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A HYBRID ITERATIVE ALGORITHM FOR A SPLIT MIXED EQUILIBRIUM PROBLEM AND A HIERARCHICAL FIXED POINT PROBLEM

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Abstract. In this paper, we suggest and analyze a hybrid algorithm for finding a common solution of a split mixed equilibrium problem and a hierarcchical fixed point problem for a finite family of nonexpansive mappings. We prove the strong convergence of the iterative method under some mild conditions and derive some applications. Finally, we give a numerical example to justify the main results.

Keywords. Hierarchical fixed point problem; Split mixed equilibrium problem; Nonexpansive mapping; Strong convergence.

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1. Introduction

Let H_1 and H_2 be real Hilbert spaces and their inner products and induced norms be denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex sets. A mapping $S : C \to C$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall\, x, y \in C.$$

We denote the set of fixed points of S by $Fix(S)$, i.e., $Fix(S) := \{x \in C : Sx = x\}$.

Let $U : C \to H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed point problem: find $x^* \in F(T)$ such that

$$\langle x^* - Ux^*, x - x^* \rangle \geq 0, \quad \forall\, x \in F(T).$$

For solving a convex minimization problem, hybrid iterative methods are in the spotlight of optimization theory; see [1, 10, 11, 12, 13, 14, 16, 19, 20] and the references therein. In 2001, Yamada [23] considered the following hybrid steepest-descent iterative method:

$$x_{m+1} = Sx_m - \mu \lambda_m T(Sx_m),$$

where T is a l-Lipschitzian continuous and η-strongly monotone operator with $l > 0$, $\eta > 0$ and $0 < \mu < \frac{2\eta}{l^2}$. Under some appropriate conditions, he proved that the sequence $\{x_m\}$ defined by the descent method converges strongly to the unique solution of the variational inequality

$$\langle T(x^*), x - x^* \rangle \geq 0, \quad \forall\, x \in Fix(S).$$

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In 2014, Zhang and Yang [24] proposed an explicit iterative algorithm based on the viscosity method for finding a solution for a class of variational inequalities over the common fixed point set of a finite family of nonexpansive mappings.

Theorem 1.1. *Let H be a real Hilbert space and let $T : H \to H$ be l-Lipschitzian continuous and η-strongly monotone mapping with $l > 0$ and $\eta > 0$. Let $\{S_j\}_{j=1}^M$ be M-nonexpansive mappings such that $\Xi = \bigcap_{j=1}^M Fix(S_j) \neq \emptyset$ and let V be ρ-Lipschitzian continuous with $\rho > 0$. For any point $x_0 \in H$, define a sequence $\{x_m\}$ as:*

$$x_{m+1} = \alpha_m \gamma V(x_m) + (I - \alpha_m \mu T) S_M^m S_{M-1}^m \dots S_1^m x_m, \ \forall\, m \geq 0,$$

where $0 < \gamma\rho < \tau$ with $\tau = \mu(2\eta - \mu l^2)$, $0 < \mu < \frac{2\eta}{l^2}$, $S_j^m = (1 - \varpi_m^j) I + \varpi_m^j S_j$ for $j = 1, 2, \dots, M$ and $\varpi_m^j \in (\zeta_1, \zeta_2)$ for some $\zeta_1, \zeta_2 \in (0,1)$. If $\lim_{m\to\infty} \alpha_m = 0$, $\sum_{m=1}^\infty \alpha_m = \infty$ and $\lim_{k\to\infty} |\varpi_m^k - \varpi_m^{k+1}| = 0$, $\forall m = 1, 2, \cdot N$, then the sequence $\{x_m\}$ converges strongly to the unique solution $x^* \in \Xi$ of the variational inequality:

$$\langle (\mu T - \gamma V) x^*, x - x^* \rangle \geq 0, \ \forall\, x \in \bigcap_{j=1}^M Fix(S_j).$$

Now we consider the following hierarchical fixed point problem (HFPP) for a finite family of nonexpansive mappings $\{S_j\}_{j=1}^M : C \to C$ with $\bigcap_{j=1}^M Fix(S_j) \neq \emptyset$ with respect to another nonexpansive nonself mapping $U : C \to H_1$: Find $x^* \in \bigcap_{j=1}^M Fix(S_j)$ such that

$$\langle x^* - Ux^*, x^* - x \rangle \leq 0, \ \forall\, x \in \bigcap_{j=1}^M Fix(S_j). \tag{1.1}$$

The solution set of the HFPP (1.1) is denoted by Φ.

If we set $S_j = S$ for $j = 1, 2, \dots, M$, a self nonexpansive mapping on C, then the HFPP (1.1) reduces to the following HFPP which is considered and studied by Moudafi and Maingé [17]: Find $x^* \in Fix(S)$ such that

$$\langle x^* - Ux^*, x^* - x \rangle \leq 0, \quad \forall\, x \in Fix(S). \tag{1.2}$$

In 1994, Blum and Oettli [3] introduced and studied the equilibrium problem (EP), which is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall\, y \in C. \tag{1.3}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction. The solution set of problem (1.3) is denoted by $Sol(EP)$. Problem (1.1) contains many problems, such as, Nash Equilibria problems, complementarity problems, fixed point problems and variational inequality problems as special cases [1, 2, 6, 7, 8, 12, 14, 19, 20].

Now, we introduce the following split mixed equilibrium problem (SMEP): Find $x^* \in C$ such that

$$F(x^*, x) + \langle f x^*, x - x^* \rangle \geq 0, \quad \forall\, x \in C, \tag{1.4}$$

and

$$y^* = Ax^* \in Q \text{ which solves } G(y^*, y) + \langle g y^*, y - y^* \rangle \geq 0, \quad \forall\, y \in Q, \tag{1.5}$$

where $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ are two bifunctions, $A : H_1 \to H_2$ is a bounded linear operator and $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be two nonlinear mappings.

The solution set of mixed equilibrium problem (1.4) is denoted by $Sol(MEP)$ and the solution set of the SMEP (1.4)-(1.5) is denoted by Ω. If $f = g = 0$ in the SMEP (1.4)-(1.5), then it is reduced to the following split equilibrium problem (SEP): Find $x^* \in C$ such that

$$F(x^*, x) \geq 0, \quad \forall\, x \in C, \tag{1.6}$$

and

$$y^* = Ax^* \in Q \text{ which solves } G(y^*, y) \geq 0, \; \forall\, y \in Q. \tag{1.7}$$

The SEP (1.6)-(1.7) was initially given by Moudafi [18] and further studied by Kazmi and Rizvi [15]. The solution set of SEP (1.6)-(1.7) is denoted by Ω_1.

If $F = G = 0$, then SMEP (1.4)-(1.5) is reduced to the split variational inequality problem (SVIP): Find $x^* \in C$ such that

$$\langle fx^*, x - x^* \rangle \geq 0, \quad \forall\, x \in C, \tag{1.8}$$

and

$$y^* = Ax^* \in Q \text{ which solves } \langle gy^*, y - y^* \rangle \geq 0, \; \forall\, y \in Q. \tag{1.9}$$

The SVIP (1.8)-(1.9) was introduced and studied by Censor, Gibali and Reich [5]. The solution set of SVIP(1.8)-(1.9) is denoted by Ω_2.

Recently, Moudafi [18] considered the following split monotone variational inclusion problem (SMVIP): Find $x^* \in H_1$ such that

$$0 \in f(x^*) + N(x^*), \tag{1.10}$$

and

$$y^* = Ax^* \in H_2 \text{ which solves } 0 \in g(y^*) + P(y^*), \tag{1.11}$$

where $N : H_1 \to 2^{H_1}$ and $P : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings. The solution set of SMVIP (1.10)-(1.11) is denoted by Ω_3.

We observe that the problems the SMEP (1.4)-(1.5), the SEP (1.6)-(1.7) and the SVIP (1.8)-(1.9) can be deduced from the SMVIP (1.10)-(1.11).

If $f = g = 0$, then the SMVIP (1.10)-(1.11) is reduced to the following split null point problem (SNPP): Find $x^* \in H_1$ such that

$$0 \in N(x^*), \tag{1.12}$$

and

$$y^* = Ax^* \in H_2 \text{ which solves } 0 \in P(y^*). \tag{1.13}$$

Byrne, Gibali and Reich [4] studied the weak and strong convergence theorems of some iterative methods for the SNPP (1.12)-(1.13).

In 2017, Kazmi, Ali and Furkan [14] analyzed a Krasnoselski-Mann type iterative method to approximate a solution of a hierarchical fixed point problem (1.2) for nonexpansive mappings and split mixed equilibrium problem (1.4)-(1.5). They proved weak convergence theorems and also proposed a hybrid iterative method for split monotone variational inclusion problem (1.10)-(1.11) and hierarchical fixed point problem (1.1). They proved that the sequences generated by their proposed hybrid iterative method is strongly convergent in real Hilbert spaces. The weak and strong convergence are different in infinite dimensional Hilbert spaces and the strong convergence is usually more desirable than the weak convergence. To prove strong convergence of algorithms for the SMEP (1.4)-(1.5) and the HFPP (1.1) is a more general and interesting problem which motivates our work.

In this paper, we introduce a hybrid iterative method for finding a common solution of split mixed equilibrium problem the SMEP (1.4)-(1.5) and hierarchical fixed point problem (1.1)) for a finite family of nonexpansive mappings. We prove a strong convergence theorem for the proposed iterative algorithm. We give some applications of the convergence results. We also have given a numerical example. The results and methods discussed in this paper extend and unify various known results in this field.

