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# HYBRID ITERATIVE METHODS FOR MULTIPLE SETS SPLIT FEASIBILITY PROBLEMS

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**Abstract.** Based on viscosity techniques, we propose two hybrid iterative methods for a multiple-sets split feasibility problem. Under appropriate conditions, we establish two strong convergence theorems in a Hilbert spaces.

**Keywords.** Multiple-sets split feasibility problems; Split feasibility problem; Iterative method; Viscosity approximation method; Fixed point.

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## 1. INTRODUCTION

The split feasibility problem (SFP), which was proposed by Censor and Elfving [1], is to find

$$x \in C \text{ such that } Ax \in Q,$$

where $A : H_1 \to H_2$ be a bounded linear operator, $C$ and $Q$ are nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively. This problem arises in signal processing, image reconstruction, and many other important applied fields. A number of image reconstruction problems can be formulated as the SFP and many iterative algorithms have been introduced to solve the SFP; see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and references therein.

In [1], Censor and Elfving used multidistance ideas to study the SFP. Their algorithms involve matrix inverses at each iteration. In [2], Byrne presented a projection method called the CQ algorithm for solving the SFP that does not involve matrix inverses as follows:

$$x_{n+1} = P_C[I - \gamma A^*(I - P_Q)A]x_n, \ \forall\, x_0 \in H,$$

where $0 < \gamma < \frac{2}{\rho(A^*A)}$, $\rho(A^*A)$ is the spectral radius of $A^*A$.

In 2010, Xu [12] further studied the CQ algorithm and its convergence via fixed point methods. Xu [12], and Qin and Yao [10] proved that the problem is equivalent to a fixed point problem of the operator $P_C[I - \gamma A^*(I - P_Q)A]$. They proved that a point $x^*$ solves SFP if and only if $x^* = P_C[I - \gamma A^*(I - P_Q)A]$. Mann's iterative method have been applied to solve the SFP. However, Mann's method is only weakly

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convergent in an infinite dimensional space. Indeed, strong convergence is more important in many engineering fields. To obtain strong convergence theorems, Sitthithakerngkiet et al. [13] studied the following fixed point algorithm for the SFP

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) P_C [I - \gamma A^* (I - P_Q) A] x_n$$

where $f$ is a contraction on $H$ and $B$ is a strongly positive bounded linear self-adjoint operator on $H$ with coefficient $\overline{\gamma} > 0$, $\alpha_n \subset (0,1)$ is a slowly vanishing sequence and $\gamma > 0$ is a constant. Under appropriate conditions, they proved $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$, which is also the unique solution of some monotone variational inequality. As an extension of the split feasibility problem, the multiple-sets split feasibility problem (MSFP), which was recently introduced [14], is formulated as finding a point $x$ with the property:

$$x \in \bigcap_{i=1}^{N} C_i \text{ and } Ax \in \bigcap_{j=1}^{M} Q_j. \tag{1.1}$$

The MSSFP (1.1) with $N = M = 1$ is the split feasibility problem. The multiple-sets split feasibility problem arises in many practical fields, such as, image reconstruction, signal processing, intensity-modulated radiation therapy (IMRT) and so on. Recently, the MSFP received much attention and many researchers proposed fixed point algorithms for solving it; see, [11, 15, 16, 17, 18, 19] and references therein.

As a direct extension of the CQ algorithm, Wang and Xu [20] gave a cyclic algorithm to solve the MSSFP:

$$x_{n+1} = P_{C_{[n]}} [I - \gamma A^* (I - P_{Q_{[n]}}) A] x_n$$

where $[n] := n(mod p)$, (mod function take values in $\{1, 2, \cdots, p\}$). They showed that the sequence $\{x_n\}$ convergence weakly to a solution of then MSSFP whenever its solution set is nonempty.

In [21], Tang and Liu proposed simultaneous and cyclic iterative algorithms for solving a split common fixed point problem and applied their main results to the multiple-sets split feasibility problem. Up to our knowledge, many weak convergence theorems of solutions were established in Hilbert spaces and Banach spaces. In many subjects, strong convergence is more applicable.

Motivated by the above related results in this field, we propose two hybrid iterative method for solving the multiple-set split feasibility problem and establish two strong convergence theorems. Our solution also uniquely solve some monotone variational inequality. The rest of the paper is organized as follows. In Section 2, we give some basic definitions, propositions and lemmas. In Section 3, we presents our hybrid iterative methods to solve the MSSFP and obtain strong convergence theorems of solutions.

## 2. Preliminaries

Throughout the paper, let $H_1$ and $H_2$ be an infinite dimensional real Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. $\to$ and $\rightharpoonup$ denote the strong convergence and weak convergence, respectively. In addition, $F(T)$ and $\omega_w(x_n)$ denote the fixed point set of $T$ and the weak $\omega$-limit set of the sequence $\{x_n\}$, respectively, that is, $F(T) = \{x : Tx = x\}$ and $\omega_w(x_n) = \{u : \exists x_{n_j} \rightharpoonup u\}$. Below we gather some basic definitions and results which are needed in the subsequent section.

Recall that a mapping $T : H \to H$ is said to be $\rho$-Lipschitzian with $\rho > 0$ if

$$\|Tx - Ty\| \leq \rho \|x - y\|, \quad \forall x, y \in H.$$

If $0 < \rho < 1$, then $T$ is a $\rho$-contraction. If $\rho = 1$, then $T$ is a nonexpansive mapping.

