Appl. Set-Valued Anal. Optim. 1 (2019), No. 2, pp. 187-201
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.1.2019.2.07

SOME HOMOGENEOUS q-DIFFERENCE OPERATORS AND THE ASSOCIATED GENERALIZED HAHN POLYNOMIALS

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Abstract. In this paper, we first construct the homogeneous q-shift operator $\widetilde{E}(a,b;D_q)$ and the homogeneous q-difference operator $\widetilde{L}(a,b;\Theta_{x,y})$. We then apply these operators in order to represent and investigate a family of generalized Cauchy polynomials and a general form of the q-Hahn polynomials. We derive some q-identities such as generating functions, extended generating functions, Mehler's formula and Rogers' formula for these q-polynomials. Relevant connections of the q-identities presented here with a number of known or new results associated with various specialized families of q-polynomials are also considered.

Keywords. Cauchy polynomials; Hahn polynomials; Rogers' formula; Cauchy polynomials; Generating functions.

2010 Mathematics Subject Classification. 05A30, 33D15, 33D45.

1. INTRODUCTION

We adopt the commonly used conventions and notations for basic (or q-) series and basic (or q-) polynomials. For the convenience of the reader, we provide a summary of the mathematical notations and definitions to be used in this paper. We refer the reader to the general references (see, for example, [7, 11, 16, 18]) for the definitions and notations.

Throughout this paper, we assume $0 < q < 1$. For a complex numbers a, the q-shifted factorials are defined by

$$(a;q)_0 := 1 \qquad \text{and} \qquad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \qquad \big(n \in \mathbb{N} := \{1,2,3,\cdots\}\big) \tag{1.1}$$

and, for large n, we have

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k). \tag{1.2}$$

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Received April 7, 2019; Accepted June 23, 2019.

The q-numbers and the q-factorials are defined as follows:

$$[n]_q := \frac{1-q^n}{1-q}, \quad [n]_q! := \prod_{k=1}^{n}\left(\frac{1-q^k}{1-q}\right) \qquad \text{and} \qquad [0]_q! := 1. \tag{1.3}$$

The q-binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q. \tag{1.4}$$

The basic (or q-) hypergeometric function of the variable z and with $\mathfrak{r}$ numerator and $\mathfrak{s}$ denominator parameters (see, for details, the monographs by Slater [16, Chapter 3] and by Srivastava and Karlsson [18, p. 347, Eq. (272)]; see also [11]) is defined as follows:

$${}_{\mathfrak{r}}\Phi_{\mathfrak{s}}\left[\begin{matrix} a_1, a_2, \cdots, a_{\mathfrak{r}}; \\ \\ b_1, b_2, \cdots, b_{\mathfrak{s}}; \end{matrix} \; q; z\right] = \sum_{n=0}^{\infty}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} \frac{(a_1, a_2, \cdots, a_{\mathfrak{r}}; q)_n}{(b_1, b_2, \cdots, b_{\mathfrak{s}}; q)_n} \frac{z^n}{(q;q)_n}. \tag{1.5}$$

Basic or q-hypergeometric series and various associated families of q-polynomials are useful in a wide variety of fields including, for example, the theory of partitions, number theory, combinatorial analysis, finite vector spaces, Lie theory, particle physics, non-linear electric circuit theory, mechanical engineering, theory of heat conduction, quantum mechanics, cosmology, and statistics (see [18, pp. 346–351] and the references cited therein).

Here, in our present investigation, we are mainly concerned with the Cauchy polynomials $p_n(x,y)$ as given below (see [7]):

$$p_n(x,y) := (x-y)(x-qy)\cdots(x-q^{n-1}y) = (y/x;q)_n\, x^n \tag{1.6}$$

with the following generating function ([5]):

$$\sum_{k=0}^{\infty} p_n(x,y)\, \frac{t^n}{(q;q)_k} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \tag{1.7}$$

where [5]

$$p_n(x,y) = (-1)^n\, q^{\binom{n}{2}}\, p_n(y, q^{1-n}x) \tag{1.8}$$

and

$$p_{n-k}(x, q^{1-n}y) = (-1)^{n-k}\, q^{\binom{k}{2}-\binom{n}{2}}\, p_{n-k}(y, q^k x). \tag{1.9}$$

The Cauchy polynomials $p_n(x,y)$ arise naturally in the q-umbral calculus (see, for details, the works by Andrews [2], Araci *et al.* [3], Goldman and Rota [8], Ihrig and Ismail [9], Johnson [10] and Roman [12]). The generating function (1.7) is also the homogeneous version of the Cauchy identity or the following q-binomial theorem (see [7]):

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k}\, z^k = {}_1\Phi_0\left[\begin{matrix} a; \\ \\ —; \end{matrix} \; q; z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{1.10}$$

Putting $a = 0$, the q-binomial theorem (1.10) becomes Euler's identity given by (see [7])

$$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \tag{1.11}$$

and its inverse relation (see [7]):

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} z^k}{(q;q)_k} = (z;q)_\infty. \tag{1.12}$$

