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FROM PROGRESSIVE DECOUPLING OF LINKAGES IN VARIATIONAL INEQUALITIES TO FIXED-POINT PROBLEMS

ABDELLATIF MOUDAFI[1,*], NOUREDDINE LEHDILI[2,3]

[1]*Aix Marseille Université, CNRS-L.I.S UMR 7296. Domaine Universitaire de Saint-Jérôme, Avenue Escadrille Normandie-Niemen, 13397 Marseille, France*
[2]*Mohamed VI Polytechnic University, Hay Moulay Rachid 43150, Ben Guerir-Morocco*
[3]*Natixis, Enterprise Risk Management Market and Counterparty Risk Modelling, 30, Avenue Pierre Mendes-France, 75013 Paris, France*

Abstract. Based on a very recent work by R. T. Rockafellar and the Yosida approximation, we propose a regularization approach of the idea and the associated notions presented in [16] (see also [17]). Next, we develop a similar analysis for fixed-point problems. In this setting, we introduce various elicitation notions that permit us to go beyond the limitation of the usual assumptions of the pseudo-contraction or the firm non-expansiveness, and we present the related progressive decoupling algorithms. Their convergence properties are investigated under global elicitability assumptions and an application to decomposition and splitting problems is also briefly mentioned.

Keywords. Linkage, Elicitation, Maximal monotone operator; Partial inverse; Firm nonexpansiveness.

1. Introduction and preliminaries

To begin with, let us recall the following concepts, which are of common use in the context of convex and nonlinear analysis; see, for example, [1]. Throughout this paper, H is a real Hilbert space, $\langle\cdot,\cdot\rangle$ denotes the associated scalar product and $\|\cdot\|$ stands for the corresponding norm. An operator $T : H \to 2^H$ is said to be monotone if

$$\langle u-v, x-y\rangle \geq 0 \quad \text{whenever} \quad u \in T(x), v \in T(y). \tag{1.1}$$

It is said to be maximal monotone if, in addition, its graph, $gphT := \{(x,y) \in H \times H : y \in T(x)\}$, is not properly contained in the graph of any other monotone operator. Local maximal monotonicity around $(\bar{x},\bar{y}) \in gphT$ is then defined when these properties hold relative to some neighborhood of $(\bar{x},\bar{y})$ (see, Pannanen [15]). Recall also that the σ-strong monotonicity is obtained by replacing relation (1.1) by

$$\langle u-v, x-y\rangle \geq 0 \geq \sigma\|x-y\|^2 \quad \forall (x,u),(y,v) \in gphT, \tag{1.2}$$

for some $\sigma > 0$. It is well-known that, for each $x \in H$ and $\gamma > 0$, there is a unique $z \in H$ such that

$$x \in (I+\gamma T)z. \tag{1.3}$$

*Corresponding author.
E-mail addresses: abdellatif.moudafi@univ-amu.fr (A. Moudafi), noureddine.lehdili@natixis.com (N. Lehdili).
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The single-valued operator $J_\gamma^T := (I+\gamma T)^{-1}$ is called the resolvent of T of parameter γ. It is a nonexpansive mapping, which is everywhere defined and is related to its Yosida approximate, namely, $T_\gamma(x) := \frac{x - J_\gamma^T(x)}{\gamma}$, by the relation

$$T_\gamma(x) \in T(J_\gamma^T(x)). \tag{1.4}$$

The latter is Lipschitz continuous with constant $\frac{1}{\gamma}$. Recall also that the inverse T^{-1} of T is the operator defined by $x \in T^{-1}(y)$ if and only if $T(x)$. On the other hand, remember that the Moreau's approximate of a given proper l.s.c convex function φ is defined by

$$\varphi_\gamma(x) = \inf_{y\in H}\{\varphi(y) + \frac{1}{2\gamma}\|y-x\|^2\},$$

and enjoys the nice property that it is a continuously differentiable function. The proximal mapping is defined as

$$prox_{\gamma\varphi}(x) := \arg\min_{y\in H}\{\varphi(y) + \frac{1}{2\gamma}\|y-x\|^2\}$$

and its localised version by

$$prox^U_{\gamma\varphi}(x) := \arg\min_{y\in U}\{\varphi(y) + \frac{1}{2\gamma}\|y-x\|^2\},$$

where U stands for a subset of H and $x \in U$. Given a proper lower-semicontinuous convex function $\varphi : H \to I\!R \cup \{+\infty\}$, the subdifferential of φ at x is the set

$$\partial\varphi(x) = \{z \in H:\ \varphi(y) \geq \varphi(x) + \langle z, y-x\rangle \text{ for all } y \in H\}.$$

Now, given a nonempty closed convex set $C \subset H$, recall also that its indicator function is defined as $i_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise. The normal cone to C at x is defined as

$$N_C(x) = \{y \in H:\ \langle y, c-x\rangle \leq 0 \text{ for all } c \in C\},$$

if $x \in C$ and $\emptyset$ otherwise. Observe that $\partial i_C = N_C$ and that its proximal mapping reduces to the projection onto the set C.

The present paper is in the spirit of a very recent work by Rockafellar. For readers' convenience, we propose first a short tour in the program developed by Rockafellar in [16] (see also [17]). Next, unlike the methodology used there, we will see that ours relies heavily on regularization. More precisely, our main purpose is twofold. Firstly, by connecting the analysis developed in [16] with partial Moreau/Yosida regularization, we revisit some fundamental notions proposed in [16] in the context of variational inequalities and optimization. Secondly, taking advantage of this approach, we introduce elicitability's notions and devise progressive decoupling algorithms in two fixed-point settings. Their convergence properties are then also investigated. An application to decomposition problems is briefly mentioned without more developments.

Let H be a Hilbert space, $T : H \to H$ a set-valued mapping and $L, L^\perp \subset H$ two complementary subspaces of H. In [16], Rockafellar considered the following problem

$$(\mathcal{L}) \qquad \text{find} \quad \bar{x} \in L \text{ and } \bar{y} \in L^\perp \text{ such that } \bar{y} \in T(\bar{x}),$$

the subspace L stands for *linkage relations*. The case when T is maximal monotone was considered in Spingarn [19] and was solved by the celebrate *partial inverse method*. The optimization case is then obtained by setting $T = \partial\varphi$, $\varphi : H \to (-\infty, +\infty]$ with φ a proper, l.s.c. convex function. In this case, under a qualification condition, problem $(\mathcal{L})$ amounts to minimizing the function φ under the subspace L. Rockafellar's challenge was then to obtain global and local solution methods without monotonicity or convexity assumptions.

Based on the relevant observation that by replacing T by $T + eP_{L^\perp}$, for some $e \geq 0$ with $P_{L^\perp}$ the orthogonal projection onto $L^\perp$, the set of solutions to $(\mathcal{L})$ does not change, Rockafellar considered the problem

$$(\mathcal{L}_e) \qquad \text{find} \quad \bar{x} \in L \text{ and } \bar{y} \in L^\perp \text{ such that } \bar{y} \in T_e(\bar{x}),$$

with $T_e = T + eP_\perp$.