Given a nonlinear mapping $F : C \to H$. Recall that $F$ is said to be monotone if

$$\langle x-y, Fx-Fy\rangle \geq 0, \quad \forall x,y \in C.$$

$B$ is said to be $\alpha$-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x-y, Fx-Fy\rangle \geq \alpha\|x-y\|^2, \quad \forall x,y \in C.$$

$B$ is said to be $\alpha$-inverse strongly monotone (for short, $\alpha$-ism) if there exists $\alpha > 0$ such that

$$\langle x-y, Fx-Fy\rangle \geq \alpha\|Fx-Fy\|^2, \quad \forall x,y \in C.$$

We can easily see that

(i) if $F$ is nonexpansive, then $I-F$ is monotone;

(ii) if $F$ is an $\alpha$-inverse-strongly monotone mapping, then it must be $\frac{1}{\alpha}$-Lipschitz. Moreover, $I-rF$ is nonexpansive when $0 < r \leq 2\alpha$.

Recall that $P_C$ is the metric projection from $H$ into $C$. Then for each point $x \in H$, the unique point $P_C x \in C$ satisfies the property:

$$\|x-P_C x\| = \inf_{y\in C}\|x-y\| =: d(x,C).$$

**Lemma 2.1.** *([22]) For a given $x \in H$, we have*

*(i) $z = P_C x$ if and only if $\langle x-z, z-y\rangle \geq 0, \forall y \in C$;*

*(ii) $z = P_C x$ if and only if $\|x-z\|^2 \leq \|x-y\|^2 - \|y-z\|^2$;*

*(iii) $\langle P_C x - P_C y, x-y\rangle \geq \|P_C x - P_C y\|^2, \forall x,y \in H$.*

It is obvious that $P_C$ is nonexpansive and monotone.

**Lemma 2.2.** *(Demiclosedness principle, [23]) Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in $C$ that converges weakly to $x$ and $\{(I-T)x_n\}$ converges strongly to $y$, then $(I-T)x = y$. In particular, if $y = 0$, then $x \in Fix(T)$.*

**Lemma 2.3.** *([24]) Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $X$. Let $S_1$ and $S_2$ be two nonexpansivemappings from $C$ into itself with a common fixed point. Define a mapping $S : C \to C$ by*

$$Sx = \delta S_1 x + (1-\delta)S_2 x, \ \forall x \in C,$$

*where $\delta$ is a constant in $(0,1)$. Then $S$ is nonexpansive and $F(S) = F(S_1)\bigcap F(S_2)$.*

Recall that $T : H \to H$ is said to be firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$\langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2, \quad x,y \in H.$$

Alternatively, $T$ is firmly nonexpansive if and only if $T$ can be expressed as

$$T = \frac{I+S}{2},$$

where $S : H \to H$ is nonexpansive.

**Definition 2.1.** A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity $I$ and a nonexpansive mapping, that is,

$$T = (1-\alpha)I + \alpha S, \tag{2.1}$$

where $\alpha \in (0,1)$ and $S : H \to H$ is nonexpansive. More precisely, if (2.1) holds, we say that $T$ is $\alpha$-averaged (for short, $\alpha$-av).

Clearly, a firmly nonexpansive mapping (in particular, a projection) is $\frac{1}{2}$-averaged.

**Proposition 2.1.** *(Basic properties of averaged mappings, [3]) Let $S, T$ and $V$ be mappings on $H$. Then*

*(i) if $T = (1-\alpha)S + \alpha V$, where $\alpha \in (0,1)$, $S$ is averaged and $V$ is nonexpansive, then $T$ is averaged;*

*(ii) $T$ is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive;*

*(iii) If $T = (1-\alpha)S + \alpha V$, where $\alpha \in (0,1)$, $S$ is firmly nonexpansive and $V$ is nonexpansive, then $T$ is averaged;*

*(iv) the composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \dots T_N$. In particular, if $T_1$ is $\alpha_1$-averaged and $T_2$ is $\alpha_2$-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite $T_1T_2$ is $\alpha$-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$;*

*(v) if $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then $\bigcap_{i=1}^N Fix(T_i) = Fix(T_1 \dots T_N)$;*

*(vi) if $T$ is $\alpha$-averaged, then*

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \frac{1-\alpha}{\alpha}\|Tx - x\|^2, \;\; x \in H, \; z \in Fix(T).$$

The following proposition summarizes some results on the relations between averaged mappings and inverse strongly monotone operators.

**Proposition 2.2.** *([3]) Let $T : H \to H$ be a mapping. Then*

*(i) $T$ is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism;*

*(ii) if $T$ is $\nu$-ism, then for $\gamma > 0$, $\gamma T$ is $\frac{\nu}{\gamma}$-ism;*

*(iii) $T$ is averaged if and only if the complement $I - T$ is $\nu$-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, $T$ is $\alpha$-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

**Lemma 2.4.** *([25]) Let $H_1$ and $H_2$ be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $A^*$ be the adjoint of $A$. Let $C$ be a nonempty closed convex subset of $H_2$, and let $G : H_2 \to H_2$ be a firmly nonexpansive mapping. Then $A^*(I-G)A$ is a $\frac{1}{\|A\|^2}$-ism, that is,*

$$\frac{1}{\|A\|^2}\|A^*(I-G)Ax - A^*(I-G)Ay\|^2 \leq \langle x - y, A^*(I-G)Ax - A^*(I-G)Ay\rangle$$

*for all $x, y \in H_1$.*

**Lemma 2.5.** *([26]) If $h : H \to R$ has an $L$-Lipschitz continuous gradient $\nabla h$, then $\nabla h$ is $\frac{1}{L}$-ism.*

We know that a linear bounded operator $A : H \to H$ is said to be strongly positive if and only if there exists $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2$ for all $x \in H$. We call such $A$ a strongly positive operator with coefficient $\overline{\gamma}$.

