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A MODIFIED INERTIAL PROJECTION AND CONTRACTION ALGORITHMS FOR QUASI-VARIATIONAL INEQUALITIES

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Abstract. In this paper, we propose a modified intertial projection and contraction algorithm for solving quasi variational inequalities. A weak convergence theorem is established in Hilbert spaces. Numerical examples are provided to demonstrate the validity of our proposed algorithm.
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1. INTRODUCTION

Let $C \subseteq H$ be a nonempty, closed and convex set in a real Hilbert space H. $\langle \cdot, \cdot \rangle$ denotes the inner product in H. Let S be a mapping on H. The fixed point set of S is denoted by $Fix(S)$. Recall that S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

The class of nonexpansive mappings is an important class of nonlinear mappings in nonlinear functional analysis. Recall that S is said to be monotone iff

$$\langle Sx - Sy, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

S is said to be strongly monotone iff there exists a positive real constant such that

$$\langle Sx - Sy, x - y \rangle \geq \lambda \|x - y\|, \quad \forall x, y \in C.$$

S is said to be inverse strongly monotone iff there exists a positive real constant such that

$$\langle Sx - Sy, x - y \rangle \geq \lambda \|Sx - Sy\|, \quad \forall x, y \in C.$$

In this paper, we consider the quasi-variational inequality (QVI), which consists of finding a point $x^* \in Fix(S)$ such that

$$\langle F(x^*), x - x^* \rangle \geqslant 0, \forall x \in C, \tag{1.1}$$

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where S is a nonexpansive mapping on H and F is a monotone mapping on H. In this paper, the solution set of quasi-variational inequality (1.1) is denoted by $SOL(C,S,F)$. In particular, if $S=I$, where I is an identity mapping, the quasi-variational inequality is reduced to the following variational inequality (VI): find $x^* \in C$ such that

$$\langle F(x^*), x-x^*\rangle \geqslant 0, \forall x \in C. \tag{1.2}$$

We denote by $SOL(C,F)$ the solution set of (1.2) in this paper.

For solving the QVI and the VI, many results were obtained via iteration methods in infinite dimensional spaces; see, e.g., [2, 3, 9, 16, 18, 23] and the references therein. An efficient method, known as the gradient-projected method, is as follows

$$x_{k+1} = P_C(x_k - \tau F(x_k)), \tag{1.3}$$

where τ is some positive real number and P_C is the metric projection from H onto C. If problem (1.1) has solutions and F is strongly monotone. The $\{x_n\}$ generated by (1.3) converges to a solution of (VI). The strong monotonicity is quite strong. In order to overcome this drawback, Korpelevich [17] proposed the following known extragradient algorithm (EgA)

$$\begin{cases} y_k = P_C(x_k - \tau F(x_k)) \\ x_{k+1} = P_C(x_k - \tau F(y_k)), \quad \forall k \geq 1, \end{cases}$$

where τ is an appropriate constant and F is Lipschitz continuous and monotone. Since 1976, many authors studied the extragradient algorithm and its modifications via various techniques; see, e.g., [10, 13, 14, 21] and the references therein. It should be noted that each iteration of the extragradient method needs to compute two orthogonal projections P_C. There are no analytic expressions for the metric projection in most cases. Hence, the extragradient algorithm is not very convenient and efficient in practical calculations. In 2011, Censor, Gibali and Reich [10] introduced the following subgradient extragradient algorithm (SEgA)

$$\begin{cases} y_k = P_C(x_k - \tau F(x_k)) \\ T_k := \{w \in H | \langle x_k - \tau F(x_k) - y_k, w - y_k\rangle \leq 0\} \\ x_{k+1} = P_{T_k}(x_k - \tau F(y_k)), \quad \forall k \geq 1, \end{cases}$$

for each $k \geq 1$. This method reduce the projections onto C by a projection onto the half-space T_k. They established strong convergence of the above subgradient extragradient algorithm in Hilbert spaces.

The inertial type algorithms was originally from the heavy ball method (an implicit discretization) of second-order dynamical systems in time [1, 6]. The main features of them are that the next iterate is defined by making use of the previous two iterates. Recently, there are increasing interests in studying inertial type algorithms; see [4, 5, 19] and the references therein. For finding zero points of maximally monotone operators, Bot and Csetnek [8] proposed a so-called inertial hybrid proximal-extragradient algorithm, which combines inertial type ideas and hybrid proximal-extragradient ideas; see, e.g., [15, 20, 22]. Their algorithm includes the following algorithm [11] as a special case

$$x_{k+1} = P_C(x_k - c_k F(x_k) + \alpha_k(x_k - x_{k-1})). \tag{1.4}$$

Algorithm (1.4) can be seen as the projection algorithm (1.2) with inertial effects and also can be seen as a bounded perturbation of projection algorithm (1.2); see, e.g., [15]. Recently, Dong, Cho and Zhong

[12] introduced the inertial projection and contraction algorithm (IPCA):

$$\begin{cases} w_k = x_k + \alpha_k(x_k - x_{k-1}), \\ y_k = P_C(w_k - \tau F(w_k)), \\ d(w_k, y_k) = (w_k - y_k) - \tau(F(w_k) - F(y_k)), \\ x_{k+1} = w_k - \gamma\beta_k d(w_k, y_k), \quad k \geq 1. \end{cases} \tag{1.5}$$

In this paper, we study a modified inertial projection and contraction algorithm and analyze its convergence in a Hilbert space H. Numerical examples are presented to illustrate the efficiency of our algorithm.

