Appl. Set-Valued Anal. Optim. 2 (2020), No. 2, pp. 205-222
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.2.2020.2.06

STRONG CONVERGENCE THEOREMS OF A SPLIT COMMON NULL POINT PROBLEM AND A FIXED POINT PROBLEM IN HILBERT SPACES

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Abstract. In this paper, we introduce a new iterative method for finding a common solution of the split common null point problem and the fixed point problem in Hilbert spaces. We obtain the strong convergence of the new iterative method. We also give some applications and numerical experiments to support our main convergence results.

Keywords. Split common null point problem; Monotone operator; Metric projection; Nonexpansive mapping.

1. INTRODUCTION

Let C and Q be nonempty closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $T: H_1 \to H_2$ be a bounded linear operator and let $T^*: H_2 \to H_1$ be the adjoint of T. The split feasibility problem (SFP) is formulated as follows:

$$\text{find an element} \quad x^* \in S = C \cap T^{-1}(Q) \tag{1.1}$$

(1.1), which was first introduced by Censor and Elfving [5] for modeling inverse problems, plays an important role in the real world, such as, in medical image reconstruction and signal processing [2, 3, 4]. Recently, (1.1) has been under the spotlight of research. For recent celebrated results, we refer to [4, 5, 15, 22, 23, 28, 29] and the references therein.

A special case of (1.1) is called the convex constrained linear inverse problem [7], which consists of finding an element $x^* \in H_1$ such that

$$x^* \in C \text{ and } Tx^* = b \in Q.$$

If C is a closed convex subset of a Hilbert space H, then C is the set of null points of the maximal monotone operator A, which is defined by $A = \partial i_C$, where i_C is the indicator function of C and ∂i_C is the subdifferential operator of i_C. So, (1.1) becomes a special case of the split common null point problem (SCNPP), which consists of finding a point $x^* \in H_1$ such that

$$0 \in A_1(x^*) \text{ and } 0 \in A_2(Tx^*),$$

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Received April 24, 2020; Accepted May 31, 2020.

where $A_i: H_i \to 2^{H_i}$, $i = 1,2$ are maximal monotone operators. This problem has been studied by many authors; see, e.g., [6, 9, 10, 24, 25, 27].

In 2008, Moudafi [13] introduced and studied the following Mann-type iterative method [12] for finding a common element of the set of solutions of a mixed equilibrium problem and the set of fixed points of a nonexpansive mapping:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= T_{r_n}^F(x_n - r_n f x_n), \\ x_{n+1} &= \beta_n x_n + (1-\beta_n) S y_n, \end{aligned}$$

where $\{r_n\} \subset (0,\infty)$, $\{\beta_n\} \subset (0,1)$, $F: C \times C \to \mathbb{R}$ is a bifunction, C is a closed and convex subset of H_1, $S: C \to C$ is a nonexpansive mapping and

$$T_r^F x = \left\{ z \in C: F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0 \ \forall y \in C \right\}.$$

Recently, Takahashi and Takahashi [20] extended Moudafi's result in another direction by combining the Mann iterative method and the Halpern iterative method [8] as follows:

$$\begin{aligned} x_0 &= x \in C, \ u \in C, \\ y_n &= T_{r_n}^F(x_n - r_n f x_n), \\ x_{n+1} &= \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) y_n]. \end{aligned}$$

Under some suitable conditions on the sequences $\{r_n\}$, $\{\beta_n\}$, and $\{\alpha_n\}$, they proved the strong convergence of the sequence $\{x_n\}$ generated by the above iterative method.

Motivated by the work of Moudafi [13] and Takahashi and Takahashi [20], we introduce a iterative methods based on the Mann iterative method and the Halpern/viscosity iterative method for finding a common solution of the split common null point problem and the fixed point problem of a nonexpansive mapping. The split minimum point problem, the split feasibility problem, the split equilibrium problem, and the split variational inequality problem are also investigated as the applications of our main results. Finally, two numerical examples are also given to illustrate the effectiveness of the proposed algorithms.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a Hilbert space H. For each $x \in H$, there is a unique $P_C^H x \in C$ such that

$$\|x - P_C^H x\| = \inf_{u \in C} \|x - u\|, \tag{2.1}$$

and the mapping $P_C^H: H \to C$ defined by (2.1) is called the metric projection from H onto C. Moreover, we have

$$\langle x - P_C^H x, y - P_C^H x\rangle \leq 0, \ \forall x \in H, \ y \in C. \tag{2.2}$$

Recall that a mapping $S: C \to C$ is said to be nonexpansive mapping if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

We denote by $\mathrm{Fix}(S)$ the set of fixed points of S, i.e., $\mathrm{Fix}(S) = \{x \in C: Sx = x\}$.

