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# CONVERGENCE ANALYSIS OF A MANN-LIKE ITERATIVE ALGORITHM IN REFLEXIVE BANACH SPACES

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**Abstract.** We introduce and analyze viscosity Mann-like iterative algorithms for solving a general system of variational inequalities involving an infinite family of nonexpansive mappings and an $m$-accretive mapping. It is proved that the sequence generated in the Mann-like iterative algorithm is norm convergent in a real reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure.

**Keywords.** Accretive mapping; Fixed point; Iterative method; Nonexpansive mapping; Viscosity algorithm.

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## 1. INTRODUCTION

Let $X$ be a real Banach space whose dual space is denoted by $X^*$. Recall that the normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{\phi \in X^* : \langle x, \phi \rangle = \|x\|^2 = \|\phi\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is a nonempty set for each $x \in X$. Let $C$ be a nonempty closed convex subset of $X$. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote by $\mathrm{Fix}(T)$ the set of fixed points of $T$. A mapping $f : C \to C$ is called a contraction on $C$ if there exists a constant $\rho \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C.$$

Throughout this paper, we use the notation $\Xi_C$ to denote the collection of all contractions on $C$, i.e., $\Xi_C = \{f : C \to C \text{ is a contraction}\}$. Note that each $f$ in $\Xi_C$ has a unique fixed point in $C$.

Let $U = \{x \in X : \|x\| = 1\}$ denote the unit sphere of $X$. Then the norm of $X$ is said to be Gateaux differentiable if the limit

$$\lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for each $x, y \in U$. In this case, $X$ is said to be smooth. The norm of $X$ is said to be uniformly Gateaux differentiable, if for each $y \in U$, limit (1.1) is attained uniformly for $x \in U$. The norm of $X$ is said to be Frechet differentiable, if for each $x \in U$, limit (1.1) is attained uniformly for $y \in U$. The

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norm of $X$ is said to be uniformly Frechet differentiable, if limit (1.1) is attained uniformly for $x,y\in U$. It is well known that (uniform) Frechet differentiability of the norm of $X$ implies (uniform) Gateaux differentiability of the norm of $X$.

A Banach space $X$ is said to be strictly convex, if, for $x,y\in U$ with $x\neq y$, one has $\|(1-\lambda)x+\lambda y\|<1$, $\forall\lambda\in(0,1)$. $X$ is said to be uniformly convex if, for each $\varepsilon\in(0,2]$, there exists $\delta>0$ such that, for any $x,y\in U$,

$$\|x-y\|\geq\varepsilon\Rightarrow\|\frac{x+y}{2}\|\leq 1-\delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. Also, it is known that if a Banach space $X$ is reflexive, then $X$ is strictly convex if and only if $X^*$ is smooth as well as $X$ is smooth if and only if $X^*$ is strictly convex. Moreover, if $X$ is smooth, then the normalized duality mapping $J$ is single-valued; if the norm of $X$ is uniformly Gateaux differentiable, then $J$ is norm-to-weak$^*$ uniformly continuous on every bounded subset of $X$; and if the norm of $X$ is uniformly Frechet differentiable, then $J$ is norm-to-norm uniformly continuous on every bounded subset of $X$.

Let $X$ be a smooth Banach space. Let $B_1,B_2:C\to X$ be two nonlinear mappings and $\nu_1,\nu_2$ be two positive constants. The general system of variational inequalities (GSVI) is to find $(x^*,y^*)\in C\times C$ such that

$$\begin{cases}\langle \nu_1B_1y^*+x^*-y^*,J(x-x^*)\rangle\geq 0, & \forall x\in C,\\ \langle \nu_2B_2x^*+y^*-x^*,J(x-y^*)\rangle\geq 0, & \forall x\in C.\end{cases}\tag{1.2}$$

The equivalence between the GSVI (1.2) and the fixed point problem of some nonexpansive mapping defined on a real 2-uniformly smooth Banach space was established by Cai and Bu [4]. They introduced and analyzed a modified extragradient method for solving the GSVI (1.2) based on the equivalence, and a strong convergence theorem in a real uniformly convex and 2-uniformly smooth Banach space. In addition, Ceng, Gupta and Ansari [5] also proposed and analyzed Mann-like implicit and explicit algorithms for solving GSVI (1.2).

If $X$ is a real Hilbert space $H$, then the GSVI (1.2) was considered and studied by Ceng, Wang and Yao [6]. If $A=B$, it was considered by Verma [22] (see also [23]). Further, if $x^*=y^*$, then problem (1.2) is reduced to the following classical variational inequality (VI) of finding $x^*\in C$ such that

$$\langle Ax^*,x-x^*\rangle\geq 0,\quad \forall x\in C.\tag{1.3}$$

This problem is a fundamental problem in the variational analysis; in particular, in the optimization theory and mechanics. A large number of algorithms for solving this problem are essentially projection algorithms that employ projections onto the feasible set $C$ of the VI to iteratively reach a solution. In particular, Korpelevich [15] proposed an algorithm for solving the VI in Euclidean space, known as the extragradient method (see also [10]). In case of Banach space setting, that is, if $A=B$ and $x^*=y^*$, then

$$\langle Ax^*,J(x-x^*)\rangle\geq 0,\quad \forall x\in C.\tag{1.4}$$

In [1], Aoyama, Iiduka and Takahashi proposed an iterative scheme to find approximate solutions of (1.4) and proved the weak convergence of the sequences generated by their proposed scheme. In [25], Yamada assumed that the feasible set is the set of common fixed points of a finite family of nonexpansive mappings and introduced a hybrid steepest-descent method. In this case, the variational inequality defined on such feasible set is also called a hierarchical variational inequality (HVI). Yamada's method is subsequently extended and modified to solve more complex problems, see, e.g., [3] and references therein.

Recently, Ceng and Yao [7] introduced and analyzed relaxed implicit and explicit viscosity approximation methods for solving a hierarchical variational inequality defined over the common fixed point set of a countable family of nonexpansive mappings in a real strictly convex and reflexive Banach space with the uniformly Gateaux differentiable norm. In [4], Cai and Bu constructed an iterative algorithm for solving a GSVI (1.2) and a fixed point problem of an infinite family of nonexpansive mappings in a uniformly convex and 2-uniformly smooth Banach space $X$. They proved the strong convergence of the proposed algorithm under some mild assumptions. In [8], Ceng and Wen proposed hybrid implicit and explicit extragradient methods for solving a zero point problem of an accretive operator, the GSVI (1.2) and a common fixed point problem of an infinite family of nonexpansive mappings in a uniformly convex Banach space which has a uniformly Gateaux differentiable norm. In [9], Ceng, Al-Otaibi, Ansari and Latif introduced some relaxed and composite viscosity methods for solving a zero point problem of an accretive operator, the GSVI (1.2) and a common fixed point problem of an infinite family of nonexpansive mappings in a uniformly convex Banach space which is also 2-uniformly smooth or has a uniformly Gateaux differentiable norm. we remark that the restrictions imposed on an infinite family of nonexpansive mappings in [8] are very differently from those in [9].

Let $\{T_k\}_{k=1}^{\infty}$ be an infinite family of nonexpansive mappings on a nonempty closed convex subset $C$ of a real Banach space $X$. Let $\{\rho_k\}_{k=1}^{\infty}$ be a sequence in $[0,1]$. Consider the nonexpansive mapping $W_k$ defined by $U_{k,k+1} = I$ and

$$\begin{cases} U_{k,k} = (1-\rho_k)I + \rho_k T_k U_{k,k+1}, \\ \dots \\ U_{k,i} = (1-\rho_i)I + \rho_i T_i U_{k,i+1}, \\ \dots \\ U_{k,2} = (1-\rho_2)I + \rho_2 T_2 U_{k,3}, \\ W_k = U_{k,1} = (1-\rho_1)I + \rho_1 T_1 U_{k,2}, \quad \forall k \geq 1. \end{cases} \tag{1.5}$$

The mapping $W_k$ is called a $W$-mapping, generated by $T_k, T_{k-1}, ..., T_1$ and $\rho_k, \rho_{k-1}, ..., \rho_1$. If $X = H$ a real Hilbert space, Takahashi [21] first introduced such a $W$-mapping to find a common fixed point of $\{T_k\}_{k=1}^{\infty}$ (see also [20] for more details).

In the case that the feasible set is the common fixed point set of an infinite family of nonexpansive mappings on $H$, based on the $W$-mapping (see [21]) and Moudafi's viscosity approximation method (see [16]), Kikkawa and Takahashi [13, 14] studied an implicit iteration scheme. Recently, based on a $V$-mapping, which is simpler than the $W$-mapping, Buong and Phong [3] introduced two new implicit iterative algorithms, which converge strongly in Banach spaces without weakly continuous duality mappings. These methods are two different combinations of the steepest-descent method with the $V$-mapping.