Then, he defined the *elicitation* in problem $(\mathcal{L})$ at level $e \geq 0$ *globally* by

$$\exists e > 0 \text{ such that } T_e := T + eP_\perp \text{ is maximal monotone,}$$

and *locally* if $T_e := T + eP_\perp$ is maximal monotone around a solution $(\bar{x}, \bar{y})$. Similarly, he defined the elicitation of strong maximal monotone operators. Finally, motivated by progressive hedging algorithm for solving problems in convex stochastic programming, see [18], Rockafellar proposed the following algorithm:

Decoupling algorithm with parameters $r > e \geq 0$

- In iteration ν with $x^\nu \in L$ and $y^\nu \in L^\perp$, compute $\hat{x}^\nu$ solution of

$$0 \in T^\nu(\hat{x}^\nu), \text{ where } T^\nu(x) = T(x) - y^\nu + r(x - x^\nu).$$

Then update by the rule

$$x^{\nu+1} = P_L(\hat{x}^\nu) \text{ and } y^{\nu+1} = y^\nu - (r-e)P_{L^\perp}(\hat{x}^\nu).$$

In the special case when $T = \partial\varphi$, φ being a proper, l.s.c. convex function, the linkage problem reduces to

$$(\mathcal{L}) \qquad \text{find} \quad \bar{x} \in L \text{ and } \bar{y} \in L^\perp \text{ such that } \bar{y} \in \partial\varphi(\bar{x}),$$

which amounts to minimizing the function φ on the subspace L.

The algorithm's subproblem takes the following form:

$$0 \in \partial\varphi^\nu(\hat{x}^\nu), \text{ where } \varphi^\nu(x) = \varphi(x) - \langle y^\nu, x\rangle + \frac{r}{2}\|x - x^\nu\|^2,$$

in other words

$$\hat{x}^\nu = argmin_{x \in H}\varphi^\nu(x) \text{ (global case) and } \hat{x}^\nu = argmin_{x \in U}\varphi^\nu(x) \text{ (local case),}$$

U being a neighborhood of a solution.

Then, Rockafellar derived that the *elicit monotonicity*, in this case, means that

$$\exists e \geq 0 \text{ such that } \partial\varphi + eP_\perp = \partial\varphi_e \text{ is maximal monotone,} \tag{1.5}$$

with $\varphi_e := \varphi + \frac{e}{2}dist_L^2$. This leads to the observation that

$$\partial\varphi + eP_\perp \text{ is maximal monotone} \Leftrightarrow \varphi_e \text{ is proper, l.s.c. and convex.} \tag{1.6}$$

Local elicitation at $(\bar{x}, \bar{y}) \in (L \times L^\perp) \cap gph\partial\varphi$ is obtained when there exist $e \geq 0$ and a convex neighborhood $U \times V$ of $(\bar{x}, \bar{y})$ such that $\partial\varphi + eP_\perp$ is maximal monotone in $U \times V$. It is then derived that

$$\partial\varphi + eP_\perp \text{ is maximal monotone } \in U \times H \Leftrightarrow \varphi_e \text{ is a proper, l.s.c. convex on } U.$$

It is also observed that strong elicitation corresponds in this setting to strong convexity. Furthermore, it is noted that if $\varphi \in \mathcal{C}^2$, then

$$(\bar{x}, \bar{y}) \text{ is a solution of } (\mathcal{L}) \text{ means that } \bar{x} \in L \text{ and } \bar{y} = \nabla\varphi(\bar{x}) \in L^{\perp},$$

and

$$\nabla^2\varphi(\bar{x}) \text{ is positive-definite relative to } L \text{ implies strong local elicitability}.$$

A broader elicitability criterion was then provided and applications to decomposition problems and splitting nonconvex optimization were proposed.

Now, we will use regularization to highlight the essence of the idea above.

2. A REGULARIZATION POINT OF VIEW

Now, we are ready to make a link with a partial regularization of Spingarn's partial inverse problem (we refer to [19]). To that end, we observe first that

$$(\mathcal{L}) \qquad \text{find} \quad \bar{x} \in L \text{ and } \bar{y} \in L^{\perp} \text{ such that } \bar{y} \in T(\bar{x}),$$

amounts to

$$(\tilde{\mathcal{L}}) \qquad \text{finding} \quad \bar{x} \in H \text{ which solves } 0 \in (T + N_L)(\bar{x}),$$

N_L being the normal cone to the subspace L.

On the other hand, by noting that

$$(N_L)_{e^{-1}} = \frac{I - J^{N_L}_{e^{-1}}}{e^{-1}} = e(I - P_L) = eP_{L^{\perp}}, \tag{2.1}$$

problem $(\mathcal{L}_e)$ can clearly be viewed as a partial Yosida-regularization of $(\tilde{\mathcal{L}})$ and thus of the initial problem $(\mathcal{L})$. More precisely, $(\mathcal{L}_e)$ amounts to searching $\bar{x} \in L$ and $\bar{y} \in L^{\perp}$ satisfying

$$(\mathcal{L}_e) \qquad \bar{y} \in (T + (N_L)_{e^{-1}})(\bar{x}), \ \text{ for any } e > 0,$$

$(N_L)_{e^{-1}}$ the Yosida approximate of the normal cone to L with constant e^{-1}. Observe that the subproblem's algorithm can reformulate by using the resolvent or the proximal map, when the resolvent associated to T is well-defined (for instance, in the case of weakly monotone operators or prox-regular functions).

The algorithm's subproblem takes the following form:
Given $x^{\nu} \in L$ and $y^{\nu} \in L^{\perp}$ at iteration ν, compute $\hat{x}^{\nu}$ a solution of the inclusion

$$0 \in T(x) - y^{\nu} + r(x - x^{\nu}) \Leftrightarrow \hat{x}^{\nu} = J^T_{\frac{1}{r}}(x^{\nu} + \frac{1}{r}y^{\nu}),$$

in the global case. The algorithm will be efficient if the resolvent of T can be computed efficiently and the projections onto L and $L^{\perp}$ relatively convenient to execute. The local case can be obtained by a localisation in a neighborhood U of a solution.