**Lemma 2.6.** ([27]) *Let $H$ be a Hilbert space and let $A$ be a strongly positive bounded linear operator on $H$ with coefficient $\overline{\gamma} > 0$. If $0 < \delta \leq \|A\|^{-1}$, then $\|I - \delta A\| \leq 1 - \delta\overline{\gamma}$.*

**Lemma 2.7.** ([28]) *Let $H$ be a Hilbert space. Let $f : H \to H$ be a $\rho$-Lipschitzian mapping and let $A : H \to H$ be a strongly positive bounded linear operator with coefficient $\delta > 0$. If $\mu\delta > \eta\rho$, then*

$$\langle(\mu A-\eta f)x-(\mu A-\eta f)y, x-y\rangle \geq (\mu\delta-\eta\rho)\|x-y\|^2, \quad x,y \in H,$$

*that is, $\mu A-\eta f$ is strongly monotone with coefficient $\mu\delta-\eta\rho$.*

**Lemma 2.8.** ([29]) *The following inequality holds in a Hilbert space $H$*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y \in H.$$

**Lemma 2.9.** ([31]) *Let $\{a_n\}$ be a sequence of non-negative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ with $a_{n_j} < a_{n_j+1}$ for all $j \in N$. Then, there exists a nondecreasing sequence $\{m_k\}$ of $N$ such that $\lim_{k\to\infty} m_k = \infty$, and the following properties are satisfied by all (sufficiently large) number $k \in N$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

Indeed, $m_k$ is the largest number $n$ in the set $\{1,2,\cdots,k\}$ such that $a_n < a_{n+1}$.

**Lemma 2.10.** *([32]) Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-b_n)a_n + c_n,$$

*where $b_n$ is a sequence in $(0,1)$ and $\{c_n\}$ is a sequence such that*

*(i)* $\sum_{n=1}^{\infty} b_n = \infty$;

*(ii)* $\limsup_{n\to\infty} \frac{c_n}{b_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |c_n| < \infty$.

*Then* $\lim_{n\to\infty} a_n = 0$.

## 3. Main results

**Lemma 3.1.** *Let $H$ be a Hilbert space. Let $f : H \to H$ be a $\rho$-Lipschitzian mapping and let $B : H \to H$ be a strongly positive bounded linear operator with coefficient $\delta > 0$. Let $C$ be nonempty, closed and convex subsets of $H$. If $\eta\delta > \gamma\rho$, then the following variational inequality*

$$\langle(\eta B-\gamma f)\hat{x}, \hat{x}-x\rangle \leq 0, \quad \forall x \in C. \tag{3.1}$$

*has a unique solution. Equivalently, $\hat{x} = P_C(I-\eta B+\gamma f)\hat{x}$.*

*Proof.* We show it by contradiction. Suppose that $\hat{x} \in C$ and $\tilde{x} \in C$ are two solution of (3.1) with $\hat{x} \neq \tilde{x}$. Then

$$\langle(\eta B-\gamma f)\hat{x}, \hat{x}-\tilde{x}\rangle \leq 0$$

and

$$\langle(\eta B-\gamma f)\tilde{x}, \tilde{x}-\hat{x}\rangle \leq 0.$$

It follows that

$$\langle(\eta B-\gamma f)\hat{x}-(\eta B-\gamma f)\tilde{x}, \hat{x}-\tilde{x}\rangle \leq 0.$$

From $\eta\delta > \gamma\rho$ and Lemma 2.7, we obtain

$$\langle(\eta B-\gamma f)\hat{x}-(\eta B-\gamma f)\tilde{x}, \hat{x}-\tilde{x}\rangle \geq (\eta\delta-\gamma\rho)\|\hat{x}-\tilde{x}\|^2 \geq 0.$$

This leads to a contradiction. Hence, variational inequality (3.1) has a unique solution and denote it by $\hat{x} \in C$. Since

$$\langle(\eta B-\gamma f)\hat{x},\hat{x}-x\rangle \leq 0 \Leftrightarrow \langle \hat{x}-(I-\eta B+\gamma f)\hat{x},\hat{x}-x\rangle \leq 0, \quad \forall x \in C,$$

we can obtain from Lemma 2.1 that $\hat{x} = P_C(I-\eta B+\gamma f)\hat{x}$. □

**Theorem 3.1.** *Let $H_1$ and $H_2$ be two real Hilbert space. Let $C_i$ and $Q_i$ be nonempty, closed and convex subsets of $H_1$ and $H_2$, respectively, for each $1 \leq i \leq N$. Let $A : H_1 \to H_2$ be a bounded linear operator and let $A^* : H_2 \to H_1$ be the adjoint of $A$. Suppose that MSSFP (1.1) has a nonempty solution set $\Gamma$. Let $f : H_1 \to H_1$ be a Lipschitzian mapping with coefficient $\rho \geq 0$. Let $B : H_1 \to H_1$ be a strongly positive bounded linear operator with coefficient $\delta > 0$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_1 \in H_1, \\ y_n = \sum_{i=1}^N \mu_{n,i} P_{C_i}(x_n - \xi_i A^*(I-P_{Q_i})Ax_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n \eta B)y_n. \end{cases} \tag{3.2}$$