2. PRELIMINARIES

We recall some important concepts and results, which will be used later. Let the symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

For every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall\, y \in C.$$

The mapping P_C is called the metric projection of H_1 onto C. It is known that P_C is nonexpansive and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall\, x \in H_1.$$

Further, $P_C x$ is characterized by the fact $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall\, y \in C,$$

which implies that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall\, x \in H_1,\ y \in C.$$

Lemma 2.1. *For all $x, y \in H_1$, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Definition 2.1. A mapping $f : H_1 \to H_1$ is said to be

(i) monotone if

$$\langle fx - fy, x - y\rangle \geq 0, \quad \forall\, x, y \in H_1;$$

(ii) α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle fx - fy, x - y\rangle \geq \alpha \|fx - fy\|^2, \quad \forall\, x, y \in H_1;$$

(iii) β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|fx - fy\| \leq \beta \|x - y\|, \quad \forall\, x, y \in H_1.$$

(iv) ϖ-averaged if there exists $\varpi \in (0,1)$ such that $S = (1-\varpi)I + \varpi U$, where $I : H_1 \to H_1$ is the identity mapping and $U : H_1 \to H_1$ is nonexpansive.

If f is α-inverse strongly monotone mapping, then f is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Lemma 2.2. [9] *Let S be a nonexpansive mapping on H_1. Then S is demiclosed at the origin, that is, $\{x_m\}$ converges weakly to $x \in H_1$ and $\{x_m - Sx_m\}$ converges strongly to 0, then $x \in Fix(S)$.*

Lemma 2.3. [2] *Let $C \subset H_1$ be a nonempty, closed and convex set and let $S : C \to H_1$ be a nonexpansive mapping. Then $Fix(S)$ is closed and convex.*

Assumpion 2.1. The bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions

(i) $F(x,x)=0, \quad \forall\, x\in C$;

(ii) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0, \quad \forall\, x\in C$;

(iii) For each $x,y,z\in C$, $\limsup_{t\to 0}F(tz+(1-t)x,y)\leq F(x,y)$;

(iv) For each $x\in C,\ y\to F(x,y)$ is convex and lower semicontinuous.

Lemma 2.4. [8] *Let C be a nonempty closed convex subset of H_1. Assume that $F: C\times C\to\mathbb{R}$ satisfying Assumption 2.1. Then, for all $r>0$ and for all $x\in H_1$, define the resolvent operator $S_r: H_1\to C$ by*

$$S_r(x)=\left\{z\in C: F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0, \quad \forall\, y\in C\right\}$$

Then the following holds:

(i) *for each $x\in H_1,\ S_r(x)\neq\emptyset$;*

(ii) *S_r is single-valued;*

(iii) *S_r is firmly nonexpansive, i.e.,*

$$\|S_rx-S_ry\|^2\leq\langle S_rx-S_ry,x-y\rangle,\ \forall\, x,y\in H_1;$$

(iv) *$Fix(S_r)=Sol(EP)$;*

(v) *$Fix(S_r)$ is closed and convex.*

Lemma 2.5. *Let $F_1: C\times C\to\mathbb{R}$ be a bifunction with Assumption 2.1 and let S_r be defined as in Lemma 2.4. Let $x,y\in H_1$ and $r_1,r_2>0$. Then*

$$\|S_{r_2}(y)-S_{r_1}(x)\|\leq\|y-x\|+\left|\frac{r_2-r_1}{r_2}\right|\|S_{r_2}(y)-y\|.$$

Lemma 2.6. [23] *Suppose that $\lambda\in(0,1)$ and $\mu>0$. Let $T: C\to C$ be a l-Lipschitzian continuous and η-strongly monotone mapping with $l>0$ and $\eta>0$. Define a mapping $S^{\lambda}: C\to H_1$ by*

$$S^{\lambda}x=Sx-\lambda\mu T(Sx), \quad \forall\, x\in C,$$

where S is a nonexpansive mapping on C. Then S^{λ} is a contraction provided $\mu<\frac{2\eta}{l^2}$, i.e.,

$$\|S^{\lambda}x-S^{\lambda}y\|\leq(1-\lambda\tau)\|x-y\|,\ \forall\, x,y\in C,$$

where $\tau=1-\sqrt{1-\mu(2\eta-\mu l^2)}$.

Lemma 2.7. [21] *Let $\{x_m\}$ and $\{y_m\}$ be bounded sequences in a Hilbert space H and let $\{\beta_m\}$ be a sequence in $[0,1]$ with $0<\liminf_{m\to\infty}\beta_m\leq\limsup_{m\to\infty}\beta_m<1$. Suppose $x_{m+1}=\beta_m x_m+(1-\beta_m)y_m$ for all integers $m\geq 0$ and $\limsup_{m\to\infty}(\|y_{m+1}-y_m\|-\|x_{m+1}-x_m\|)\leq 0$. Then $\lim_{m\to\infty}\|y_m-x_m\|=0$.*

Lemma 2.8. [22] *Assume that $\{\delta_m\}$ is a sequence of nonnegative real numbers such that*

$$\delta_{m+1}\leq(1-\gamma_m)\delta_m+\alpha_m,$$

where $\{\gamma_m\}$ is a sequence in $(0,1)$ and α_m is a sequence such that

(i) *$\sum_{m=1}^{\infty}\gamma_m=\infty$;*

(ii) *$\limsup_{m\to\infty}\frac{\alpha_m}{\gamma_m}\leq 0$ or $\sum_{m=1}^{\infty}|\alpha_m|<\infty$.*

Then $\lim_{m\to\infty}\delta_m=0$.

Lemma 2.9. *Let $\{S_j\}_{j=1}^{M}$ be averaged mappings with common fixed points. Then $\bigcap_{j=1}^{M}Fix(S_j)=Fix(S_1\ldots S_M)$.*

3. MAIN RESULTS

We prove a strong convergence theorem for the SMEP (1.3)-(1.4) and the HFPP (1.1) in this section.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.1 and let G be upper semicontinuous. Let mappings $f : C \to H_1$ and $g : Q \to H_2$ be θ_1-inverse strongly monotone and θ_2-inverse strongly monotone, respectively. Let $S_j : C \to C$ be a nonexpansive mapping for each $j = 1,2,\ldots,M$. Let $V : C \to C$ be a l-lipschitzian continuous and η-strongly monotone mapping with $l > 0$ and $\eta > 0$ and let $U : C \to C$ be a τ-lipschitzian continuous mapping with $\tau > 0$. Let $0 < \mu < \frac{2\eta}{l^2}$ and $0 < \rho\tau < \zeta$, where $\zeta = 1 - \sqrt{1-\mu(2\eta - \mu l^2)}$. Assume that $\Gamma = \Omega \bigcap \Phi \neq \emptyset$. Let the iterative sequences $\{z_m\}$ and $\{x_m\}$ be generated via the hybrid iterative algorithm:*

$$\begin{cases} x_0 \in C; \\ y_m = X(x_m); w_m = Y(Ay_m); \\ z_m = y_m + \gamma A^*(w_m - Ay_m); \\ x_{m+1} = \alpha_m \rho U(x_m) + \beta_m x_m + [(1-\beta_m)I - \alpha_m \mu V] S_M^m S_{M-1}^m \ldots S_1^m z_m, \forall m \geq 0 \end{cases} \tag{3.1}$$

where $S_j^m = (1-\varpi_m^j)I + \varpi_m^j S_j$, $X = S_{r_m}^F(I - r_m f)$, $Y = S_{r_m}^G(I - r_m g)$ and $\gamma \in \left(0, \frac{1}{\|A\|^2}\right), \varpi_m^j \in (\xi_1, \xi_2)$ for some $\xi_1, \xi_2 \in (0,1)$, $\{\alpha_m\}$ and $\{\beta_m\}$ are two real sequences in $(0,1)$ and $\{r_m\} \subset (0,\alpha)$, where $\alpha = 2\min\{\theta_1, \theta_2\}$, satisfy the conditions:

- (i) $\lim_{m\to\infty} \alpha_m = 0$ and $\sum_{m=1}^{\infty} \alpha_m = \infty$.
- (ii) $0 < \liminf_{m\to\infty} \beta_m \leq \limsup_{m\to\infty} \beta_m < 1$.
- (iii) $\lim_{m\to\infty} \inf r_m > 0$.
- (iv) $\lim_{m\to\infty} |\varpi_{m+1}^j - \varpi_m^j| = 0$ for $j = 1,2,\ldots,M$.

Then the sequence $\{x_m\}$ converges strongly to $\bar{p} \in \Gamma$.

Proof. We divide the proof into five steps.

