily be optimal. To rigorously verify weak LU-efficient solution, we must also consider the sufficient optimality conditions. Under appropriate convexity assumptions, these sufficient conditions offer definitive guarantees of optimality.

Theorem 3.2. *Let $\bar{x} \in E$ be a feasible solution to problem* (MFIOP)*, and assume that there exist $\lambda_k^L, \lambda_k^U > 0$ and $\mu_s^L, \mu_s^U \geq 0$ such that*

(a) $0 \in \sum_{k=1}^m \lambda_k^L(\partial f_k^L(\bar{x}) - z_k \partial g_k^U(\bar{x})) + \sum_{k=1}^m \lambda_k^U(\partial f_k^U(\bar{x}) - c_k \partial g_k^L(\bar{x})) + \sum_{s=1}^p \mu_s^L \partial h_s^L(\bar{x}) + \sum_{s=1}^p \mu_s^U \partial h_s^U(\bar{x})$,

(b) $\mu_s h_s(\bar{x}) = [0,0]$, $s \in S$;

(c) $f_k^L(x) - z_k g_k^U(x), f_k^U(x) - c_k g_k^L(x), h_s^L(x), h_s^U(x)$ *are all geodesic convexity at $\bar{x}$,*

where $z_k = \frac{f_k^L}{g_k^U}(\bar{x})$ and $c_k = \frac{f_k^U}{g_k^L}(\bar{x})$. Then, $\bar{x}$ is a weak LU-efficient solution to the problem (MFIOP).

Proof. Using condition (a), one has $\rho_k^L \in \partial f_k^L(\bar{x}) - z_k\partial g_k^U(\bar{x}), \rho_k^U \in \partial f_k^U(\bar{x}) - c_k\partial g_k^L(\bar{x}), w_s^L \in \partial h_s^L(\bar{x}), w_s^U \in \partial h_s^U(\bar{x})$ such that $0 = \sum_{k=1}^m \lambda_k^L \rho_k^L(\bar{x}) + \sum_{k=1}^m \lambda_k^U \rho_k^U(\bar{x}) + \sum_{s=1}^p \mu_s^L w_s^L(\bar{x}) + \sum_{s=1}^p \mu_s^U w_s^U(\bar{x})$. It follows that

$$\begin{aligned}\sum_{k=1}^m \lambda_k^L \langle \rho_k^L, \exp_x^{-1}(\bar{x})\rangle_x + \sum_{k=1}^m \lambda_k^U \langle \rho_k^U, \exp_x^{-1}(\bar{x})\rangle_x \\ + \sum_{s=1}^p \mu_s^L \langle w_s^L, \exp_x^{-1}(\bar{x})\rangle_x + \sum_{s=1}^p \mu_s^U \langle w_s^U, \exp_x^{-1}(\bar{x})\rangle_x = 0.\end{aligned} \tag{3.1}$$

Assume that $\bar{x}$ is not a weak LU-efficient solution to the problem(MFIOP). There exists $\tilde{x} \in D$ such that $\frac{f_k(\tilde{x})}{g_k(\tilde{x})} <_{LU} \frac{f_k(\bar{x})}{g_k(\bar{x})}$, which implies that $\frac{f_k^L(\tilde{x})}{g_k^U(\tilde{x})} - \frac{f_k^L(\bar{x})}{g_k^U(\bar{x})} < 0$ and $\frac{f_k^U(\tilde{x})}{g_k^L(\tilde{x})} - \frac{f_k^U(\bar{x})}{g_k^L(\bar{x})} < 0$. Therefore, we conclude

$$f_k^L(\tilde{x}) - z_k g_k^U(\tilde{x}) = \left[\frac{f_k^L(\tilde{x})}{g_k^U(\tilde{x})} - \frac{f_k^L(\bar{x})}{g_k^U(\bar{x})}\right] g_k^U(\tilde{x}) < 0 = f_k^L(\bar{x}) - z_k g_k^U(\bar{x}).$$

Similarly, we can obtain $f_k^U(\tilde{x}) - c_k g_k^L(\tilde{x}) < f_k^U(\bar{x}) - c_k g_k^L(\bar{x})$. From $\lambda_k^L, \lambda_k^U > 0$, we obtain

$$\sum_{k=1}^m \lambda_k^L \left(f_k^L(\tilde{x}) - z_k g_k^U(\tilde{x})\right) < \sum_{k=1}^m \lambda_k^L (f_k^L(\bar{x}) - z_k g_k^U(\bar{x})),$$

$$\sum_{k=1}^m \lambda_k^U \left(f_k^U(\tilde{x}) - c_k g_k^L(\tilde{x})\right) < \sum_{k=1}^m \lambda_k^U (f_k^U(\bar{x}) - c_k g_k^L(\bar{x})),$$