In the optimization setting, we successively have

$$(N_L)_{e^{-1}} = (\partial i_L)_{e^{-1}} = \partial(i_L)_{e^{-1}} = \nabla\big(\frac{e}{2}dist^2_L\big). \tag{2.2}$$

This combined with the additivity of the subdifferentials, leads to

$$\partial\varphi(\bar{x}) + \nabla\big(\frac{e}{2}dist^2_L(\bar{x})\big) = \partial\big(\varphi + \frac{e}{2}dist^2_L\big)(\bar{x}),$$

and we retrieve exactly the function $\varphi_e := \varphi + \frac{e}{2} dist_L^2$ defined in the first section. Regarding the subproblem's algorithm, it reduces using the proximal map to

The algorithm's subproblem:

Given $x^\nu \in L$ and $y^\nu \in L^\perp$ at iteration ν, compute $\hat{x}^\nu$ a solution of the inclusion

$$0 \in \partial\varphi(x) - y^\nu + r(x - x^\nu),$$

equivalently,

$$\hat{x}^\nu = prox_{\frac{1}{r}\varphi}(x^\nu + \frac{1}{r}y^\nu) \text{ (global case) and } \hat{x}^\nu = prox^U_{\frac{1}{r}\varphi}(x^\nu + \frac{1}{r}y^\nu)\text{(local case)},$$

U standing for a neighborhood of a solution. Likewise, the efficiency of the algorithm will depend on the easy computation of the proximity mapping of the function φ and the relative convenience of the execution of the projections onto the two subspaces L and/or $L^\perp$.

3. Fixed-point problems

The extension of the partial inverse method to primal-dual fixed point problems was justified in [11] by the fact that there exist (firmly) nonexpansive mappings that are not proximal mappings and that are not resolvent operators, which is the case for instance, for the periodic problems considered in [10]. Now, to begin with, let us recall that a mapping $Q : D \subset H \to H$ is said to be a *local pseudo-contraction* if each point $z \in D$ has a neighborhood N for which

$$\langle Q(x) - Q(y), x - y\rangle \leq \|x - y\|^2 \text{ for all } x, y \in N. \tag{3.1}$$

It is easy to verify that

$$\|Q(x) - Q(y)\| \leq \|x - y\| \text{ for all } x, y \in N,$$

that defines local nonexpensiveness of Q. We remark also that Q is *locally pseudo-contractive* if and only if, $I - Q$ is *locally monotone*; see, for instance, [4]. Furthermore, we would like to mention, as was observed by Bogin [2] (see also [7]), that a mapping Q is *strict pseudo-contractive* if and only if $I - Q$ is *strongly monotone*. Remember also that a mapping is *locally firmly nonexpansive* if each point $z \in D$ has a neighborhood N for which

$$\langle Q(x) - Q(y), x - y\rangle \geq \|Q(x) - Q(y)\|^2 \text{ for all } x, y \in N. \tag{3.2}$$

The global versions of these notions are obtained when N is all of D. Moreover, in this case, it can easily checked that the σ-*strong monotonicity* of $I - Q$ is equivalent to $(1+\sigma)$-*inverse strong monotonicity* of the mapping Q.

The main interest in *pseudo-contractive mappings* stems from its firm connection with the important class of monotone operators (more generally the class of accretive operators in Banach space settings). Thus the mapping theory for monotone operators is closely related to the fixed-point theory of pseudo-contractive mappings. In subsection 3.2, we will utilize this correspondence. Before, we will use, in subsection 3.1, a Minty's Fact that provides another correspondence with *firmly nonexpansive mappings*. The latter can in turn be characterized by using *nonexpansive mappings*. Indeed, it is well known that

$$Q \text{ is firmly nonexpansive} \Leftrightarrow R_Q = 2Q - I \text{ is nonexpansive},$$

where R_Q is the so-called reflected operator associated to Q. Therefore, all results in [16] have counterparts that can be formulated for pseudo-contractive and firmly nonexpansive mappings. We will focus on these counterparts based on the analysis developed in the present paper.

In what follows, before stating the corresponding formulations that turned out to be elegant, we would like also to emphasize that, in [5], the fundamental technique for approximating fixed points of nonexpansive mappings was extend to the locally nonexpansive case relying on a condition which is known to assure the existence of fixed points. More precisely, Kirk and Morales obtained the following [5]-Theorem 2: Let X be an arbitrary Banach space, D an open subset of X, and $Q:\bar{D}\to X$ a continuous mapping, which is locally nonexpansive on D and such that $Q(\bar{D})$ is precompact. Suppose $x_1\in D$ for which

$$\|x_1-Q(x_1)\|<\|x-Q(x)\| \text{ for all } x\in\partial D, \tag{3.3}$$

and let $\{t_\nu\}\subset I\!R$ satisfy $\sum_{\nu=1}^{\infty}t_\nu=\infty$ and $0\le t_\nu\le b<1,\ \nu=1,2,\cdots$. Then the sequence $\{x^\nu\}$ generated by the following rule

$$x^{\nu+1}=(1-t_\nu)x^\nu+t_\nu Q(x^\nu),\ \nu=1,2,\cdots \tag{3.4}$$

lies in D and converges to a fixed point of the mapping Q in D ($\bar{D}$ and ∂D stand respectively for the closure and the boundary of D). The convergence of the unique continuous path $t\mapsto x_t\in D$, $t\in[0,1[$, satisfying $x_t=tQ(x_t)+(1-t)z$ with $z\in D$, to a fixed point of Q was also obtained. An extension of this version to locally pseudo-contractive mappings was obtained in [9] in uniformly convex Banach spaces. The convergence of the sequence was obtained in a subset D of uniformly smooth Banach spaces, in [6] and [7], for a local strict pseudo-contractive mapping, provided that the range of the mapping Q is bounded and $FixQ\neq\emptyset$ together with slow convergence of the parameters to 0, namely $\sum_{\nu=1}^{\infty}t_\nu=\infty$ with $\lim_{\nu\to+\infty}t_\nu=0$. We finally state the following lemma, which will be needed in the sequel.

Lemma 3.1. [20, Lemma 3.2] *Let S be a closed convex nonempty set of a real Hilbert space H and let (x^ν) be a sequence in H. Suppose that for all $u\in S$*

$$\|x^{\nu+1}-u\|\le\|x^\nu-u\| \textit{ for every } \nu=0,1,2,\cdots.$$

Then, the sequence $\big(P_S(x^\nu)\big)$ strongly converges to some $z\in S$.

3.1. **(Partial inverse)-fixed point problems.** *Minty's Fact* [8]*:*

$$T \text{ is firmly nonexpansive} \Leftrightarrow T^{-1}-I \text{ is maximal monotone.}$$

In [11], Moudafi introduced the following new fixed-point problem

$$(\mathcal{L})\quad \text{find}\quad \bar{x}\in L \text{ and } \bar{y}\in L^\perp \text{ such that } \bar{x}=Q(\bar{x}+\bar{y}),$$

where $Q:H\to H$ is a single-valued mapping and $L,L^\perp\subset H$ are complementary subspaces of H.

Inspired by Spingarn [19], Moudafi introduced in [11] the concept of the partial complement of Q with respect to the subspace L by:

$$S_L^Q:=P_L\circ Q+P_{L^\perp}\circ(I-Q),$$

and showed that $(\mathcal{L})$ is equivalent to

$$\text{find}\quad \bar{x}\in L \text{ and } \bar{y}\in L^\perp \text{ such that } \bar{x}+\bar{y}=S_L^Q(\bar{x}+\bar{y}). \tag{3.5}$$

This suggests the following strategy. To solve $(\mathcal{L})$, we have to find a fixed point of S_L^Q after the project onto L (resp. onto $L^\perp$), which provides x (resp. y).