Recently, Saad and Sukhi [15] introduced the following q-exponential operator $R(bD_q)$:

$$R(bD_q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} (bD_q)^k, \tag{1.13}$$

where the q-difference operator or the q-derivative, acting upon the variable a, is defined by (see [6] and [14])

$$D_q\{f(a)\} = \frac{f(a) - f(qa)}{a}. \tag{1.14}$$

Evidently, we have

$$\lim_{q \to 1-} \left\{ \frac{D_q\{f(a)\}}{1-q} \right\} = f'(a),$$

provided that the derivative $f'(a)$ exists. Moreover, it is easily seen that

$$R(yD_q)\{x^n\} = p_n(x,y). \tag{1.15}$$

Suppose that the operator $R(yD_q)$ acts upon the variable x. We then have the following consequence (see [15] and [17]):

$$R(yD_q)\left\{\frac{1}{(xt;q)_\infty}\right\} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \tag{1.16}$$

$$R(yD_q)\left\{\frac{1}{(xt,xs;q)_\infty}\right\} = \frac{(ys;q)_\infty}{(xt,xs;q)_\infty} {}_1\Phi_1\left[\begin{matrix} xs; \\ ys; \end{matrix} \; q;yt\right], \tag{1.17}$$

$$R(yD_q)\left\{\frac{(xv;q)_\infty}{(xt,xs;q)_\infty}\right\} = \frac{(ys;q)_\infty}{(xs;q)_\infty} {}_2\Phi_1\left[\begin{matrix} v/t, y/x; \\ ys; \end{matrix} \; q;xt\right]. \tag{1.18}$$

Srivastava and Abdlhusein [17] showed by setting $v = 0$ in (1.18), that

$$R(yD_q)\left\{\frac{1}{(xt,xs;q)_\infty}\right\} = \frac{(ys;q)_\infty}{(xs;q)_\infty} {}_2\Phi_1\left[\begin{matrix} y/x, 0; \\ ys; \end{matrix} \; q;xt\right]. \tag{1.19}$$

Comparing (1.19) with (1.17), we get the following q-hypergeometric transformation:

$${}_2\Phi_1\left[\begin{matrix} y/x, 0; \\ ys; \end{matrix} \; q;xt\right] = \frac{1}{(xt;q)_\infty} {}_1\Phi_1\left[\begin{matrix} xs; \\ ys; \end{matrix} \; q;yt\right]. \tag{1.20}$$

Motivated by the recent investigations by (for example) [4], Saad and Sukhi [15] and Srivastava and Abdlhusein [17], we aim here to introduce the (presumably new) homogeneous q-difference operators $\widetilde{E}(a,b;D_q)$ and $\widetilde{T}(a,b;\Theta_{x,y})$.

Definition 1.1. The first homogeneous q-difference operator $\widetilde{E}(a,b;D_q)$ is defined by

$$\widetilde{E}(a,b;D_q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (a;q)_k}{(q;q)_k} (bD_q)^k. \tag{1.21}$$

Remark 1.1. When compared with $R(yD_q)$, the homogeneous q-difference operator $\widetilde{E}(a,b;D_q)$ given by (1.21) involves two parameters. Clearly, the operator $R(yD_q)$ can be considered as a special case of this operator in (1.21) for $a=0$, that is,

$$\widetilde{E}(0,b;D_q)=R(bD_q). \tag{1.22}$$

Definition 1.2. The second homogeneous q-difference operator $\widetilde{L}(a,b;\Theta_{x,y})$ is defined by

$$\widetilde{L}(a,b;\Theta_{x,y})=\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}\,(a;q)_k}{(q;q)_k}\,(b\Theta_{x,y})^k, \tag{1.23}$$

where (see [6] and [14])

$$\Theta_{x,y}\{f(x,y)\}=\frac{f(q^{-1}x,y)-f(x,qy)}{q^{-1}x-y}, \tag{1.24}$$

which acts upon functions of suitable variables x and y (see [17] and [14]), and

$$\Theta_{x,y}^k\{p_n(y,x)\}=(-1)^k\,\frac{(q;q)_n}{(q;q)_{n-k}}\,p_{n-k}(y,x) \qquad \text{and} \qquad \Theta_{x,y}^k\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\}=(-t)^k\,\frac{(xt;q)_\infty}{(yt;q)_\infty}. \tag{1.25}$$

Remark 1.2. The homogeneous q-difference operator $\widetilde{L}(a,b;\Theta_{x,y})$ given by (1.23) involves two parameters. It can be considered as a generalization of the homogeneous q-difference operator $L(b\Theta_{x,y})$, which was introduced by Saad and Sukhi [14]. In its special case when $a=0$, we have

$$\widetilde{L}(0,b;\Theta_{x,y})=L(b\Theta_{x,y}). \tag{1.26}$$

The definition (1.23) is motivated by the natural question of extending such generating functions as (for example) Mehler's formula and Rogers-type formula for a general form for the Hahn polynomials $h_n(x,y,a,b|q)$. The operators defined here (1.21) and (1.23) turn out to be suitable for dealing with the generalized Cauchy polynomials $p_n(x,y,a)$ as well as the generalized Hahn polynomials $h_n(x,y,a,b|q)$. They are then applied in order to represent and investigate such q-identities as generating functions, extended generating functions, Mehler's formula and Rogers-type formula for the polynomials $p_n(x,y,a)$ and $h_n(x,y,a,b|q)$.