In this paper, we introduce and analyze viscosity Mann-like algorithm for solving the GSVI (1.2), and a common fixed point problem of a countable family of nonexpansive mappings, and a zero point problem of an $m$-accretive operator. We establish strong convergence theorems for the proposed algorithms via the $V$-mapping in a real reflexive Banach space $X$ with the uniformly Gateaux differentiable norm and the normal structure. The results presented in this paper improve, extend, and develop the corresponding results announced by some others, e.g., [3, 4, 8, 9] and the references therein.

## 2. PRELIMINARIES

Let $X$ be a real Banach space and let $C$ be its closed convex subset. Let $T : C \to C$ be a nonlinear mapping.

Recall that $T$ is said to be

(i) strongly pseudocontractive if there exists a constant $\alpha \in (0,1)$ and some $j(x-y) \in J(x-y)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \le \alpha\|x-y\|^2, \quad \forall x,y \in C;$$

(ii) pseudocontractive if there exists some $j(x-y) \in J(x-y)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2, \quad \forall x,y \in C.$$

Next, we list two existence results of fixed points.

**Proposition 2.1.** [12] *Let $C$ be a nonempty, bounded, closed and convex subset of a reflexive Banach space $X$ which also has the normal structure. Let $T$ be a nonexpansive self-mapping on $C$. Then,* $\mathrm{Fix}(T) \neq \emptyset$.

**Proposition 2.2.** [11] *Let $C$ be a nonempty, closed and convex subset of a Banach space $X$, and $T : C \to C$ be a continuous and strong pseudocontraction. Then $T$ has a unique fixed point in $C$.*

**Lemma 2.1.** [24] *Let $\{s_k\}$ be a sequence of nonnegative real numbers satisfying*
$$s_{k+1} \le (1-\alpha_k)s_k + \alpha_k\beta_k + \gamma_k, \quad \forall k \ge 1,$$
*where $\{\alpha_k\}, \{\beta_k\}$ and $\{\gamma_k\}$ satisfy the following conditions:*

*(i)* $\{\alpha_k\} \subset [0,1]$ *and* $\sum_{k=1}^{\infty} \alpha_k = \infty$*;*

*(ii)* $\limsup_{k\to\infty} \beta_k \le 0$*;*

*(iii)* $\gamma_k \ge 0$ *for all $k \ge 1$, and* $\sum_{k=1}^{\infty} \gamma_k < \infty$.

*Then* $\lim_{k\to\infty} s_k = 0$.

The following lemma is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$.

**Lemma 2.2.** [10] *Let $X$ be a real Banach space and $J$ be the normalized duality map on $X$. Then, for all $x,y \in X$ one has*

*(i)* $\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \forall j(x+y) \in J(x+y)$*;*

*(ii)* $\|x+y\|^2 \ge \|x\|^2 + 2\langle y, j(x)\rangle, \forall j(x) \in J(x)$.

Let $D$ be a subset of $C$ and let $\Pi$ be a mapping of $C$ into $D$. Then $\Pi$ is said to be sunny if
$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x),$$
whenever
$$\Pi(x) + t(x - \Pi(x)) \in C$$
for $x \in C$ and $t \ge 0$. A mapping $\Pi$ of $C$ into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping $\Pi$ of $C$ into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of $\Pi$. A subset $D$ of $C$ is called a sunny nonexpansive retract of $C$ if there exists a sunny nonexpansive retraction from $C$ onto $D$.

**Proposition 2.3.** [18] *Let $C$ be a nonempty closed convex subset of a smooth Banach space $X$. Let $D$ be a nonempty subset of $C$ and $\Pi$ be a retraction of $C$ onto $D$. Then, the following are equivalent:*

*(i)* $\Pi$ *is sunny and nonexpansive;*
*(ii)* $\|\Pi(x)-\Pi(y)\|^2 \le \langle x-y, J(\Pi(x)-\Pi(y))\rangle, \forall x,y \in C$;
*(iii)* $\langle x-\Pi(x), J(y-\Pi(x))\rangle \le 0, \forall x \in C, y \in D$.

It is well known that if $X$ is a Hilbert space, then a sunny nonexpansive retraction $\Pi_C$ coincides with the metric projection from $X$ onto $C$. Let $C$ be a nonempty closed convex subset of a reflexive and strictly convex Banach space $X$ with a uniformly Gateaux differentiable norm. Recall that a closed convex subset $C$ of a Banach space $X$ is said to have a normal structure if for each bounded convex subset $K$ of $C$ which contains at least two points, there exists an element $x$ of $K$ which is not a diametral point of $K$, i.e., $\sup\{\|x-y\| : y \in K\} < d(K)$, where $d(K)$ is the diameter of $K$. It is well known that a closed convex subset of a uniformly convex Banach space has the normal structure and a compact convex subset of a Banach space has the normal structure; see [2] for more details.

**Lemma 2.3.** [17] *Let $X$ be a real reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure, and let $C$ be a nonempty closed convex subset of $X$. Let $T : C \to C$ be a continuous and bounded pseudocontraction with* $\mathrm{Fix}(T) \neq \emptyset$*, and let $f : C \to C$ be a fixed continuous and bounded strong pseudocontraction with coefficient $\alpha \in (0,1)$. Let $\{x_t\}$ be the net generated by the following $x_t = tf(x_t) + (1-t)Tx_t$, $\forall t \in (0,1)$, Then $\{x_t\}$ converges strongly as $t \to 0$ to a point $x^*$ in* $\mathrm{Fix}(T)$*, which is the unique solution in* $\mathrm{Fix}(T)$ *to the following VI:*

$$\langle (I-f)x^*, J(x^*-p)\rangle \le 0, \quad \forall p \in \mathrm{Fix}(T).$$

Recall that a mapping $F$ with domain $D(F)$ and range $R(F)$ in a real Banach space $X$ is said to be

(a) accretive if, for each $x,y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx-Fy, j(x-y)\rangle \ge 0;$$

(b) $\delta$-strongly accretive if, for each $x,y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx-Fy, j(x-y)\rangle \ge \delta\|x-y\|^2 \quad \text{for some} \quad \delta \in (0,1);$$

(c) $\alpha$-inverse-strongly accretive if, for each $x,y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx-Fy, j(x-y)\rangle \ge \alpha\|Fx-Fy\|^2 \quad \text{for some} \quad \alpha \in (0,1);$$

(d) $\zeta$-strictly pseudocontractive if, for each $x,y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx-Fy, j(x-y)\rangle \le \|x-y\|^2 - \zeta\|x-y-(Fx-Fy)\|^2 \quad \text{for some} \quad \zeta \in (0,1).$$

It is easy to see that the last inequality can be rewritten as

$$\langle (I-F)x-(I-F)y, j(x-y)\rangle \ge \zeta\|(I-F)x-(I-F)y\|^2, \tag{2.1}$$

where $I$ denotes the identity mapping of $X$. Clearly, if $F$ is $\zeta$-strictly pseudocontractive with $\zeta = 0$, then it is said to be pseudocontractive. It is not hard to find that every nonexpansive mapping is pseudocontractive.

**Lemma 2.4.** [19] *Let $\{x_k\}$ and $\{z_k\}$ be bounded sequences in a Banach space $X$ and let $\{\alpha_k\}$ be a sequence in $[0,1]$ such that $0 < \liminf_{k\to\infty} \alpha_k \leq \limsup_{k\to\infty} \alpha_k < 1$. Suppose that $x_{k+1} = \alpha_k x_k + (1-\alpha_k)z_k, \forall k \geq 1$, and*

$$\limsup_{k\to\infty}(\|z_{k+1}-z_k\| - \|x_{k+1}-x_k\|) \leq 0.$$

*Then $\lim_{k\to\infty} \|z_k - x_k\| = 0$.*

**Proposition 2.4.** [1] *Let $C$ be a nonempty closed convex subset of a smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$ and $A: C \to X$ be an accretive mapping. Then for all $\lambda > 0$,*

$$\mathrm{VI}(C,A) = \mathrm{Fix}(\Pi_C(I-\lambda A)).$$

**Proposition 2.5.** [8] *Let $C$ be a nonempty closed convex subset of a smooth Banach space $X$ and let the mapping $F: C \to X$ be $\zeta$-strictly pseudocontractive and $\delta$-strongly accretive with $\delta + \zeta \geq 1$. Then, for $\lambda \in (0,1]$, we have*

$$\|(I-\lambda F)x - (I-\lambda F)y\| \leq \{\sqrt{\frac{1-\delta}{\zeta}} + (1-\lambda)(1+\frac{1}{\zeta})\}\|x-y\|, \quad \forall x,y \in C.$$