To approximate such solutions, the first author of this paper developed an algorithm, which he called the method of partial complement. This procedure amounts to applying the Mann algorithm to S_L^Q. If Q is firmly nonexpansive on H, he proved that S_L^Q is firmly nonexpansive. This can easily also be seen by noting that this amounts to proving that $2S_L^Q - I$ is nonexpansive. This follows immediately from the fact that

$$2S_L^Q - I = (2P_L - I) \circ (2Q - I) \text{ and } 2P_L - I, 2Q - I \text{ are both nonexpansives,}$$

since P_L and Q are firmly nonexpansives and the composition of two nonexpansive mappings is still nonexpansive. Observe that by replacing Q by

$$Q_e := \left(Q^{-1} + eP_{L^\perp}\right)^{-1} \text{ for some } e \geq 0,$$

which sounds nearly like a *Bott-Duffin inverse*, the set of solutions to problem $(\mathcal{L})$ does not change again. This offers a nice strategy to define both an elicitation notion and a progressive decoupling algorithm in this fixed-point context.

Now, we are in a position to present our algorithm with parameters $r > e \geq 0$, which can be derived by localizing and/or rescaling the method above.

The algorithm proceeds iteratively as follows:

Given $x^\nu \in L$ and $y^\nu \in L^\perp$ at iteration ν, compute $\hat{x}^\nu$, which solves $x = Q^\nu(x)$, where Q^ν is given by

$$Q^\nu(x) = Q\big((1-r)x + y^\nu + rx^\nu\big).$$

Then, update by the rule

$$x^{\nu+1} = P_L(\hat{x}^\nu) \text{ and } y^{\nu+1} = y^\nu - (r-e)P_{L^\perp}(\hat{x}^\nu).$$

It is worth mentioning that in [11] the version for $e = 0$ and $r = 1$ was treated for fully firmly nonexpansive mapping Q.

Likewise, we define the *firm monotone elicitation* of a mapping Q at a level e by:

$$\exists e > 0 \text{ such that } Q_e := \left(Q^{-1} + eP_{L^\perp}\right)^{-1} \text{ is firmly nonexpansive.}$$

or equivalently, relying on the inverse operator,

$$\exists e > 0 \text{ such that } (Q_e)^{-1} = Q^{-1} + eP_{L^\perp} \text{ is } 1-\text{strongly monotone.} \tag{3.6}$$

This leads easily to the following key explicit definition. Q is *elicit firmly nonexpansive* at a level e if and only if, for all $x, y \in H$,

$$\langle Q(x) - Q(y), x - y\rangle \geq \|Q(x) - Q(y)\|^2 - e\|P_{L^\perp}\big(Q(x)\big) - P_{L^\perp}\big(Q(y)\big)\|^2. \tag{3.7}$$

The *local elicit firm monotonicity* is then obtained when the latter property is satisfied in a neighborhood of a solution. Similarly, elicitation for $(1+\sigma)$-inverse strong-monotone mappings, for $\sigma > 0$, can be defined. We wish to point out that the convergence results can most be derived from those obtained in the global case through localizing and/or rescaling the classical methods. Our priority in this paper is given first to elicitation notions, which are very promising since they permit to go beyond the pseudo-contraction and the firm nonexpansiveness, whether globally or locally, where their availability are not evident. Indeed, the elicitation allows the freedom to select the value of the elicitation parameter e in order to enhance properties of mapping Q

for algorithmic purposes. Our interest is also in presenting the convergence properties of our progressive decoupling algorithms. However, for simplicity and clarity, we mainly confine our attention in what follows to the global elictation cases.

Theorem 3.1. *Let Q be a globally elicited firm nonexpansive mapping at a level $e \geq 0$ and assume that problem $(\mathcal{L})$ is solvable. Then, for any $r > e$, starting from any $x^0 \in L$ and $y^0 \in L^{\perp}$, the sequence (x^ν, y^ν) generated by the algorithm is Féjer monotone with respect to the solution set Γ and is also asymptotically regular. Specifically, we have*

$$|(x^{\nu+1}, y^{\nu+1}) - (\bar{x}, \bar{y})|_{r,e} \leq |(x^\nu, y^\nu) - (\bar{x}, \bar{y})|_{r,e} \text{ with } (\bar{x}, \bar{y}) \in \Gamma,$$

and

$$\lim_{\nu \to +\infty} |(x^{\nu+1}, y^{\nu+1}) - (x^\nu, y^\nu)|_{r,e} = 0, \text{ where } |(x,y)|_{r,e} = \|x\|^2 + \frac{1}{r(r-e)}\|y\|^2.$$

Furthermore, the sequence $\left(P_\Gamma^{r,e}(x^\nu, y^\nu)\right)$ converges strongly to some point in Γ. $P^{r,e}$ being the metric projection in the Hilbert space $(H, |\cdot|_{r,e})$.

Proof. First, we observe that

$$y^{\nu+1} = y^\nu - (r-e)P_{L^\perp}(\hat{x}^\nu) \Leftrightarrow P_{L^\perp}(\hat{x}^\nu) = (r-e)^{-1}(y^\nu - y^{\nu+1}),$$

which implies that

$$u^{\nu+1} := -(r-e)^{-1}(y^\nu - y^{\nu+1}) \in L^\perp \Rightarrow y^{\nu+1} = y^\nu - (r-e)u^{\nu+1} \in L^\perp.$$

Note also that $\hat{x}^\nu = x^{\nu+1} + u^{\nu+1}$. Hence, the subproblem algorithm can be rewritten as

$$\begin{aligned} x^{\nu+1} + u^{\nu+1} &= Q\big((1-r)(x^{\nu+1} + u^{\nu+1}) + y^\nu + rx^\nu\big) \\ &= Q\big((1-r)(x^{\nu+1} + u^{\nu+1}) + y^{\nu+1} + (r-e)u^{\nu+1} + rx^\nu\big). \end{aligned}$$

Consequently,

$$x^{\nu+1} + u^{\nu+1} = Q\big(x^{\nu+1} + y^{\nu+1} + (1-e)u^{\nu+1} - r(x^{\nu+1} - x^\nu)\big). \tag{3.8}$$