2. Generalized Cauchy polynomials $p_n(x,y,a)$

In this section, we introduce the generalized Cauchy polynomials $p_n(x,y,a)$. We then represent these polynomials $p_n(x,y,a)$ by means of the homogeneous q-difference operator $\widetilde{E}(a,b;D_q)$ and derive their generating function. We also use the operational formula for $p_n(x,y,a)$ in order to establish an extended generating function, Mehler's formula and the Rogers formula for the generalized Cauchy polynomials $p_n(x,y,a)$.

Let us start by recalling following Leibniz rule (see [13]):

$$D_q^n\{f(x)g(x)\}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{k(k-n)}\,D_q^k\{f(x)\}\,D_q^{n-k}\{g(q^kx)\}. \tag{2.1}$$

For $f(x)=x^k$, $f(x)=(xt;q)_\infty$ and

$$f(x)={}_1\Phi_1\left[\begin{matrix}a;\\ & q;bx\\ 0;\end{matrix}\right],$$

we have the following identities:

$$D_q^n\{x^k\} = (q^{k-n+1};q)_n\, x^{k-n}, \quad D_q^n\{(xt;q)_\infty\} = q^{\binom{n}{2}}\, t^n\, \frac{(xt;q)_\infty}{(xt;q)_n} \tag{2.2}$$

and

$$D_q^n\left\{ {}_1\Phi_1 \left[\begin{matrix} a; \\ & q; bx \\ 0; \end{matrix} \right] \right\} = (-1)^n\, b^n\, (a;q)_n\, q^{\binom{n}{2}}\, {}_1\Phi_1 \left[\begin{matrix} aq^n; \\ & q; bxq^n \\ 0; \end{matrix} \right]. \tag{2.3}$$

Suppose now that the operator D_q acts upon the variable s. If we set $f(s) = s^n$ and $g(s) = s^m$ or, alternatively,

$$f(s) = {}_1\Phi_1 \left[\begin{matrix} a; \\ & q; bsq^k \\ 0; \end{matrix} \right]$$

and

$$g(s) = \frac{1}{(xsq^k;q)_\infty},$$

and make use of (2.2) and (2.1), we get the following identities to be used in our investigation:

$$\left(q^{n+m-k+1};q\right)_k = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{j(j-k+m)}\, \left(q^{n-j+1};q\right)_j\, \left(q^{m-k+j+1};q\right)_{k-j} \tag{2.4}$$

and

$$D_q^n \left\{ \frac{{}_1\Phi_1 \left[\begin{matrix} a; \\ & q; bsq^k \\ 0; \end{matrix} \right]}{(xsq^k;q)_\infty} \right\}$$

$$= \frac{1}{(xs;q)_\infty} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j-n+k)+\binom{j}{2}}\, x^j\, (xs;q)_j\, D_q^{n-j} \left\{ {}_1\Phi_1 \left[\begin{matrix} a; \\ & q; bsq^{j+k} \\ 0; \end{matrix} \right] \right\}. \tag{2.5}$$

Next, recalling that the homogeneous q-difference operator $\widetilde{E}(a,b;D_q)$ is defined by (1.21) (see Definition 1.1) as well as its special case when $a = 0$ by Remark 1.1, we are led to Proposition 2.1 below.

Proposition 2.1. *Suppose that the operator D_q acts upon the variable x. Then*

$$\widetilde{E}(a,y;D_q)\left\{\frac{1}{(xt;q)_\infty}\right\} = \frac{1}{(xt;q)_\infty}\, {}_1\Phi_1 \left[\begin{matrix} a; \\ & q; yt \\ 0; \end{matrix} \right], \tag{2.6}$$

$$\widetilde{E}(a,y;D_q)\left\{\frac{1}{(xt,xs;q)_\infty}\right\}$$

$$= \frac{1}{(xt,xs;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n\, q^{\binom{n}{2}}\, (a;q)_n\, (sy)^n}{(q;q)_n}\, {}_2\Phi_1 \left[\begin{matrix} aq^n, xs; \\ & q; q^n\, yt \\ 0; \end{matrix} \right] \tag{2.7}$$

and

$$\widetilde{E}(a,y;D_q)\left\{\frac{x^k}{(xt;q)_\infty}\right\} = \frac{x^k}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(a;q)_n (yt)^n}{(q;q)_n}\,{}_3\Phi_2\left[\begin{array}{c} q^{-k},aq^n,xt; \\ 0,0; \end{array} q;\frac{y}{x}q^{n+k}\right]. \tag{2.8}$$

Remark 2.1. Upon setting $k=0$ in the assertion (2.8), if we make use of the following identity:

$${}_3\Phi_2\left[\begin{array}{c} 1,aq^n,xt; \\ 0,0; \end{array} q;\frac{y}{x}\,q^n\right]=1,$$

we get (2.6).