From conditions (c), we see that

$$\left[f_k^L(\tilde{x}) - z_k g_k^U(\tilde{x})\right] - \left[f_k^L(\bar{x}) - z_k g_k^U(\bar{x})\right] \geq \langle \rho_k^L, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}},$$

$$\left[f_k^U(\tilde{x}) - c_k g_k^L(\tilde{x})\right] - \left[f_k^U(\bar{x}) - c_k g_k^L(\bar{x})\right] \geq \langle \rho_k^U, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}}.$$

Hence, we have

$$0 > \sum_{k=1}^m \lambda_k^L \left\{\left[f_k^L(\tilde{x}) - z_k g_k^U(\tilde{x})\right] - \left[f_k^L(\bar{x}) - z_k g_k^U(\bar{x})\right]\right\} \geq \sum_{k=1}^m \lambda_k^L \langle \rho_k^L, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}},$$

$$0 > \sum_{k=1}^m \lambda_k^U \left\{\left[f_k^U(\tilde{x}) - c_k g_k^L(\tilde{x})\right] - \left[f_k^U(\bar{x}) - c_k g_k^L(\bar{x})\right]\right\} \geq \sum_{k=1}^m \lambda_k^U \langle \rho_k^U, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}}.$$

It follows that $\sum_{k=1}^m \lambda_k^L \langle \rho_k^L, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}} + \sum_{k=1}^m \lambda_k^U \langle \rho_k^U, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}} < 0$. Due to $\sum_{s=1}^p \mu_s^L h_s^L(\tilde{x}) \leq 0$ and $\sum_{s=1}^p \mu_s^U h_s^U(\tilde{x}) \leq 0$ under conditions (b) and (c), we deduce $\sum_{s=1}^p \mu_s^L \langle w_s^L, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}} \leq 0$ and $\sum_{s=1}^p \mu_s^U \langle w_s^U, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}} \leq 0$. Thus

$$\begin{aligned}\sum_{k=1}^m \lambda_k^L \langle \rho_k^L, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}} + \sum_{k=1}^m \lambda_k^U \langle \rho_k^U, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}} \\ + \sum_{s=1}^p \mu_s^L \langle w_s^L, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}} + \sum_{s=1}^p \mu_s^U \langle w_s^U, \exp_{\bar{x}}^{-1}(\tilde{x})\rangle_{\bar{x}} < 0.\end{aligned}$$

which leads to a contradiction to (3.1). The theorem is proved.

Example 3.2. Let $\mathscr{M} \subset R^2$ be defined as $\mathscr{M} := \{x = (x_1, x_2) \in R^2, x > 0\}$. Thus $\mathscr{M}$ is a Riemannian manifold equipped with a certain metric. A bilinear form is defined by

$$\langle v_1, v_2 \rangle_x = \langle \rho(x) v_1, v_2 \rangle_x, v_1, v_2 \in T_x\mathscr{M}, x \in \mathscr{M},$$

where $\rho(x) = \begin{pmatrix} \frac{1}{x_1} & 0 \\ 0 & \frac{1}{x_2} \end{pmatrix}$. Let $\mathscr{M}$ be termed a Hadamard manifold. For any $x \in \mathscr{M}, v \in T_x\mathscr{M}$, we define the exponential map $\exp_x : T_x\mathscr{M} \to \mathscr{M}$ by $\exp_x(v) = (x_1 e^{\frac{v_1}{x_1}}, x_2 e^{\frac{v_2}{x_2}})$ for all $v = (v_1, v_2) \in \mathscr{M}$. For any given $x, y \subset \mathscr{M}$, we have $\exp_x^{-1} y = \left(x_1 \ln \frac{y_1}{x_1}, x_2 \ln \frac{y_2}{x_2} \right)$. Let $f(x), g(x), h(x) : \mathscr{M} \to M_I$ be defined as functions. Consider the following optimization problem over the Hadamard manifold (P2):

$$\begin{aligned} \min \quad & \frac{f(x)}{g(x)} = \frac{[|x_1 - 1| + x_1 - 1, |x_1 - 1| + x_1]}{[x_1, x_1 + 1]} \\ \text{s.t.} \quad & h(x) = [x_1 - 1, x_1 + x_2] \leq_{LU} [0, 0]. \end{aligned}$$