Step 1. Since $f : C \to H_1$ is θ_1-inverse strongly monotone, we have

$$\begin{aligned} \|(I - r_m f)x - (I - r_m f)y\|^2 &= \|(x-y) - r_m(fx - fy)\|^2 \\ &\leq \|x-y\|^2 - r_m(2\theta_1 - r_m)\|fx - fy\|^2 \\ &\leq \|x-y\|^2, \end{aligned}$$

which shows that $(I - r_m f)$ is nonexpansive. Similarly, we can show that $(I - r_m g)$ is nonexpansive too. Hence $X = S_{r_m}^F(I - r_m f), Y = S_{r_m}^G(I - r_m g)$ are nonexpansive. Since $\Gamma \neq \emptyset$, it follows from Lemma 2.3 that $Fix(S_{r_m}^F(I - r_m f))$ and $Fix(S_{r_m}^G(I - r_m g))$ are closed and convex sets. So, Ω is closed and convex. Since $\Phi = \bigcap_{j=1}^M Fix(S_j) \neq \emptyset$, we have hat Φ is closed and convex. Thus, Γ is nonempty, closed and convex. Let $p \in \Gamma$ then $p \in \Omega$. Then $S_{r_m}^F(I - r_m f)p = p$ and $S_{r_m}^G(I - r_m g)Ap = Ap$. From (3.1), we have

$$\begin{aligned} \|y_m - p\|^2 &\leq \|(x_m - p) - r_m(fx_m - fp)\|^2 \\ &\leq \|x_m - p\|^2 - r_m(2\theta_1 - r_m)\|fx_m - fp\|^2 \\ &\leq \|x_m - p\|^2 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\|w_m - Ap\|^2 &\leq \|(Ay_m - Ap) - r_m(gAy_m - gAp)\|^2 \\ &\leq \|Ay_m - Ap\|^2 - r_m(2\theta_2 - r_m)\|gAy_m - gAp\|^2 \\ &\leq \|Ay_m - Ap\|^2. \end{aligned} \tag{3.3}$$

From (3.1) and (3.2), we evaluate

$$\begin{aligned}\|z_m - p\|^2 &= \|y_m - p\|^2 + \|\gamma A^*(w_m - Ay_m)\|^2 + 2\gamma\langle y_m - p, A^*(w_m - Ay_m)\rangle \\ &\leq \|y_m - p\|^2 + \gamma^2\|A^*\|^2\|w_m - Ay_m\|^2 \\ &\quad + 2\gamma\langle A(y_m - p) + (w_m - Ay_m) - (w_m - Ay_m), w_m - Ay_m\rangle \\ &= \|y_m - p\|^2 + \gamma^2\|A^*\|^2\|w_m - Ay_m\|^2 \\ &\quad + \gamma\Big[\|w_m - Ap\|^2 + \|w_m - Ay_m\|^2 - \|Ay_m - Ap\|^2 - 2\|w_m - Ay_m\|^2\Big] \\ &\leq \|y_m - p\|^2 - \gamma(1 - \gamma\|A^*\|^2)\|w_m - Ay_m\|^2 \\ &\leq \|x_m - p\|^2 - \gamma(1 - \gamma\|A^*\|^2)\|w_m - Ay_m\|^2 \\ &\leq \|x_m - p\|^2. \end{aligned} \tag{3.4}$$

Using Lemma 2.6, we get

$$\begin{aligned}\|x_{m+1} - p\| &= \|\alpha_m(\rho U(x_m) - \mu V(p)) + \beta_m(x_m - p) + [(1-\beta_m)I - \alpha_m\mu V]S_M^m S_{M-1}^m \dots S_1^m z_m \\ &\quad - [(1-\beta_m)I - \alpha_m\mu V]S_M^m S_{M-1}^m \dots S_1^m p\| \\ &\leq \alpha_m\|\rho U(x_m) - \mu V(p)\| + \beta_m\|x_m - p\| + \|[(1-\beta_m)I - \alpha_m\mu V]S_M^m S_{M-1}^m \dots S_1^m z_m \\ &\quad - [(1-\beta_m)I - \alpha_m\mu V]S_M^m S_{m-1}^m \dots S_1^m p\| \\ &= \alpha_m\|\rho U(x_m) - \mu V(p)\| + \beta_m\|x_m - p\| + (1-\beta_m)\left\|\left(I - \frac{\alpha_m\mu V}{1-\beta_m}\right)S_M^m S_{M-1}^m \dots S_1^m z_m \right. \\ &\quad \left. - \left(I - \frac{\alpha_m\mu V}{1-\beta_m}\right)S_M^m S_{M-1}^m \dots S_1^m p\right\| \\ &\leq \alpha_m\|\rho U(x_m) - \mu V(p)\| + \beta_m\|x_m - p\| + (1-\beta_m)\left(1 - \frac{\alpha_m\zeta}{1-\beta_m}\right)\|z_m - p\| \\ &\leq \alpha_m\|\rho U(x_m) - \mu V(p)\| + \beta_m\|x_m - p\| + (1 - \beta_m - \alpha_m\zeta)\|x_m - p\| \\ &\leq \alpha_m\rho\|U(x_m) - U(p)\| + \alpha_m\|\rho U(p) - \mu V(p)\| + (1 - \alpha_m\zeta)\|x_m - p\| \\ &\leq \alpha_m\rho\tau\|x_m - p\| + \alpha_m\|\rho U(p) - \mu V(p)\| + (1 - \alpha_m\zeta)\|x_m - p\| \\ &\leq (1 - \alpha_m(\zeta - \rho\tau))\|x_m - p\| + \alpha_m\|\rho U(p) - \mu V(p)\| \\ &\leq (1 - \alpha_m(\zeta - \rho\tau))\|x_m - p\| + \alpha_m(\zeta - \rho\tau)\frac{\|\rho U(p) - \mu V(p)\|}{\zeta - \rho\tau} \\ &\leq \max\left\{\|x_m - p\|, \frac{\|\rho U(p) - \mu V(p)\|}{\zeta - \rho\tau}\right\}. \end{aligned}$$

By induction on m, we have

$$\|x_m - p\| \leq \max\left\{\|x_o - p\|, \frac{\rho U(p) - \mu V(p)\|}{\zeta - \rho\tau}\right\}, m = 1, 2, \dots.$$

Therefore, $\{x_m\}$ is bounded and further it follows that $\{y_m\}, \{w_m\}$ and $\{z_m\}$ are also bounded.

Step 2. We show that

$$\lim_{m\to\infty}\|x_{m+1}-x_m\|=0,\quad \lim_{m\to\infty}\|x_m-y_m\|=0,$$
$$\lim_{m\to\infty}\|x_m-z_m\|=0,\quad \lim_{m\to\infty}\|x_m-w_m\|=0.$$

Setting $x_{m+1}=\beta_m x_m+(1-\beta_m)w_m$, $\forall m\geq 1$, we have

$$\begin{aligned}
w_{m+1}-w_m &= \frac{x_{m+2}-\beta_{m+1}x_{m+1}}{1-\beta_{m+1}}-\frac{x_{m+1}-\beta_m x_m}{1-\beta_m}\\
&= \frac{\alpha_{m+1}\rho U(x_{m+1})+((1-\beta_{m+1})I-\alpha_{m+1}\mu V)S_M^{m+1}S_{M-1}^{m+1}\dots S_1^{m+1}z_{m+1}}{1-\beta_{m+1}}\\
&\quad -\frac{\alpha_m\rho U(x_m)+((1-\beta_m)I-\alpha_m\mu V)S_M^m S_{M-1}^m\dots S_1^m z_m}{1-\beta_m}\\
&= \frac{\alpha_{m+1}}{1-\beta_{m+1}}\Big[\rho U(x_{m+1})-\mu V(S_M^{m+1}S_{M-1}^{m+1}\dots S_1^{m+1}z_{m+1})\Big]\\
&\quad +\frac{\alpha_m}{1-\beta_m}\Big[\mu V(S_M^m S_{M-1}^m\dots S_1^m z_m)-\rho U(x_m)\Big]\\
&\quad +(S_M^{m+1}S_{M-1}^{m+1}\dots S_1^{m+1}z_{m+1}-S_M^m S_{M-1}^m\dots S_1^m z_m)\\
&= \frac{\alpha_{m+1}}{1-\beta_{m+1}}\Big[\rho U(x_{m+1})-\mu V(S_M^{m+1}S_{M-1}^{m+1}\dots S_1^{m+1}z_{m+1})\Big]\\
&\quad +\frac{\alpha_m}{1-\beta_m}\Big[\mu V(S_M^m S_{M-1}^m\dots S_1^m z_m)-\rho U(x_m)\Big]\\
&\quad +(S_M^{m+1}S_{M-1}^{m+1}\dots S_1^{m+1}z_{m+1}-S_M^{m+1}S_{M-1}^{m+1}\dots S_1^{m+1}z_m\\
&\quad +S_M^{m+1}S_{M-1}^{m+1}\dots S_1^{m+1}z_m-S_M^m S_{M-1}^m\dots S_1^m z_m).
\end{aligned}$$

It follows

$$\begin{aligned}
\|w_{m+1}-w_m\|-\|x_{m+1}-x_m\| &\leq \frac{\alpha_{m+1}}{1-\beta_{m+1}}\Big[\rho\|U(x_{m+1})\|+\mu\|V(S_M^{m+1}S_{M-1}^{m+1}\dots S_1^{m+1}z_{m+1})\|\Big]\\
&\quad +\frac{\alpha_m}{1-\beta_m}\Big[\mu\|V(S_M^m S_{M-1}^m\dots S_1^m z_m)\|+\rho\|U(x_m)\|\Big]\\
&\quad +\|S_M^{m+1}S_{M-1}^{m+1}\dots S_1^{m+1}z_m-S_M^m S_{M-1}^m\dots S_1^m z_m\|\\
&\quad +\|z_{m+1}-z_m\|-\|x_{m+1}-x_m\|.
\end{aligned}\tag{3.5}$$

From the definition of S_j^m, we have

$$\begin{aligned}
\|S_2^m S_1^m z_m-S_2^{m-1}S_1^{m-1}z_m\| &\leq \|S_2^m S_1^m z_m-S_2^m S_1^{m-1}z_m\|+\|S_2^m S_1^{m-1}z_m-S_2^{m-1}s_1^{m-1}z_m\|\\
&\leq \|S_1^m z_m-S_1^{m-1}z_m\|+\|S_2^m S_1^{m-1}z_m-S_2^{m-1}S_1^{m-1}z_m\|\\
&\leq \|(1-\varpi_m^1)z_m+\varpi_m^1 S_1 z_m-(1-\varpi_{m-1}^1)z_m-\varpi_{m-1}^1 S_1 z_m\|\\
&\quad +\|(1-\varpi_m^2)S_1^{m-1}z_m+\varpi_m^2 S_2 S_1^{m-1}z_m\\
&\quad -(1-\varpi_{m-1}^2)S_1^{m-1}z_m-\varpi_{m-1}^2 S_2 S_1^{m-1}z_m\|\\
&\leq |\varpi_m^1-\varpi_{m-1}^1|(\|z_m\|+\|S_1 z_m\|)\\
&\quad +|\varpi_m^2-\varpi_{m-1}^2|(\|S_1^{m-1}z_m\|+\|S_2 S_1^{m-1}z_m\|).
\end{aligned}\tag{3.6}$$