*Assume that the following conditions are satisfied:*

*(i)* $\eta\delta > \gamma\rho$;
*(ii)* $0 < \xi_i < \frac{2}{\|A\|^2}$, $\forall i \in \{1,\cdots,N\}$;
*(iii)* $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
*(iv)* $\sum_{i=1}^N \mu_{n,i} = 1$, $\sum_{n=0}^\infty \sum_{i=1}^N |\mu_{n,i} - \mu_{(n-1),i}| < \infty$ *and* $\mu_{n,i} > 0$, *for* $i \in \{1,\cdots,N\}$.

*Then $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$, which is the unique solution of the following variational inequality*

$$\langle(\eta B-\gamma f)x^*,x^*-x\rangle \leq 0, \ \ \forall x \in \Gamma. \tag{3.3}$$

*Equivalently, $x^* = P_\Gamma(I-\eta B+\gamma f)x^*$.*

*Proof.* The proof is split into five steps.

Step 1. We show that $\{x_n\}$ is bounded.

Let $y_n = \sum_{i=1}^N \mu_{n,i} S_i x_n = T_n x_n$, where

$$S_i = P_{C_i}(I-\xi_i A^*(I-P_{Q_i})A).$$

For nay $p \in \Gamma$, we have $S_i p = p$, $\forall i \in \{1,\cdots,N\}$ and $T_n p = p$. Since $P_{Q_i}$ is firmly nonexpansive, we have from Lemma 2.4 that $A^*(I-P_{Q_i})A$ is $\frac{1}{\|A\|^2}$-ism. From $0 < \xi_i < \frac{2}{\|A\|^2}$, we have that $I-\xi_i A^*(I-P_{Q_i})A$ is nonexpansive. Then $S_i$ is nonexpansive. Obviously, $T_n$ is also nonexpansive. Then

$$\|y_n - p\| = \|T_n x_n - T_n p\| \leq \|x_n - p\|.$$

From the condition $\lim_{n\to\infty}\alpha_n = 0$, we may assume that, without loss of generality, $\alpha_n < \frac{1}{\eta\|B\|}$ for all $n$. It follows from Lemma 2.6 that

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+(I-\alpha_n\eta B)y_n-p\| \\
&= \|\alpha_n(\gamma f(x_n)-\eta Bp)+(I-\alpha_n\eta B)(y_n-p)\| \\
&\leq \alpha_n\|\gamma f(x_n)-\eta Bp\|+\|I-\alpha_n\eta B\|\|y_n-p\| \\
&= \alpha_n\|\gamma f(x_n)-\gamma f(p)+\gamma f(p)-\eta Bp\|+\|1-\alpha_n\eta B\|\|y_n-p\| \\
&\leq \alpha_n\gamma\rho\|x_n-p\|+\alpha_n\|\gamma f(p)-\eta Bp\|+(1-\alpha_n\eta\delta)\|x_n-p\| \\
&= [1-\alpha_n(\eta\delta-\gamma\rho)]\|x_n-p\|+\alpha_n\|\gamma f(p)-\eta Bp\| \\
&= [1-\alpha_n(\eta\delta-\gamma\rho)]\|x_n-p\|+\alpha_n(\eta\delta-\gamma\rho)\frac{\|\gamma f(p)-\eta Bp\|}{\eta\delta-\gamma\rho} \\
&\leq \max\{\|x_n-p\|,\frac{\|\gamma f(p)-\eta Bp\|}{\eta\delta-\gamma\rho}\} \\
&\leq \cdots \\
&\leq \max\{\|x_0-p\|,\frac{\|\gamma f(p)-\eta Bp\|}{\eta\delta-\gamma\rho}\}.
\end{aligned} \tag{3.4}$$

Therefore, $\{x_n\}$ is bounded. $\{y_n\}$, $\{f(x_n)\}$, $\{S_ix_n\}$, $\{T_nx_n\}$ and $\{BT_nx_n\}$ are also bounded.

Step 2. We show that $\|x_n-T_nx_n\|\to 0$ as $n\to\infty$.

Observe that

$$\|x_{n+1}-T_nx_n\| = \alpha_n\|\gamma f(x_n)-\eta BT_nx_n\|\to 0\ (as\ n\to\infty).$$