Let $f_k^L(x), f_k^U(x), g_k^L(x), g_k^U(x), h_s^L(x), h_s^U(x) : E \subset \mathscr{M} \to R$ be a locally Lipschitz function at $x = (1, 1)$, and be geodesic convex. We obtain

$$\partial f(x) = \begin{cases} 2x_1^2, & x > 1, \\ 2x_1^2 t, t \in [0, 1], & x = 1, \\ 0, & x < 1. \end{cases} \qquad \partial g(x) = 1.$$

$$z = \frac{f^L(1,1)}{g^U(1,1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad c = \frac{f^U(1,1)}{g^L(1,1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\partial f^L(1,1) - z\partial g^U(1,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \partial f^U(1,1) - c\partial g^L(1,1) = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

$$\partial h^L(1,1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \partial h^U(1,1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Setting $\lambda^L = 1, \lambda^U = 1, \mu^L = 1, \mu^U = 0$, we obtain

$$0 \in \lambda^L(\partial f^L(1,1) - z\partial g^U(1,1)) + \lambda^U(\partial f^U(1,1) - c\partial g^L(1,1)) + \mu^L \partial h^L(1,1) + \mu^U \partial h^U(1,1),$$

$$\mu h(1,1) = [0,0].$$

We conclude that $(1,1)$ is a weak LU-efficient solution to problem (P2). It is easy to verify

$$\frac{f(1,1)}{g(1,1)} \ominus_{gh} \frac{f(x)}{g(x)} = \left[-\frac{|x_1 - 1| + x_1 - 1}{x_1 + 1}, -\frac{|x_1 - 1|}{x_1} \right] \leq_{LU} [0,0].$$

4. DUALITY THEORY

In this section, we define the Mond-Weir type dual problem associated with the problem (MFIOP). We explore the connection between the primal problem (MFIOP) and its corresponding dual (MWD), and prove the duality results.

We consider the Mond-Weir type dual problem (MWD):

$$\begin{aligned} \max \quad & \frac{f(y)}{g(y)} = \left(\frac{f_1(y)}{g_1(y)}, \ldots, \frac{f_m(y)}{g_m(y)} \right), \\ \text{s.t.} \quad & (y, \lambda_k^L, \lambda_k^U, \mu_s^L, \mu_s^U) \in W, \end{aligned}$$

$$\begin{aligned} W = \Big\{ & (y, \lambda_k^L, \lambda_k^U, \mu_s^L, \mu_s^U) : 0 \in \sum_{k=1}^m \lambda_k^L (\partial f_k^L(y) - z_k \partial g_k^U(y) + \sum_{k=1}^m \lambda_k^U (\partial f_k^U(y) - c_k \partial g_k^L(y)) \\ & + \sum_{s=1}^p \mu_s^L \partial h_s^L(y) + \sum_{s=1}^p \mu_s^U \partial h_s^U(y), \sum_{s=1}^p \mu_s h_s(y) \geq [0,0], \lambda_k^L > 0, \lambda_k^U > 0, \mu_s^L \geq 0, \\ & \mu_s^U \geq 0, k \in K, s \in S \Big\}, \end{aligned}$$

which denotes the entire feasible region for (MWD). In addition, We also assume $Y = \{y \in E : (y, \lambda_k^L, \lambda_k^U, \mu_s^L, \mu_s^U) \in W\}$. $(\bar{y}, \lambda_k^L, \lambda_k^U, \mu_s^L, \mu_s^U)$ is a weak LU-efficient solution to problem (MWD) if any $(y, \lambda_k^L, \lambda_k^U, \mu_s^L, \mu_s^U) \in W$ does not satisfy $\frac{f_k(\bar{y})}{g_k(\bar{y})} <_{LU} \frac{f_k(y)}{g_k(y)}$. $(\bar{y}, \lambda_k^L, \lambda_k^U, \mu_s^L, \mu_s^U)$ is an LU-efficient solution to problem (MWD) if any $(y, \lambda_k^L, \lambda_k^U, \mu_s^L, \mu_s^U) \in W$ does not satisfy $\frac{f_k(\bar{y})}{g_k(\bar{y})} \leq_{LU} \frac{f_k(y)}{g_k(y)}$.