Theorem 2.1. *Suppose that the operator D_q acts upon the variable s. Then*

$$\widetilde{E}(a,t;D_q)\left\{\frac{(ys;q)_\infty}{(xs;q)_\infty}\,{}_1\Phi_1\left[\begin{array}{c} a; \\ 0; \end{array} q;bs\right]\right\} = \frac{(ys;q)_\infty}{(xs;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+k} q^{\binom{k}{2}+\binom{n}{2}+\binom{n+j+k}{2}}}{(q;q)_k\,(q;q)_n\,(q;q)_j} \cdot\frac{(a;q)_n\,(a;q)_{n+j+k}\,(xs;q)_k}{(ys;q)_k}\,{}_1\Phi_1\left[\begin{array}{c} aq^n; \\ 0; \end{array} q;bsq^{j+k+n}\right]t^{n+j+k}. \tag{2.9}$$

Proof. We observe that

$$\widetilde{E}(a,t;D_q)\left\{\frac{(ys;q)_\infty}{}(xs;q)_\infty\,{}_1\Phi_1\left[\begin{array}{c} a; \\ 0; \end{array} q;bs\right]\right\} = \sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(a;q)_n\,t^n}{(q;q)_n}\,D_q^n\left\{\frac{(ys;q)_\infty}{(xs;q)_\infty}\,{}_1\Phi_1\left[\begin{array}{c} a; \\ 0; \end{array} q;bs\right]\right\}. \tag{2.10}$$

By applying (2.1), the right-hand side of (2.10) becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(a;q)_n\,t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)}\,D_q^k\{(ys;q)_\infty\}D_q^{n-k}\left\{\frac{{}_1\Phi_1\left[\begin{array}{c} a; \\ 0; \end{array} q;bsq^k\right]}{(xsq^k;q)_\infty}\right\} \tag{2.11}$$

or, equivalently,

$$\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{(-1)^n\, q^{\binom{n}{2}+k(k-n)+\binom{k}{2}}\, t^n\, y^k (a;q)_n}{(q;q)_k\,(q;q)_{n-k}}\,(ysq^k;q)_\infty\, D_q^{n-k}\left\{\frac{{}_1\Phi_1\begin{bmatrix} a; & \\ & q; bsq^k \\ 0; & \end{bmatrix}}{(xsq^k;q)_\infty}\right\}, \tag{2.12}$$

which, upon replacing n by $n+k$, yields

$$(ys;q)_\infty\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{n+k}\, q^{\binom{n+k}{2}-kn+\binom{k}{2}}\, t^{n+k}\, y^k\,(a;q)_{n+k}}{(ys;q)_k\,(q;q)_k\,(q;q)_n}\,D_q^{n}\left\{\frac{{}_1\Phi_1\begin{bmatrix} a; & \\ & q; bsq^k \\ 0; & \end{bmatrix}}{(xsq^k;q)_\infty}\right\}. \tag{2.13}$$

Now, by using (2.5) in (2.13), we find that

$$\begin{aligned}
&\frac{(ys;q)_\infty}{(xs;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n}\frac{(-1)^{n+k}\, q^{-kn+j(j-n+k)+\binom{k}{2}+\binom{n+k}{2}}\,(a;q)_{n+k}\,(xs;q)_k\, y^k\, t^{n+k}\, x^j}{(ys;q)_k\,(q;q)_k\,(q;q)_{n-j}\,(q;q)_j}\\
&\qquad\cdot D_q^{n-j}\left\{{}_1\Phi_1\begin{bmatrix} a; & \\ & q; bsq^{j+k} \\ 0; & \end{bmatrix}\right\}\\
&=\frac{(ys;q)_\infty}{(xs;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{n+j+k}\, q^{-kn-jn+\binom{k}{2}+\binom{n+j+k}{2}}\,(a;q)_{n+j+k}\,(xs;q)_k\, y^k t^{n+j+k}\, x^j}{(ys;q)_k\,(q;q)_k\,(q;q)_n\,(q;q)_j}\\
&\qquad\cdot D_q^{n}\left\{{}_1\Phi_1\begin{bmatrix} a; & \\ & q; bsq^{j+k} \\ 0; & \end{bmatrix}\right\},
\end{aligned} \tag{2.14}$$

in which the right-hand side can be simplified to the following form:

$$\begin{aligned}
&\frac{(ys;q)_\infty}{(xs;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+k}\, q^{\binom{k}{2}+\binom{n}{2}+\binom{n+j+k}{2}}\,(a;q)_n\,(a;q)_{n+j+k}\,(xs;q)_k\, b^n\, y^k\, t^{n+j+k}\, x^j}{(ys;q)_k\,(q;q)_k\,(q;q)_n\,(q;q)_j}\\
&\qquad\cdot {}_1\Phi_1\begin{bmatrix} aq^n; & \\ & q; bsq^{j+k+n} \\ 0; & \end{bmatrix}.
\end{aligned} \tag{2.15}$$

By appropriately combining the above observations, we get the assertion (2.9) of Theorem 2.1. □

Definition 2.1. In terms of the q-shifted factorial, the generalized Cauchy polynomials $p_n(x,y,a)$ are defined by

$$p_n(x,y,a)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k\, q^{\binom{k}{2}}\,(a;q)_k\, x^{n-k}\, y^k \tag{2.16}$$

with

$$p_n(0,y,a)=(-1)^n\, q^{\binom{n}{2}}\,(a;q)_n\, y^n,\quad p_n(x,0,a)=x^n \tag{2.17}$$

and

$$p_n(x,y,0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} x^{n-k} y^k. \tag{2.18}$$