So, we only need to prove $\|x_{n+1}-x_n\|\to 0$. Indeed,

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n\gamma f(x_n)+(I-\alpha_n\eta B)y_n-\alpha_{n-1}\gamma f(x_{n-1})-(I-\alpha_{n-1}\eta B)y_{n-1}\| \\
&= \|\alpha_n\gamma f(x_n)-\alpha_n\gamma f(x_{n-1})+\alpha_n\gamma f(x_{n-1})-\alpha_{n-1}\gamma f(x_{n-1})+(I-\alpha_n\eta B)T_nx_n \\
&\quad -(I-\alpha_n\eta B)T_{n-1}x_{n-1}+(I-\alpha_n\eta B)T_{n-1}x_{n-1}-(I-\alpha_{n-1}\eta B)T_{n-1}x_{n-1}\| \\
&\leq \alpha_n\gamma\|f(x_n)-f(x_{n-1})\|+|\alpha_n-\alpha_{n-1}|\|\gamma f(x_{n-1})\| \\
&\quad +\|I-\alpha_n\eta B\|\|T_nx_n-T_{n-1}x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\eta BT_{n-1}x_{n-1}\| \\
&\leq \alpha_n\gamma\rho\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\gamma f(x_{n-1})\| \\
&\quad +(1-\alpha_n\eta\delta)\|T_nx_n-T_{n-1}x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\eta BT_{n-1}x_{n-1}\| \\
&\leq \alpha_n\gamma\rho\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\gamma f(x_{n-1})\| \\
&\quad +(1-\alpha_n\eta\delta)(\|T_nx_n-T_nx_{n-1}\|+\|T_nx_{n-1}-T_{n-1}x_{n-1}\|) \\
&\quad +|\alpha_n-\alpha_{n-1}|\|\eta BT_{n-1}x_{n-1}\| \\
&\leq \alpha_n\gamma\rho\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|\gamma f(x_{n-1})\| \\
&\quad +(1-\alpha_n\eta\delta)(\|x_n-x_{n-1}\|+\|\sum_{i=1}^{N}\mu_{n,i}S_ix_{n-1}-\sum_{i=1}^{N}\mu_{(n-1),i}S_ix_{n-1}\|) \\
&\quad +|\alpha_n-\alpha_{n-1}|\|\eta BT_{n-1}x_{n-1}\|
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_n\gamma\rho\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\gamma f(x_{n-1})\| \\
&\quad + (1-\alpha_n\eta\delta)(\|x_n - x_{n-1}\| + \sum_{i=1}^{N}|\mu_{n,i} - \mu_{(n-1),i}|\|S_i x_{n-1}\|) \\
&\quad + |\alpha_n - \alpha_{n-1}|\|\eta B T_{n-1}x_{n-1}\| \\
&= [1-\alpha_n(\eta\delta - \gamma\rho)]\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|\gamma f(x_{n-1})\| + \|\eta B T_{n-1}x_{n-1}\|) \\
&\quad + (1-\alpha_n\eta\delta)\sum_{i=1}^{N}|\mu_{n,i} - \mu_{(n-1),i}|\|S_i x_{n-1}\| \\
&\le [1-\alpha_n(\eta\delta - \gamma\rho)]\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|\gamma f(x_{n-1})\| + \|\eta B T_{n-1}x_{n-1}\|) \\
&\quad + \sum_{i=1}^{N}|\mu_{n,i} - \mu_{(n-1),i}|\|S_i x_{n-1}\|.
\end{aligned}$$

Let

$$M_1 = \sup_{n,i}\{\|\gamma f(x_{n-1})\| + \|\eta B T_{n-1}x_{n-1}\|, \|S_i x_{n-1}\|\}.$$

It follows that

$$\|x_{n+1} - x_n\| \le [1-\alpha_n(\eta\delta - \gamma\rho)]\|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + \sum_{i=1}^{N}|\mu_{n,i} - \mu_{(n-1),i}|)M_1.$$

From the conditions $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\sum_{n=0}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=0}^{\infty}\sum_{i=1}^{N}|\mu_{n,i} - \mu_{(n-1),i}| < \infty$ and Lemma 2.10, we obtain

$$\|x_{n+1} - x_n\| \to 0 \quad (as\ n \to \infty).$$

Therefore, $\|x_n - T_n x_n\| \to 0$, as $n \to \infty$.

Step 3. We show that $\omega_\omega(x_n) \subseteq \Gamma$.

To see this, we take $q \in \omega_\omega(x_n)$ and assume that $x_{n_l} \rightharpoonup q$ as $l \to \infty$ for some subsequence $\{x_{n_l}\}$ of $\{x_n\}$. We know that $T_n = \sum_{i=1}^{N}\mu_{n,i}S_i$. From the conditions $\mu_{n,i} > 0$ and $\sum_{i=1}^{N}\mu_{n,i} = 1$, for $\forall i \in \{1, \cdots, N\}$, we may assume, with no loss of generality, that

$$\mu_{n_l,i} \to \mu_i \ \ (as\ l \to \infty),\ \forall 1 \le i \le N.$$

It is obvious that each $\mu_i > 0$ and $\sum_{i=1}^{N}\mu_i = 1$. And we also have

$$T_{n_l}x \to Tx, \quad \forall x \in H_1, \quad (as\ l \to \infty),$$

where

$$T = \sum_{i=1}^{N}\mu_i S_i.$$

By using Lemma 2.3, we have that $T$ is nonexpansive and $F(T) = \bigcap_{i=1}^{N}F(S_i) = \Gamma$. It follows that

$$\begin{aligned}
\|x_{n_l} - Tx_{n_l}\| &\le \|x_{n_l} - T_{n_l}x_{n_l}\| + \|T_{n_l}x_{n_l} - Tx_{n_l}\| \\
&\le \|x_{n_l} - T_{n_l}x_{n_l}\| + \sum_{i=1}^{N}|\mu_{n_l,i} - \mu_i|\|S_i x_{n_l}\| \\
&\to 0\ (l \to \infty).
\end{aligned}$$

It follows from Lemma 2.3 that $q \in F(T) = \Gamma$, that is, $\omega_\omega(x_n) \subseteq \Gamma$.