Taking $(\bar{x}, \bar{y}) \in \Gamma$ (i.e., $\bar{x} \in L, \bar{y} \in L^\perp; \bar{x} = Q(\bar{x} + \bar{y})$) and using the elicitable firm nonexpansiveness of Q at a level e, we obtain

$$\begin{aligned} &\langle x^{\nu+1} + u^{\nu+1} - \bar{x}, x^{\nu+1} + y^{\nu+1} + (1-e)u^{\nu+1} - r(x^{\nu+1} - x^\nu) - \bar{x} - \bar{y}\rangle \\ &\geq \|x^{\nu+1} + u^{\nu+1} - \bar{x}\|^2 - e\|u^{\nu+1}\|^2, \end{aligned} \tag{3.9}$$

because $u^{\nu+1} \in L^\perp$ and $\bar{x} \in L$. The first hand of the last inequality can be rewritten as

$$\begin{aligned} \langle \quad & x^{\nu+1} - \bar{x} + u^{\nu+1}, (x^{\nu+1} - \bar{x} - r(x^{\nu+1} - x^\nu)) + (y^{\nu+1} - \bar{y} + (1-e)u^{\nu+1})\rangle \\ &= \langle x^{\nu+1} - \bar{x}, x^{\nu+1} - \bar{x} - r(x^{\nu+1} - x^\nu)\rangle + \langle u^{\nu+1}, y^{\nu+1-\bar{y}} + (1-e)u^{\nu+1})\rangle. \end{aligned}$$

By reporting in relation (3.9), this leads to

$$\begin{aligned} \|x^{\nu+1} - \bar{x}\|^2 + r\langle x^\nu - x^{\nu+1}, x^{\nu+1} - \bar{x}\rangle &+ (1-e)\|u^{\nu+1}\|^2 + \langle u^{\nu+1}, y^{\nu+1} - \bar{y}\rangle \\ &\geq \|x^{\nu+1} - \bar{x}\|^2 + (1-e)\|u^{\nu+1}\|^2. \end{aligned}$$

Consequently,

$$r\langle x^\nu - x^{\nu+1}, x^{\nu+1} - \bar{x}\rangle + \langle u^{\nu+1}, y^{\nu+1} - \bar{y}\rangle \geq 0. \tag{3.10}$$

On the other hand, an elementary computation gives

$$2\langle x^\nu - x^{\nu+1}, x^{\nu+1} - \bar{x}\rangle = \|x^\nu - \bar{x}\|^2 - \|x^{\nu+1} - x^\nu\|^2 - \|x^{\nu+1} - \bar{x}\|^2. \tag{3.11}$$

Similarly, we derive

$$2\langle u^{\nu+1}, y^{\nu+1}-\bar{y}\rangle = (r-e)^{-1}(\|y^{\nu}-\bar{y}\|^2 - \|y^{\nu+1}-y^{\nu}\|^2) - \|y^{\nu+1}-\bar{y}\|^2), \tag{3.12}$$

since $u^{\nu+1} = -(r-e)^{-1}(y^{\nu}-y^{\nu+1})$. Therefore, a substitution in (3.10) leads to

$$\begin{aligned} r\big(\|x^{\nu}-\bar{x}\|^2 \quad &- \quad \|x^{\nu+1}-x^{\nu}\|^2 - \|x^{\nu+1}-\bar{x}\|^2) \\ &+ \quad (r-e)^{-1}\big(\|y^{\nu}-\bar{y}\|^2 - \|y^{\nu+1}-y^{\nu}\|^2 - \|y^{\nu+1}-\bar{y}\|^2\big) \geq 0, \end{aligned}$$

from which we deduce the following key property

$$\begin{aligned} \|x^{\nu+1}-\bar{x}\|^2 + \frac{1}{r(r-e)}\|y^{\nu+1}-\bar{y}\|^2 \quad &+ \quad \|x^{\nu+1}-x^{\nu}\|^2 + \frac{1}{r(r-e)}\|y^{\nu+1}-y^{\nu}\|^2 \\ &\leq \quad \|x^{\nu}-\bar{x}\|^2 + \frac{1}{r(r-e)}\|y^{\nu}-\bar{y}\|^2. \end{aligned}$$

The latter can be rewritten as

$$|(x^{\nu+1}, y^{\nu+1}) - (\bar{x},\bar{y})|_{r,e} + |(x^{\nu+1}, y^{\nu+1}) - (x^{\nu}, y^{\nu})|_{r,e} \leq |(x^{\nu}, y^{\nu}) - (\bar{x},\bar{y})|_{r,e}. \tag{3.13}$$

This ensures that the sequence (x^{ν}, y^{ν}) is Féjer monotone with respect to Γ. The sequence $(|(x^{\nu}-\bar{x}, y^{\nu}-\bar{y})|_{r,e})$ is decreasing and lower bounded by 0. Consequently, it converges to some finite limit, say $l(\bar{x},\bar{y})$. By passing to the limit in (3.13), we obtain that

$$\lim_{\nu\to+\infty} |(x^{\nu+1}, y^{\nu+1}) - (x^{\nu}, y^{\nu})|_{r,e} = 0.$$

Finally, since Γ is a nonempty closed convex subset of H, the sequence (x^{ν}, y^{ν}) is Féjer monotone with respect to Γ in $(H, |\cdot|_{r,e})$, the desired result follows by invoking Lemma 3.1. □

Proposition 3.1. *Let H be a finite dimensional Hilbert space. Suppose that the mapping Q is continuous, globally elicited firm nonexpansive at a level $e \geq 0$ and that the solution set $\Gamma \neq \emptyset$. Then, the sequence (x^{ν}, y^{ν}) generated by the algorithm converges to some element $(x^*, y^*) \in \Gamma$. Furthermore,*

$$(x^*, y^*) = \lim_{\nu\to+\infty} P_{\Gamma}^{r,e}(x^{\nu}, y^{\nu}).$$

Proof. First, observe that the continuity assumption falls away when only a small value of the elicitation level e is needed. Indeed, relation (3.7) together with the Cauchy-Shwarz inequality and the nonexpansiveness of the projection operator implies the Lipschitz continuity of the mapping Q. Suppose further $0 \leq e < 1$. Furthermore, the fact that $(|(x^{\nu}-\bar{x}, y^{\nu}-\bar{y})|_{r,e})$ converges to a finite limit ensures that both the sequences (x^{ν}) and (y^{ν}) are bounded. Let $\tilde{x}$ and $\tilde{y}$ be respectively cluster points of the sequences (x^{ν}) and (y^{ν}). By passing to the limit in (3.8) and taking into account the asymptotical regularity of both (x^{ν}) and (y^{ν}), the fact that $\lim_{\nu\to+\infty} u^{\nu} = 0$ together with the continuity of Q yields that

$$\tilde{x} = Q(\tilde{x}+\tilde{y}), \text{ that is } (\tilde{x},\tilde{y}) \in \Gamma.$$