2. Preliminaries

In this section, we recall some known facts and necessary tools. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty, closed and convex subset of H. The notation $x_k \rightharpoonup x (x_k \to x)$ is used to indicate that the sequence $\{x_k\}_{k\in N}$ converges weakly (strongly) to x.

The following is a known equality in a Hilbert space

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \quad \forall x, y \in H, \tag{2.1}$$

where λ is a real number in $(0,1)$.

Definition 2.1. Let $A : H \to H$ be a point-to-set operator defined on a real Hilbert space H. A is said to be

(i) monotone iff

$$\langle A(x) - A(y), x - y \rangle \geq 0, \quad \forall x, y \in H.$$

(i) maximal monotone iff A is monotone, and the graph $G(A)$ of A,

$$G(A) := (x,u) \in H \times H : u \in A(x),$$

is not properly contained in the graph of any other monotone operator.

It is known that a monotone mapping A is maximal if and only if, for any $(x,u) \in H \times H$, $\langle u - v, x - y \rangle \geq 0$, $\forall (v,y) \in G(A) \Rightarrow u \in A(x)$.

Lemma 2.1. [7] *Let C be a nonempty set of a Hilbert space H and let $\{x_k\}_{k\in N}$ be a sequence in H such that the following two conditions hold:*
(i) for every $x \in C$, $\lim_{k\to+\infty} \|x_k \to x\|$ exists;
(ii) every sequential weak cluster point of $\{x_k\}_{k\in N}$ lies in C.
Then $\{x_k\}_{k\in N}$ converges weakly to a point in C.

Lemma 2.2. [1] *Let $\{\varphi_k\}, \{\delta_k\}$ and $\{\alpha_k\}$ be the sequences in $[0,+\infty)$ such that, for each $k \geq 1$,*

$$\varphi_{k+1} \leq \varphi_k + \alpha_k(\varphi_k - \varphi_{k-1}) + \delta_k, \sum \delta_k < +\infty$$

and there exists a real number α with $0 \leq \alpha_k \leq \alpha < 1$ for all $k \geq 1$. Then the following conclusions hold:
(i) $\sum[\varphi_k - \varphi_{k-1}]_+ < +\infty$, where $[t]_+ = max\{t, 0\}$;
(ii) there exists $\varphi^* \in [0,+\infty)$ such that $\lim_{k\to+\infty} \varphi_k = \varphi^*$.

Lemma 2.3. [11] *Let K be a closed convex subset of real Hilbert space H and let P_K be the metric (nearest point) projection from H onto K (i.e., for $x \in H$, $\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}$. Then, for any $x \in H$ and $z = P_K x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in K.$$

3. The modified inertial projection and contraction algorithms

Modified by the inertial extragradient algorithm [11], we we introduce the following algorithm

Algorithm 3.1.

$$\begin{cases} w_k = x_k + \alpha_k(x_k - x_{k-1}), \\ y_k = P_C(w_k - b_k F(w_k)), \\ d(w_k, y_k) = (w_k - y_k) - b_k(F(w_k) - F(y_k)), \\ x_{k+1} = (1 - \theta_k)w_k + \theta_k S(w_k - \gamma\beta_k d(w_k, y_k)). \end{cases} \tag{3.1}$$

where $\{b_k\}$ is a positive real number sequence such that

$$b_k\|F(w_k) - F(y_k)\| \leq \mu\|w_k - y_k\|, \quad \forall k \geq 1,$$

$\gamma \in (0,2)$, $\mu \in (0,1)$ and

$$\beta_k = \begin{cases} \varphi(w_k, y_k)/\|d(w_k, y_k)\|^2, & \text{if } d(w_k, y_k) \neq 0, \\ 0, & \text{if } d(w_k, y_k) = 0, \end{cases} \tag{3.2}$$

where

$$\varphi(w_k, y_k) := \langle w_k - y_k, d(w_k, y_k)\rangle,$$

and $\{\alpha_k\}$ is nondecreasing sequence such that

$$\alpha_1 = 0, 0 \leq \alpha_k \leq \alpha < \frac{1}{3},$$

and $0 < \theta < \theta_k \leq \frac{1}{2}$. If $y_k = w_k$ or $d(w_k, y_k) = 0$, then the iterative process stops; otherwise, we set $k := k + 1$ and go on to (3.1) to evaluate the next iterate x_{k+2}.

In this paper, we assume the solution set of the QVI is nonempty, i.e., $SOL(C, S, F) \neq \varnothing$, and set $v_k := w_k - \gamma\beta_k d(w_k, y_k)$ for each $k \geq 1$.