*In particular, if $1 - \frac{\zeta}{1+\zeta}(1-\sqrt{\frac{1-\delta}{\zeta}}) \leq \lambda \leq 1$, then $I - \lambda F$ is nonexpansive.*

**Proposition 2.6.** [8] *Let $C$ be a nonempty closed convex subset of a smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$ and let the mapping $B_i: C \to X$ be $\zeta_i$-strictly pseudocontractive and $\delta_i$-strongly accretive with $\delta_i + \zeta_i \geq 1$ for $i = 1,2$. Let $G: C \to C$ be the mapping defined by*

$$G(x) = \Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)x, \quad \forall x \in C.$$

*If $1 - \frac{\zeta_i}{1+\zeta_i}(1-\sqrt{\frac{1-\delta_i}{\zeta_i}}) \leq \mu_i \leq 1$ for $i = 1,2$, then $G: C \to C$ is nonexpansive.*

**Proposition 2.7.** [8] *Let $C$ be a nonempty closed convex subset of a smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let $B_1, B_2: C \to X$ be two nonlinear mappings and $\mu_1, \mu_2$ be two positive numbers. For given $x^*, y^* \in C$, $(x^*, y^*)$ is a solution of the GSVI (1.2) if and only if $x^* \in \Omega$ where $\Omega$ is the set of fixed points of the mapping $G := \Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)$ and $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

**Remark 2.1.** In [4, Lemma 2.10], Cai and Bu established the equivalence between the GSVI (1.2) and the fixed point problem of the mapping $G := \Pi_C(I-\lambda A)\Pi_C(I-\mu B)$ in a real 2-uniformly smooth Banach space $X$. Thus, Proposition 2.7 is more general than [4, Lemma 2.10] because the 2-uniform smoothness of $X$ is replaced by the weaker condition, i.e., the smoothness of $X$.

Let $C$ be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let the mapping $B_i: C \to X$ be $\xi_i$-inverse-strongly accretive for $i = 1,2$. Let $F: C \to X$ be $\delta$-strongly accretive and $\zeta$-strictly pseudocontractive with $\delta + \zeta > 1$. Assume that $\nu_i \in (0, \frac{\xi_i}{\kappa^2}), i = 1,2$, where $\kappa$ is the 2-uniformly smooth constant of $X$. Recently, Ceng, Gupta and Ansari [5] introduced the following iterative algorithm of Mann-like in order to solve GSVI (1.2).

**Algorithm 2.1.** [5] Put $G := \Pi_C(I-\nu_1 B_1)\Pi_C(I-\nu_2 B_2)$. For arbitrarily given $x_0 \in C$, let the sequence $\{x_k\}$ be generated iteratively by

$$x_{k+1} = \beta_k x_k + \gamma_k G x_k + (1-\beta_k-\gamma_k)\Pi_C(I-\lambda_k F)Gx_k,$$

where $\{\lambda_k\},\{\beta_k\}$ and $\{\gamma_k\}$ are three sequences in $[0,1]$ such that $\beta_k+\gamma_k \le 1, \forall k \ge 0$.

It was proven in [5] that under appropriate conditions, $\{x_k\}$ converges in norm, as $k \to \infty$, to the unique solution $x^* \in \Omega$ to the following VI:

$$\langle F(x^*), J(x-x^*)\rangle \ge 0, \quad \forall x \in \Omega. \tag{2.2}$$

Recall that a possibly multivalued operator $A \subset X \times X$ with domain $D(A)$ and range $R(A)$ in $X$ is accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i=1,2)$, there exists $j(x_1-x_2) \in J(x_1-x_2)$ such that $\langle y_1-y_2, j(x_1-x_2)\rangle \ge 0$. An accretive operator $A$ is said to satisfy the range condition if $\overline{D(A)} \subset R(I+rA)$ for all $r>0$. An accretive operator $A$ is $m$-accretive if $R(I+rA)=X$ for each $r>0$. If $A$ is an accretive operator which satisfies the range condition, then we can define, for each $r>0$, a mapping $J_r : R(I+rA) \to D(A)$ by $J_r = (I+rA)^{-1}$, which is called the resolvent of $A$. It is well known that $J_r$ is nonexpansive and $\mathrm{Fix}(J_r) = A^{-1}0$ for all $r>0$. Hence,

$$\mathrm{Fix}(J_r) = A^{-1}0 = \{z \in D(A) : 0 \in Az\}.$$

If $A^{-1}0 \ne \emptyset$, then the inclusion $0 \in Az$ is solvable.

The following resolvent identity is well-known.

**Proposition 2.8.** *(resolvent identity). For $\lambda > 0$, $\mu > 0$, and $x \in X$,*

$$J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + \Big(1-\frac{\mu}{\lambda}\Big)J_\lambda x\Big).$$

Recently, Ceng and Wen [8] proposed another explicit iterative scheme for finding a point $x^* \in \mathscr{F} = \bigcap_{i=1}^{\infty}\mathrm{Fix}(S_i)\cap\Omega\cap A^{-1}0$ in a uniformly convex Banach space $X$ which has a uniformly Gateaux differentiable norm:

$$\begin{cases} y_k = \alpha_k f(x_k) + \beta_k x_k + \gamma_k J_{r_k} x_k + \delta_k S_k x_k, \\ x_{k+1} = \sigma_k y_k + (1-\sigma_k)G y_k, \quad \forall k \ge 1. \end{cases} \tag{2.3}$$

Under approximate conditions on the parameter sequences, they proved the strong convergence of the sequence $\{x_k\}$ generated by (2.3) to the unique solution $x^* \in \mathscr{F}$ to the VI

$$\langle (I-f)x^*, J(x^*-p)\rangle \le 0, \quad \forall p \in \mathscr{F}.$$

Let $C$ be a nonempty closed convex subset of a smooth Banach space $X$ and let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on $C$. Let $\mathscr{F} := \bigcap_{i=1}^{\infty}\mathrm{Fix}(T_i)$. Recently, Buong and Phuong [3] considered the following HVI with $C = X$: find $x^* \in \mathscr{F}$ such that

$$\langle F(x^*), J(x-x^*)\rangle \ge 0, \quad \forall x \in \mathscr{F}. \tag{2.4}$$

By introducing a mapping $V_k$, defined by

$$V_k = V_k^1, \quad V_k^i = T^i T^{i+1}\cdots T^k, \quad T^i = (1-\alpha_i)I + \alpha_i T_i, \quad i = 1,2,\dots,k, \tag{2.5}$$

where

$$\alpha_i \in (0,1) \quad \text{and} \quad \sum_{i=1}^{\infty}\alpha_i < \infty, \tag{2.6}$$

Buong and Phuong [3] designed two implicit methods for the problem.

Let LIM be a continuous linear functional on $l^\infty$ and $s=(a_1,a_2,\ldots)\in l^\infty$. We write $\mathrm{LIM}_k a_k$ instead of $\mathrm{LIM}(s)$. LIM is called a Banach limit if LIM satisfies $\|\mathrm{LIM}\|=\mathrm{LIM}_k 1=1$ and $\mathrm{LIM}_k a_{k+1}=\mathrm{LIM}_k a_k$ for all $(a_1,a_2,\ldots)\in l^\infty$. If LIM is a Banach limit, then there hold the following:

(i) for all $k\geq 1$, $a_k\leq c_k$ implies $\mathrm{LIM}_k a_k\leq \mathrm{LIM}_k c_k$;
(ii) $\mathrm{LIM}_k a_{k+m}=\mathrm{LIM}_k a_k$ for any fixed positive integer $m$;
(iii) $\liminf_{k\to\infty} a_k\leq \mathrm{LIM}_k a_k\leq \limsup_{k\to\infty} a_k$ for all $(a_1,a_2,\ldots)\in l^\infty$.