Proposition 2.2. *The following operational formula holds true*:

$$\widetilde{E}(a,y;D_q)\{x^n\} = p_n(x,y,a). \tag{2.19}$$

Proof. In order to demonstrate the assertion (2.19) of Proposition 2.2, we see that

$$\begin{aligned}\widetilde{E}(a,y;D_q)\{x^n\} &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (a;q)_k}{(q;q)_k} (yD_q)^k \{x^n\} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (a;q)_k}{(q;q)_k} \frac{(q;q)_n}{(q;q)_{n-k}} y^k x^{n-k} = p_n(x,y,a).\end{aligned} \tag{2.20}$$

□

Theorem 2.2. $\big[$Generating function for $p_n(x,y,a)\big]$ *The following generating function holds true*:

$$\sum_{n=0}^{\infty} p_n(x,y,a) \frac{t^n}{(q;q)_n} = \frac{1}{(xt;q)_\infty} {}_1\Phi_1 \left[\begin{matrix} a; \\ 0; \end{matrix} \; q; yt \right] \qquad (|xt| < 1). \tag{2.21}$$

Proof. Let us suppose that the operator D_q acts upon the variable x. We then find that

$$\begin{aligned}\sum_{n=0}^{\infty} p_n(x,y,a) \frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty} \widetilde{E}(a,y;D_q)\{x^n\} \frac{t^n}{(q;q)_n} \\ &= \widetilde{E}(a,y;D_q) \left\{ \sum_{n=0}^{\infty} \frac{(xt)^n}{(q;q)_n} \right\} \\ &= \widetilde{E}(a,y;D_q) \left\{ \frac{1}{(xt;q)_\infty} \right\} \\ &= \frac{1}{(xt;q)_\infty} {}_1\Phi_1 \left[\begin{matrix} a; \\ \\ 0; \end{matrix} \; q; yt \right] \qquad (|xt| < 1),\end{aligned} \tag{2.22}$$

which evidently completes the proof of the assertion (2.21) of Theorem 2.2. □

Theorem 2.3. $\big[$Extended generating function for $p_n(x,y,a)\big]$ *It is asserted that*

$$\begin{aligned}&\sum_{n=0}^{\infty} p_{n+k}(x,y,a) \frac{t^n}{(q;q)_n} \\ &\quad = \frac{x^k}{(xt;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (a;q)_n (yt)^n}{(q;q)_n} {}_3\Phi_2 \left[\begin{matrix} q^{-k}, aq^n, xt; \\ \\ 0, 0; \end{matrix} \; q; \frac{y}{x} q^{n+k} \right] \qquad (|xt| < 1).\end{aligned} \tag{2.23}$$

Proof. We suppose that the operator D_q acts upon the variable s. We then obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p_{n+k}(x,y,a)\,\frac{t^n}{(q;q)_n} \\ &= \sum_{n=0}^{\infty} \widetilde{E}(a,y;D_q)\left\{x^{n+k}\right\}\frac{t^n}{(q;q)_n} \\ &= \widetilde{E}(a,y;D_q)\left\{x^k\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\right\} \\ &= \widetilde{E}(a,y;D_q)\left\{\frac{x^k}{(xt;q)_\infty}\right\}. \end{aligned} \tag{2.24}$$

The proof of the assertion (2.23) of Theorem 2.3 is thus completed by applying the relation (2.8). □

Remark 2.2. Upon setting $k=0$ in Theorem 2.3, we get the generating function (2.21).

Theorem 2.4. [Rogers-type formula for $p_n(x,y,a)$] *The following Rogers-type formula holds true*:

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{n+m}(x,y,a)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m} \\ &= \frac{1}{(xt,xs;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n\,q^{\binom{n}{2}}\,(a;q)_n(sy)^n}{(q;q)_n}\,{}_2\Phi_1\left[\begin{matrix} aq^n, xs; \\ 0; \end{matrix}\; q; ytq^n\right] \end{aligned} \tag{2.25}$$

$$(\max\{|xt|,|xs|\}<1).$$

Proof. We suppose that the operator D_q acts upon the variable x. We then find that

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p_{n+m}(x,y,a)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\widetilde{E}(a,y;D_q)\left\{x^{n+m}\right\}\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m} \\ &= \widetilde{E}(a,y;D_q)\left\{\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\sum_{n=0}^{\infty}\frac{(xs)^m}{(q;q)_m}\right\} \\ &= \widetilde{E}(a,y;D_q)\left\{\frac{1}{(xt,xs;q)_\infty}\right\} \qquad (max\{|xt|,|xs|\}<1). \end{aligned} \tag{2.26}$$

The proof of Theorem 2.4 is now completed by making use of the relation (2.7). □

We aim now to present an operator approach to Mehler's formula for the generalized Cauchy polynomials $p_n(x,y,a)$.