From (3.6), we have

$$\begin{aligned}
&\|S_3^m S_2^m S_1^m z_m - S_3^{m-1} S_2^{m-1} S_1^{m-1} z_m\| \\
\leq\ & \|S_3^m S_2^m S_1^m z_m - S_3^m S_2^{m-1} S_1^{m-1} z_m\| + \|S_3^m S_2^{m-1} S_1^{m-1} z_m - S_3^{m-1} S_2^{m-1} S_1^{m-1} z_m\| \\
\leq\ & \|S_2^m S_1^m z_m - S_2^{m-1} S_1^{m-1} z_m\| + \|(1-\varpi_m^3) S_2^{m-1} S_1^{m-1} z_m + \varpi_m^3 S_3 S_2^{m-1} S_1^{m-1} z_m \\
& -(1-\varpi_{m-1}^3) S_2^{m-1} S_1^{m-1} z_m - \varpi_{m-1}^3 S_3 S_2^{m-1} S_1^{m-1} z_m\| \\
\leq\ & |\varpi_m^1 - \varpi_{m-1}^1|(\|z_m\| + \|S_1 z_m\|) + |\varpi_m^2 - \varpi_{m-1}^2|(\|S_1^{m-1} z_m\| + \|S_2 S_1^{m-1} z_m\|) \\
& +|\varpi_m^3 - \varpi_{m-1}^3|(\|S_2^{m-1} S_1^{m-1} z_m\| + \|S_3 S_2^{m-1} S_1^{m-1} z_m\|).
\end{aligned}$$

By induction on M, we have

$$\begin{aligned}
&\|S_M^m S_{M-1}^m \ldots S_1^m z_m - S_M^{m-1} S_{M-1}^{m-1} \ldots S_1^{m-1} z_m\| \\
\leq\ & |\varpi_m^1 - \varpi_{m-1}^1|(\|z_m\| + \|S_1 z_m\|) + |\varpi_m^2 - \varpi_{m-1}^2|(\|S_1^{m-1} z_m\| + \|S_2 S_1^{m-1} z_m\|) \\
& + \ldots + |\varpi_m^M - \varpi_{m-1}^M|(\|S_{M-1}^{m-1} \ldots S_1^{m-1} z_m\| \\
& +\|S_M S_{M-1}^{m-1} \ldots S_1^{m-1} z_m\|). \qquad (3.7)
\end{aligned}$$

Since $\lim_{m\to\infty} |\varpi_{m+1}^j - \varpi_m^j| = 0$ for $j = 1, 2, \ldots, M$ and $\{z_m\}$, $\{Sz_m\}$, $\{S_1 z_m\}$ and $\|S_1^m z_m\|$ are all bounded, we get from condition (iv) that

$$\lim_{m\to\infty} \|S_M^m S_{M-1}^m \ldots S_1^m z_m - S_M^{m-1} S_{M-1}^{m-1} \ldots S_1^{m-1} z_m\| = 0. \qquad (3.8)$$

Further, we evaluate

$$\begin{aligned}
\|z_{m+1} - z_m\| &= \|y_{m+1} + \gamma A^*(w_{m+1} - Ay_{m+1}) - y_m - \gamma A^*(w_m - Ay_m)\| \\
&\leq \|y_{m+1} - y_m\| + \gamma\|A\|(\|w_{m+1} - w_m\| - \|Ay_{m+1} - Ay_m\|). \qquad (3.9)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_{m+1} - y_m\|^2 &= \|X(x_{m+1}) - X(x_m)\|^2 \\
&= \|S_{r_m}^F (I - r_m f) x_{m+1} - S_{r_m}^F (I - r_m f) x_m\|^2 \\
&\leq \|(x_{m+1} - x_m) - r_m(f x_{m+1} - f x_m)\|^2 \\
&\leq \|x_{m+1} - x_m\|^2 - r_m(2\theta_1 - r_m)\|f x_{m+1} - f x_m\|^2 \\
&\leq \|x_{m+1} - x_m\|^2 \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
\|w_{m+1} - w_m\|^2 &= \|Y(Ay_{m+1}) - Y(Ay_m)\|^2 \\
&= \|S_{r_m}^G (I - r_m g) Ay_{m+1} - S_{r_m}^G (I - r_m g) Ay_m\|^2 \\
&\leq \|Ay_{m+1} - Ay_m\|^2 - r_m(2\theta_2 - r_m)\|gAy_{m+1} - gAy_m\|^2. \qquad (3.11)
\end{aligned}$$

From (3.3), we have

$$\begin{aligned}
\|gAy_m - gAp\|^2 &\leq [r_m(2\theta_2 - r_m)]^{-1}(\|Ay_m - Ap\|^2 - \|w_m - Ap\|^2) \\
&\leq [r_m(2\theta_2 - r_m)]^{-1}(\|Ay_m - Ap\| + \|w_m - Ap\|)\|Ay_m - w_m\| \\
&\leq 2[r_m(2\theta_2 - r_m)]^{-1}\|A\|\|y_m - p\|\|Ay_m - w_m\|.
\end{aligned}$$

From (3.4), we get

$$\begin{aligned}\gamma(1-\gamma\|A^*\|^2)\|w_m-Ay_m\|^2 &\leq \|x_m-p\|^2-\|z_m-p\|^2\\ &\leq \|x_m-z_m\|(\|x_m-p\|+\|z_m-p\|)\\ &\leq R_1\|x_m-z_m\|.\end{aligned}$$

where $R_1 := \sup_m\{\|x_m-p\|+\|z_m-p\|\}$. Note that

$$\|x_m-z_m\| \leq \|x_m-y_m\|+\|y_m-z_m\|. \tag{3.12}$$

Hence

$$\begin{aligned}\|z_m-p\|^2 &= \|y_m+\gamma A^*(w_m-Ay_m)-p\|^2\\ &= \langle y_m+\gamma A^*(w_m-Ay_m)-p, z_m-p\rangle\\ &= \frac{1}{2}[\|(y_m-p)+\gamma A^*(w_m-Ay_m)\|^2+\|z_m-p\|^2-\|(y_m-z_m)\\ &\quad +\gamma A^*(w_m-Ay_m)\|^2]\\ &= \frac{1}{2}[\|y_m-p\|^2+\|z_m-p\|^2+\|\gamma A^*(w_m-Ay_m)\|^2\\ &\quad +2\gamma\langle Ay_m-Ap, w_m-Ay_m\rangle-\|(z_m-y_m)+\gamma A^*(w_m-Ay_m)\|^2]\\ &\leq \frac{1}{2}[\|y_m-p\|^2+\|z_m-p\|^2+\|\gamma A^*(w_m-Ay_m)\|^2\\ &\quad +2\gamma\|Ay_m-Ap\|\|w_m-Ay_m\|-\|z_m-y_m\|^2-\|\gamma A^*(w_m-Ay_m)\|^2\\ &\quad -2\gamma\langle z_m-y_m, A^*(w_m-Ay_m)\rangle]\\ &\leq \|y_m-p\|^2-\|z_m-y_m\|^2+2\gamma\|Ay_m-Ap\|\|w_m-Ay_m\|\\ &\quad +2\gamma\|z_m-y_m\|\|A^*\|\|w_m-Ay_m\|\\ &\leq \|y_m-p\|^2-\|z_m-y_m\|^2+2\gamma\|w_m-Ay_m\|(\|Ay_m-Ap\|\\ &\quad +\|A^*\|\|z_m-y_m\|)\end{aligned}$$

Using (3.2) and (3.4), we get

$$\begin{aligned}\|z_m-y_m\|^2 &\leq \|y_m-p\|^2-\|z_m-p\|^2+2\gamma\|Ay_m-Ap\|\|w_m-Ay_m\|\\ &\quad +2\gamma\|z_m-y_m\|\|A^*\|\|w_m-Ay_m\|\\ &\leq \|x_m-p\|^2-\|z_m-p\|^2+2\gamma\|Ay_m-Ap\|\|w_m-Ay_m\|\\ &\quad +2\gamma\|z_m-y_m\|\|A^*\|\|w_m-Ay_m\|\\ &\leq 2\gamma\|Ay_m-Ap\|\|w_m-Ay_m\|+2\gamma\|z_m-y_m\|\|A^*\|\|w_m-Ay_m\|\\ &\leq 2\gamma\|w_m-Ay_m\|(\|Ay_m-Ap\|+\|A^*\|\|z_m-y_m\|)\\ &\leq 2\gamma Q_1\|w_m-Ay_m\| \end{aligned}\tag{3.13}$$

where $Q_1=\sup_m\{\|Ay_m-Ap\|+\|A^*\|\|z_m-y_m\|)$. Further, from (3.4), we have

$$\begin{aligned}\gamma(1-\gamma\|A^*\|^2)\|w_m-Ay_m\|^2 &\leq \|x_m-p\|^2-\|z_m-p\|^2\\ &\leq \|x_m-p\|^2-\|x_m-p\|^2=0.\end{aligned}$$