**Lemma 2.5.** [26] *Let $a\in\mathbf{R}$ be a real number and a sequence $\{a_k\}\in l^\infty$ satisfy the condition $\mathrm{LIM}_k a_k\leq a$ for all Banach limit LIM. If $\limsup_{k\to\infty}(a_{k+m}-a_k)\leq 0$, then $\limsup_{k\to\infty} a_k\leq a$.*

**Lemma 2.6.** [3] *Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $X$ and let $\{T_i\}_{i=1}^k$, $k\geq 1$, be $k$ nonexpansive self-mappings on $C$ such that the set of common fixed points $\mathscr{F}:=\bigcap_{i=1}^k \mathrm{Fix}(T_i)\neq\emptyset$. Let $a,b$ and $\alpha_i$, $i=1,2,\ldots,k$, be real numbers such that $0<a\leq\alpha_i\leq b<1$, and let $V_k$ be a mapping defined by (2.5) for all $k\geq 1$. Then, $\mathrm{Fix}(V_k)=\mathscr{F}$.*

**Lemma 2.7.** [3] *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and let $\{T_i\}_{i=1}^\infty$ be an infinite family of nonexpansive self-mappings on $C$ such that the set of common fixed points $\mathscr{F}:=\bigcap_{i=1}^\infty \mathrm{Fix}(T_i)\neq\emptyset$. Let $V_k$ be a mapping defined by (2.5), and let $\alpha_i$ satisfy (2.6). Then, for each $x\in C$ and $i\geq 1$, $\lim_{k\to\infty} V_k^i x$ exists.*

**Remark 2.2.** (i) We can define the mappings

$$V_\infty^i x:=\lim_{k\to\infty} V_k^i x \quad \text{and} \quad Vx:=V_\infty^1 x=\lim_{k\to\infty} V_k x, \quad \forall x\in C. \tag{2.7}$$

(ii) It can be readily seen from the proof of Lemma 2.7 that if $D$ is a nonempty and bounded subset of $C$, then the following holds:

$$\lim_{k\to\infty}\sup_{x\in D}\|V_k^i x-V_\infty^i x\|=0, \quad \forall i\geq 1.$$

In particular, whenever $i=1$, we have

$$\lim_{k\to\infty}\sup_{x\in D}\|V_k x-Vx\|=0.$$

**Proposition 2.9.** *Let $C$ be a nonempty closed convex subset of a strictly convex and smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let $B_i: C\to X$ be $\zeta_i$-strictly pseudocontractive and $\delta_i$-strongly accretive with $\delta_i+\zeta_i\geq 1$ for each $i=1,2$. Define the mapping $G: C\to C$ by $G=\Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)$, where $1-\frac{\zeta_i}{1+\zeta_i}(1-\sqrt{\frac{1-\delta_i}{\zeta_i}})\leq\mu_i\leq 1$ for $i=1,2$. Let $\{T_i\}_{i=1}^\infty$ be a countable family of nonexpansive self-mappings on $C$ such that $\mathscr{F}:=\bigcap_{i=1}^\infty \mathrm{Fix}(T_i)\cap\mathrm{Fix}(G)\neq\emptyset$. Let $\alpha_i$ satisfy (2.6). Then, $\mathrm{Fix}(V\circ G)=\mathscr{F}$.*

*Proof.* By Proposition 2.6, we know that $G: C\to C$ is nonexpansive. Let $p\in\mathscr{F}$. Then it is obvious that $Gp=p$ and $V_k^i p=p$ for all integers $i,k\geq 1$ with $k\geq i$. So, we have $V_\infty^i Gp=p$ for all integer $i\geq 1$. In particular, we have $(V\circ G)p=V_\infty^1 Gp$ and hence $\mathscr{F}\subset\mathrm{Fix}(V\circ G)$.

Next, we prove that $\mathrm{Fix}(V\circ G)\subset\mathscr{F}$. Let $x\in\mathrm{Fix}(V\circ G)$ and $y\in\mathscr{F}$. Then,

$$\begin{aligned}\|V_kGx-V_kGy\| &= \|V_k^1Gx-V_k^1Gy\|\\ &= \|(1-\alpha_1)(V_k^2Gx-V_k^2Gy)+\alpha_1(T_1V_k^2Gx-T_1V_k^2Gy)\|\\ &\le (1-\alpha_1)\|V_k^2Gx-V_k^2Gy\|+\alpha_1\|V_k^2Gx-V_k^2Gy\|\\ &= \|V_k^2Gx-V_k^2Gy\|\\ &\le \|V_k^{i+1}Gx-V_k^{i+1}Gy\|\\ &\le \|V_k^kGx-V_k^kGy\|\\ &\le \|Gx-Gy\|\\ &\le \|x-y\|,\end{aligned}$$

which together with $\|(V\circ G)x-(V\circ G)y\|=\|x-y\|$ implies that

$$\|V_\infty^iGx-V_\infty^iGy\|=\|V_\infty^{i+1}Gx-V_\infty^{i+1}Gy\|=\|Gx-y\|.$$

Therefore, we have

$$\begin{aligned}&\|(1-\alpha_i)(V_\infty^{i+1}Gx-V_\infty^{i+1}Gy)+\alpha_i(T_iV_\infty^{i+1}Gx-T_iV_\infty^{i+1}Gy)\|\\ &=\|V_\infty^{i+1}Gx-V_\infty^{i+1}Gy\|\\ &=\|Gx-y\|,\end{aligned}$$

for every $i\ge 1$. Since $X$ is strictly convex, $0<\alpha_i<1$, and $y\in\mathscr{F}$, we have $Gx-y=T_iV_\infty^{i+1}Gx-T_iV_\infty^{i+1}Gy=T_iV_\infty^{i+1}Gx-y$ and $Gx-y=V_\infty^{i+1}Gx-V_\infty^{i+1}Gy=V_\infty^{i+1}Gx-y$. Hence, $Gx=T_iV_\infty^{i+1}Gx$ and $Gx=V_\infty^{i+1}Gx$ for every $i\ge 1$. Consequently, for every $i\ge 1$, we have $Gx=T_iGx$. In particular, when $i=1$, we have that $Gx=T_1V_\infty^2Gx$ and $Gx=V_\infty^2Gx$. So, it follows that

$$x=(V\circ G)x=(1-\alpha_1)V_\infty^2Gx+\alpha_1T_1V_\infty^2Gx=Gx,$$

which together with $Gx=T_iGx,\forall i\ge 1$, implies that, for every $i\ge 1$, $x=T_ix$. It means that $x\in\mathscr{F}$. This completes the proof. $\square$

## 3. Main results

**Theorem 3.1.** *Let $X$ be a real reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure, and let $C$ be a nonempty closed convex subset of $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let $B_i: C\to X$ be $\zeta_i$-strictly pseudocontractive and $\delta_i$-strongly accretive with $\delta_i+\zeta_i\ge 1$ for each $i=1,2$. Define the mapping $G: C\to C$ by $G=\Pi_C(I-\mu_1B_1)\Pi_C(I-\mu_2B_2)$, where $1-\frac{\zeta_i}{1+\zeta_i}(1-\sqrt{\frac{1-\delta_i}{\zeta_i}})\le\mu_i\le 1$ for $i=1,2$. Let $f: C\to C$ be a fixed continuous bounded and strong pseudocontraction with coefficient $\alpha\in(0,1)$. Let $\{T_i\}_{i=1}^\infty$ be a countable family of nonexpansive self-mappings on $C$ such that $\mathscr{F}=\bigcap_{i=1}^\infty\mathrm{Fix}(T_i)\cap\mathrm{Fix}(G)\ne\emptyset$. Let $\{V_k\}_{k=1}^\infty$ be defined by (2.5) and (2.6) such that $\mathrm{Fix}(V\circ G)=\mathscr{F}$. Let $\{x_k\}_{k=1}^\infty$ be a sequence generated in the implicit manner*

$$x_k=\gamma_kf(x_k)+(1-\gamma_k)V_kG[(1-\beta_k)x_k+\beta_kV_kGx_k],\quad\forall k\ge 1,\tag{3.1}$$

*where $\{\beta_k\}$ is a sequence in $[0,1]$ and $\{\gamma_k\}$ is a sequence in $(0,1)$ such that $\lim_{k\to\infty}\gamma_k=0$ and $\lim_{k\to\infty}\beta_k/\gamma_k=0$. Then $\{x_k\}$ defined by (3.1) converges strongly to a point $x^*\in\mathscr{F}$ which is the unique solution of the* *following VI*

$$\langle (I-f)x^*, J(x^*-p)\rangle \le 0, \quad \forall p \in \mathscr{F}. \tag{3.2}$$

*Proof.* For each $k \ge 1$, define a mapping $U_k : C \to C$ by

$$U_k x = \gamma_k f(x) + (1-\gamma_k)V_k G[(1-\beta_k)x + \beta_k V_k Gx].$$

Then $U_k : C \to C$ is a continuous, strong pseudocontraction for each $k \ge 1$. Indeed, since $f$ is a strong pseudocontraction with coefficient $\alpha \in (0,1)$, we have

$$\begin{aligned}
&\langle U_k x - U_k y, J(x-y)\rangle \\
&\le \gamma_k \alpha \|x-y\|^2 + (1-\gamma_k)\|V_k G[(1-\beta_k)x + \beta_k V_k Gx] - V_k G[(1-\beta_k)y + \beta_k V_k Gy]\| \|x-y\| \\
&\le \gamma_k \alpha \|x-y\|^2 + (1-\gamma_k)\|(1-\beta_k)(x-y) + \beta_k (V_k Gx - V_k Gy)\| \|x-y\| \\
&\le \gamma_k \alpha \|x-y\|^2 + (1-\gamma_k)[(1-\beta_k)\|x-y\| + \beta_k \|V_k Gx - V_k Gy\|] \|x-y\| \\
&\le (1-\gamma_k(1-\alpha))\|x-y\|^2, \quad \forall x, y \in C.
\end{aligned}$$