Theorem 4.1. *(Weak Duality): Let x and $(y, \lambda_k^L, \lambda_k^U, \mu_s^L, \mu_s^U)$ represent feasible solutions to the problems (*MFIOP*) and (*MWD*). If $f_k^L(x) - z_k g_k^U(x), f_k^U(x) - c_k g_k^L(x), h_s^L(x), h_s^U(x)$ are the geodesic convex functions on $D \cup Y$, then it does not hold that $\frac{f_k(x)}{g_k(x)} <_{LU} \frac{f_k(y)}{g_k(y)}$.*

Proof. Since $(y, \lambda_k^L, \lambda_k^U, \mu_s^L, \mu_s^U)$ represent feasible solutions to problems (MWD), there exist $\lambda_k^L, \lambda_k^U > 0$, $\mu_s^L, \mu_s^U \geq 0$ and $\rho_k^L \in \partial f_k^L(y) - z_k \partial g_k^U(y)$, $\rho_k^U \in \partial f_k^U(y) - c_k \partial g_k^L(y)$, $w_s^L \in \partial h_s^L(y)$, $w_s^U \in \partial h_s^U(y)$ such that $0 = \sum_{k=1}^m \lambda_k^L \rho_k^L(y) + \sum_{k=1}^m \lambda_k^U \rho_k^U(y) + \sum_{s=1}^p \mu_s^L w_s^L(y) + \sum_{s=1}^p \mu_s^U w_s^U(y)$. It follows that

$$\begin{aligned} & \sum_{k=1}^m \lambda_k^L \langle \rho_k^L, \exp_y^{-1}(x) \rangle_y + \sum_{k=1}^m \lambda_k^U \langle \rho_k^U, \exp_y^{-1}(x) \rangle_y \\ & + \sum_{s=1}^p \mu_s^L \langle w_s^L, \exp_y^{-1}(x) \rangle_y + \sum_{s=1}^p \mu_s^U \langle w_s^U, \exp_y^{-1}(x) \rangle_y = 0. \end{aligned} \tag{4.1}$$

Equality $x = y$ immediately implies that the weak duality holds. If $x \neq y$, we suppose $\frac{f_k(x)}{g_k(x)} <_{LU} \frac{f_k(y)}{g_k(y)}$, which implies that $\frac{f_k^L(x)}{g_k^U(x)} - \frac{f_k^L(y)}{g_k^U(y)} < 0$ and $\frac{f_k^U(x)}{g_k^L(x)} - \frac{f_k^U(y)}{g_k^L(y)} < 0$. Thus

$$f_k^L(y) - z_k g_k^U(y) = \left[\frac{f_k^L(y)}{g_k^U(y)} - \frac{f_k^L(\bar{x})}{g_k^U(\bar{x})} \right] g_k^U(y) > 0 \geq \left[\frac{f_k^L(x)}{g_k^U(x)} - \frac{f_k^L(\bar{x})}{g_k^U(\bar{x})} \right] g_k^U(x) = f_k^L(x) - z_k g_k^U(x).$$

Similarly, we can obtain $f_k^U(y) - c_k g_k^L(y) > f_k^U(x) - c_k g_k^L(x)$. From $\lambda_k^L, \lambda_k^U > 0$, we see that

$$\begin{aligned} \sum_{k=1}^m \lambda_k^L \left(f_k^L(y) - z_k g_k^U(y) \right) &> \sum_{k=1}^m \lambda_k^L (f_k^L(x) - z_k g_k^U(x)), \\ \sum_{k=1}^m \lambda_k^U \left(f_k^U(y) - c_k g_k^L(y) \right) &> \sum_{k=1}^m \lambda_k^U (f_k^U(x) - c_k g_k^L(x)). \end{aligned}$$

Since the functions $f_k^L(x)-z_kg_k^U(x), f_k^U(x)-t_kg_k^L(x), h_s^L(x), h_s^U(x)$ are geodesic convex function on $D\cup Y$, Lemma 2.2 directly yields

$$\left[f_k^L(x)-z_kg_k^U(x)\right]-\left[f_k^L(y)-z_kg_k^U(y)\right]\geq\langle\rho_k^L,\exp_y^{-1}(x)\rangle_y,,\quad k\in K,$$

$$\left[f_k^U(x)-c_kg_k^L(x)\right]-\left[f_k^U(y)-c_kg_k^L(y)\right]\geq\langle\rho_k^U,\exp_y^{-1}(x)\rangle_y,\quad k\in K,$$

$$h_s^L(x)-h_s^L(y)\geq\langle w_s^L,\exp_y^{-1}(x)\rangle_y,\ h_s^U(x)-h_s^U(y)\geq\langle w_s^U,\exp_y^{-1}(x)\rangle_y,\quad s\in S,$$