Lemma 3.1. *If $y_k = w_k$ or $d(w_k, y_k) = 0$ in (3.1), then $v_k, w_k, y_k \in SOL(C, F)$.*

Proof. Observe that

$$\begin{aligned} \|d(w_k, y_k)\| &= \|(w_k - y_k) - b_k(F(w_k) - F(y_k))\| \\ &\geq \|w_k - y_k\| - b_k\|F(w_k) - F(y_k)\| \\ &\geq (1 - \mu)\|w_k - y_k\|, \end{aligned}$$

and

$$\begin{aligned} \|d(w_k, y_k)\| &= \|(w_k - y_k) - b_k(F(w_k) - F(y_k))\| \\ &\leq \|w_k - y_k\| + b_k\|F(w_k) - F(y_k)\| \\ &\leq (1 + \mu)\|w_k - y_k\|. \end{aligned}$$

So, $d(w_k, y_k) = 0$ if and only if $w_k = y_k$. If $d(w_k, y_k) = 0$, we conclude from (3.1) and (3.2) that $v_k = y_k$ and

$$y_k = P_C(y_k - b_k F(y_k)).$$

Using Lemma 2.3 yields that $v_k, w_k, y_k \in SOL(C, F)$. This completes the proof. □

Lemma 3.2. *Assume that F is monotone on H and $d(w_k, y_k) \neq 0$. Let $u \in SOL(C, F)$. Then the following assertions hold:*

(i)

$$\|v_k - u\|^2 \leq \|w_k - u\|^2 - \frac{2-\gamma}{\gamma}\|v_k - w_k\|^2, \tag{3.3}$$

(ii)

$$\|w_k - y_k\|^2 \leq \frac{1+\mu^2}{(1-\mu)\gamma^2}\|v_k - w_k\|^2. \tag{3.4}$$

Proof. (i) From the Cauchy-Schwarz inequality and $b_k\|F(w_k) - F(y_k)\| \leq \mu\|w_k - y_k\|$, it follows

$$\begin{aligned} \varphi(w_k, y_k) &= \langle w_k - y_k, d(w_k, y_k)\rangle \\ &= \langle w_k - y_k, (w_k - y_k) - b_k(F(w_k) - F(y_k))\rangle \\ &= \|w_k - y_k\|^2 - b_k\langle w_k - y_k, F(w_k) - F(y_k)\rangle \\ &\geq \|w_k - y_k\|^2 - b_k\|w_k - y_k\|\|F(w_k) - F(y_k)\| \\ &\geq (1-\mu)\|w_k - y_k\|^2. \end{aligned} \tag{3.5}$$

Since F is monotone on H, we have $\langle F(w_k) - F(y_k), w_k - y_k\rangle \geq 0$. It follows that

$$\begin{aligned} \|d(w_k, y_k)\|^2 &= \|(w_k - y_k) - b_k(F(w_k) - F(y_k))\|^2 \\ &= \|w_k - y_k\|^2 + b_k^2\|F(w_k) - F(y_k)\|^2 - 2b_k\langle w_k - y_k, F(w_k) - F(y_k)\rangle \\ &\leq \|w_k - y_k\|^2 + b_k^2\|F(w_k) - F(y_k)\|^2 \\ &\leq (1+\mu^2)\|w_k - y_k\|^2. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$\beta_k = \frac{\varphi(w_k, y_k)}{\|d(w_k, y_k)\|^2} \geq \frac{1-\mu}{1+\mu^2}. \tag{3.7}$$

From $v_k = w_k - \gamma\beta_k d(w_k, y_k)$, we have

$$\begin{aligned} \|v_k - u\|^2 &= \|(w_k - u) - \gamma\beta_k d(w_k, y_k)\|^2 \\ &= \|w_k - u\|^2 - 2\gamma\beta_k\langle w_k - u, d(w_k, y_k)\rangle + \gamma^2\beta_k^2\|d(w_k, y_k)\|^2. \end{aligned} \tag{3.8}$$

Note that

$$\langle w_k - u, d(w_k, y_k)\rangle = \langle w_k - y_k, d(w_k, y_k)\rangle + \langle y_k - u, d(w_k, y_k)\rangle. \tag{3.9}$$

By the definition of $\{y_k\}$ and Lemma 2.3, we have

$$\langle y_k - u, w_k - y_k - b_k F(w_k)\rangle \geq 0. \tag{3.10}$$

Since F is monotone, one has

$$\langle y_k - u, b_k F(y_k) - b_k F(u)\rangle \geq 0. \tag{3.11}$$

Since $u \in SOL(C,F)$ and $y_k \in C$, it follows from (1.1) that

$$\langle y_k - u, b_k F(u)\rangle \geq 0. \tag{3.12}$$

From (3.10), (3.11) and (3.12), we have

$$\langle y_k - u, d(w_k, y_k)\rangle = \langle y_k - u, w_k - y_k - b_k(F(w_k) - F(y_k))\rangle \geq 0. \tag{3.13}$$

Combined (3.9) with (3.13), we obtain

$$\langle w_k - u, d(w_k, y_k)\rangle \geq \langle w_k - y_k, d(w_k, y_k)\rangle = \varphi(w_k, y_k). \tag{3.14}$$

It follows from (3.14) , (3.8) and $\beta_k = \frac{\varphi(w_k,y_k)}{\|d(w_k,y_k)\|^2}$ that

$$\begin{aligned} \|v_k - u\|^2 &\leq \|w_k - u\|^2 - 2\gamma\beta_k\varphi(w_k, y_k) + \gamma^2\beta_k^2\|d(w_k, y_k)\|^2 \\ &= \|w_k - u\|^2 - \gamma(2-\gamma)\beta_k\varphi(w_k, y_k). \end{aligned} \tag{3.15}$$

Using (3.2) and $v_k = w_k - \gamma\beta_k d(w_k, y_k)$, we have

$$\beta_k\varphi(w_k, y_k) = \|\beta_k d(w_k, y_k)\|^2 = \frac{1}{\gamma^2}\|v_k - w_k\|^2. \tag{3.16}$$

Therefore (3.3) follows from (3.15) and (3.16) directly.