The uniqueness of the cluster point follows by the celebrate Opial's lemma [14]. The proof is presented here for completeness. Indeed, let (x^*, y^*) be another cluster point of (x^{ν}, y^{ν}), we show that $(x^*, y^*) = (\tilde{x},\tilde{y})$. From

$$\begin{aligned} \|(x^{\nu}, y^{\nu}) - (x^*, y^*)\|_{r,e}^2 \quad &= \quad \|(x^{\nu}, y^{\nu}) - (\tilde{x},\tilde{y})\|_{r,e}^2 + \|(\tilde{x},\tilde{y}) - (x^*, y^*)\|_{r,e}^2 \\ &+2 \quad \langle (x^{\nu}, y^{\nu}) - (\tilde{x},\tilde{y}), (\tilde{x},\tilde{y}) - (x^*, y^*)\rangle_{r,e}, \end{aligned}$$

we see that the limit of $(\langle (x^\nu,y^\nu)-(\tilde{x},\tilde{y}),(\tilde{x},\tilde{y})-(x^*,y^*)\rangle_{r,e})$ must exists and has to be zero because $(\tilde{x},\tilde{y})$ is a cluster point of (x^ν,y^ν). This leads, at the limit, to

$$l(x^*,y^*)=l(\tilde{x},\tilde{y})+\|(\tilde{x},\tilde{y})-(x^*,y^*)\|^2_{r,e}.$$

Reversing the role of (x^*,y^*) and $(\tilde{x},\tilde{y})$, we also have

$$l(\tilde{x},\tilde{y})=l(x^*,y^*)+\|(\tilde{x},\tilde{y})-(x^*,y^*)\|^2_{r,e}.$$

Combining the last two equalities, we get $(x^*,y^*)=(\tilde{x},\tilde{y})$. Consequently, the whole sequence (x^ν,y^ν) converges to some element $(x^*,y^*)\in\Gamma$.

Now, since Γ is a closed convex nonempty subset of H, the characterization of the orthogonal projection leads to

$$\langle (x,y)-P^{r,e}_\Gamma(x^\nu,y^\nu),(x^\nu,y^\nu)-P^{r,e}_\Gamma(x^\nu,y^\nu)\rangle_{r,e}\le 0\ \forall\nu,\forall(x,y)\in\Gamma. \tag{3.14}$$

Taking $(x,y)=(x^*,y^*)\in\Gamma$ in (3.14), we obtain

$$\langle (x^*,y^*)-P^{r,e}_\Gamma(x^\nu,y^\nu),(x^\nu,y^\nu)-P^{r,e}_\Gamma(x^\nu,y^\nu)\rangle_{r,e}\le 0 \text{ for all } \nu,$$

By passing to the limit in the last inequality and by taking into account the fact that $(P^{r,e}_\Gamma(x^\nu,y^\nu))$ converges to some $(\tilde{x},\tilde{y})$, we obtain

$$\langle (x^*-\tilde{x},y^*-\tilde{y}),(x^*-\tilde{x},y^*-\tilde{y})\rangle_{r,e}\le 0,$$

which means that

$$\|x^*-\tilde{x}\|^2+\frac{1}{r(r-e)}\|x^*-\tilde{y}\|^2=0. \tag{3.15}$$

Therefore, we have $(x^*,y^*)=(\tilde{x},\tilde{y})$, which completes the proof. □

Let us now illustrate briefly that the decoupling of linkages opens the way to problem decomposition based on the block separability.

- Decomposable structure

Suppose that $H=H_1\times\cdots\times H_n$ for n copies of an Hilbert space H_0 and that $Q(x)=\prod_{i=1}^n Q_i(x_i)$ for given mappings $Q_i:H_i\to H_i$. Problem $(\mathcal{L})$ reads as

$$\text{find } (\bar{x}_1,\cdots,\bar{x}_n)\in S,(\bar{y}_1,\cdots,\bar{y}_n)\in S^\perp \text{ such that } \bar{x}_i=Q_i(\bar{x}_i+\bar{y}_i). \tag{3.16}$$

With respect to an elicitation $e\ge 0$ (global or local) and a proximal parameter $r>e$, the decomposable progressive decoupling algorithm proceeds as follows:

Decomposable progressive decoupling algorithm:

i) Iteration ν: given $(x_1,\cdots,x^\nu_n)\in S$ and $(y^\nu_1,\cdots,y^\nu_n)\in S^\perp$ with $\sum_{i=1}^n y^\nu_i=0$, $i=1\cdots n$, compute $\hat{x}^\nu_i\in H_i$ solution of the fixed-point problem $x_i=Q_i(x_i+y^\nu_i)$ $i=1,\cdots,n$.

ii) Update by: $x^{\nu+1}=P_L(\hat{x}^\nu)$ and $y^{\nu+1}_i=y^\nu_i-(r-e)(\hat{x}^\nu_i-x^{\nu+1})$, $i=1,\cdots,n$.

Another illustration of decoupling comes from taking the spaces in the block-separable structure to be the same and making a special choice of the space L.

- Splitting structure

Suppose that $H=H_0\times\cdots\times H_0$ for n copies of an Hilbert space H_0 and that $Q(x)=\prod_{i=1}^n Q_i(x_i)$ for given mappings $Q_i:H_0\to H_0$ and choose the subspace $L:=\{u=(x_1,\cdots,x_n),x_1=x_2,\cdots,=$

$x_n)\}$ of which the orthogonal subspace is $L^\perp = \{v = (y_1, \cdots, y_n);\ \sum_{i=1}^n y_i = 0\}$. Problem $(\mathcal{L})$ reads then as

$$\text{find } \bar{x}, \bar{y}_1, \cdots, \bar{y}_m \in H_0; \sum_{i=1}^m \bar{y}_i = 0 \text{ such that } \bar{x} = Q_i(\bar{x} + \bar{y}_i). \tag{3.17}$$

With respect to an elicitation $e \geq 0$ (global or local) and a proximal parameter $r > e$, the splitting progressive decoupling algorithm proceeds as follows:

Splitting progressive decoupling algorithm:

i) Iteration ν: having $x^\nu \in H_0$ and $y_1^\nu, \cdots, y_n^\nu \in H_0$ with $\sum_{i=1}^n y_i^\nu = 0$, $i = 1 \cdots n$, compute $\hat{x}_i^\nu$ solution of the fixed-point problem $x = Q_i(x + y_i^\nu)$.

ii) Update by: $x^{\nu+1} = \sum_{i=1}^n \hat{x}_i^\nu$ and $y_i^{\nu+1} = y_i^\nu - (r-e)(\hat{x}_i^\nu - x^{\nu+1}),\ i = 1, \cdots, n$.