where $\rho_k^L\in\partial f_k^L(\bar{x})-z_k\partial g_k^U(\bar{x}),\rho_k^U\in\partial f_k^U(\bar{x})-c_k\partial g_k^L(\bar{x}),w_s^L\in\partial h_s^L(\bar{x}),w_s^U\in\partial h_s^U(\bar{x})$. Thus

$$0>\sum_{k=1}^m\lambda_k^L\left\{\left[f_k^L(x)-z_kg_k^U(x)\right]-\left[f_k^L(y)-z_kg_k^U(y)\right]\right\}\geq\sum_{k=1}^m\lambda_k^L\langle\rho_k^L,\exp_y^{-1}(x)\rangle_y,$$

$$0>\sum_{k=1}^m\lambda_k^U\left\{\left[f_k^U(x)-c_kg_k^L(x)\right]-\left[f_k^U(y)-c_kg_k^L(y)\right]\right\}\geq\sum_{k=1}^m\lambda_k^U\langle\rho_k^U,\exp_y^{-1}(x)\rangle_y,$$

$$\sum_{s=1}^p\mu_s^Lh_s^L(x)-\sum_{s=1}^p\mu_s^Lh_s^L(y)\geq\sum_{s=1}^p\mu_s^L\langle w_s^L,\exp_y^{-1}(x)\rangle_y,$$

$$\sum_{s=1}^p\mu_s^Uh_s^U(x)-\sum_{s=1}^p\mu_s^Uh_s^U(y)\geq\sum_{s=1}^p\mu_s^U\langle w_s^U,\exp_y^{-1}(x)\rangle_y.$$

Thus $\sum_{k=1}^m\lambda_k^L\langle\rho_k^L,\exp_y^{-1}(x)\rangle_y+\sum_{k=1}^m\lambda_k^U\langle\rho_k^U,\exp_y^{-1}(x)\rangle_y<0$. Since x and $(y,\lambda_k^L,\lambda_k^U,\mu_s^L,\mu_s^U)$ are feasible solutions of problem (MFIOP) and (MWD), respectively, we have $\sum_{s=1}^p\mu_s^Lh_s^L(x)-\sum_{s=1}^p\mu_s^Lh_s^L(y)\leq 0$ and $\sum_{s=1}^p\mu_s^Uh_s^U(x)-\sum_{s=1}^p\mu_s^Uh_s^U(y)\leq 0$. Hence $\sum_{s=1}^p\mu_s^L\langle w_s^L,\exp_y^{-1}(x)\rangle_y\leq 0$ and $\sum_{s=1}^p\mu_s^U\langle w_s^U,\exp_y^{-1}(x)\rangle_y\leq 0$. It follows that

$$\sum_{k=1}^m\lambda_k^L\langle\rho_k^L,\exp_y^{-1}(x)\rangle_y+\sum_{k=1}^m\lambda_k^U\langle\rho_k^U,\exp_y^{-1}(x)\rangle_y$$
$$+\sum_{s=1}^p\mu_s^L\langle w_s^L,\exp_y^{-1}(x)\rangle_y+\sum_{s=1}^p\mu_s^U\langle w_s^U,\exp_y^{-1}(x)\rangle_y<0,$$

which contradicts (4.1). This completes the proof.

Theorem 4.2. *(Strong Duality) Let $\bar{x}$ be a weak LU-efficient solution to problem (*MFIOP*), and assume that the Slater constraint qualification is satisfied at $\bar{x}$. Then, there exist $\bar{\lambda}_k^L,\bar{\lambda}_k^U,\bar{\mu}_s^L,\bar{\mu}_s^U$ such that $(\bar{y},\bar{\lambda}_k^L,\bar{\lambda}_k^U,\bar{\mu}_s^L,\bar{\mu}_s^U)$ is feasible for problem (*MWD*), and the objective functions of problems (*MFIOP*) and (*MWD*) coincide at $\bar{x}$. If every assumption of the weak duality theorem holds, then $(\bar{x},\bar{\lambda}_k^L,\bar{\lambda}_k^U,\bar{\mu}_s^L,\bar{\mu}_s^U)$ is a weak LU-efficient solution to problem (*MWD*).*