Theorem 2.5. [Mehler's formula for $p_n(x,y,a)$] *The following Mehler-type bilinear generating function holds true for the generalized Cauchy polynomials* $p_n(x,y,a)$:

$$\begin{aligned}\sum_{n=0}^{\infty} p_n(x,y,a)\,p_n(u,v,\alpha)\,\frac{t^n}{(q;q)_n} \\ &= \widetilde{E}(a,y;D_q)\left\{\frac{1}{(uxt;q)_\infty}\,{}_1\Phi_1\left[\begin{matrix}\alpha; \\ 0;\end{matrix}\; q; vxt\right]\right\} \qquad (|uxt|<1). \end{aligned} \tag{2.27}$$

Proof. We observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_n(x,y,a)\, p_n(u,v,\alpha)\, \frac{t^n}{(q;q)_n} \\
&\quad = \sum_{n=0}^{\infty} \widetilde{E}(a,y;D_q)\{x^n\} p_n(u,v,\alpha)\, \frac{t^n}{(q;q)_n} \\
&\quad = \widetilde{E}(a,y;D_q) \left\{ \sum_{n=0}^{\infty} p_n(u,v,\alpha)\, \frac{(xt)^n}{(q;q)_n} \right\} \\
&\quad = \widetilde{E}(a,y;D_q) \left\{ \frac{1}{(uxt;q)_\infty}\, {}_1\Phi_1 \left[\begin{matrix} \alpha; \\ & q; vxt \\ 0; \end{matrix} \right] \right\} \qquad (|uxt| < 1),
\end{aligned} \tag{2.28}$$

which evidently completes our demonstration of th the Mehler-type bilinear generating function (2.27) for the generalized Cauchy polynomials $p_n(x,y,a)$. □

3. The homogeneous q-difference operator $\widetilde{L}(a,b;\Theta_{x,y})$ and the generalized Hahn polynomials

In this section, we first define the generalized Hahn polynomials $h_n(x,y,a,b|q)$ in terms of the homogeneous q-difference operator $\widetilde{L}(a,b;\Theta_{x,y})$ given by Definition 1.2. We then derive an extended generating function, a Mehler's formula and a Rogers-type formula for the generalized Hahn polynomials $h_n(x,y,a,b|q)$.

Remark 3.1. In terms of the q-hypergeometric function ${}_r\Phi_s$ in (1.5), the homogeneous q-difference operator $\widetilde{L}(a,b;\Theta_{x,y})$ defined by (1.23) can be written as follows:

$$\widetilde{L}(a,b;\Theta_{x,y}) = {}_1\Phi_1 \left[\begin{matrix} a; \\ & q; -b\Theta_{x,y} \\ 0; \end{matrix} \right]. \tag{3.1}$$

Theorem 3.1. *The following operational formula holds true*:

$$\widetilde{L}(a,b;\Theta_{x,y}) \left\{ \frac{(xt;q)_\infty}{(yt;q)_\infty} \right\} = \frac{(xt;q)_\infty}{(yt;q)_\infty}\, {}_1\Phi_1 \left[\begin{matrix} a; \\ & q; bt \\ 0; \end{matrix} \right]. \tag{3.2}$$

Proof. Let the function $f_q(x,y;a,b;t)$ be defined by

$$f_q(x,y;a,b;t) = \widetilde{L}(a,b;\Theta_{x,y}) \left\{ \frac{(xt;q)_\infty}{(yt;q)_\infty} \right\}. \tag{3.3}$$

Then, by using (1.23) and (1.25), we have

$$f_q(x,y;a,b;t) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}\,(a;q)_k\,b^k}{(q;q)_k}\,\Theta_{x,y}^k \left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\}$$
$$= \frac{(xt;q)_\infty}{(yt;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k\,q^{\binom{k}{2}}\,(a;q)_k}{(q;q)_k}\,(bt)^k$$
$$= \frac{(xt;q)_\infty}{(yt;q)_\infty}\,{}_1\Phi_1\left[\begin{matrix} a; & \\ & q;bt \\ 0; & \end{matrix}\right], \tag{3.4}$$

which completes the proof of the assertion (3.2) of Theorem 3.1. □

Definition 3.1. In terms of the q-shifted factorial and the Cauchy polynomials $p_n(x,y)$, the generalized Hahn polynomials $h_n(x,y,a,b|q)$ is defined by

$$h_n(x,y,a,b|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k\,q^{\binom{k}{2}}\,b^k\,(a;q)_k\,p_{n-k}(y,x). \tag{3.5}$$

Remark 3.2. In the special case when $a=0$ and $b=z$, the q-polynomials $h_n(x,y,a,b|q)$ reduce to the following known trivariate q-polynomials $F_n(x,y,z;q)$, which were investigated earlier by Mohammed (see, for details, [1]):

$$F_n(x,y,z;q) := (-1)^n\,q^{-\binom{n}{2}}\,h_n(x,y,0,z|q). \tag{3.6}$$

The q-polynomials $h_n(x,y,a,b|q)$ is a general form of the Hahn polynomials $\psi_n^{(a)}(x|q)$, for which we need to set $a=0$, $b=1$ and $y=ax$. Also, if we let $a=0$, $y=ax$ and $b=y$, we get the bivariate Hahn polynomials $\psi_n^{(a)}(x,y|q)$. We thus have

$$h_n(x,ax,0,1|q) = (-1)^n\,q^{\binom{n}{2}}\,\psi_n^{(a)}(x|q) \qquad \text{and} \qquad h_n(x,ax,0,y|q) = (-1)^n\,q^{\binom{n}{2}}\,\psi_n^{(a)}(x,y|q). \tag{3.7}$$