Therefore,

$$\lim_{m\to\infty} \|w_m - Ay_m\| = 0. \tag{3.14}$$

Hence from (3.13), we get

$$\lim_{m\to\infty} \|z_m - y_m\| = 0. \tag{3.15}$$

Next, we estimate

$$\begin{aligned}
\|y_m - p\|^2 &= \|X(x_m) - p\|^2 \\
&= \|S^F_{r_m}(I - r_m f)x_m - S^F_{r_m}(I - r_m f)p\|^2 \\
&\leq \langle (I - r_m f)x_m - (I - r_m f)p, y_m - p\rangle \\
&= \frac{1}{2}[\|(I - r_m f)x_m - (I - r_m f)p\|^2 + \|y_m - p\|^2 \\
&\quad -\|x_m - y_m - r_m(fx_m - fp)\|^2] \\
&\leq \frac{1}{2}[\|x_m - p\|^2 + \|y_m - p\|^2 - \|x_m - y_m\|^2 \\
&\quad +2r_m\langle x_m - y_m, fx_m - fp\rangle - r_m^2\|fx_m - fp\|^2] \\
&\leq \frac{1}{2}[\|x_m - p\|^2 + \|y_m - p\|^2 - \|x_m - y_m\|^2 \\
&\quad +2r_m\|x_m - y_m\|\|fx_m - fp\|] \\
&\leq \|x_m - p\|^2 - \|x_m - y_m\|^2 + 2r_m\|x_m - y_m\|\|fx_m - fp\| \qquad (3.16)
\end{aligned}$$

From (3.2) and (3.16), we get

$$\begin{aligned}
\|x_m - y_m\|^2 &\leq \|x_m - p\|^2 - \|y_m - p\|^2 + 2r_m\|x_m - y_m\|\|fx_m - fp\| \\
&\leq 2r_m\|x_m - y_m\|\|fx_m - fp\|
\end{aligned}$$

Further, we have

$$\|x_m - y_m\| \leq 2r_m\|fx_m - fp\| \tag{3.17}$$

From (3.2), we get

$$r_m(2\theta_1 - r_m)\|fx_m - fp\|^2 \leq \|x_m - p\|^2 - \|y_m - p\|^2$$

Therefore, we have $\lim_{m\to\infty}\|fx_m - fp\| = 0$. From (3.17), we get

$$\lim_{m\to\infty} \|x_m - y_m\| = 0. \tag{3.18}$$

Suttituting (3.15) and (3.18) into (3.12), we get

$$\lim_{m\to\infty} \|x_m - z_m\| = 0. \tag{3.19}$$

From (3.7), we get

$$\lim_{m\to\infty} \|gAy_{m+1} - gAy_m\| = 0.$$

From (3.9) and (3.11), we get $\|z_{m+1} - z_m\| \leq \|y_{m+1} - y_m\|$ and $\|w_{m+1} - w_m\| \leq \|Ay_{m+1} - Ay_m\|$. From (3.10), we get $\|z_{m+1} - z_m\| \leq \|x_{m+1} - x_m\|$ From (3.5), (3.8) and using conditions (i-ii), we get

$$\lim_{m\to\infty} \sup\Big(\|w_{m+1} - w_m\| - \|x_{m+1} - x_m\|\Big) \leq 0.$$

Using Lemma 2.7, we have

$$\lim_{m\to\infty} \|w_m - x_m\| = 0.$$

Further using condition (ii), we have

$$\lim_{m\to\infty}\|x_{m+1}-x_m\| = \lim_{m\to\infty}(1-\beta_m)\|w_m-x_m\| = 0. \tag{3.20}$$

Step 3. We show that

$$\lim_{m\to\infty}\|x_m - S_M^m S_{M-1}^m \dots S_1^m x_m\| = 0.$$

From (3.1), we get

$$\begin{aligned}
\|x_m - S_M^m S_{M-1}^m \dots S_1^m x_m\| &\le \|x_m - x_{m+1}\| + \|x_{m+1} - S_M^m S_{M-1}^m \dots S_1^m x_m\| \\
&\le \|x_m - x_{m+1}\| + \|\alpha_m \rho U(x_m) + \beta_m x_m \\
&\quad + [(1-\beta_m)I - \alpha_m \mu V] S_M^m S_{M-1}^m \dots S_1^m z_m \\
&\quad - S_M^m S_{M-1}^m \dots S_1^m x_m\| \\
&\le \|x_m - x_{m+1}\| + \alpha_m \|\rho U(x_m) - \mu V\left(S_M^m S_{M-1}^m \dots S_1^m z_m\right)\| \\
&\quad + \beta_m \|x_m - S_M^m S_{M-1}^m \dots S_1^m z_m\| \\
&\quad + \|S_M^m S_{M-1}^m \dots S_1^m z_m - S_M^m S_{M-1}^m \dots S_1^m x_m\| \\
&\le \|x_m - x_{m+1}\| + \alpha_m \|\rho U(x_m) - \mu V\left(S_M^m S_{M-1}^m \dots S_1^m z_m\right)\| \\
&\quad + \beta_m \|x_m - S_M^m S_{M-1}^m \dots S_1^m z_m\| + \|z_m - x_m\|
\end{aligned}$$

Next, we estimate

$$\begin{aligned}
\|x_m - S_M^m S_{M-1}^m \dots S_1^m z_m\| &\le \|x_m - x_{m+1}\| + \|x_{m+1} - S_M^m S_{M-1}^m \dots S_1^m z_m\| \\
&\le \|x_m - x_{m+1}\| + \|\alpha_m \rho U(x_m) + \beta_m x_m \\
&\quad + [(1-\beta_m)I - \alpha_m \mu V] S_M^m S_{M-1}^m \dots S_1^m z_m \\
&\quad - S_M^m S_{M-1}^m \dots S_1^m z_m\| \\
&\le \|x_m - x_{m+1}\| + \alpha_m \|\rho U(x_m) - \mu V(S_M^m S_{M-1}^m \dots S_1^m z_m)\| \\
&\quad + \beta_m \|x_m - S_M^m S_{M-1}^m \dots S_1^m z_m\|
\end{aligned}$$

Further, we have

$$\begin{aligned}
\|x_m - S_M^m S_{M-1}^m \dots S_1^m z_m\| &\le \frac{1}{1-\beta_m}\|x_m - x_{m+1}\| \\
&\quad + \frac{\alpha_m}{1-\beta_m}\|\rho U(x_m) - \mu V(S_M^m S_{M-1}^m \dots S_1^m z_m)\|
\end{aligned}$$

Using condition (i-ii), we have

$$\lim_{m\to\infty}\|x_m - S_M^m S_{M-1}^m \dots S_1^m z_m\| = 0. \tag{3.21}$$

From (3.19), (3.20), (3.21) and using condition (i-ii), we have

$$\lim_{m\to\infty}\|x_m - S_M^m S_{M-1}^m \dots S_1^m x_m\| = 0. \tag{3.22}$$

Step 4. We show that $\bar{p} \in \Gamma$.

First, we show that $\bar{p} \in \Omega$. Since $y_m = X(x_m) = S_{r_m}^F(I - r_m f)x_m$, we have

$$F(y_m, q) + \langle f x_m, q - y_m\rangle + \frac{1}{r_m}\langle q - y_m, y_m - x_m\rangle \ge 0, \quad \forall\, q \in C.$$

Since F is monotone, we have

$$\langle fx_m, q-y_m\rangle + \frac{1}{r_m}\langle q-y_m, y_m-x_m\rangle \geq F(q,y_m), \quad \forall\, q\in C. \tag{3.23}$$

Hence replacing m with m_v in (3.23), we have

$$\langle fx_{m_v}, q-y_{m_v}\rangle + \frac{1}{r_{m_v}}\langle q-y_{m_v}, y_{m_v}-x_{m_v}\rangle \geq F(q,y_{m_v})\ \forall\, q\in C. \tag{3.24}$$

Let $y_i = iy+(1-i)\bar{p}\in C$ with $0<i\leq 1$. So, from (3.24) we have

$$\begin{aligned}
\langle y_i-y_{m_v}, fy_i\rangle &\geq \langle y_i-y_{m_v}, fy_i\rangle - \langle y_i-y_{m_v}, fx_{m_v}\rangle \\
&\quad -\left\langle y_i-y_{m_v}, \frac{y_{m_v}-x_{m_v}}{r_{m_v}}\right\rangle + F(y_i,y_{m_v}) \\
&= \langle y_i-y_{m_v}, fy_i-fy_{m_v}\rangle + \langle y_i-y_{m_v}, fy_{m_v}-fx_{m_v}\rangle \\
&\quad -\left\langle y_i-y_{m_v}, \frac{y_{m_v}-x_{m_v}}{r_{m_v}}\right\rangle + F(y_i,y_{m_v})
\end{aligned} \tag{3.25}$$

Since the sequences $\{x_m\},\{y_m\},\{z_m\}$ and $\{w_m\}$ have the same behaviour, so there exists subsequences $\{y_{m_v}\}$ of $\{y_m\}$, $\{z_{m_v}\}$ of $\{z_m\}$, $\{w_{m_v}\}$ of $\{w_m\}$ and $\{x_{m_v}\}$ of $\{x_m\}$ such that $z_{m_v}\rightharpoonup\bar{p}, w_{m_v}\rightharpoonup\bar{p}$, $x_{m_v}\rightharpoonup\bar{p}$ and $y_{m_v}\rightharpoonup\bar{p}$. Since $\lim_{v\to\infty}\|y_{m_v}-x_{m_v}\|=0$ and f is lipschitz continuous, we have

$$\lim_{v\to\infty}\|fy_{m_v}-fx_{m_v}\|=0.$$

Further since $\liminf_{v\to\infty} r_{m_v}>0$, there exists a number $r>0$ such that $\liminf_{v\to\infty} r_{m_v}=r$. It follows that

$$\begin{aligned}
\lim_{v\to\infty}\frac{\|y_{m_v}-x_{m_v}\|}{r_{m_v}} &\leq \frac{\lim_{v\to\infty}\|y_{m_v}-x_{m_v}\|}{\lim_{v\to\infty}\inf r_{m_v}} \\
&= \frac{1}{r}\lim_{v\to\infty}\|y_{m_v}-x_{m_v}\|=0.
\end{aligned}$$