For an operator $A:\ H \to 2^H$, we define its domain, range, and graph as follows:

$$D(A) = \{x \in H:\ A(x) \neq \emptyset\},$$
$$R(A) = \cup\{Az:\ z \in D(A)\},$$

and

$$G(A) = \{(x,y) \in H \times H:\ x \in D(A),\ y \in A(x)\},$$

respectively. The inverse A^{-1} of A is defined by

$$x \in A^{-1}(y) \text{ if and only if } y \in A(x).$$

The operator A is said to be monotone if, for each $x,y \in D(A)$, $\langle u-v, x-y\rangle \geq 0$ for all $u \in A(x)$ and $v \in A(y)$. We denote by I^H the identity operator on H. A monotone operator A is said to be maximal monotone if there is no proper monotone extension of A or $R(I^H + \lambda A) = H$ for all $\lambda > 0$. If A is monotone, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda^A:\ R(I^H + \lambda A) \to D(A)$ by

$$J_\lambda^A = (I^H + \lambda A)^{-1},$$

which is called the resolvent of A.

A monotone operator A is said to satisfy the range condition if $\overline{D(A)} \subset R(I^H + \lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A. For a monotone operator A, which satisfies the range condition, we have $A^{-1}(0) = \mathrm{Fix}(J_\lambda^A)$ for all $\lambda > 0$. If A is a maximal monotone operator, then A satisfies the range condition.

The following lemmas will be needed in what follows for the proof of the main results in this paper.

Lemma 2.1. [1] *Let $A:\ D(A) \subset H \to 2^H$ be a monotone operator. Then, for $\lambda,\ \mu > 0$, and $x \in D(A)$, we have*

$$J_\lambda^A x = J_\mu^A\left(\frac{\mu}{\lambda}x + \big(1-\frac{\mu}{\lambda}\big)J_\lambda^A x\right).$$

Lemma 2.2. *Let $A:\ D(A) \subset H \to 2^H$ be a monotone operator. Then, we have the following statements:*

i) *for $r \geq s > 0$, we have*

$$\|x - J_s^A x\| \leq 2\|x - J_r^A x\|,$$

for all $x \in R(I^H + rA) \cap R(I^H + sA)$;

ii) *for all $r > 0$ and for all $x,y \in R(I^H + rA)$, we have*

$$\langle x-y, J_r^A x - J_r^A y\rangle \geq \|J_r^A x - J_r^A y\|^2;$$

iii) *for all $r > 0$ and for all $x,y \in R(I^H + rA)$, we have*

$$\langle (I^H - J_r^A)x - (I^H - J_r^A)y, x-y\rangle \geq \|(I^H - J_r^A)x - (I^H - J_r^A)y\|^2;$$

iv) *if $\Omega = A^{-1}(0) \neq \emptyset$, then, for all $x^* \in \Omega$ and for all $x \in R(I^H + rA)$,*

$$\|J_r^A x - x^*\|^2 \leq \|x - x^*\|^2 - \|x - J_r^A x\|^2.$$

Proof. i) From Lemma 2.1, we have

$$\begin{aligned}\|x-J_s^A x\| &\leq \|x-J_r^A x\|+\|J_r^A x-J_s^A x\| \\ &\leq \|x-J_r^A x\|+(1-\frac{s}{r})\|x-J_r^A x\| \\ &\leq 2\|x-J_r^A x\|.\end{aligned}$$

ii) Let $u=J_r^A x$ and $v=J_r^A y$. From the definition of J_r^A, we have $x \in u+rA(u)$ and $y \in v+rA(v)$. Thus, it follows from the monotonicity of A that

$$\frac{1}{r}\langle u-v, x-u-(y-v)\rangle \geq 0.$$

So,

$$\langle x-y, u-v\rangle \geq \|u-v\|^2,$$

that is,

$$\langle x-y, J_r^A x-J_r^A y\rangle \geq \|J_r^A x-J_r^A y\|^2.$$

iii) Note that

$$\begin{aligned}\langle (I^H-J_r^A)x-(I^H-J_r^A)y, x-y\rangle &= \|(I^H-J_r^A)x-(I^H-J_r^A)y\|^2 \\ &\quad +\langle (I^H-J_r^A)x-(I^H-J_r^A)y, J_r^A x-J_r^A y\rangle \\ &= \|(I^H-J_r^A)x-(I^H-J_r^A)y\|^2 \\ &\quad +\langle x-y, J_r^A x-J_r^A y\rangle-\|J_r^A x-J_r^A y\|^2.\end{aligned}$$

From ii), we get

$$\langle (I^H-J_r^A)x-(I^H-J_r^A)y, x-y\rangle \geq \|(I^H-J_r^A)x-(I^H-J_r^A)y\|^2.$$

iv) Since $x^* \in A^{-1}(0)$, we have $x^* \in \operatorname{Fix}(J_r^A)$. It follows from iv) that

$$\begin{aligned}\|J_r^A x-x^*\|^2 &= \|x-x^*\|^2+\|x-J_r^A x\|^2+2\langle J_r^A x-x, x-x^*\rangle \\ &= \|x-x^*\|^2+\|x-J_r^A x\|^2-2\langle (I^H-J_r^A)x-(I^H-J_r^A)x^*, x-x^*\rangle \\ &\leq \|x-x^*\|^2-\|x-J_r^A x\|^2.\end{aligned}$$