(ii). Note that

$$\varphi(w_k, y_k) \overset{(3.16)}{=} \frac{1}{\beta_k\gamma^2}\|v_k - w_k\|^2 \overset{(3.7)}{\leq} \frac{1+\mu^2}{(1-\mu)\gamma^2}\|v_k - w_k\|^2.$$

(3.5) yields

$$\|w_k - y_k\|^2 \leq \frac{1}{1-\mu}\varphi(w_k, y_k) \leq \frac{1+\mu^2}{(1-\mu)\gamma^2}\|v_k - w_k\|^2.$$

This completes the proof. □

Theorem 3.1. *Assume that F is a monotone mapping on H. Let $\{\alpha_k\} \subset [c,d] \in (0,1)$. Then the sequence $\{x_k\}_{k\in N}$ generated by (3.1) converges weakly to a solution $u \in SOL(C,S,F)$.*

Proof. Fixing $u \in SOL(C,S,F)$, we have $u \in Fix(S)$ and $u \in SOL(C,F)$. From (2.1), we have

$$\begin{aligned} \|w_k - u\|^2 &= \|(1+\alpha_k)(x_k - u) - \alpha_k(x_{k-1} - u)\|^2 \\ &= (1+\alpha_k)\|x_k - u\|^2 - \alpha_k\|x_{k-1} - u\|^2 + \alpha_k(1+\alpha_k)\|x_k - x_{k-1}\|^2. \end{aligned} \tag{3.17}$$

We also have

$$\begin{aligned}
\|x_{k+1}-u\|^2 &= \|(1-\theta_k)(w_k-u)+\theta_k(S(v_k)-u)\|^2 \\
&= (1-\theta_k)\|w_k-u\|^2+\theta_k\|S(v_k)-u\|^2-\theta_k(1-\theta_k)\|w_k-S(v_k)\|^2 \\
&\le (1-\theta_k)\|w_k-u\|^2+\theta_k\|v_k-u\|^2-\frac{1-\theta_k}{\theta_k}\|x_{k+1}-w_k\|^2 \\
&\le \|w_k-u\|^2-\theta_k\frac{2-\gamma}{\gamma}\|w_k-v_k\|^2-\frac{1-\theta_k}{\theta_k}\|x_{k+1}-w_k\|^2 \\
&\le \|w_k-u\|^2-\frac{1-\theta_k}{\theta_k}\|x_{k+1}-w_k\|^2.
\end{aligned} \tag{3.18}$$

From $\theta_k \le \frac{1}{2}$, we obtain $\frac{1-\theta_k}{\theta_k} \ge 1$. It follows that

$$\|x_{k+1}-u\|^2 \le \|w_k-u\|^2-\|x_{k+1}-w_k\|^2. \tag{3.19}$$

On the other hand, we have

$$\begin{aligned}
\|x_{k+1}-w_k\|^2 &= \|x_{k+1}-x_k-\alpha_k(x_k-x_{k-1})\|^2 \\
&= \|x_{k+1}-x_k\|^2+\alpha_k{}^2\|x_k-x_{k-1}\|^2-2\alpha_k\langle x_{k+1}-x_k, x_k-x_{k-1}\rangle \\
&\ge \|x_{k+1}-x_k\|^2+\alpha_k{}^2\|x_k-x_{k-1}\|^2-2\alpha_k\|x_{k+1}-x_k\|\,\|x_k-x_{k-1}\| \\
&\ge (1-\alpha_k)\|x_{k+1}-x_k\|^2-\alpha_k(1-\alpha_k)\|x_k-x_{k-1}\|^2.
\end{aligned} \tag{3.20}$$

Combining (3.19), (3.17) with (3.20), we obtain

$$\begin{aligned}
\|x_{k+1}-u\|^2 &\le (1+\alpha_k)\|x_k-u\|^2-\alpha_k\|x_{k-1}-u\|^2+\alpha_k(1+\alpha_k)\|x_k-x_{k-1}\|^2 \\
&\quad -(1-\alpha_k)\|x_{k+1}-x_k\|^2-\alpha_k(\alpha_k-1)\|x_k-x_{k-1}\|^2 \\
&= (1+\alpha_k)\|x_k-u\|^2-\alpha_k\|x_{k-1}-u\|^2-(1-\alpha_k)\|x_{k+1}-x_k\|^2+2\alpha_k\|x_k-x_{k-1}\|^2 \\
&\le (1+\alpha_{k+1})\|x_k-u\|^2-\alpha_k\|x_{k-1}-u\|^2-(1-\alpha_k)\|x_{k+1}-x_k\|^2+2\alpha_k\|x_k-x_{k-1}\|^2,
\end{aligned}$$