From the monotonicity of f and lower semicontinuity of F, we have from (3.25) that

$$\langle y_i-\bar{p}, fy_i\rangle \geq F(y_i,\bar{p})\ as\ v\to\infty$$

and

$$\begin{aligned}
0 &\leq F(y_i,y_i) \\
&\leq iF(y_i,q)+(1-i)F(y_i,\bar{p}) \\
&\leq iF(y_i,q)+(1-i)\langle y_i-\bar{p}, fy_i\rangle \\
&= iF(y_i,q)+(1-i)i\langle q-\bar{p}, fy_i\rangle.
\end{aligned}$$

Hence,

$$0\leq F(y_i,q)+(1-i)\langle q-\bar{p}, fy_i\rangle.$$

Letting $i\to 0_+$, we have

$$F(\bar{p},q)+\langle q-\bar{p}, f\bar{p}\rangle \geq 0, \quad \forall q\in C.$$

This implies that $\bar{p}$ solves problem (1.3). Since A is bounded linear operator, we have $Ay_{m_v}\rightharpoonup A\bar{p}$. Now setting $w_{m_v}=Ay_{m_v}-YAy_{m_v}$, it follows from (3.14) that $\lim_{v\to\infty}w_{m_v}=0$ and $Ay_{m_v}-w_{m_v}=YAy_{m_v}$.

Therefore from Lemma 2.4, we have

$$\begin{aligned} & G(Ay_{m_v} - w_{m_v}, r) + \langle gAy_{m_v}, r - (Ay_{m_v} - w_{m_v})\rangle \\ & + \frac{1}{r_{m_v}}\left\langle r - (Ay_{m_v} - w_{m_v}), Ay_{m_v} - w_{m_v} - Ay_{m_v}\right\rangle \geq 0, \quad \forall\, r \in Q. \end{aligned} \tag{3.26}$$

Note that G is upper semicontinuous in the first argument. Taking lim sup in (3.26) as $v \to \infty$ and using $\liminf_{v\to\infty} r_{m_v} > 0$, we get

$$G(A\bar{p}, r) + \langle r - A\bar{p}, fA\bar{p}\rangle \geq 0, \quad \forall\, r \in Q.$$

which implies that $\bar{q} = A\bar{p}$ solves problem (1.4) which shows that $\bar{p} \in \Omega$ and thus $\bar{p} \in \Gamma$.

Next we show that $\bar{p} \in \Phi$. Since $\{\varpi_v^j\}$ is bounded for $j = 1,2,\ldots,M$, we assume that $\varpi_{v_t}^j \to \varpi_\infty^j$ as $t \to \infty$, where $0 < \varpi_\infty^j < 1$ for $j = 1,2,\ldots,M$. Let $S_j^\infty = (1-\varpi_\infty^j)I + \varpi_\infty^j S_j$, for $j = 1,2,\ldots,M$. Then

$$Fix(S_j^\infty) = Fix(S_j) \text{ for } j = 1,2,\ldots,M.$$

Since

$$\begin{aligned} \|S_j^{v_t} p - S_j^\infty p\| &= \|(1-\varpi_{v_t}^j)p + \varpi_{v_t}^j S_j p - (1-\varpi_\infty^j)p - \varpi_\infty^j S_j p\| \\ &\leq |\varpi_{v_t}^j - \varpi_\infty^j|(\|p\| + \|S_j p\|), \end{aligned}$$

we get

$$\lim_{t\to\infty} \sup_{p\in D} \|S_j^{v_t} p - S_j^\infty p\| = 0, \tag{3.27}$$

where D is an arbitrary bounded subset of H_1. Since S_j^∞ is ϖ_∞^j-averaged for $j = 1,2,\ldots,M$, we have from Lemma 2.9 that $\bigcap_{j=1}^M Fix(S_j^\infty) = Fix(S_1^\infty S_2^\infty \ldots S_M^\infty)$. Since $\{x_m\}$ is bounded, there exists a subsequence $\{x_{m_t}\}$ of $\{x_m\}$ such that $x_{m_t} \rightharpoonup y$ as $t \to \infty$. Further, we have

$$\begin{aligned} \|x_{m_t} - S_M^\infty S_{M-1}^\infty \ldots S_1^\infty x_{m_t}\| &\leq \|x_{m_t} - S_M^{m_t} S_{M-1}^{m_t} \ldots S_1^{m_t} x_{m_t}\| \\ &\quad + \|S_M^{m_t} S_{M-1}^{m_t} \ldots S_1^{m_t} x_{m_t} - S_M^\infty S_{M-1}^\infty \ldots S_1^{m_t} x_{m_t}\| \\ &\quad + \|S_M^\infty S_{M-1}^\infty \ldots S_1^{m_t} x_{m_t} - S_M^\infty S_{M-1}^\infty \ldots S_1^\infty x_{m_t}\| \\ &\leq \|x_{m_t} - S_M^{m_t} S_{M-1}^{m_t} \ldots S_1^{m_t} x_{m_t}\| \\ &\quad + \|S_M^{m_t} S_{M-1}^{m_t} \ldots S_1^{m_t} x_{m_t} - S_M^\infty S_{M-1}^\infty \ldots S_1^{m_t} x_{m_t}\| \\ &\quad + \|S_1^{m_t} x_{m_t} - S_1^\infty x_{m_t}\| \\ &\leq \|x_{m_t} - S_M^{m_t} S_{M-1}^{m_t} \ldots S_1^{m_t} x_{m_t}\| \\ &\quad + \sup_{p\in D'} \|S_M^{m_t} p - S_M^\infty p\| + \sup_{p\in D''} \|S_1^{m_t} p - S_1^\infty p\|, \end{aligned} \tag{3.28}$$

where D' is a bounded subset including $\{S_1^{m_t} x_{m_t}\}$ and D'' is a bounded subset including $\{x_{m_t}\}$. From (3.22) and (3.27), we get

$$\lim_{m\to\infty} \|x_{m_t} - S_M^\infty S_{M-1}^\infty \ldots S_1^\infty x_{m_t}\| = 0.$$

From Lemma 2.2, we have $y \in Fix(S_M^\infty S_{M-1}^\infty \ldots S_1^\infty)$.

Step 5. Finally, we claim that

$$\lim_{m\to\infty} \sup \langle(\mu V - \rho Y)\bar{p}, \bar{p} - x_m\rangle \leq 0.$$

We have

$$\begin{aligned}\limsup_{m\to\infty}\langle(\mu V-\rho U)\bar{p},\bar{p}-x_m\rangle &= \lim_{m\to\infty}\langle(\mu V-\rho U)\bar{p},\bar{p}-x_{m_v}\rangle\\ &= \langle(\mu V-\rho U)\bar{p},\bar{p}-y\rangle\\ &\leq 0.\end{aligned}$$

Next, we show that $x_m\to\bar{p}$ as $m\to\infty$.