In view of Proposition 2.2, we know that $U_k$ has a unique fixed point $x_k$ in $C$ for each $k \ge 1$. Hence (3.1) is well defined. Next, we show that $\{x_k\}$ is bounded. For any $p \in \mathscr{F}$ and $k \ge 1$, we obtain

$$\begin{aligned}
\|x_k - p\|^2 &\le \gamma_k \alpha \|x_k - p\|^2 + \gamma_k \langle f(p) - p, J(x_k - p)\rangle \\
&\quad + (1-\gamma_k)\|V_k G[(1-\beta_k)x_k + \beta_k V_k Gx_k] - p\| \|x_k - p\| \\
&\le \gamma_k \alpha \|x_k - p\|^2 + \gamma_k \langle f(p) - p, J(x_k - p)\rangle \\
&\quad + (1-\gamma_k)\|(1-\beta_k)(x_k - p) + \beta_k (V_k Gx_k - p)\| \|x_k - p\| \\
&\le \gamma_k \alpha \|x_k - p\|^2 + \gamma_k \langle f(p) - p, J(x_k - p)\rangle \\
&\quad + (1-\gamma_k)[(1-\beta_k)\|x_k - p\| + \beta_k \|V_k Gx_k - p\|] \|x_k - p\| \\
&\le (1-\gamma_k(1-\alpha))\|x_k - p\|^2 + \gamma_k \langle f(p) - p, J(x_k - p)\rangle,
\end{aligned}$$

which implies that

$$\|x_k - p\|^2 \le \frac{1}{1-\alpha}\langle f(p) - p, J(x_k - p)\rangle. \tag{3.3}$$

It follows that

$$\|x_k - p\| \le \frac{1}{1-\alpha}\|f(p) - p\|.$$

Thus, $\{x_k\}$ is bounded, and so are the sequences $\{f(x_k)\}, \{y_k\}, \{V_k Gx_k\}$ and $\{V_k Gy_k\}$, where $y_k = (1-\beta_k)x_k + \beta_k V_k Gx_k$. Since $V : C \to C$ is a nonexpansive mapping, it follows that $V \circ G : C \to C$ is also nonexpansive. By Proposition 2.2, we deduce that, for each $n \ge 1$, there exists a unique element $z_n \in C$ such that

$$z_n = \frac{1}{n} f(z_n) + (1 - \frac{1}{n})VGz_n. \tag{3.4}$$

From Lemma 2.3, we conclude that $z_n \to x^* \in \mathrm{Fix}(V \circ G) = \mathscr{F}$ as $n \to \infty$, where $x^*$ is the unique solution in $\mathscr{F}$ to the VI (3.2). Furthermore, for each $k \ge 1$, we rewrite (3.1) as follows:

$$\begin{aligned}
x_k &= \gamma_k f(x_k) + (1-\gamma_k)V_k G[(1-\beta_k)x_k + \beta_k V_k Gx_k] \\
&= \gamma_k f(x_k) + (1-\gamma_k)V_k Gy_k,
\end{aligned}$$

where $y_k = (1-\beta_k)x_k + \beta_k V_k G x_k$. Since $f$ and $\{x_k\}$ are bounded, $\{f(x_k)\}$ is also bounded. Moreover, for every $k,n \geq 1$, we have

$$\begin{aligned}
\|x_k - VGz_n\| &\leq \gamma_k \|f(x_k) - VGz_n\| + (1-\gamma_k)\|V_k G y_k - V_k G z_n\| + (1-\gamma_k)\|V_k G z_n - VGz_n\| \\
&\leq \gamma_k \|f(x_k) - VGz_n\| + \|y_k - z_n\| + \|V_k G z_n - VGz_n\| \\
&\leq \gamma_k \|f(x_k) - VGz_n\| + (1-\beta_k)\|x_k - z_n\| + \beta_k \|V_k G x_k - z_n\| + \|V_k G z_n - VGz_n\| \\
&\leq \gamma_k \|f(x_k) - VGz_n\| + \beta_k \|V_k G x_k - z_n\| + \|x_k - z_n\| + \|V_k G z_n - VGz_n\|.
\end{aligned} \tag{3.5}$$

If $D$ is a nonempty and bounded subset of $C$, then, for $\varepsilon > 0$, there exists $m_0 > i$ such that, for all $k > m_0$,

$$\sup_{x\in D} \|V_k^i x - V_\infty^i x\| \leq \varepsilon. \tag{3.6}$$

Taking $D = \{Gz_n : n \geq 1\} \cup \{Gx_k : k \geq 1\}$ and setting $i = 1$, we have find from (3.6) that

$$\|V_k G z_n - VGz_n\| \leq \sup_{x\in D}\|V_k x - Vx\| \leq \varepsilon \quad \text{and} \quad \|V_k G x_k - VGx_k\| \leq \sup_{x\in D}\|V_k x - Vx\| \leq \varepsilon,$$

which immediately implies that

$$\lim_{k\to\infty} \|V_k G z_n - VGz_n\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|V_k G x_k - VGx_k\| = 0. \tag{3.7}$$

Note that $\lim_{k\to\infty}\gamma_k = \lim_{k\to\infty}\beta_k = 0$. Let LIM be a Banach limit. Then from (3.5), we get

$$\mathrm{LIM}_k \|x_k - VGz_n\|^2 \leq \mathrm{LIM}_k \|x_k - z_n\|^2. \tag{3.8}$$

Again from (3.4), we have

$$x_k - z_n = \frac{1}{n}(x_k - f(z_n)) + (1-\frac{1}{n})(x_k - VGz_n),$$

that is,

$$(1-\frac{1}{n})(x_k - VGz_n) = x_k - z_n - \frac{1}{n}(x_k - f(z_n)).$$

It follows from Lemma 2.2 (ii) that

$$\begin{aligned}
(1-\frac{1}{n})^2\|x_k - VGz_n\|^2 &\geq \|x_k - z_n\|^2 - \frac{2}{n}\langle x_k - z_n + z_n - f(z_n), J(x_k - z_n)\rangle \\
&= (1-\frac{2}{n})\|x_k - z_n\|^2 + \frac{2}{n}\langle f(z_n) - z_n, J(x_k - z_n)\rangle.
\end{aligned} \tag{3.9}$$

By using (3.8) and (3.9), we have

$$(1-\frac{1}{n})^2 \mathrm{LIM}_k \|x_k - z_n\|^2 \geq (1-\frac{2}{n})\mathrm{LIM}_k\|x_k - z_n\|^2 + \frac{2}{n}\mathrm{LIM}_k\langle f(z_n) - z_n, J(x_k - z_n)\rangle,$$

and hence

$$\frac{1}{n^2}\mathrm{LIM}_k\|x_k - z_n\|^2 \geq \frac{2}{n}\mathrm{LIM}_k\langle f(z_n) - z_n, J(x_k - z_n)\rangle.$$

This implies that $\frac{1}{2n}\mathrm{LIM}_k\|x_k - z_n\|^2 \geq \mathrm{LIM}_k\langle f(z_n) - z_n, J(x_k - z_n)\rangle$. Since $z_n \to x^* \in \mathrm{Fix}(V\circ G) = \mathscr{F}$ as $n \to \infty$, by the uniformly Gateaux differentiability of the norm of $X$, we have

$$\mathrm{LIM}_k\langle f(x^*) - x^*, J(x_k - x^*)\rangle \leq 0. \tag{3.10}$$

Since $x_k = \gamma_k f(x_k) + (1-\gamma_k)V_k G y_k$, where $y_k = (1-\beta_k)x_k + \beta_k V_k G x_k$, we have

$$\begin{aligned}
(I-f)x_k &= (1-\gamma_k)(V_k G y_k - f(x_k)) \\
&= (1-\gamma_k)(V_k G y_k - V_k G x_k + V_k G x_k - x_k + x_k - f(x_k)) \\
&= (1-\gamma_k)(V_k G y_k - V_k G x_k) - (1-\gamma_k)(I - V_k G)x_k + (1-\gamma_k)(I-f)x_k,
\end{aligned}$$

which implies that

$$(I-f)x_k = \frac{1-\gamma_k}{\gamma_k}(V_kGy_k - V_kGx_k) - \frac{1-\gamma_k}{\gamma_k}(I-V_kG)x_k.$$