The polynomials in (3.5) can be represented by the homogeneous q-difference operator in Definition 1.2 (1.23) as follows:

$$h_n(x,y,a,b|q) = \widetilde{L}(a,b;\Theta_{x,y})\{p_n(y,x)\}, \tag{3.8}$$

since

$$\widetilde{L}(a,b;\Theta_{x,y})\{p_n(y,x)\}$$
$$= \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}\,(a;q)_k}{(q;q)_k}\,(b\,\Theta_{x,y})^k\,\{p_n(y,x)\}$$
$$= \sum_{k=0}^{n} \frac{(-1)^k\,q^{\binom{k}{2}}\,(a;q)_k\,b^k}{(q;q)_k}\,\frac{(q;q)_n}{(q;q)_{n-k}}\,p_{n-k}(y,x)$$
$$= h_n(x,y,a,b|q). \tag{3.9}$$

Theorem 3.2. $\big[$Generating function for $h_n(x,y,a,b|q)\big]$ *The following generating function holds true for the q-polynomials $h_n(x,y,a,b|q)$:*

$$\sum_{n=0}^{\infty} h_n(x,y,a,b,|q)\,\frac{t^n}{(q;q)_n} = \frac{(xt;q)_\infty}{(yt;q)_\infty}\,{}_1\Phi_1\left[\begin{matrix} a; & \\ & q;bt \\ 0; & \end{matrix}\right] \qquad (\max\{|xt|,|yt|\}<1). \tag{3.10}$$

Proof. It is easily seen that

$$\begin{aligned}\sum_{n=0}^{\infty} h_n(x,y,a,b|q)\,\frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty} \widetilde{L}(a,b;\Theta_{x,y})\{p_n(y,x)\}\,\frac{t^n}{(q;q)_n} \\ &= \widetilde{L}(a,b;\Theta_{x,y})\left\{\sum_{n=0}^{\infty} p_n(y,x)\,\frac{t^n}{(q;q)_n}\right\} \\ &= \widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\right\} \\ &= \frac{(xt;q)_\infty}{(yt;q)_\infty}\,{}_1\Phi_1\left[\begin{matrix} a; \\ & q;bt \\ 0; \end{matrix}\right], \end{aligned} \tag{3.11}$$

which evidently completes the proof of the assertion (3.10) of Theorem 3.2. □

Theorem 3.3. [Extended generating function for $h_n(x,y,a,b|q)$] *The following extended generating function holds true for the q-polynomials* $h_n(x,y,a,b|q)$:

$$\sum_{n=0}^{\infty} h_{n+k}(x,y,a,b|q)\,\frac{t^n}{(q;q)_n} = \widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xt;q)_\infty}{(xt;q)_k\,(yt;q)_\infty}\,p_k(y,x)\right\}. \tag{3.12}$$

Proof. We observe that

$$\begin{aligned}&\sum_{n=0}^{\infty} h_{n+k}(x,y,a,b|q)\,\frac{t^n}{(q;q)_n} \\ &\quad= \sum_{n=0}^{\infty} \widetilde{L}(a,b;\Theta_{x,y})\{p_{n+k}(y,x)\}\,\frac{t^n}{(q;q)_n} \\ &\quad= \widetilde{L}(a,b;\Theta_{x,y})\left\{p_k(y,x)\,\sum_{n=0}^{\infty} p_n(y,q^kx)\,\frac{t^n}{(q;q)_n}\right\} \\ &\quad= \widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xtq^k;q)_\infty}{(yt;q)_\infty}\,p_k(y,x)\right\} \\ &\quad= \widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xt;q)_\infty}{(xt;q)_k\,(yt;q)_\infty}\,p_k(y,x)\right\}, \end{aligned} \tag{3.13}$$

which evidently completes the proof of the assertion (3.10) of Theorem 3.3. □

Remark 3.3. Upon setting $k=0$ in (3.12), we get the generating function (3.10) for the q-polynomials $h_n(x,y,a,b|q)$.

Theorem 3.4. $\big[$Rogers-type formula for $h_n(x,y,a,b|q)\big]$ *The following Rogers-type formula holds true for the q-polynomials* $h_n(x,y,a,b|q)$:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y,a,b|q)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}$$
$$=\widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xs;q)_\infty}{(ys;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}x/y,0;\\ \\ xs;\end{matrix}\quad q;yt\right]\right\}$$
$$=\widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xs;q)_\infty}{(yt,ys;q)_\infty}\,{}_1\Phi_1\left[\begin{matrix}ys;\\ \\ xs;\end{matrix}\quad q;xt\right]\right\}. \tag{3.14}$$