3.2. **(Hierarchical)-Fixed point problems.** *Browder's Fact* [3]

$$T \text{ is maximal monotone} \Leftrightarrow \text{its dual mapping } I - T \text{ is pseudo-contractive.}$$

In what follows, relying on this connection we take up the notion of the hierarchical fixed-point problem initiated by Moudafi and Maingé; see, for example, [12]. We consider the problem of finding $\bar{x} \in L$ such that

$$(\mathcal{L}) \qquad 0 = (I - Q)(\bar{x}) + N_L(\bar{x}), \text{ for a given a mapping } Q,$$

which amounts to finding $\bar{x} \in L, \bar{y} \in L^\perp$ such that $\bar{x} = \bar{y} + Q(\tilde{x})$ (this clearly reduces to the classical fixed-point problem when $L = H$). The corresponding partial regularization version deals with the main operator $Q_e := Q - eP_{L^\perp}$, which is pseudo-contractive if, for all $x, y \in H$,

$$\langle Q_e(x) - Q_e(y), x - y\rangle \leq \|x - y\|^2.$$

This is equivalent to

$$\langle \big(Q + eP_L\big)(x) - \big(Q + eP_L\big)(y), x - y\rangle \leq (1+e)\|x - y\|^2,\ \forall x, y \in H,$$

in other words

$$\langle \big(\frac{1}{1+e}Q + \frac{e}{1+e}P_L\big)(x) - \big(\frac{1}{1+e}Q + \frac{e}{1+e}P_L\big)(y), x - y\rangle \leq \|x - y\|^2,\ \forall x, y \in H.$$

Consequently,

$$Q_e \text{ is pseudo-contractive } \Leftrightarrow \frac{1}{1+e}Q + \frac{e}{1+e}P_L \text{ is pseudo-contractive .}$$

In the light of the above analysis, we define the *pseudo-contractive elicitation* as:

$$\exists e > 0 \text{ such that } \frac{1}{1+e}Q + \frac{e}{1+e}P_L \text{ pseudo-contractive.} \tag{3.18}$$

This leads to

$$\langle Q(x) - Q(y), x - y\rangle \leq (1+e)\|x - y\|^2 - e\|P_L(x) - P_L(y)\|^2,$$

which is clearly equivalent to the key inequality

$$\langle Q(x) - Q(y), x - y\rangle \leq \|x - y\|^2 + e\|P_{L^\perp}(x) - P_{L^\perp}(y)\|^2. \tag{3.19}$$

The *local pseudo-contractive elicitation* is then obtained when the latter property is satisfied in a neighborhood of a solution. Similarly, the *elicitation for strict pseudo-contraction*, whether global or local can be defined.

Now, let us formulate the progressive decoupling algorithm with parameters $r > e \geq 0$.
The algorithm proceeds iteratively as follow:

Given $x^\nu \in L$ and $y^\nu \in L^\perp$ at iteration ν, compute $\hat{x}^\nu$ that solves $x = Q^\nu(x)$, where Q^ν is given by

$$Q^\nu(x) = \frac{1}{1+r}(y^\nu + Q(x)) + \frac{r}{1+r}x^\nu.$$

Then, update by setting

$$x^{\nu+1} = P_L(\hat{x}^\nu) \text{ and } y^{\nu+1} = y^\nu - (r-e)P_{L^\perp}(\hat{x}^\nu).$$

We are ready now to present the convergence properties of this algorithm.

Theorem 3.2. *Let Q be a globally elicited pseudo-contractive mapping at a level $e \geq 0$ and assume that problem $(\mathcal{L})$ is solvable. Then, for any $r > e$, starting from any $x^0 \in L$ and $y^0 \in L^\perp$, the sequence (x^ν, y^ν) generated by the algorithm is both Féjer monotone with respect to the solution set Γ and asymptotically regular, in the sense*

$$|(x^{\nu+1}, y^{\nu+1}) - (\bar{x}, \bar{y})|_{r,e} \leq |(x^\nu, y^\nu) - (\bar{x}, \bar{y})|_{r,e} \text{ with } (\bar{x}, \bar{y}) \in \Gamma,$$

and

$$\lim_{\nu \to +\infty} |(x^{\nu+1}, y^{\nu+1}) - (x^\nu, y^\nu)|_{r,e} = 0, \text{ where } |(x,y)|_{r,e} := \|x\|^2 + \frac{1}{r(r-e)}\|y\|^2.$$

Furthermore, the sequence $\left(P_\Gamma^{r,e}(x^\nu, y^\nu)\right)$ converges strongly to a point in Γ.

Proof. Set $u^{\nu+1} := -(r-e)^{-1}(y^\nu - y^{\nu+1}) \in L^\perp$ and observe that $y^{\nu+1} = y^\nu - (r-e)u^{\nu+1} \in L^\perp$. Note also that $\hat{x}^\nu = x^{\nu+1} + u^{\nu+1}$. Therefore, the subproblem algorithm can be rewritten as

$$Q(x^{\nu+1} + u^{\nu+1}) = (1+r)(x^{\nu+1} + u^{\nu+1}) - (y^\nu + rx^\nu). \tag{3.20}$$

Now, let $(\bar{x}, \bar{y}) \in \Gamma$ (i.e., $\bar{x} \in L, \bar{y} \in L^\perp$ such that $Q(\bar{x}) = \bar{x} - \bar{y}$). Taking into account (3.20) and applying (3.19) with $x = x^{\nu+1} + u^{\nu+1}$ and $y = \bar{x}$, we obtain

$$\begin{aligned} \langle (1+r)(x^{\nu+1} + u^{\nu+1}) &- (y^\nu + rx^\nu) - (\bar{x} - \bar{y}), (x^{\nu+1} + u^{\nu+1}) - \bar{x} \rangle \\ &\leq \|x^{\nu+1} + u^{\nu+1} - \bar{x}\|^2 + e\|u^{\nu+1}\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle x^{\nu+1} - y^{\nu+1} &+ r(x^{\nu+1} - x^\nu) + (1+e)u^{\nu+1} - \bar{x} + \bar{y}, (x^{\nu+1} - \bar{x}) + u^{\nu+1} \rangle \\ &\leq \|x^{\nu+1} - \bar{x}\|^2 + (1+e)\|u^{\nu+1})\|^2, \end{aligned}$$

in other words

$$\begin{aligned} \langle (x^{\nu+1} - \bar{x}) &+ r(x^{\nu+1} - x^\nu) - (y^{\nu+1} - \bar{y}) + (1+e)u^{\nu+1}, (x^{\nu+1} - \bar{x}) + u^{\nu+1} \rangle \\ &\leq \|x^{\nu+1} - \bar{x}\|^2 + (1+e)\|u^{\nu+1})\|^2. \end{aligned}$$

This leads to

$$\begin{aligned} \|x^{\nu+1} - \bar{x}\|^2 + r\langle x^{\nu+1} - x^\nu, x^{\nu+1} - \bar{x} \rangle &- \langle u^{\nu+1}, y^{\nu+1} - \bar{y} \rangle + (1+e)\|u^{\nu+1}\|^2 \\ &\leq \|x^{\nu+1} - \bar{x}\|^2 + (1+e)\|u^{\nu+1})\|^2. \end{aligned}$$