Consequently, for $x^* \in \mathrm{Fix}(V\circ G) = \mathscr{F}$, we conclude that

$$\begin{aligned}
\langle (I-f)x_k, J(x_k-x^*)\rangle &= \frac{1-\gamma_k}{\gamma_k}\langle V_kGy_k - V_kGx_k, J(x_k-x^*)\rangle \\
&\quad - \frac{1-\gamma_k}{\gamma_k}\langle (I-V_kG)x_k - (I-V_kG)x^*, J(x_k-x^*)\rangle \\
&\le \frac{1-\gamma_k}{\gamma_k}\langle V_kGy_k - V_kGx_k, J(x_k-x^*)\rangle \\
&\le \frac{1-\gamma_k}{\gamma_k}\|V_kGy_k - V_kGx_k\|\|x_k-x^*\| \\
&\le \frac{1-\gamma_k}{\gamma_k}\|y_k - x_k\|\|x_k-x^*\| \\
&= \frac{1-\gamma_k}{\gamma_k}\beta_k\|V_kGx_k - x_k\|\|x_k-x^*\| \\
&\le \frac{\beta_k}{\gamma_k}\|V_kGx_k - x_k\|\|x_k-x^*\|.
\end{aligned} \tag{3.11}$$

Let us show that $\|x_k - VGx_k\| \to 0$ as $k\to\infty$. Indeed, from (3.1), we have

$$\begin{aligned}
\|x_k - V_kGx_k\| &\le \|x_k - V_kGy_k\| + \|V_kGy_k - V_kGx_k\| \\
&\le \gamma_k\|f(x_k) - V_kGy_k\| + \|y_k - x_k\| \\
&= \gamma_k\|f(x_k) - V_kGy_k\| + \beta_k\|V_kGx_k - x_k\|,
\end{aligned}$$

which together with $\gamma_k\to 0$ and $\beta_k \to 0$, yields that $\lim_{k\to\infty}\|x_k - V_kGx_k\| = 0$. Since

$$\|x_k - VGx_k\| \le \|x_k - V_kGx_k\| + \|V_kGx_k - VGx_k\|,$$

we obtain from (3.7) that

$$\lim_{k\to\infty}\|x_k - VGx_k\| = 0. \tag{3.12}$$

On the other hand, observe that

$$\begin{aligned}
\langle (I-f)x_k, J(x_k-x^*)\rangle &= \|x_k-x^*\|^2 + \langle x^* - f(x^*), J(x_k-x^*)\rangle + \langle f(x^*) - f(x_k), J(x_k-x^*)\rangle \\
&\ge (1-\alpha)\|x_k-x^*\|^2 + \langle x^* - f(x^*), J(x_k-x^*)\rangle.
\end{aligned} \tag{3.13}$$

It follows from (3.11) and (3.13) that

$$\|x_k-x^*\|^2 \le \frac{1}{1-\alpha}\left(\langle f(x^*) - x^*, J(x_k-x^*)\rangle + \frac{\beta_k}{\gamma_k}\|V_kGx_k - x_k\|\|x_k-x^*\|\right).$$

This together with (3.10), implies that $\mathrm{LIM}_k\|x_k-x^*\|^2 \le 0$, i.e., $\mathrm{LIM}_k\|x_k-x^*\|^2 = 0$. Thus, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ which converges strongly to $x^* \in \mathrm{Fix}(V\circ G) = \mathscr{F}$.

Now, assume that there exists another subsequence $\{x_{m_j}\}$ of $\{x_k\}$ such that $x_{m_j} \to \tilde{x} \in C$. According to (3.12), we know that $\tilde{x} \in \mathrm{Fix}(V\circ G) = \mathscr{F}$. Then we have that $\|(I-f)x_{m_j} - (I-f)\tilde{x}\| \to 0$ as $j\to\infty$. We claim that $\tilde{x}$ is a solution in $\mathscr{F}$ to the VI (3.2). Indeed, since for any $p\in\mathscr{F}$ the sequences $\{x_{m_j} - p\}$

and $\{x_{m_j} - f(x_{m_j})\}$ are bounded and $J$ is norm-to-weak$^*$ uniformly continuous on every bounded subset of $X$, we deduce that as $j \to \infty$

$$\begin{aligned}
&|\langle (I-f)x_{m_j}, J(x_{m_j}-p)\rangle - \langle (I-f)\tilde{x}, J(\tilde{x}-p)\rangle| \\
&\le \|(I-f)x_{m_j} - (I-f)\tilde{x}\| \|x_{m_j}-p\| + |\langle (I-f)\tilde{x}, J(x_{m_j}-p) - J(\tilde{x}-p)\rangle| \to 0.
\end{aligned}$$

Thus it follows from (3.11) that, for any $p \in \mathscr{F}$,

$$\langle f(\tilde{x}) - \tilde{x}, J(p-\tilde{x})\rangle = \lim_{j\to\infty} \langle f(x_{m_j}) - x_{m_j}, J(p - x_{m_j})\rangle \le 0.$$

This is, $\tilde{x} \in \mathscr{F}$ is a solution of the VI (3.2). Hence, $\tilde{x} = x^*$. Therefore, each cluster point of $\{x_k\}$ is $x^*$. So $\{x_k\}$ converges strongly to $x^*$, which is the unique solution in $\mathscr{F}$ to the VI (3.2). This completes the proof. □

Now we state and prove the strong convergence theorem for Mann-type explicit viscosity iterative algorithm.

**Theorem 3.2.** *Let $X$ be a real reflexive Banach space with the uniformly Gateaux differentiable norm and the normal structure, and let $C$ be a nonempty closed convex subset of $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let $B_i : C \to X$ be $\zeta_i$-strictly pseudocontractive and $\delta_i$-strongly accretive with $\delta_i + \zeta_i \ge 1$ for each $i = 1,2$. Define the mapping $G : C \to C$ by $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$, where $1 - \frac{\zeta_i}{1+\zeta_i}(1 - \sqrt{\frac{1-\delta_i}{\zeta_i}}) \le \mu_i \le 1$ for $i = 1,2$. Let $f : C \to C$ be a fixed contraction with coefficient $\alpha \in (0,1)$. Let $\{T_i\}_{i=1}^\infty$ be a countable family of nonexpansive self-mappings on $C$ such that $\mathscr{F} = \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \cap \mathrm{Fix}(G) \ne \emptyset$. Let $\{V_k\}_{k=1}^\infty$ be defined by (2.5) and (2.6) such that $\mathrm{Fix}(V \circ G) = \mathscr{F}$. For any given $x_1 \in C$, let $\{x_k\}_{k=1}^\infty$ be a sequence defined by*

$$\begin{cases} x_{k+1} = (1-\varepsilon_k-\beta_k)x_k + \varepsilon_k f(x_k) + \beta_k V_k G y_k, \\ y_k = (1-\gamma_k)x_k + \gamma_k V_k G x_k, \quad \forall k \ge 1, \end{cases} \tag{3.14}$$

*where $\{\varepsilon_k\}$ and $\{\beta_k\}$ are two sequences in $(0,1)$ with $\varepsilon_k + \beta_k \le 1, \forall k \ge 1$, and $\{\gamma_k\}$ is a sequence in $[0,1]$. Assume that*

*(i)* $\lim_{k\to\infty} \varepsilon_k = 0$, $\sum_{k=1}^\infty \varepsilon_k = \infty$ *and* $0 < \liminf_{k\to\infty} \beta_k \le \limsup_{k\to\infty} \beta_k < 1$*;*
*(ii)* $\lim_{k\to\infty} |\gamma_{k+1} - \gamma_k| = 0$ *and* $\limsup_{k\to\infty} \gamma_k < 1$.

*Then there hold the following assertions:*

*(I)* $\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0$*;*
*(II) the sequence $\{x_k\}$ converges strongly to a point $x^* \in \mathscr{F}$ which is the unique solution in $\mathscr{F}$ to the VI (3.2) provided $\lim_{k\to\infty} \gamma_k = 0$ and $\beta_k \equiv \beta$ for some fixed $\beta \in (0,1)$.*

*Proof.* Step 1. The proof of conclusion (I).