where the last inequality follows from the fact that $\{\alpha_k\}$ is nondecreasing. Therefore,

$$\begin{aligned}
&\|x_{k+1}-u\|^2-\alpha_{k+1}\|x_k-u\|^2+2\alpha_{k+1}\|x_{k+1}-x_k\|^2 \\
&\le \|x_k-u\|^2-\alpha_k\|x_{k-1}-u\|^2+2\alpha_k\|x_k-x_{k-1}\|^2 \\
&\quad +2\alpha_{k+1}\|x_{k+1}-x_k\|^2-(1-\alpha_k)\|x_{k+1}-x_k\|^2
\end{aligned}$$

Set $\Gamma_k := \|x_k-u\|^2-\alpha_k\|x_{k-1}-u\|^2+2\alpha_k\|x_k-x_{k-1}\|^2$. Then

$$\Gamma_{k+1}-\Gamma_k \le -(1-\alpha_k-2\alpha_{k+1})\|x_k-x_{k-1}\|^2.$$

It follows from $\alpha_k \le \alpha < \frac{1}{3}$ and the fact that $\{\alpha_k\}$ is nondecreasing that

$$\Gamma_{k+1}-\Gamma_k \le -(1-3\alpha_k)\|x_k-x_{k-1}\|^2 \le -(1-3\alpha)\|x_k-x_{k-1}\|^2 \le 0, \tag{3.21}$$

This implies that $\{\Gamma_k\}$ is nonincreasing. On the other hand, we have

$$\begin{aligned}\Gamma_k &= \|x_k-u\|^2-\alpha_k\|x_{k-1}-u\|^2+2\alpha_k\|x_k-x_{k-1}\|^2\\ &\geq \|x_k-u\|^2-\alpha_k\|x_{k-1}-u\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_k-u\|^2 &\leq \alpha_k\|x_{k-1}-u\|^2+\Gamma_k\\ &\leq \alpha\|x_{k-1}-u\|^2+\Gamma_1\\ &\vdots\\ &\leq \alpha^k\|x_0-u\|^2+\Gamma_1(\alpha^{k-1}+...+1)\\ &\leq \alpha^k\|x_0-u\|^2+\frac{\Gamma_1}{1-\alpha}. \end{aligned} \tag{3.22}$$

In addition, we also have

$$\begin{aligned}\Gamma_{k+1} &= \|x_{k+1}-u\|^2-\alpha_{k+1}\|x_k-u\|^2+2\alpha_{k+1}\|x_{k+1}-x_k\|^2\\ &\geq -\alpha_{k+1}\|x_k-u\|^2.\end{aligned} \tag{3.23}$$

From (3.22) and (3.23), we obtain

$$-\Gamma_{k+1}\leq\alpha_{k+1}\|x_k-u\|^2\leq\alpha\|x_k-u\|^2\leq\alpha^{k+1}\|x_0-u\|^2+\frac{\alpha\Gamma_1}{1-\alpha}.$$

It follows from (3.21) that

$$\begin{aligned}(1-3\alpha)\sum_{k=1}^{n}\|x_{k+1}-x_k\|^2 &\leq \Gamma_1-\Gamma_{k+2}\leq\alpha^{k+2}\|x_0-u\|^2+\frac{\Gamma_1}{1-\alpha}\\ &\leq \|x_0-u\|^2+\frac{\Gamma_1}{1-\alpha},\end{aligned}$$

which shows that

$$\sum_{k=1}^{\infty}\|x_{k+1}-x_k\|^2<+\infty. \tag{3.24}$$

So, $\lim\limits_{k\to+\infty}\|x_{k+1}-x_k\|=0$. Taking into account that

$$\begin{aligned}\|x_{k+1}-w_k\| &= \|x_{k+1}-x_k-\alpha_k(x_k-x_{k-1})\|\\ &\leq \|x_{k+1}-x_k\|+\alpha\|x_k-x_{k-1}\|,\end{aligned} \tag{3.25}$$

one has

$$\lim_{k\to+\infty}\|x_{k+1}-w_k\|=0. \tag{3.26}$$

Using (3.18) yields that

$$\|x_{k+1}-u\|^2\leq\|w_k-u\|^2-\theta_k\frac{2-\gamma}{\gamma}\|w_k-v_k\|^2-\frac{1-\theta_k}{\theta_k}\|x_{k+1}-w_k\|^2.$$

From $\theta \leq \theta_k \leq \frac{1}{2}$, we obtain $\frac{1-\theta_k}{\theta_k} \geq 1$. So,

$$(\|x_{k+1}-w_k\|-\|w_k-u\|)^2 \leq \|w_k-u\|^2-\theta\frac{2-\gamma}{\gamma}\|w_k-v_k\|^2-\|x_{k+1}-w_k\|^2$$

and

$$\begin{aligned}\theta\frac{2-\gamma}{\gamma}\|w_k-v_k\|^2 &\leq 2\|x_{k+1}-w_k\|\|w_k-u\|-2\|x_{k+1}-w_k\|^2\\ &\leq 2\|x_{k+1}-w_k\|\|w_k-u\|.\end{aligned}$$