Proof. In order to prove the first assertion in (3.14), we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y,a,b|q)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\widetilde{L}(a,b;\Theta_{x,y})\big\{p_{n+m}(y,x)\big\}\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}$$
$$=\widetilde{L}(a,b;\Theta_{x,y})\left\{\sum_{n=0}^{\infty}p_n(y,x)\,\frac{t^n}{(q;q)_n}\,\sum_{m=0}^{\infty}p_m(y,q^nx)\,\frac{s^m}{(q;q)_m}\right\}$$
$$=\widetilde{L}(a,b;\Theta_{x,y})\left\{\sum_{n=0}^{\infty}p_n(y,x)\,\frac{t^n}{(q;q)_n}\,\frac{(xsq^n;q)_\infty}{(ys;q)_\infty}\right\}$$
$$=\widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xs;q)_\infty}{(ys;q)_\infty}\sum_{n=0}^{\infty}p_n(y,x)\,\frac{t^n}{(xs;q)_n\,(q;q)_n}\right\}$$
$$=\widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xs;q)_\infty}{(ys;q)_\infty}\sum_{n=0}^{\infty}\frac{(x/y;q)_n\,(yt)^n}{(xs;q)_n\,(q;q)_n}\right\}$$
$$=\widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xs;q)_\infty}{(ys;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}x/y,0;\\ \\ xs;\end{matrix}\quad q;yt\right]\right\}. \tag{3.15}$$

The proof of the second assertion in (3.14) of Theorem 3.4 would follow when we replace y by x and x by y in (1.20). □

Finally, we prove a Mehler-type bilinear generating function for the generalized Hahn polynomias $h_n(x,y,a,b|q)$.

Theorem 3.5. $\big[$Mehler's formula for $h_n(x,y,a,b|q)\big]$ *The following Mehler-type bilinear generating function holds true for* $h_n(x,y,a,b|q)$:

$$\sum_{n=0}^{\infty} h_n(x,y,a,b|q)\,h_n(u,v,c,d|q)\,\frac{t^n}{(q;q)_n}$$
$$=\widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\,{}_3\Phi_3\left[\begin{matrix}x/y,u/v,c;\\ \\ 0,0,xt;\end{matrix}\quad q;dvyt\right]\right\}. \tag{3.16}$$

Proof. It is observed that

$$\sum_{n=0}^{\infty} h_n(x,y,a,b|q)\, h_n(u,v,c,d|q)\, \frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} \widetilde{L}(a,b;\Theta_{x,y})\{p_n(y,x)\}\, h_n(u,v,c,d|q)\, \frac{t^n}{(q;q)_n}$$

$$= \widetilde{L}(a,b;\Theta_{x,y})\left\{\sum_{n=0}^{\infty} p_n(y,x)\, \frac{t^n}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k\, q^{\binom{k}{2}}\, d^k\, (c;q)_k\, p_{n-k}(v,u)\right\}$$

$$= \widetilde{L}(a,b;\Theta_{x,y})\left\{\sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{(-1)^k\, q^{\binom{k}{2}}\, d^k\, t^n}{(q;q)_k\, (q;q)_{n-k}}\, (c;q)_k\, p_{n-k}(v,u)\, p_n(y,x)\right\}$$

$$= \widetilde{L}(a,b;\Theta_{x,y})\left\{\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{\binom{k}{2}}\, d^k\, t^{n+k}}{(q;q)_k\, (q;q)_n}\, (c;q)_k\, p_k(v,u)\, p_{n+k}(y,x)\right\}$$

$$= \widetilde{L}(a,b;\Theta_{x,y})\left\{\sum_{k=0}^{\infty} p_k(v,u)\, p_k(y,x)\, \frac{(-1)^k\, q^{\binom{k}{2}}\, (c;q)_k\, (dt)^k}{(q;q)_k} \sum_{n=0}^{\infty} p_n(y,q^k x)\, \frac{t^n}{(q;q)_n}\right\}$$

$$= \widetilde{L}(a,b;\Theta_{x,y})\left\{\sum_{k=0}^{\infty} p_k(v,u)\, p_k(y,x)\, \frac{(-1)^k\, q^{\binom{k}{2}}\, (c;q)_k\, (dt)^k}{(q;q)_k}\, \frac{(xtq^k;q)_\infty}{(yt;q)_\infty}\right\}$$

$$= \widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{\binom{k}{2}}\, (x/y,u/v,c;q)_k}{(xt;q)_k}\, \frac{(dvyt)^k}{(q;q)_k}\right\}$$

$$= \widetilde{L}(a,b;\Theta_{x,y})\left\{\frac{(xt;q)_\infty}{(yt;q)_\infty}\, {}_3\Phi_3\left[\begin{matrix} x/y,u/v,c; \\ \\ 0,0,xt; \end{matrix}\; q;dvyt\right]\right\}, \tag{3.17}$$

which evidently completes our demonstration of Theorem 3.5. □

4. Concluding Remarks and Observations

In our present investigation, we have constructed a pair of potentially useful homogeneous q-operators, namely, the homogeneous q-shift operator $\widetilde{E}(a,b;D_q)$ and the homogeneous q-difference operator $\widetilde{L}(a,b;\Theta_{x,y})$. We then have successfully applied each of these homogeneous q-operators in order to represent and investigate a family of generalized Cauchy polynomials and a general form of the q-Hahn polynomials. In particular, we have derived several q-identities such as generating functions, extended generating functions, Mehler-type bilinear generating functions and Rogers-type formulas for the general families of q-polynomials which we have introduced in this paper. We have also considered relevant connections of the q-identities presented here with a number of known or new results associated with various specialized families of q-polynomials.

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