First, we claim that $\{x_k\}$ is bounded. Indeed, take a fixed $p \in \mathscr{F}$ arbitrarily. Observe that

$$\begin{aligned}
\|x_{k+1} - p\| &\le (1-\varepsilon_k-\beta_k)\|x_k - p\| + \varepsilon_k \|f(x_k) - p\| + \beta_k \|V_k G y_k - p\| \\
&\le (1-\varepsilon_k-\beta_k)\|x_k - p\| + \varepsilon_k \|f(x_k) - f(p)\| + \varepsilon_k \|f(p) - p\| + \beta_k \|y_k - p\| \\
&\le (1-\varepsilon_k-\beta_k)\|x_k - p\| + \alpha\varepsilon_k \|x_k - p\| + \beta_k \|y_k - p\| + \varepsilon_k \|f(p) - p\|,
\end{aligned}$$

and

$$\begin{aligned}\|y_k - p\| &\le \|(1-\gamma_k)\|x_k - p\| + \gamma_k\|V_kGx_k - p\| \\ &\le \|(1-\gamma_k)\|x_k - p\| + \gamma_k\|x_k - p\| \\ &= \|x_k - p\|.\end{aligned}$$

Combining these two inequalities, we have

$$\begin{aligned}\|x_{k+1} - p\| &\le (1-\varepsilon_k-\beta_k)\|x_k - p\| + \alpha\varepsilon_k\|x_k - p\| + \beta_k\|y_k - p\| + \varepsilon_k\|f(p) - p\| \\ &\le (1-\varepsilon_k-\beta_k)\|x_k - p\| + \alpha\varepsilon_k\|x_k - p\| + \beta_k\|x_k - p\| + \varepsilon_k\|f(p) - p\| \\ &= (1-(1-\alpha)\varepsilon_k)\|x_k - p\| + \varepsilon_k\|f(p) - p\| \\ &\le \max\{\|x_k - p\|, \frac{\|f(p) - p\|}{1-\alpha}\}.\end{aligned}$$

By induction,

$$\|x_k - p\| \le \max\{\|x_1 - p\|, \frac{\|f(p) - p\|}{1-\alpha}\}, \quad \forall k \ge 1.$$

Hence it follows that $\{x_k\}$ is bounded, so are $\{y_k\}, \{V_kGx_k\}, \{V_kGy_k\}$ and $f(x_k)$.

Second, we claim that $\|x_{k+1} - x_k\| \to 0$ as $k \to \infty$. Indeed, define a sequence $\{w_k\}$ by

$$x_{k+1} = \rho_k x_k + (1-\rho_k)w_k, \quad \forall k \ge 1,$$

where $\rho_k = 1 - \varepsilon_k - \beta_k, \forall k \ge 1$. Then we have

$$\begin{aligned}w_{k+1} - w_k &= \frac{x_{k+2} - \rho_{k+1}x_{k+1}}{1-\rho_{k+1}} - \frac{x_{k+1} - \rho_k x_k}{1-\rho_k} \\ &= \frac{\varepsilon_{k+1}f(x_{k+1}) + \beta_{k+1}V_{k+1}Gy_{k+1}}{1-\rho_{k+1}} - \frac{\varepsilon_k f(x_k) + \beta_k V_kGy_k}{1-\rho_k} \\ &= \frac{\varepsilon_{k+1}}{1-\rho_{k+1}}f(x_{k+1}) - \frac{\varepsilon_k}{1-\rho_k}f(x_k) + \frac{\beta_{k+1}}{1-\rho_{k+1}}(V_{k+1}Gy_{k+1} - V_{k+1}Gy_k) \\ &\quad + V_{k+1}Gy_k - V_kGy_k + \frac{\varepsilon_k}{1-\rho_k}V_kGy_k - \frac{\varepsilon_{k+1}}{1-\rho_{k+1}}V_{k+1}Gy_k,\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\|y_{k+1} - y_k\| &\le (1-\gamma_{k+1})\|x_{k+1} - x_k\| + |\gamma_{k+1} - \gamma_k|\|x_k\| \\ &\quad + \gamma_{k+1}\|V_{k+1}Gx_{k+1} - V_kGx_k\| + |\gamma_{k+1} - \gamma_k|\|V_kGx_k\| \\ &\le (1-\gamma_{k+1})\|x_{k+1} - x_k\| + |\gamma_{k+1} - \gamma_k|\|x_k\| + \gamma_{k+1}(\|V_{k+1}Gx_{k+1} - V_{k+1}Gx_k\| \\ &\quad + \|V_{k+1}Gx_k - V_kGx_k\|) + |\gamma_{k+1} - \gamma_k|\|V_kGx_k\| \\ &\le \|x_{k+1} - x_k\| + |\gamma_{k+1} - \gamma_k|\|x_k\| + \gamma_{k+1}\alpha_{k+1}\|T_{k+1}Gx_k - Gx_k\| \\ &\quad + |\gamma_{k+1} - \gamma_k|\|V_kGx_k\| \\ &= \|x_{k+1} - x_k\| + |\gamma_{k+1} - \gamma_k|(\|x_k\| + \|V_kGx_k\|) + \gamma_{k+1}\alpha_{k+1}\|T_{k+1}Gx_k - Gx_k\|.\end{aligned} \tag{3.16}$$

Combining (3.15) with (3.16), we obtain

$$
\begin{aligned}
&\|w_{k+1}-w_k\|-\|x_{k+1}-x_k\| \\
&\le \frac{\varepsilon_{k+1}}{1-\rho_{k+1}}(\|f(x_{k+1})\|+\|V_{k+1}Gy_k\|)+\frac{\varepsilon_k}{1-\rho_k}(\|f(x_k)\|+\|V_kGy_k\|) \\
&\quad+\frac{\beta_{k+1}}{1-\rho_{k+1}}\|V_{k+1}Gy_{k+1}-V_{k+1}Gy_k\|+\|V_{k+1}Gy_k-V_kGy_k\|-\|x_{k+1}-x_k\| \\
&\le \frac{\varepsilon_{k+1}}{1-\rho_{k+1}}(\|f(x_{k+1})\|+\|V_{k+1}Gy_k\|)+\frac{\varepsilon_k}{1-\rho_k}(\|f(x_k)\|+\|V_kGy_k\|) \\
&\quad+\frac{\beta_{k+1}}{1-\rho_{k+1}}\{\|x_{k+1}-x_k\|+|\gamma_{k+1}-\gamma_k|(\|x_k\|+\|V_kGx_k\|) \\
&\quad+\gamma_{k+1}\alpha_{k+1}\|T_{k+1}Gx_k-Gx_k\|\}+\alpha_{k+1}\|T_{k+1}Gy_k-Gy_k\|-\|x_{k+1}-x_k\| \\
&\le \frac{\varepsilon_{k+1}}{1-\rho_{k+1}}(\|f(x_{k+1})\|+\|V_{k+1}Gy_k\|)+\frac{\varepsilon_k}{1-\rho_k}(\|f(x_k)\|+\|V_kGy_k\|) \\
&\quad+\frac{\beta_{k+1}}{1-\rho_{k+1}}\{|\gamma_{k+1}-\gamma_k|(\|x_k\|+\|V_kGx_k\|)+\gamma_{k+1}\alpha_{k+1}\|T_{k+1}Gx_k-Gx_k\|\} \\
&\quad+\alpha_{k+1}\|T_{k+1}Gy_k-Gy_k\|.
\end{aligned}
\tag{3.17}
$$

So, it follows from (3.17), $\alpha_k \to 0$ and conditions (i), (ii) that

$$\limsup_{k\to\infty}(\|w_{k+1}-w_k\|-\|x_{k+1}-x_k\|)\le 0.$$

Since $\lim_{k\to\infty}\varepsilon_k=0$ and $0<\liminf_{k\to\infty}\beta_k\le\limsup_{k\to\infty}\beta_k<1$, we have

$$0<\liminf_{k\to\infty}\rho_k\le\limsup_{k\to\infty}\rho_k<1.$$

From Lemma 2.4, we get $\lim_{k\to\infty}\|w_k-x_k\|=0$. Consequently,

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|=\lim_{k\to\infty}(1-\rho_k)\|w_k-x_k\|=0. \tag{3.18}$$

Step 2. The proof of conclusion (II).

Suppose that $\lim_{k\to\infty}\gamma_k=0$ and $\beta_k\equiv\beta$ for some fixed $\beta\in(0,1)$. In this case, conditions (i), (ii) are still satisfied. Let $\{z_n\}$ be defined by (3.4) such that $z_n\to x^*\in \mathrm{Fix}(V\circ G)=\mathscr{F}$, where $x^*$ is the unique solution in $\mathscr{F}$ to the VI (3.2). Observe that, for every $k,n\ge 1$,

$$
\begin{aligned}
\|x_{k+1}-VGz_n\| &\le (1-\varepsilon_k-\beta)\|x_k-VGz_n\|+\varepsilon_k\|f(x_k)-VGz_n\| \\
&\quad+\beta[\|V_kGy_k-V_kGz_n\|+\|V_kGz_n-VGz_n\|] \\
&\le (1-\varepsilon_k-\beta)\|x_k-VGz_n\|+\varepsilon_k\|f(x_k)-VGz_n\| \\
&\quad+\beta[\|y_k-z_n\|+\|V_kGz_n-VGz_n\|] \\
&\le (1-\varepsilon_k-\beta)\|x_k-VGz_n\|+\varepsilon_k\|f(x_k)-VGz_n\| \\
&\quad+\beta[\|x_k-z_n\|+\|y_k-x_k\|+\|V_kGz_n-VGz_n\|] \\
&= (1-\varepsilon_k-\beta)\|x_k-VGz_n\|+\varepsilon_k\|f(x_k)-VGz_n\| \\
&\quad+\beta[\|x_k-z_n\|+\gamma_k\|V_kGx_k-x_k\|+\|V_kGz_n-VGz_n\|] \\
&\le \sigma_k+(1-\beta)\|x_k-VGz_n\|+\beta\|x_k-z_n\|,
\end{aligned}
\tag{3.19}
$$

where

$$\sigma_k=\varepsilon_k\|f(x_k)-VGz_n\|+\beta[\gamma_k\|V_kGx_k-x_k\|+\|V_kGz_n-VGz_n\|].$$