Proof. $\bar{x}$ is classified as a weakly LU-efficient solution of (MFIOP). Then, there exist $\bar{\lambda}_k^L,\bar{\lambda}_k^U>0$ and $\bar{\mu}_s^L,\bar{\mu}_s^U\geq 0$ for $k\in K$ and $s\in S$ such that $\bar{\mu}_sh_s(\bar{x})=[0,0]$ and

$$0\in\sum_{k=1}^m\bar{\lambda}_k^L(\partial f_k^L(\bar{x})-z_k\partial g_k^U(\bar{x}))+\sum_{k=1}^m\bar{\lambda}_k^U(\partial f_k^U(\bar{x})-c_k\partial g_k^L(\bar{x}))+\sum_{s=1}^p\bar{\mu}_s^L\partial h_s^L(\bar{x})+\sum_{s=1}^p\bar{\mu}_s^U\partial h_s^U(\bar{x}).$$

Thus $(\bar{x},\bar{\lambda}_k^L,\bar{\lambda}_k^U,\bar{\mu}_s^L,\bar{\mu}_s^U)$ is a feasible solution of (MWD). Thus, the objective functions of (MFIOP) and (MWD) have the same value . According to the weak duality theorem, $\frac{f_k(\bar{x})}{g_k(\bar{x})}<_{LU}\frac{f_k(y)}{g_k(y)}$ does not hold. Therefore, $(\bar{x},\bar{\lambda}_k^L,\bar{\lambda}_k^U,\bar{\mu}_s^L,\bar{\mu}_s^U)$ is a weak LU-efficient solution to problem (MWD).

Theorem 4.3. *(Inverse duality) Let $(\bar{y},\bar{\lambda}_k^L,\bar{\lambda}_k^U,\bar{\mu}_s^L,\bar{\mu}_s^U)$ be a weakly LU-efficient solution of the problem (*MWD*), with $\bar{y}\in D$. If $f_k^L(x)-z_kg_k^U(x), f_k^U(x)-c_kg_k^L(x), h_s^L(x)$, $h_s^U(x)$ are geodesic convex functions on $D\cup Y$, then $\bar{y}$ is a weak LU-efficient solution of the problem (*MFIOP*).*

Proof. Since $(\bar{y},\bar{\lambda}_k^L,\bar{\lambda}_k^U,\bar{\mu}_s^L,\bar{\mu}_s^U)$ constitutes a weakly LU-efficient solution of (MWD), we have $\frac{f_k(y)}{g_k(y)}\leq_{LU}\frac{f_k(\bar{y})}{g_k(\bar{y})}$. From Theorem 4.1, we see that $\frac{f_k(y)}{g_k(y)}\leq_{LU}\frac{f_k(x)}{g_k(x)}$. Since $\bar{y}\in D$, we obtain $\frac{f_k(\bar{y})}{g_k(\bar{y})}\leq_{LU}\frac{f_k(x)}{g_k(x)}$. Therefore, $\bar{y}$ is a weak LU-efficient solution of (MFIOP).

Theorem 4.4. *(Restricted Inverse Duality) Let $(\bar{y},\bar{\lambda}_k^L,\bar{\lambda}_k^U,\bar{\mu}_s^L,\bar{\mu}_s^U)$ be a feasible solution of (*MWD*). If $f_k^L(x)-z_kg_k^U(x)$, $f_k^U(x)-c_kg_k^L(x)$, $h_s^L(x)$, $h_s^U(x)$ are geodesic convex functions on $D\cup Y$, and there exists a point $\bar{x}\in D$ such that $\frac{f_k(\bar{y})}{g_k(\bar{y})}=\frac{f_k(\bar{x})}{g_k(\bar{x})}$, then $\bar{y}$ is a weak LU-efficient solution to (*MFIOP*).*

Proof. Suppose $\bar{y}$ is not a weak LU-efficient solution of the problem (MFIOP). Then There exists $\tilde{x}\in D$ such that $\frac{f_k(\tilde{x})}{g_k(\tilde{x})}<_{LU}\frac{f_k(\bar{y})}{g_k(\bar{y})}$, which implies $\frac{f_k^L(\tilde{x})}{g_k^U(\tilde{x})}-\frac{f_k^L(\bar{y})}{g_k^U(\bar{y})}<0$ and $\frac{f_k^U(\tilde{x})}{g_k^L(\tilde{x})}-\frac{f_k^U(\bar{y})}{g_k^L(\bar{y})}<0$. Therefore,

$$f_k^L(\tilde{x})-z_kg_k^U(\tilde{x})=\left[\frac{f_k^L(\tilde{x})}{g_k^U(\tilde{x})}-\frac{f_k^L(\bar{y})}{g_k^U(\bar{y})}\right]g_k^U(\tilde{x})<0=f_k^L(\bar{y})-z_kg_k^U(\bar{y}).$$