Step 4. We show that

$$\limsup_{n\to\infty}\langle(\eta B-\gamma f)x^*,x^*-x_n\rangle \le 0. \tag{3.5}$$

where $x^*$ is the unique solution of variational inequality (3.1).

Indeed, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\eta B-\gamma f)x^*,x^*-x_n\rangle = \lim_{j\to\infty}\langle(\eta B-\gamma f)x^*,x^*-x_{n_j}\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_j} \rightharpoonup \bar{x} \in \Gamma$. Then

$$\limsup_{n\to\infty}\langle(\eta B-\gamma f)x^*,x^*-x_n\rangle = \langle(\eta B-\gamma f)x^*,x^*-\bar{x}\rangle \le 0.$$

Step 5. We show that $x_n \to x^*$ $(n\to\infty)$.

From Lemma 2.6 and Lemma 2.8, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n\gamma f(x_n)+(I-\alpha_n\eta B)y_n-x^*\|^2\\
&= \|\alpha_n(\gamma f(x_n)-\eta Bx^*)+(I-\alpha_n\eta B)(y_n-x^*)\|^2\\
&\le \|(I-\alpha_n\eta B)(y_n-x^*)\|^2+2\alpha_n\langle\gamma f(x_n)-\eta Bx^*,x_{n+1}-x^*\rangle\\
&\le (1-\alpha_n\eta\delta)^2\|y_n-x^*\|^2+2\alpha_n\langle\gamma f(x_n)-\eta Bx^*,x_{n+1}-x^*\rangle\\
&= (1-\alpha_n\eta\delta)^2\|y_n-x^*\|^2+2\alpha_n\langle\gamma f(x_n)-\gamma f(x^*),x_{n+1}-x^*\rangle\\
&\quad +2\alpha_n\langle\gamma f(x^*)-\eta Bx^*,x_{n+1}-x^*\rangle\\
&\le (1-\alpha_n\eta\delta)^2\|y_n-x^*\|^2+2\alpha_n\gamma\rho\|x_n-x^*\|\|x_{n+1}-x^*\|\\
&\quad +2\alpha_n\langle\gamma f(x^*)-\eta Bx^*,x_{n+1}-x^*\rangle\\
&\le (1-\alpha_n\eta\delta)^2\|x_n-x^*\|^2+\alpha_n\gamma\rho(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2)\\
&\quad +2\alpha_n\langle\gamma f(x^*)-\eta Bx^*,x_{n+1}-x^*\rangle.
\end{aligned}$$

Thus

$$(1-\alpha_n\gamma\rho)\|x_{n+1}-x^*\|^2 \le [(1-\alpha_n\eta\delta)^2+\alpha_n\gamma\rho]\|x_n-x^*\|^2+2\alpha_n\langle\gamma f(x^*)-\eta Bx^*,x_{n+1}-x^*\rangle.$$

Since $\eta\delta>\gamma\rho$ and

$$0<\alpha_n\le\frac{1}{\eta\|B\|}\le\frac{1}{\eta\delta},$$

we have $1-\alpha_n\gamma\rho>1-\alpha_n\eta\delta\ge 0$. Hence

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \frac{(1-\alpha_n\eta\delta)^2+\alpha_n\gamma\rho}{1-\alpha_n\gamma\rho}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\rho}\langle\gamma f(x^*)-\eta Bx^*,x_{n+1}-x^*\rangle\\
&= [1-\frac{2\alpha_n(\eta\delta-\gamma\rho)}{1-\alpha_n\gamma\rho}]\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\rho}\langle\gamma f(x^*)-\eta Bx^*,x_{n+1}-x^*\rangle\\
&\quad +\frac{\alpha_n^2\eta^2\delta^2}{1-\alpha_n\gamma\rho}\|x_n-x^*\|^2\\
&\le [1-\frac{2\alpha_n(\eta\delta-\gamma\rho)}{1-\alpha_n\gamma\rho}]\|x_n-x^*\|^2\\
&\quad +\frac{2\alpha_n(\eta\delta-\gamma\rho)}{1-\alpha_n\gamma\rho}(\frac{\langle\gamma f(x^*)-\eta Bx^*,x_{n+1}-x^*\rangle}{\eta\delta-\gamma\rho}+\alpha_nM_2).
\end{aligned}$$

where $M_2$ is a constant satisfying

$$M_2 = \sup_{n\geq 0}\{\frac{\eta^2\delta^2}{2(\eta\delta-\gamma\rho)}\|x_n - x^*\|^2\}.$$

From the condition $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$ and (3.5), we have

$$\sum_{n=0}^{\infty}\frac{2\alpha_n(\eta\delta-\gamma\rho)}{1-\alpha_n\gamma\rho} > \sum_{n=0}^{\infty}2\alpha_n(\eta\delta-\gamma\rho) = \infty.$$

and

$$\limsup_{n\to\infty}(\frac{\langle\gamma f(x^*)-\eta Bx^*, x_{n+1}-x^*\rangle}{\eta\delta-\gamma\rho} + \alpha_n M_2) \leq 0.$$

From Lemma 2.10, we can obtain that $\|x_n - x^*\| \to 0$ as $n\to\infty$. □

Next, we give the other strong convergence theorem in Hilbert spaces.