Repeating the same arguments as those of (3.7) in the proof of Theorem 3.1, we obtain

$$\lim_{k\to\infty}\|V_kGz_n - VGz_n\| = 0 \quad \text{and} \quad \lim_{k\to\infty}\|V_kGx_k - VGx_k\| = 0. \tag{3.20}$$

Since $\lim_{k\to\infty}\varepsilon_k = \lim_{k\to\infty}\gamma_k = \lim_{k\to\infty}\|V_kGz_n - VGz_n\| = 0$, we know that $\sigma_k \to 0$ as $k\to\infty$. From (3.19), we get

$$\begin{aligned}\|x_{k+1}-VGz_n\|^2 &\le ((1-\beta)\|x_k - VGz_n\| + \beta\|x_k - z_n\|)^2\\ &\quad + \sigma_k[2((1-\beta)\|x_k - VGz_n\| + \beta\|x_k - z_n\|) + \sigma_k]\\ &= (1-\beta)^2\|x_k - VGz_n\|^2 + \beta^2\|x_k - z_n\|^2\\ &\quad + 2\beta(1-\beta)\|x_k - VGz_n\|\|x_k - z_n\| + \theta_k\\ &\le (1-\beta)^2\|x_k - VGz_n\|^2 + \beta^2\|x_k - z_n\|^2\\ &\quad + \beta(1-\beta)(\|x_k - VGz_n\|^2 + \|x_k - z_n\|^2) + \theta_k\\ &= (1-\beta)\|x_k - VGz_n\|^2 + \beta\|x_k - z_n\|^2 + \theta_k,\end{aligned} \tag{3.21}$$

where $\theta_k = \sigma_k[2((1-\beta)\|x_k - VGz_n\| + \beta\|x_k - z_n\|) + \sigma_k] \to 0$ as $k\to\infty$. For any Banach limit LIM, we derive from (3.21) that

$$\text{LIM}_k\|x_k - VGz_n\|^2 = \text{LIM}_k\|x_{k+1} - VGz_n\|^2 \le \text{LIM}_k\|x_k - z_n\|^2.$$

Observe that $x_k - z_n = \frac{1}{n}(x_k - f(z_n)) + (1-\frac{1}{n})(x_k - VGz_n)$. By the same argument as that of (3.10) in the proof of Theorem 3.1, we get

$$\text{LIM}_k\langle f(x^*) - x^*, J(x_k - x^*)\rangle \le 0. \tag{3.22}$$

On the other hand, it follows from (3.18) that

$$\lim_{k\to\infty}|\langle f(x^*) - x^*, J(x_{k+1} - x^*)\rangle - \langle f(x^*) - x^*, J(x_k - x^*)\rangle| = 0. \tag{3.23}$$

Hence by Lemma 2.5 we deduce from (3.22) and (3.23) that

$$\limsup_{k\to\infty}\langle (f-I)x^*, J(x_k - x^*)\rangle \le 0. \tag{3.24}$$

Finally, we show that $x_k \to x^*$ as $k\to\infty$. From Lemma 2.2 (i) and (3.14) with $\beta_k \equiv \beta$, we have

$$\begin{aligned}&\|x_{k+1} - x^*\|^2\\ &= \|(1-\varepsilon_k-\beta)(x_k - x^*) + \varepsilon_k(f(x_k) - x^*) + \beta(V_kGy_k - x^*)\|^2\\ &\le \|(1-\varepsilon_k-\beta)(x_k - x^*) + \beta(V_kGy_k - x^*)\|^2 + 2\varepsilon_k\langle f(x_k) - x^*, J(x_{k+1} - x^*)\rangle\\ &\le [(1-\varepsilon_k-\beta)\|x_k - x^*\| + \beta\|V_kGy_k - x^*\|]^2 + 2\varepsilon_k\langle f(x_k) - x^*, J(x_{k+1} - x^*)\rangle\\ &\le (1-\varepsilon_k)^2\|x_k - x^*\|^2 + 2\varepsilon_k[\langle f(x_k) - f(x^*), J(x_{k+1} - x^*)\rangle + \langle f(x^*) - x^*, J(x_{k+1} - x^*)\rangle]\\ &\le (1-\varepsilon_k)^2\|x_k - x^*\|^2 + 2\varepsilon_k[\alpha\|x_k - x^*\|\|x_{k+1} - x^*\| + \langle f(x^*) - x^*, J(x_{k+1} - x^*)\rangle]\\ &\le (1-\varepsilon_k)^2\|x_k - x^*\|^2 + \alpha\varepsilon_k[\|x_k - x^*\|^2 + \|x_{k+1} - x^*\|^2] + 2\varepsilon_k\langle f(x^*) - x^*, J(x_{k+1} - x^*)\rangle,\end{aligned}$$

which implies that

$$\|x_{k+1} - x^*\|^2 \le \frac{(1-\varepsilon_k)^2 + \alpha\varepsilon_k}{1-\alpha\varepsilon_k}\|x_k - x^*\|^2 + \frac{2\varepsilon_k}{1-\alpha\varepsilon_k}\langle f(x^*) - x^*, J(x_{k+1} - x^*)\rangle. \tag{3.25}$$

Observe that for all $k \geq 1$

$$\frac{(1-\varepsilon_k)^2+\alpha\varepsilon_k}{1-\alpha\varepsilon_k} = \frac{1-\alpha\varepsilon_k-2\varepsilon_k(1-\alpha)+\varepsilon_k^2}{1-\alpha\varepsilon_k} = 1-\frac{2(1-\alpha)\varepsilon_k}{1-\alpha\varepsilon_k}+\frac{\varepsilon_k^2}{1-\alpha\varepsilon_k}.$$

Since

$$\lim_{k\to\infty}\left(\frac{2(1-\alpha)\varepsilon_k}{1-\alpha\varepsilon_k}-\frac{\varepsilon_k^2}{1-\alpha\varepsilon_k}\right)/\varepsilon_k = 2(1-\alpha) > 1-\alpha,$$

we may assume, without loss of generality, that for all $k \geq 1$

$$\frac{2(1-\alpha)\varepsilon_k}{1-\alpha\varepsilon_k}-\frac{\varepsilon_k^2}{1-\alpha\varepsilon_k} \geq (1-\alpha)\varepsilon_k.$$

This, together with (3.25), leads to

$$\begin{aligned}
&\|x_{k+1}-x^*\|^2 \\
&\leq \left[1-\left(\frac{2(1-\alpha)\varepsilon_k}{1-\alpha\varepsilon_k}-\frac{\varepsilon_k^2}{1-\alpha\varepsilon_k}\right)\right]\|x_k-x^*\|^2+\frac{2\varepsilon_k}{1-\alpha\varepsilon_k}\langle f(x^*)-x^*, J(x_{k+1}-x^*)\rangle \\
&\leq [1-(1-\alpha)\varepsilon_k]\|x_k-x^*\|^2+\frac{2\varepsilon_k}{1-\alpha\varepsilon_k}\langle f(x^*)-x^*, J(x_{k+1}-x^*)\rangle \\
&= [1-(1-\alpha)\varepsilon_k]\|x_k-x^*\|^2+(1-\alpha)\varepsilon_k\cdot\frac{2}{(1-\alpha\varepsilon_k)(1-\alpha)}\langle f(x^*)-x^*, J(x_{k+1}-x^*)\rangle.
\end{aligned} \tag{3.26}$$

Since $\sum_{k=1}^{\infty}\varepsilon_k = \infty$, $\lim_{k\to\infty}\varepsilon_k = 0$ and $\limsup_{k\to\infty}\langle f(x^*)-x^*, J(x_{k+1}-x^*)\rangle \leq 0$ (due to (3.24)), we know that $\sum_{k=1}^{\infty}(1-\alpha)\varepsilon_k = \infty$ and

$$\limsup_{k\to\infty}\frac{2}{(1-\alpha\varepsilon_k)(1-\alpha)}\langle f(x^*)-x^*, J(x_{k+1}-x^*)\rangle \leq 0.$$

Therefore, applying Lemma 2.1 to (3.26) we conclude that $\lim_{k\to\infty}\|x_k-x^*\| = 0$. This completes the proof. □


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