Similarly, we have $f_k^U(\tilde{x})-c_kg_k^L(\tilde{x})<f_k^U(\bar{y})-c_kg_k^L(\bar{y})$. From $\lambda_k^L,\lambda_k^U>0$, we obtain

$$\sum_{k=1}^m\lambda_k^L\left(f_k^L(\tilde{x})-z_kg_k^U(\tilde{x})\right)<\sum_{k=1}^m\lambda_k^L(f_k^L(\bar{y})-z_kg_k^U(\bar{y})),$$

$$\sum_{k=1}^m\lambda_k^U\left(f_k^U(\tilde{x})-c_kg_k^L(\tilde{x})\right)<\sum_{k=1}^m\lambda_k^U(f_k^U(\bar{y})-c_kg_k^L(\bar{y})),$$

Note that $f_k^L(x)-s_kg_k^U(x),f_k^U(x)-t_kg_k^L(x),h_s^L(x),h_s^U(x)$ are geodesic convex functions on $D\cup Y$. Lemma 2.2 yields

$$\left[f_k^L(\tilde{x})-z_kg_k^U(\tilde{x})\right]-\left[f_k^L(\bar{y})-z_kg_k^U(\bar{y})\right]\geq\langle\rho_k^L,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}},\quad k\in K,$$

$$\left[f_k^U(\tilde{x})-c_kg_k^L(\tilde{x})\right]-\left[f_k^U(\bar{y})-c_kg_k^L(\bar{y})\right]\geq\langle\rho_k^U,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}},\quad k\in K,$$

$$h_s^L(\tilde{x})-h_s^L(\bar{y})\geq\langle w_s^L,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}},\ h_s^U(\tilde{x})-h_s^U(\bar{y})\geq\langle w_s^U,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}},\quad s\in S.$$

We can derive

$$0>\sum_{k=1}^m\lambda_k^L\left\{\left[f_k^L(\tilde{x})-z_kg_k^U(\tilde{x})\right]-\left[f_k^L(\bar{y})-z_kg_k^U(\bar{y})\right]\right\}\geq\sum_{k=1}^m\lambda_k^L\langle\rho_k^L,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}},$$

$$0>\sum_{k=1}^m\lambda_k^U\left\{\left[f_k^U(\tilde{x})-c_kg_k^L(\tilde{x})\right]-\left[f_k^U(\bar{y})-c_kg_k^L(\bar{y})\right]\right\}\geq\sum_{k=1}^m\lambda_k^U\langle\rho_k^U,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}},$$

$$\sum_{s=1}^p\bar{\mu}_s^Lh_s^L(\tilde{x})-\sum_{s=1}^p\bar{\mu}_s^Lh_s^L(\bar{y})\geq\sum_{s=1}^p\bar{\mu}_s^L\langle w_s^L,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}},$$

$$\sum_{s=1}^p\bar{\mu}_s^Uh_s^U(\tilde{x})-\sum_{s=1}^p\bar{\mu}_s^Uh_s^U(\bar{y})\geq\sum_{s=1}^p\bar{\mu}_s^U\langle w_s^U,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}}.$$

Thus $\sum_{k=1}^{m}\lambda_k^L\langle\rho_k^L,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}}+\sum_{k=1}^{m}\lambda_k^U\langle\rho_k^U,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}}<0$. Since $(\bar{y},\bar{\lambda}_k^L,\bar{\lambda}_k^U,\bar{\mu}_s^L,\bar{\mu}_s^U)\in W$ and $\tilde{x}\in D$, we have

$$\sum_{k=1}^{m}\bar{\lambda}_k^L\langle\rho_k^L,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}}+\sum_{k=1}^{m}\bar{\lambda}_k^U\langle\rho_k^U,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}}$$
$$+\sum_{s=1}^{p}\bar{\mu}_s^L\langle w_s^L,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}}+\sum_{s=1}^{p}\bar{\mu}_s^U\langle w_s^U,\exp_{\bar{y}}^{-1}(\tilde{x})\rangle_{\bar{y}}<0.$$

This contradicts the constraint conditions of (MWD), thereby proving the theorem.