Using, as in the proof of Theorem 3.1, the elementary equation $2\langle a-c, c-b\rangle = \|a-b\|^2 - \|a-c\|^2 - \|c-b\|^2$ and the fact that $u^{\nu+1} := -(r-e)^{-1}(y^\nu - y^{\nu+1})$, we obtain

$$\begin{aligned}\|x^{\nu+1}-\bar{x}\|^2 + \frac{1}{r(r-e)}\|y^{\nu+1}-\bar{y}\|^2 \;&+\; \|x^{\nu+1}-x^\nu\|^2 + \frac{1}{r(r-e)}\|y^{\nu+1}-y^\nu\|^2 \\ &\le\; \|x^\nu-\bar{x}\|^2 + \frac{1}{r(r-e)}\|y^\nu-\bar{y}\|^2,\end{aligned}$$

which is nothing else than

$$|(x^{\nu+1},y^{\nu+1})-(\bar{x},\bar{y})|_{r,e} + |(x^{\nu+1},y^{\nu+1})-(x^\nu,y^\nu)|_{r,e} \le |(x^\nu,y^\nu)-(\bar{x},\bar{y})|_{r,e}. \tag{3.21}$$

This clearly implies that the sequence (x^ν, y^ν) is Féjer monotone with respect to the solution set Γ. Moreover, the sequence $(|(x^\nu-\bar{x}, y^\nu-\bar{y})|_{r,e})$ is decreasing and lower bounded by 0, which ensures that it converges to some finite limit, say $l(\bar{x},\bar{y})$. By passing to the limit in (3.21), this leads to the asymptotical regularity of (x^ν, y^ν), namely,

$$\lim_{\nu\to+\infty} |(x^{\nu+1},y^{\nu+1})-(x^\nu,y^\nu)|_{r,e} = 0. \tag{3.22}$$

Γ is a nonempty closed convex subset of H, and the sequence (x^ν, y^ν) is Féjer monotone with respect to Γ in $(H, |\cdot|_{r,e})$. From Lemma 3.1, we can obtain the desired conclusion immediately. □

It is worth mentioning that the convergence properties in the classical case (see, for instance, [5] and [6]) require a continuity or a Lipschitz continuity assumption. Observe also that in this case, Q_e and Q share these two continuity properties.

Proposition 3.2. *Let H be a finite dimensional Hilbert space. Suppose that the solution set $\Gamma \neq \emptyset$ and that the mapping Q is globally elicited pseudo-contractive at a level $e \geq 0$. Suppose further that the mapping Q is continuous. Then, the sequence (x^ν, y^ν) generated by the algorithm converges to some element $(x^*, y^*) \in \Gamma$. Moreover, we have $(x^*, y^*) = \lim_{\nu\to+\infty} P_\Gamma^{r,e}(x^\nu, y^\nu)$.*

Proof. The fact that $(|(x^\nu-\bar{x}, y^\nu-\bar{y})|_{r,e})$ converges to a finite limit ensures that both the sequences (x^ν) and (y^ν) are bounded. Let $\tilde{x}$ and $\tilde{y}$ be respectively cluster points of the sequences (x^ν) and (y^ν). By passing to the limit in (3.20) and taking into account the asymptotical regularity of both the sequences (x^ν) and (y^ν), the fact that $\lim_{\nu\to+\infty} u^\nu = 0$ and the continuity of Q yield that

$$\tilde{x}-\tilde{y} = Q(\tilde{x}), \text{ that is } (\tilde{x},\tilde{y}) \in \Gamma. \tag{3.23}$$

The uniqueness of the cluster point follows again by appealing Opial's lemma [14]. This guarantees the convergence of the whole sequence. The rest of the proof is the same as that developed in the end of the proof of Proposition 3.1. □

4. CONCLUSION

In this paper, we first highlighted the essence of the idea presented by Rockafellar in [16] (see also [17]) from a Moreau-Yosida regularization point of view and its great applicability in variational inequalities and optimization. Second, we shown that this offers us a powerful elicitation tool when translated to challenging fixed-point problems. In these two settings, we introduced elicitation notions, designed two progressive decoupling algorithms and established their convergence properties. The fact that the decoupling of linkages opens the way to the decomposition and splitting problems were also briefly illustrated. We would like to emphasize that elicitation notions are very promising, since they allows to go beyond the pseudo-contraction and the firm nonexpansiveness whether globally or locally, where their availability are not evident. We took advantage of the freedom to select the value of parameter's elicitation in order to enhance properties of mappings for algorithmic purposes. For simplicity and clarity, we mainly confined our attention to the global elictation. However, the convergence results will be still valid under local elicitations around a solution $(\tilde{x},\tilde{y})$ and if furthermore the iterations start with x^0+y^0 and get close enough to $\tilde{x}+\tilde{y}$, and localization is attended to in computing the intermediate step $\hat{x}^{\nu}$. Moreover, this allows us to envision the possibility of developing convergence results under local elicitation assumptions around a solution with most probably condition (3.3) which is known to assure the existence of fixed points (where the mapping Q, this time, is only asked to be pseudo-contractive or firmly nonexpansive relatively to a nonempty closed convex set D in the sense of definitions (3.1)-(3.2) with a proper convex closed subset D of H). It is intimately related to the localized versions of the classical methods (see, for example, [4] and [5]) but differs in a crucial change variables in the derivation with additional parameters to work with, which adds the flexibility. Further study and more complete analysis have to be done for theses questions. A topic for future work could be also to replace the Krasnoselskii-Mann method, that is, the hidden engine in our algorithms, by the Ishikawa iterations or other extensions for more possibilities in that direction.

Finally, having in mind the connection between operators and equilibrium functions, we may also consider the following equilibrium problems

$$(\mathcal{L}) \quad \text{find } \bar{x}\in L, \bar{y}\in L^{\perp}:\ F(\bar{x},z)\geq F(\bar{x},\bar{x})+\langle \bar{y},z-\bar{x}\rangle\ \forall z\in C,$$

where $C\subset H$ is a closed convex set and $F:C\times C\to I\!R$ a bifunction. Recall that the bifunction is called monotone if $F(x,y)+F(y,x)\leq 0$ for all $x,y\in C$. To the bifunction F, one can attach the set-valued operator A^F defined by $A^F(x)=\{y^*\in H: F(x,z)-F(x,x)\geq\langle y^*,z-x\rangle\ \forall z\in H\}$ if $x\in C$ and $\emptyset$ otherwise, which is monotone when $F(x,x)=0$ for all $x\in C$. The monotone bifunctions F, which satisfies $F(x,x)=0$ for all $x\in C$, is said to be maximal monotone if the operator A^F is maximal monotone. Observe that when $F(x,x)=0$ for all $x\in C$, $A^F(x)=\partial(F(x,\cdot)+i_C)$ for all $x\in H$ and remember that the proximal point algorithm was extended, by the first author of this paper in [13], to the class of bifunctions verifying the usual conditions: (A1) $F(x,x)=0$ for all $x,y\in C$; (A2) F is monotone; (A3) $\limsup_{t\downarrow 0}F(tz+(1-t)x,y)\leq F(x,y)$ for any $x,y,z\in C$ and (A4) for each $x\in C, y\to F(x,y)$ is convex and lower-semicontinuous. To avoid these assumptions ensuring maximal monotonicity of A^F, in the light of the analysis developed above together with [17], we may translate the elicitation concept and design a progressive decoupling algorithm to the problem of solving problem $(\mathcal{L})$. This is beyond the scope of this paper.

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