**Theorem 3.2.** *Let $H_1$ and $H_2$ be two real Hilbert space. Let $C_i$ and $Q_i$ be nonempty, closed and convex subsets of $H_1$ and $H_2$, respectively, for each $1\leq i\leq N$. Let $A: H_1\to H_2$ be a bounded linear operator and $A^*: H_2\to H_1$ be the adjoint of $A$. Suppose that MSSFP (1.1) has a nonempty solution set $\Gamma$. Let $f: H_1\to H_1$ be a Lipschitzian mapping with coefficient $\rho\geq 0$. Let $B: H_1\to H_1$ be a strongly positive bounded linear operator with coefficient $\delta>0$. Let $\{x_n\}$ be a sequence defined as follows*

$$\begin{cases} x_1\in H_1, \\ y_n = P_{C_{[n]}}(x_n - \xi_{[n]}A^*(I-P_{Q_{[n]}})Ax_n), \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I-\alpha_n\eta B)y_n. \end{cases} \tag{3.6}$$

*where $[n] = n \pmod N$ are the mod functions taking values in $\{1,\cdots,N\}$. Assume that the following conditions are satisfied:*

*(i) $\eta\delta > \gamma\rho$;*
*(ii) $0<\xi_i<\frac{2}{\|A\|^2}$, for $\forall i\in\{1,\cdots,N\}$;*
*(iii) $\{\alpha_n\}\subset(0,1)$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_n\alpha_n = \infty$.*

*Then $\{x_n\}$ converges strongly to a point $x^*\in\Gamma$, which solves variational inequality (3.1).*

*Proof.* First, we show that sequence $\{x_n\}$ is bounded. Let $y_n = T_{[n]}x_n$, where

$$T_{[n]} = P_{C_{[n]}}(I - \xi_{[n]}A^*(I-P_{Q_{[n]}})A.$$

Picking any $p\in\Gamma$, we have $T_{[n]}p = p$. Since $P_{Q_{[n]}}$ is firmly nonexpansive, it follows from Lemma 2.4 that $A^*(I-P_{Q_{[n]}})A$ is $\frac{1}{\|A\|^2}$-ism. From $0<\xi_{[n]}<\frac{2}{\|A\|^2}$, we have that $I-\xi_{[n]}A^*(I-P_{Q_{[n]}})A$ is nonexpansive. Then $T_{[n]}$ is also nonexpansive. It follows that

$$\|y_n - p\| = \|T_{[n]}x_n - T_{[n]}p\| \leq \|x_n - p\|.$$

Using (3.4), we get that $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{f(x_n)\}$ and $\{T_{[n]}x_n\}$ are also bounded.

Next, we show that $\|x_n - x^*\|\to 0$ as $n\to\infty$, where $x^*$ is the unique solution of variational inequality (3.1). Since $A^*(I-P_{Q_{[n]}})A$ is $\frac{1}{\|A\|^2}$-ism, it follows from Proposition 2.1 and Proposition 2.2 that $\xi_{[n]}A^*(I-P_{Q_{[n]}})A$ is $\frac{1}{\xi_{[n]}\|A\|^2}$-ism, $I-\xi_{[n]}A^*(I-P_{Q_{[n]}})A$ is $\frac{\xi_{[n]}\|A\|^2}{2}$-av and

$$T_{[n]} = P_{C_{[n]}}(I - \xi_{[n]}A^*(I-P_{Q_{[n]}})A$$

is $\frac{2+\xi_{[n]}\|A\|^2}{4}$-av. Then

$$\begin{aligned}\|y_n - x^*\|^2 &= \|T_{[n]}x_n - x^*\|^2 \\ &\le \|x_n - x^*\|^2 - \frac{1-\frac{2+\xi_{[n]}\|A\|^2}{4}}{\frac{2+\xi_{[n]}\|A\|^2}{4}}\|T_{[n]}x_n - x_n\|^2 \\ &= \|x_n - x^*\|^2 - \frac{2-\xi_{[n]}\|A\|^2}{2+\xi_{[n]}\|A\|^2}\|T_{[n]}x_n - x_n\|^2.\end{aligned}$$

Observe that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n\gamma f(x_n) + (I-\alpha_n\eta B)y_n - x^*\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - \eta Bx^*) + (I-\alpha_n\eta B)(y_n - x^*)\|^2 \\ &= \alpha_n^2\|\gamma f(x_n) - \eta Bx^*\|^2 + \|(I-\alpha_n\eta B)(y_n - x^*)\|^2 \\ &\quad + 2\alpha_n\langle\gamma f(x_n) - \eta Bx^*, (I-\alpha_n\eta B)(y_n - x^*)\rangle \\ &\le \alpha_n^2\|\gamma f(x_n) - \eta Bx^*\|^2 + (1-\alpha_n\eta\delta)^2\|y_n - x^*\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n\eta\delta)\|\gamma f(x_n) - \eta Bx^*\|\|y_n - x^*\| \\ &\le \alpha_n^2\|\gamma f(x_n) - \eta Bx^*\|^2 + (1-\alpha_n\eta\delta)^2(\|x_n - x^*\|^2 - \frac{2-\xi_{[n]}\|A\|^2}{2+\xi_{[n]}\|A\|^2}\|T_{[n]}x_n - x_n\|^2) \\ &\quad + 2\alpha_n(1-\alpha_n\eta\delta)\|\gamma f(x_n) - \eta Bx^*\|\|x_n - x^*\|.\end{aligned}$$