From (3.26), one has

$$\lim_{k\to+\infty}\|v_k-w_k\|=0. \tag{3.27}$$

From (3.4), one has

$$\lim_{k\to+\infty}\|y_k-w_k\|=0. \tag{3.28}$$

Since $\theta_k \geq \theta$, it follows from $v_k := w_k-\gamma\beta_k d(w_k,y_k)$ that

$$\|w_k-S(v_k)\|=\frac{1}{\theta_k}\|x_{k+1}-w_k\| \leq \frac{1}{\theta}\|x_{k+1}-w_k\|,$$

which together with (3.26) implies that

$$\lim_{k\to+\infty}\|w_k-S(v_k)\|=0. \tag{3.29}$$

Combing (3.27) and (3.29), we have

$$\|S(v_k)-v_k\| \leq \|S(v_k)-w_k\|+\|w_k-v_k\| \to 0. \tag{3.30}$$

Now we show that $\{x_k\}$ converges weakly to an element of $SOL(C,S,F)$. Due to the boundedness of $\{x_k\}$, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ which converges weakly to $\hat{x}$. By (3.26), we get $w_{k_i} \rightharpoonup \hat{x}$ and by (3.28) $y_{k_i} \rightharpoonup \hat{x}$. It follows from (3.30) that $x_k - S(x_k) \to 0$. Then $\hat{x} = Fix(S)$. Now, we only need to show that $\hat{x} \in SOL(C,F)$. Define a mapping A by

$$A(v)=\begin{cases}F(v)+N_C(v), & \text{if } v\in C,\\ \emptyset, & \text{if } v\notin C.\end{cases}$$

where $N_C(v)$ is the normal cone of C at $v \in C$, i.e.,

$$N_C(v) := \{d \in H|\langle d, y-v\rangle \leq 0, \forall y \in C\}.$$

It is known that A is a maximal monotone and $A^{-1}(0) = SOL(C,F)$. If $(v,w) \in G(A)$, then we have $w-F(v) \in N_C(v)$ since $w \in A(v) = F(v)+N_C(v)$. Thus it follows that $\langle w-F(v), v-y\rangle \geq 0$, for all $y \in C$. Since $y_{k_i} \in C$, we have

$$\langle w-F(v), v-y_{ki}\rangle \geq 0.$$

On the other hand, by the definition of $\{y_k\}$ and Lemma 2.3, we conclude that

$$\langle w_k - b_k F(w_k) - y_k, y_k - v\rangle \geq 0,$$

which further implies that

$$\langle \frac{y_k - w_k}{b_k} + F(w_k), v - y_k \rangle \geq 0.$$

Hence

$$\begin{aligned}\langle w, v - y_{ki} \rangle &\geq \langle F(v), v - y_{ki} \rangle \\ &\geq \langle F(v), v - y_{ki} \rangle - \left\langle \frac{y_k - w_k}{b_k} + F(w_k), v - y_k \right\rangle \\ &= \langle F(v) - F(y_{ki}), v - y_{ki} \rangle + \langle F(y_{ki}) - F(w_{ki}), v - y_{ki} \rangle - \left\langle \frac{y_{ki} - w_{ki}}{b_k}, v - y_k \right\rangle \\ &\geq \langle F(y_{ki}) - F(w_{ki}), v - y_{ki} \rangle - \left\langle \frac{y_{ki} - w_{ki}}{b_k}, v - y_k \right\rangle,\end{aligned}$$

which implies

$$\langle w, v - y_{ki} \rangle \geq \langle F(y_{ki}) - F(w_{ki}), v - y_{ki} \rangle - \left\langle \frac{y_{ki} - w_{ki}}{b_k}, v - y_k \right\rangle.$$

Taking the limit as $i \to \infty$ in the above inequality, we obtain $\langle w, v - \hat{x} \rangle \geq 0$. Since A is a maximal monotone operator, it follows that $\hat{x} \in A^{-1}(0) = SOL(C, F)$. This completes the proof. □

4. Numerical experiments

In this section, we present numerical experiments in support of the convergence of the proposed algorithm. In order to evaluate the performance of the proposed algorithm, we provide an example to compare the proposed algorithm with the inertial projection and contraction algorithm. We take $\theta_k = \frac{1}{2}, \gamma = 1.5, \alpha = 0.2$, for all the algorithms.