5. Conclusions

In this paper, we investigated multiobjective fractional interval-valued optimization problems on Hada-mard manifolds and presented the KKT-type optimality conditions and the Mond-Weir type duality theorems for these problems. We can transform nonconvex functions in Euclidean space into geodesic convex functions on Hadamard manifolds. Therefore, the results presented in this paper have a broader applicability than the results presented in [18]. Compared with [13], our results extend the analysis from differentiable cases to nondifferentiable cases. Furthermore, we may further investigate the optimality conditions and dual problems for nondifferentiable second-order interval-valued optimization problems on Hadamard manifolds.

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References

[1] A. Ben-Israel, P. D. Robers, A decomposition method for interval linear programming, Manag. Sci. 16 (1970), 374-387.

[2] Y. Chalco-Cano, W. A. Lodwick, A. Rufin-Lizana, Optimality conditions of type KKT for optimization problem with interval-valued objective function via generalized derivative, Fuzzy Optim. Decis. Mak. 12 (2013), 305-322.

[3] H. Ishibuchi, H. Tanaka, Multiobjective programming in optimization of the interval objective function, Eur. J. Oper. Res. 48 (1990), 219-225.

[4] C. Jiang, X. Han, G. R. Liu, G. P. Liu, A nonlinear interval number programming method for uncertain optimization problems, Eur. J. Oper. Res. 188 (2008), 1-13.

[5] H. C. Wu, The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function, Eur. J. Oper. Res. 176 (2007), 46-59.

[6] H. C. Wu, Wolfe duality for interval-valued optimization, J. Optim. Theory Appl. 138 (2008), 497-509.

[7] H. C. Wu, Duality theory for optimization problems with interval-valued objective functions, J. Optim. Theory Appl. 144 (2010), 615-628.

[8] H. C. Wu, The Karush-Kuhn-Tucker optimality conditions in multiobjective programming problems with interval-valued objective functions, Eur. J. Oper. Res. 196 (2009), 49-60.

[9] R. E. Steuer, Algorithms for linear programming problems with interval objective function coefficients, Math. Oper. Res. 6 (1981), 333-348.

[10] B. Urli, R. Nadeau, PROMISE/scenarios: An interactive method for multiobjective stochastic linear programming under partial uncertainty, Eur. J. Oper. Res. 155 (2004), 361-372.

[11] C. Udrişte, Convex functions and optimization methods on Riemannian manifolds, Springer, 2013.

[12] S. Treanta, B. B. Upadhyay, A. Ghosh, K. Nonlaopon, Optimality conditions for multiobjective mathematical programming problems with equilibrium constraints on Hadamard manifolds, Mathematics. 10 (2022), 3516.

[13] S. Chen, The KKT optimality conditions for optimization problem with interval-valued objective function on Hadamard manifolds, Optimization, 71 (2022), 613-632.

[14] B. B. Upadhyay, A. Ghosh, On Pareto Efficiency Criteria for Nonsmooth Multiobjective Mathematical Programming Problems with Equilibrium Constraints on Hadamard Manifolds via Convexificators, J. Nonlinear Convex Anal. 26 (2025), 691-714.

[15] B. B. Upadhyay, S. K. Singh, I. Stancu-Minasian, Optimality and Duality for Robust Nonsmooth Semidefinite Multiobjective Programming Problems Using Convexificators, J. Oper. Res. Soc. China (2025), doi: 10.1007/s40305-024-00570-z

[16] B. B. Upadhyay, S. K. Singh, I. Stancu-Minasian, Constraint Qualifications and Optimality Conditions for Nonsmooth Semidefinite Multiobjective Programming Problems with Mixed Constraints Using Convexificators, Mathematics, 12 (2024), 3202.

[17] S. Hosseini, M. R. Pouryayevali, Generalized gradients and characterization of epi-Lipschitz sets in Riemannian manifolds, Nonlinear Anal. 74 (2011), 3884-3895.

[18] T. Antczak, Optimality conditions and duality results for nonsmooth vector optimization problems with the multiple interval-valued objective function, Acta Math. Sci. 37 (2017), 1133-1150.

[19] N. Boumal, An Introduction to Optimization on Smooth Manifolds, Cambridge University Press, Cambridge, 2023.

[20] C. Udrişte, Convex Functions and Optimization Methods on Riemannian Manifolds, Springer, 2013.

[21] A. Barani, Generalized monotonicity and convexity for locally Lipschitz functions on Hadamard manifolds, Differ. Geom. Dyn. Syst. 15 (2013), 26-37.

[22] Z. Peng, C. Deng, Y. Zhao, J. Peng, Optimality conditions and duality for E-differentiable fractional multiobjective interval valued optimization problems with E-invexity, Appl. Set-Valued Anal. Optim. 6 (2024), 295-307.