Then

$$\begin{aligned}(1-\alpha_n\eta\delta)^2\frac{2-\xi_{[n]}\|A\|^2}{2+\xi_{[n]}\|A\|^2}\|T_{[n]}x_n - x_n\|^2 &\le (1-\alpha_n\eta\delta)^2\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad + \alpha_n^2\|\gamma f(x_n) - \eta Bx^*\|^2 + 2\alpha_n(1-\alpha_n\eta\delta) \\ &\quad \times\|\gamma f(x_n) - \eta Bx^*\|\|(x_n - x^*)\| \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n^2\eta^2\delta^2\|x_n - x^*\|^2 \\ &\quad + \alpha_n^2\|\gamma f(x_n) - \eta Bx^*\|^2 + 2\alpha_n(1-\alpha_n\eta\delta) \\ &\quad \times\|\gamma f(x_n) - \eta Bx^*\|\|x_n - x^*\|.\end{aligned} \tag{3.7}$$

Next, we analyze inequality (3.7) by considering the following two cases.

Case 1. Assume that there exists $n_0$ large enough such that $\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2$ for all $n \ge n_0$. Since $\|x_n - x^*\|^2$ is bounded, we have that $\lim_{n\to\infty}\|x_n - x^*\|^2$ exists. Since $\lim_{n\to\infty}\alpha_n = 0$, $0 < \xi_{[n]} < \frac{2}{\|A\|^2}$ for $\forall n \ge 1$, $\{x_n\}$ and $\{f(x_n)\}$ are bounded, we can obtain

$$\|T_{[n]}x_n - x_n\| \to 0 \ (n\to\infty).$$

Since

$$\|x_{n+1} - T_{[n]}x_n\| = \|x_{n+1} - y_n\| = \alpha_n\|\gamma f(x_n) - \eta By_n\| \to 0 \ (n\to\infty)$$

we have

$$\|x_{n+1} - x_n\| \to 0 \ (n\to\infty). \tag{3.8}$$

Next, we show that $\omega_\omega(x_n) \subseteq \Gamma$. To see this, we take $q \in \omega_\omega(x_n)$ and assume that $x_{n_l} \rightharpoonup q$ as $l\to\infty$ for some subsequence $\{x_{n_l}\}$ of $\{x_n\}$. We may further assume $n_l = k(modN)$ for all $l$. From (3.8), we have

$x_{n_l+j} \rightharpoonup q$ for all $j \geq 0$. Then

$$\|x_{n_l+j} - T_{[k+j]}x_{n_l+j}\| = \|x_{n_l+j} - T_{[n_l+j]}x_{n_l+j}\| \to 0 \ \ (as\ l \to \infty).$$

By Lemma 2.2, we can obtain $q \in F(T_{[k+j]})$ for all $j$. Hence, $q \in \Gamma$, that is, $\omega_\omega(x_n) \subseteq \Gamma$. The remaining of the proof is the same as that of Theorem 3.1, we omit it. Therefore, we can obtain that $x_n \to x^*$ $(n \to \infty)$.

Case 2. Assume that there exists a subsequence $\{\|x_{n_j} - x^*\|^2\}$ of $\{\|x_n - x^*\|^2\}$ such that $\|x_{n_j} - x^*\|^2 < \|x_{n_j+1} - x^*\|^2$ for all $j \in N$. It follows from Lemma 2.9 that there exists a nondecreasing sequence $\{m_k\}$ of $N$ such that $\lim_{k\to\infty} m_k = \infty$, and the following inequalities hold for all $k \in N$:

$$\|x_{m_k} - x^*\|^2 \leq \|x_{m_k+1} - x^*\|^2 \ and\ \|x_k - x^*\|^2 \leq \|x_{m_k+1} - x^*\|^2. \tag{3.9}$$

Similarly, we can get

$$\|T_{[m_k]}x_{m_k} - x_{m_k}\| \to 0 \ (n \to \infty).$$

Following an argument similar to that in Case 1, we have $\omega_\omega(x_{m_k}) \subseteq \Gamma$. Also, we have

$$\limsup_{n\to\infty} \langle (\eta B - \gamma f)x^*, x^* - x_{m_k}\rangle \leq 0$$

and

$$\|x_{m_k+1} - x^*\|^2 \leq [1 - \frac{2\alpha_{m_k}(\eta\delta - \gamma\rho)}{1 - \alpha_{m_k}\gamma\rho}]\|x_{m_k} - x^*\|^2 + \frac{2\alpha_{m_k}(\eta\delta - \gamma\rho)}{1 - \alpha_{m_k}\gamma\rho}(\frac{\langle \gamma f(x^*) - \eta Bx^*, x_{m_k+1} - x^*\rangle}{\eta\delta - \gamma\rho} + \alpha_{m_k}M_2).$$

where $M$ is a constant satisfying

$$M_2 = \sup_{k\geq 0}\{\frac{\eta^2\delta^2}{2(\eta\delta - \gamma\rho)}\|x_{m_k} - x^*\|^2\}.$$

By the same argument as in Case 1, we obtain that $\|x_{m_k} - x^*\| \to 0$ as $k \to \infty$. Using (3.9), we get $\|x_k - x^*\| \leq \|x_{m_k} - x^*\|$, $\forall k \in N$. Therefore, $x_k \to x^*$ as $k \to \infty$. This ends the proof. □

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