$$\begin{aligned}
\|x_{m+1}-\bar{p}\|^2 &= \|\alpha_m(\rho U(x_m)-\mu V(\bar{p}))+\beta_m(x_m-\bar{p})+[(1-\beta_m)I-\alpha_m\mu V]S_M^mS_{M-1}^m\dots S_1^m z_m\\
&\quad -[(1-\beta_m)I-\alpha_m\mu V]S_M^mS_{M-1}^m\dots S_1^m\bar{p}\|^2\\
&\leq \|\beta_m(x_m-\bar{p})+[(1-\beta_m)I-\alpha_m\mu V]S_M^mS_{M-1}^m\dots S_1^m z_m\\
&\quad -[(1-\beta_m)I-\alpha_m\mu V]S_M^mS_{M-1}^m\dots S_1^m\bar{p}\|^2\\
&\quad +2\alpha_m\langle\rho U(x_m)-\mu V(\bar{p}),x_{m+1}-\bar{p}\rangle\\
&\leq (\beta_m\|x_m-\bar{p}\|+(1-\beta_m)\left\|\left(I-\frac{\alpha_m\mu V}{1-\beta_m}\right)S_M^mS_{M-1}^m\dots S_1^m z_m\right.\\
&\quad \left.-\left(I-\frac{\alpha_m\mu V}{1-\beta_m}\right)S_M^mS_{M-1}^m\dots S_1^m\bar{p}\right\|^2\\
&\quad +2\alpha_m\rho\langle U(x_m)-U(\bar{q}),x_{m+1}-\bar{p}\rangle+2\alpha_m\langle\rho U(\bar{p})-\mu V(\bar{p}),x_{m+1}-\bar{p}\rangle\\
&\leq \left(\beta_m\|x_m-\bar{p}\|+(1-\beta_m)\left(1-\frac{\alpha_m\zeta}{1-\beta_m}\right)\|z_m-\bar{p}\|\right)^2\\
&\quad +2\alpha_m\rho\sigma\|x_m-\bar{p}\|\|x_{m+1}-\bar{p}\|+2\alpha_m\langle\rho U(\bar{p})-\mu V(\bar{p}),x_{m+1}-\bar{p}\rangle\\
&\leq (\beta_m\|x_m-\bar{p}\|+(1-\beta_m-\alpha_m\zeta)\|z_m-\bar{p}\|)^2+\alpha_m\rho\sigma(\|x_m-\bar{p}\|^2+\|x_{m+1}-\bar{p}\|^2)\\
&\quad +2\alpha_m\langle\rho U(\bar{p})-\mu V(\bar{p}),x_{m+1}-\bar{p}\rangle\\
&= [(1-\alpha_m\zeta)^2+\alpha_m\rho\sigma]\|x_m-\bar{p}\|^2+\alpha_m\rho\sigma\|x_{m+1}-\bar{p}\|^2\\
&\quad +2\alpha_m\langle\rho U(\bar{p})-\mu V(\bar{p}),x_{m+1}-\bar{p}\rangle\\
&\leq \left(\frac{(1-\alpha_m\zeta)^2+\alpha_m\rho\sigma}{1-\alpha_m\rho\sigma}\right)\|x_m-\bar{p}\|^2\\
&\quad +\left(\frac{2\alpha_m}{1-\alpha_m\rho\sigma}\right)\langle\rho U(\bar{p})-\mu V(\bar{p}),x_{m+1}-\bar{p}\rangle\\
&= \left(1-\frac{2(\zeta-\rho\sigma)\alpha_m}{1-\alpha_m\rho\sigma}\right)\|x_m-\bar{p}\|^2+\left(\frac{\alpha_m^2\zeta^2}{1-\alpha_m\rho\sigma}\right)\|x_m-\bar{p}\|^2\\
&\quad +\left(\frac{2\alpha_m^2\zeta^2}{1-\alpha_m\rho\sigma}\right)\|x_m-\bar{p}\|^2+\left(\frac{2\alpha_m}{1-\alpha_m\rho\sigma}\right)\langle\rho U(\bar{p})-\mu V(\bar{p}),x_{m+1}-\bar{p}\rangle\\
&\leq \left(1-\frac{2(\zeta-\rho\sigma)\alpha_m}{1-\alpha_m\rho\sigma}\right)\|x_m-\bar{p}\|^2+\left(\frac{2\alpha_m(\zeta-\rho\sigma)}{1-\alpha_m\rho\sigma}\right)\left\{\frac{\alpha_m\zeta^2}{2(\tau-\rho\sigma)}M_1\right.\\
&\quad \left.+\left(\frac{1}{\zeta-\rho\sigma}\right)\left\langle\rho U(\bar{p})-\mu V(\bar{p}),x_{m+1}-\bar{p}\right\rangle\right\}\\
&= (1-\chi_m)\|x_m-\bar{p}\|^2+\chi_m\sigma_m,
\end{aligned}$$

where $M_1 = \sup\{\|x_m - \bar{p}\|^2 | m \geq 0\}$, $\chi_m = \frac{2(\zeta-\rho\sigma)\alpha_m}{1-\alpha_m\rho\sigma}$ and

$$\sigma_m = \frac{\alpha_m \zeta^2}{2(\zeta - \rho\sigma)} M_1 + \frac{1}{\zeta - \rho\sigma}\Big\langle \rho U(\bar{p}) - \mu T(\bar{p}), x_{m+1} - \bar{p}\Big\rangle.$$

Since $\chi_m \to 0, \sum_{m=0}^{\infty} \chi_m = \infty$ and $\limsup_{m\to\infty} \sigma_m \leq 0$. By applying Lemma 2.8, we get $x_m \to \bar{p}$ as $m \to \infty$. □

Now we give some results from Theorem 3.1. First, we give an iterative method to find the common solution of the HFPP (1.1) and the SEP (1.5)-(1.6).

Corollary 3.1. *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bifunction satisfying Assumption 2.1 and let G be upper semicontinuous and let $S_j : C \to C$ be a nonexpansive mapping for each $j = 1,2,\ldots,M$. Let $V : C \to C$ be a l-lipschitzian continuous and η-strongly monotone mapping with $l > 0$ and $\eta > 0$ and let $U : C \to C$ be a τ-lipschitzian continuous mapping with $\tau > 0$. Let $0 < \mu < \frac{2\eta}{l^2}$ and $0 < \rho\tau < \zeta$, where $\zeta = 1 - \sqrt{1 - \mu(2\eta - \mu l^2)}$. Assume that $\Gamma_1 = \Omega_1 \bigcap \Phi \neq \emptyset$. Let the iterative sequences $\{z_m\}$ and $\{x_m\}$ be generated by hybrid iterative algorithm:*

$$\begin{cases} x_0 \in C; \\ z_m = S^F_{r_m}(x_m + \gamma A^*(S^G_{r_m} - I)Ax_m); \\ x_{m+1} = \alpha_m \rho U(x_m) + \beta_m x_m + [(1-\beta_m)I - \alpha_m \mu V] S^m_M S^m_{M-1} \ldots S^m_1 z_m, \forall m \geq 0 \end{cases}$$

where $S^m_j = (1 - \varpi^j_m)I + \varpi^j_m S_j$, $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, $\varpi^j_m \in (\xi_1, \xi_2)$ for some $\xi_1, \xi_2 \in (0,1)$, $\{\alpha_m\}$ and $\{\beta_m\}$ are two real sequences in $(0,1)$ and $\{r_m\} \subset (0,\infty)$ satisfy the conditions:

(1) $\lim_{m\to\infty} \alpha_m = 0$ and $\sum_{m=1}^{\infty} \alpha_m = \infty$.
(2) $0 < \liminf_{m\to\infty} \beta_m \leq \limsup_{m\to\infty} \beta_m < 1$.
(3) $\lim_{m\to\infty} \inf r_m > 0$.
(4) $\lim_{m\to\infty} |\varpi^j_{m+1} - \varpi^j_m| = 0$ for $j = 1,2,\ldots,M$.

Then the sequence $\{x_m\}$ converges strongly to $\bar{p} \in \Gamma_1$.

Proof. Setting $f = g = 0$ in Theorem 3.1, we have the conclusion immediately. □

Next, we give an iterative method to find a common solution of HFPP (1.1) and the SVIP (1.7)-(1.8).

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let mapping $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be θ_1-inverse strongly monotone and θ_2-inverse strongly monotone, respectively. Let $S_j : C \to C$ be a nonexpansive mapping for each $j = 1,2,\ldots,M$. Let $V : C \to C$ be a l-lipschitzian continuous and η-strongly monotone mapping with $l > 0$ and $\eta > 0$ and let $U : C \to C$ be a τ-lipschitzian continuous mapping with $\tau > 0$. Let $0 < \mu < \frac{2\eta}{l^2}$ and $0 < \rho\tau < \zeta$, where $\zeta = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$. Assume that $\Gamma_2 = \Omega_2 \bigcap \Phi \neq \emptyset$. Let the iterative sequences $\{z_m\}$ and $\{x_m\}$*

be generated by hybrid iterative algorithm:

$$\begin{cases} x_0 \in C; \\ y_m = X(x_m); w_m = Y(Ay_m); \\ z_m = y_m + \gamma A^*(w_m - Ay_m); \\ x_{m+1} = \alpha_m \rho U(x_m) + \beta_m x_m + [(1-\beta_m)I - \alpha_m \mu V] S_M^m S_{M-1}^m \dots S_1^m z_m, \forall m \geq 0 \end{cases}$$

where $S_j^m = (1-\varpi_m^j)I + \varpi_m^j S_j$, $X = P_C(I - r_m f), Y = P_Q(I - r_m g)$, $\gamma \in \left(0, \frac{1}{\|A\|^2}\right), \varpi_m^j \in (\xi_1, \xi_2)$ *for some* $\xi_1, \xi_2 \in (0,1)$, $\{\alpha_m\}$ *and* $\{\beta_m\}$ *are two real sequences in* $(0,1)$ *and* $\{r_m\} \subset (0, \alpha)$, *where* $\alpha = 2\min\{\theta_1, \theta_2\}$, *satisfy the conditions:*

(1) $\lim_{m\to\infty} \alpha_m = 0$ and $\sum_{m=1}^{\infty} \alpha_m = \infty$.
(2) $0 < \liminf_{m\to\infty} \beta_m \leq \limsup_{m\to\infty} \beta_m < 1$.
(3) $\lim_{m\to\infty} \inf r_m > 0$.
(4) $\lim_{m\to\infty} |\varpi_{m+1}^j - \varpi_m^j| = 0$ for $j = 1, 2, \dots, M$.

Then the sequence $\{x_m\}$ *converges strongly to* $\bar{p} \in \Gamma_2$.

Proof. Setting $F = G = 0$, we get $S_{r_m}^F = P_C$ and $S_{r_m}^G = P_Q$ in Theorem 3.1. □

Further, we give an iterative method to find a common solution of the HFPP (1.1) and the SMVIP (1.9)-(1.10).