Example 4.1. Let $F : R^2 \to R^2$ be defined by

$$F(x, y) = (2x + 2y + \sin(x), -2x + 2y + \sin(y)), \quad \forall x, y \in R.$$

Note that

$$\begin{aligned}\langle F(z_1) - F(z_2), z_1 - z_2 \rangle &= [(2x_1 + 2y_1 + \sin(x_1)) - (2x_2 + 2y_2 + \sin(x_2))](x_1 - x_2) \\ &+ [(-2x_1 + 2y_1 + \sin(y_1)) - (-2x_2 + 2y_2 + \sin(y_2))]^2 \\ &= 2(x_1 - x_2)^2 + (\sin(x_1) - \sin(x_2))(x_1 - x_2) \\ &+ 2(y_1 - y_2)^2 + (\sin(y_1) - \sin(y_2))(y_1 - y_2) \geq \|z_1 - z_2\|^2.\end{aligned}$$

Let $C = \{x \in R^2 | e_0 \leq x \leq 10e_1\}$, where $e_0 = (-10, -10)$ and $e_1 = (10, 10)$. Take the initial point $x_0 = (1, 10) \in R^2$. Since $(0, 0)$ is the unique solution of variational inequality (1.1). Denote the stopping criterion by $\| x_k \| \leq 10^{-8}$.

The numerical results for this example are shown in Fig.1, Fig.3 and Fig.5, which compare iteration numbers, average iteration numbers, and runtime of the proposed algorithm presented in this paper with the previous algorithm. We can see that the proposed algorithm in this paper is better.

Example 4.2. Let $F : R^n \to R^n$ be defined by $F(x) = Ax + b$, where $A = Z^T Z, Z = (z_{ij})_{n \times n}$ and $b = (b_i) \in R^n$, where $z_{ij} \in [1, 100]$ and $b \in [-100, 0]$ are generated randomly. Since the randomly generated

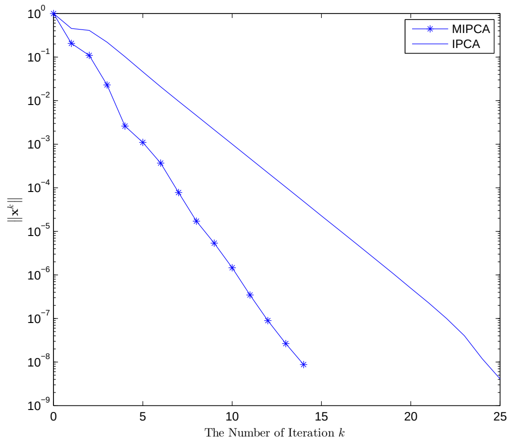

FIGURE 1. Iteration number of MIPCA and IPCA with $\theta_k = \frac{1}{2}$, $\gamma = 1.5$, $\alpha = 0.2$

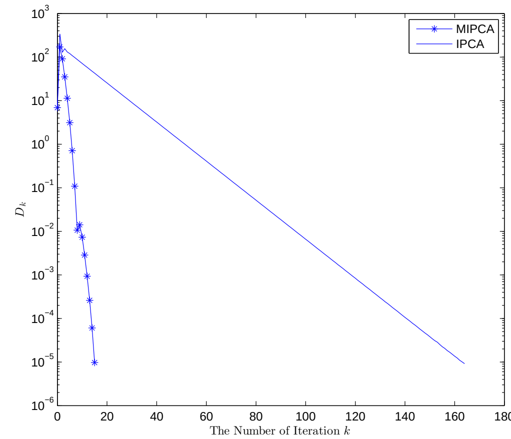

FIGURE 2. Iteration number of MIPCA and IPCA with $\theta_k = \frac{1}{2}$, $\gamma = 1.5$, $\alpha = 0.2$

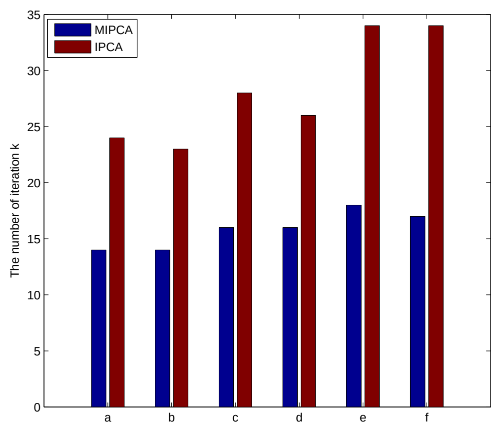

FIGURE 3. Average iteration numbers of MIPCA and IPCA for Example 4.1

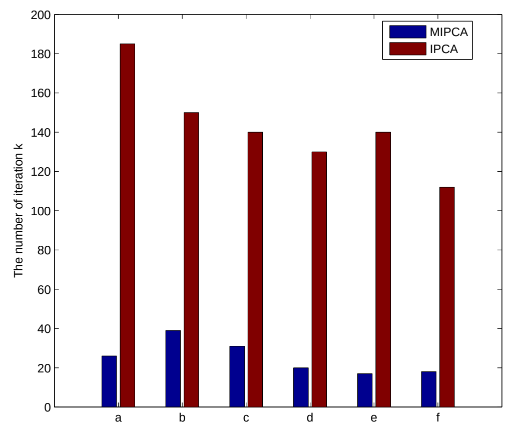

FIGURE 4. Average iteration numbers of MIPCA and IPCA for Example 4.2

Z is a non-singular matrix, matrix A must be a positive definite matrix. Since A is positive definite and symmetry, we have

$$\begin{aligned} \langle F(x_1) - F(x_2), x_1 - x_2 \rangle &= \langle A(x_1 - x_2), x_1 - x_2 \rangle \\ &= (x_1 - x_2)^T A(x_1 - x_2) \geq \min(eig(A)) \|x_1 - x_2\|^2. \end{aligned}$$

We obtain that F is η-strongly monotone with $\eta = \min(eig(A))$. In this example, set $C = \{x \in R^n | \|x - d\| \leq r\}, d \in R^n$, where r is a randomly chosen radius. Take the initial point $x_0 = (c_i) \in R^n$, and $c_i \in [0,1]$ is chosen randomly. Set $n = 100$ and denote the stopping criterion by $D_k = \|x_{k+1} - x_k\| \leq 10^{-5}$.