Corollary 3.3. *Let* H_1 *and* H_2 *be two real Hilbert spaces. Let* $C \subseteq H_1$ *and* $Q \subseteq H_2$ *be nonempty, closed and convex subsets. Let* $A : H_1 \to H_2$ *be a bounded linear operator with its adjoint operator* A^*. *Let* $N : H_1 \to 2^{H_1}$ *and* $P : H_2 \to 2^{H_2}$ *be the multi-valued maximal monotone mappings. Let mappings* $f : C \to H_1$ *and* $g : Q \to H_2$ *be* θ_1*-inverse strongly monotone and* θ_2*-inverse strongly monotone, respectively. Let* $S_j : C \to C$ *be a nonexpansive mapping for each* $j = 1, 2, \dots, M$. *Let* $V : C \to C$ *be a* l*-lipschitzian continuous and* η*-strongly monotone mapping with* $l > 0$ *and* $\eta > 0$ *and let* $U : C \to C$ *be a* τ*-lipschitzian continuous mapping with* $\tau > 0$. *Let* $0 < \mu < \frac{2\eta}{l^2}$ *and* $0 < \rho\tau < \zeta$, *where* $\zeta = 1 - \sqrt{1 - \mu(2\eta - \mu l^2)}$. *Assume that* $\Gamma_3 = \Omega_3 \bigcap \Phi \neq \emptyset$. *Let the iterative sequences* $\{z_m\}$ *and* $\{x_m\}$ *be generated by hybrid iterative algorithm:*

$$\begin{cases} x_0 \in C; \\ y_m = X(x_m); w_m = Y(Ay_m); \\ z_m = y_m + \gamma A^*(w_m - Ay_m); \\ x_{m+1} = \alpha_m \rho U(x_m) + \beta_m x_m + [(1-\beta_m)I - \alpha_m \mu V] S_M^m S_{M-1}^m \dots S_1^m z_m, \forall m \geq 0 \end{cases}$$

where $S_j^m = (1-\varpi_m^j)I + \varpi_m^j S_j$, $X = J_\lambda^N(I - \lambda f), Y = J_\lambda^P(I - \lambda g)$, $\gamma \in \left(0, \frac{1}{\|A\|^2}\right), \varpi_m^j \in (\xi_1, \xi_2)$ *for some* $\xi_1, \xi_2 \in (0,1)$, $\{\alpha_m\}$ *and* $\{\beta_m\}$ *are two real sequences in* $(0,1)$ *and* $\{\lambda\} \subset (0, \alpha)$, *where* $\alpha = 2\min\{\theta_1, \theta_2\}$, *satisfy the conditions:*

(1) $\lim_{m\to\infty} \alpha_m = 0$ and $\sum_{m=1}^{\infty} \alpha_m = \infty$.
(2) $0 < \liminf_{m\to\infty} \beta_m \leq \limsup_{m\to\infty} \beta_m < 1$.
(3) $\lim_{m\to\infty} |\varpi_{m+1}^j - \varpi_m^j| = 0$ for $j = 1, 2, \dots, M$.

Then the sequence $\{x_m\}$ *converges strongly to* $\bar{p} \in \Gamma_3$.

Last, we give an iterative method to find a common solution of the HFPP (1.1) and the SNPP (1.11)-(1.12).

Corollary 3.4. *Let H_1 and H_2 be two real Hilbert spaces. Let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let $N : H_1 \to 2^{H_1}$ and $P : H_2 \to 2^{H_2}$ be the multi-valued maximal monotone mappings and let $S_j : C \to C$ be a nonexpansive mapping for each $j = 1,2,\ldots,M$. Let $V : C \to C$ be a l-lipschitzian continuous and η-strongly monotone mapping with $l > 0$ and $\eta > 0$ and let $U : C \to C$ be a τ-lipschitzian continuous mapping with $\tau > 0$. Let $0 < \mu < \frac{2\eta}{l^2}$ and $0 < \rho\tau < \zeta$, where $\zeta = 1 - \sqrt{1-\mu(2\eta-\mu l^2)}$. Assume that $\Gamma_4 = \Omega_4 \bigcap \Phi \neq \emptyset$. Let the iterative sequences $\{z_m\}$ and $\{x_m\}$ be generated by hybrid iterative algorithm:*

$$\begin{cases} x_0 \in C; \\ z_m = J_\lambda^N(x_m + \gamma A^*(J_\lambda^P - I)Ax_m); \\ x_{m+1} = \alpha_m \rho U(x_m) + \beta_m x_m + [(1-\beta_m)I - \alpha_m \mu V] S_M^m S_{M-1}^m \ldots S_1^m z_m, \forall m \geq 0 \end{cases}$$

where $S_j^m = (1-\varpi_m^j)I + \varpi_m^j S_j$, $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, $\varpi_m^j \in (\xi_1, \xi_2)$ for some $\xi_1, \xi_2 \in (0,1)$, $\{\alpha_m\}$ and $\{\beta_m\}$ are two real sequences in $(0,1)$ and $\{\lambda\} \subset (0,\alpha)$, where $\alpha = 2\min\{\theta_1, \theta_2\}$, satisfy the conditions:

(1) $\lim_{m\to\infty} \alpha_m = 0$ and $\sum_{m=1}^\infty \alpha_m = \infty$.
(2) $0 < \liminf_{m\to\infty} \beta_m \leq \limsup_{m\to\infty} \beta_m < 1$.
(3) $\lim_{m\to\infty} |\varpi_{m+1}^j - \varpi_m^j| = 0$ for $j = 1,2,\ldots,M$.

Then the sequence $\{x_m\}$ converges strongly to $\bar{p} \in \Gamma_4$.

4. A Numerical Example

Now we give a numerical example which illustrate Theorem 3.1.

Let $H_1 = H_2 = \mathbb{R}$, the set of all real numbers with the inner product defined by $\langle x,y\rangle = xy$, $\forall x,y \in \mathbb{R}$ and standard norm $|.|$. Let $C = [0,+\infty)$, $Q = (-\infty,0]$ and let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be defined by

$$\begin{aligned} F(x,y) &= x^2 + y^2 - 3xy, \quad \forall\, x,y \in C; \\ G(u,v) &= 2u^2 - 3v^2 + 5uv, \quad \forall\, u,v \in Q. \end{aligned}$$

Let the mappings $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be defined by

$$\begin{aligned} f(x) &= x+2, \quad \forall x \in H_1, \\ g(u) &= 5u, \quad \forall u \in H_2, \end{aligned}$$

respectively. Let the mapping $A : H_1 \to H_2$ be defined by $A(x) = 2x$, $\forall x \in H_1$ and $S_j : C \to C$, $V : C \to C$ and $U : C \to C$ are defined by $S_j x = 0$ for $j = 1,2,\ldots,M$, $V(x) = 2x$ and $U(x) = \frac{x}{2} + 1$, $\forall x \in C$. It is easy to see that $\varpi = \frac{1}{2}$, $\eta = l = 2$. Hence $0 < \mu < \frac{2\eta}{l^2} = 1$. Put $\mu = 1$ we get $\zeta = 1 - \sqrt{1-\mu(2\eta-\mu l^2)} = 1$. From $0 < \rho\tau < \zeta$, we have $0 < \rho < 2$, i.e., $\rho \in (0,2)$. Without loss of generality, we put $\rho = 1$. Let

$$\alpha_m = \frac{1}{3m^2},\ \beta_m = \frac{2m-1}{3m^2},\ r_m = \frac{1}{2} \text{ and } \varpi_m^j = \frac{1}{3}$$

TABLE 1. Results for different initial values

No. of iterations	x_m	x_m	x_m
	$x_0 = -1.5$	$x_0 = 1$	$x_0 = 2$
m=1	-1.2540	0.5318	1.0637
m=5	-0.6937	0.2370	0.4741
m=10	-0.3556	0.1245	0.2489
m=15	-0.1867	0.0583	0.1165
m=20	-0.0874	0.0312	0.0623
m=25	-0.0467	0.0148	0.0295
m=30	-0.0222	0.0080	0.0159
m=35	-0.0119	0.0038	0.0076
m=40	-0.0057	0.0021	0.0041
m=45	-0.0031	0.0010	0.0020
m=50	-0.0015	0.0005	0.0011
m=55	-0.0009	0.0003	0.0005
m=60	-0.0004	0.0001	0.0003
m=65	-0.0002	0.0000	0.0001
m=70	-0.0000	0.0000	0.0000

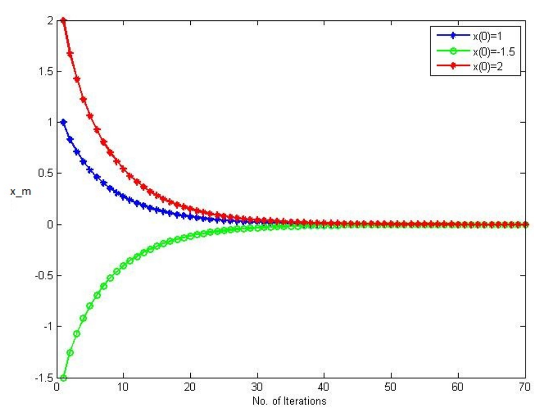

FIGURE 1. The convergence of x_m with three initial values

for each $j = 1, 2, \ldots, M$. The sequences $\{\alpha_m\}$, $\{\beta_m\}$, $\{r_m\}$ and $\{\varpi_m^j\}$ satisfy conditions (i)-(iv). Since $S_j x = 0$ for $j = 1, 2, \ldots, M$ and $Ax = 2x$ for every $x \in \mathbb{R}$, we have

$$\bigcap_{j=1}^{M} Fix(S_j) = \{0\}$$

and A is a bounded linear operator with $A^* = A$ and $\|A\| = 2$ and hence $\gamma \in (0, \frac{1}{4})$. Therefore, we choose $\gamma = 0.2$. Further f, g both are 1 and $\frac{1}{5}$ inverse strongly monotone mappings and hence $\{r_m\} \subset (0, \alpha)$, where $\alpha = 2\min\{1, \frac{1}{5}\} = \frac{2}{5}$. So we take $\lambda = \frac{1}{3}$, which yields that $\Phi = sol(HFPP) = \{0\}$.

All codes were written in Matlab. The values of $\{x_m\}$ with different m are given in Table 1.

5. Conclusion

In this paper, we derived an iterative algorithm for finding a common solution of a split mixed equilibrium problem and a hierarchical fixed point problem for a finite family of nonexpansive mappings. We proved that the iterative algorithm converges strongly in Hilbert spaces. Finally, we presented a numerical example to clarify our main result. The method and results presented in this paper generalize and improve the corresponding results announced recently.

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