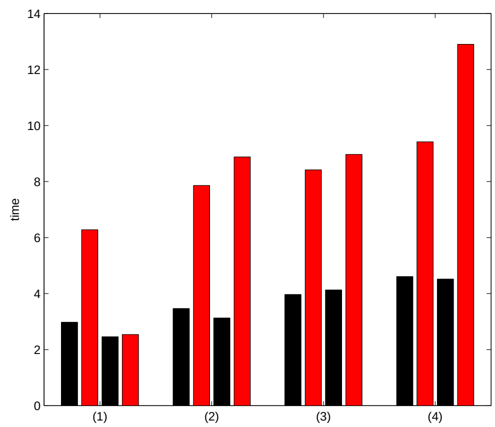

FIGURE 5. Average runtime of MIPCA and IPCA for Example 4.1

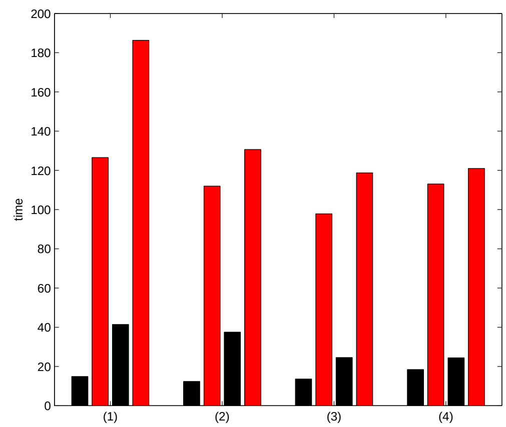

FIGURE 6. Average runtime of MIPCA and IPCA for Example 4

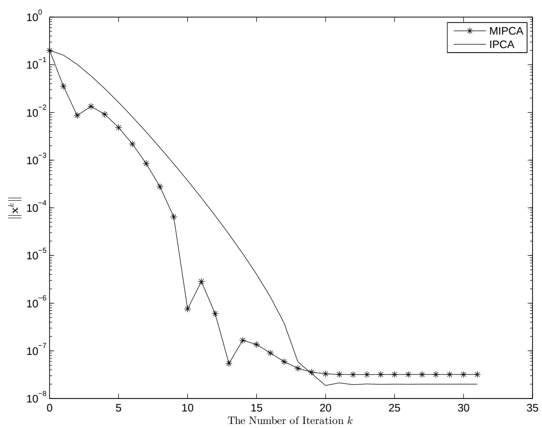

FIGURE 7. Iteration number of MIPCA and IPCA with $\theta_k = \frac{1}{2}$, $\gamma = 1.23$, $\alpha = 0.3$ for Example 4.3

The numerical results for Example are shown in Fig.2, Fig.4 and Fig.6. Fig.2 and Fig.4 compare the iteration number and the average iteration numbers of the proposed algorithm presented in this paper with the previous algorithm. We can see that the modified inertial projection and contraction algorithms converges faster. Fig.6 compares the average runtime of the modified inertial projection and contraction algorithms with the previous algorithm. We can see that the proposed algorithm in this paper is better.

In Fig. 3 and Fig. 4 (a), (c) and (e), we select the same initial point (where each initial point is the average value of the selected initial points), but $\gamma = 1.5, \alpha = 0.2; \gamma = 1.5, \alpha = 0.25; \gamma = 1.5, \alpha = 0.3$; respectively. In Figure 3 and Figure 4 (b), (d) and (f), we chose another initial point, and $\gamma = 1, \alpha = 0.2; \gamma = 1, \alpha = 0.25; \gamma = 1, \alpha = 0.3$.

In Figure 5 and Figure 6, we select four different initial points in the four groups, the black bar graph in (1), (2), (3), (4) represents the average time of MIPCA, where $\gamma = 1$ and 1.5 , respectively. The red bar graph in (1), (2), (3), (4) represents the average time of IPCA, where $\gamma = 1$ and 1.5.

Example 4.3. Let $F : R \to R$ be defined by $F(x) = x^3 + 2x$. Now, we show that F is strongly monotone. Arbitrarily taking $x, y \in C$, $C \subset [0,2]$, we have

$$\langle F(x) - F(y), x - y\rangle = (x-y)^2(x^2 + xy + y^2 + 2) \geq (x-y)^2 \geq 0.$$

The numerical results for this example are shown in Fig.7. From the figure we can see that our algorithm converges faster. We conclude from Fig.1, Fig.2 and Fig.7 that the modified inertial projection and contraction algorithms have less iterations, faster convergence, and the numerical results of three examples show that the proposed algorithm performs better .

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