This completes the proof. □

Lemma 2.3. [30] *Let T be a nonexpansive self-mapping on a nonempty closed and convex subset C of a Hilbert space H. If T has a fixed point, then I^H-T is demiclosed, that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I^H-T)(x_n)\}$ strongly converges to some y, it follows that $(I^H-T)(x)=y$.*

Lemma 2.4. [19] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Hilbert space H and $\{\beta_n\}$ be a sequence in $(0,1)$ with $0<\liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n<1$. Let $x_{n+1}=(1-\beta_n)y_n+\beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|) \leq 0$. Then $\lim_{n\to\infty}\|x_n-y_n\|=0$.*

Lemma 2.5. [11] *Let $\{s_n\}$ be a sequence of nonnegative numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ and $\{c_n\}$ be a sequence of real numbers satisfying the conditions:*

i) $s_{n+1} \leq (1-\alpha_n)s_n+\alpha_n c_n$,

ii) $\sum_{n=0}^{\infty}\alpha_n=\infty$, $\limsup_{n\to\infty}c_n \leq 0$.

Then $\lim_{n\to\infty}s_n=0$.

3. Main results

Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$ be two maximal monotone operators on H_1 and H_2, respectively. Let $S : H_1 \to H_1$ be a nonexpansive mapping and let $T : H_1 \to H_2$ be a bounded linear operator from H_1 onto H_2. Suppose that $\Omega = A^{-1}0 \cap T^{-1}(B^{-1}0) \cap \mathrm{Fix}(S) \neq \emptyset$. We consider the following problem:

$$\text{find an element } x^\dagger \in \Omega. \tag{3.1}$$

Next, we propose the main iterative method for solving Problem (3.1). For any $u,\ x_0 \in H_1$, generate a sequence $\{x_n\}$ as follows

$$\begin{aligned} y_n &= J^A_{\gamma^A_n} x_n, \\ z_n &= J^B_{\gamma^B_n}(Ty_n), \\ t_n &= y_n + \delta T^*(z_n - Ty_n), \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)[\alpha_n u + (1-\alpha_n)St_n],\ n \geq 0, \end{aligned} \tag{3.2}$$

where $\{\gamma^A_n\}$, $\{\gamma^B_n\}$, $\{\beta_n\}$, and $\{\alpha_n\}$ are sequences of positive real numbers.

We are now in a position to prove the strong convergence of the sequence $\{x_n\}$ generated in (3.2) under the following conditions:

C1) $\min\left\{\inf_n\{\gamma^A_n\}, \inf_n\{\gamma^B_n\}\right\} = r > 0$, $\lim_{n\to\infty} |\gamma^A_{n+1} - \gamma^A_n| = 0$;
C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
C3) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$;
C4) $\delta \in \left(0, \dfrac{2}{\|T\|^2}\right)$.

Theorem 3.1. *If the conditions* C1)–C4) *are satisfied, then the sequence* $\{x_n\}$ *generated by* (3.2) *converges strongly to* $x^\dagger = P^{H_1}_{\Omega} u$ *as* $n \to \infty$.

Proof. First, we prove that the sequence $\{x_n\}$ is bounded.

Fix $p \in \Omega$. From $p \in A^{-1}0$ and Lemma 2.2 iv), we have

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - J^A_{\gamma^A_n} x_n\|^2. \tag{3.3}$$

From $Tp = J^B_{\gamma^B_n}(Tp)$ and Lemma 2.2 iv), we have

$$\|z_n - Tp\|^2 \leq \|Ty_n - Tp\|^2 - \|Ty_n - J^B_{\gamma^B_n}(Ty_n)\|^2. \tag{3.4}$$

From Lemma 2.2 iii), we have

$$\begin{aligned} \|t_n - p\|^2 &= \|y_n - p\|^2 + \delta^2 \|T^*(z_n - Ty_n)\|^2 + 2\delta\langle y_n - p, T^*(z_n - Ty_n)\rangle \\ &\leq \|y_n - p\|^2 - \delta(2 - \delta\|T\|^2)\|z_n - Ty_n\|^2 \\ &\leq \|x_n - p\|^2 - \|x_n - J^A_{\gamma^A_n} x_n\|^2 - \delta(2 - \delta\|T\|^2)\|z_n - Ty_n\|^2. \end{aligned} \tag{3.5}$$

Put $d_n = \alpha_n u + (1-\alpha_n)St_n$. From $p = Sp$ and (3.5), we have

$$\begin{aligned} \|d_n - p\| &\leq \alpha_n \|u - p\| + (1-\alpha_n)\|St_n - Sp\| \\ &\leq \alpha_n \|u - p\| + (1-\alpha_n)\|x_n - p\|. \end{aligned} \tag{3.6}$$

Thus, it follows from (3.6) that

$$\begin{aligned}\|x_{n+1}-p\| &\le \beta_n\|x_n-p\|+(1-\beta_n)\|d_n-p\|\\ &\le [1-\alpha_n(1-\beta_n)]\|x_n-p\|+\alpha_n(1-\beta_n)\|u-p\|\\ &\le \max\{\|x_n-p\|,\|u-p\|\}\\ &\quad\vdots\\ &\le \max\{\|x_0-p\|,\|u-p\|\}.\end{aligned}$$

Hence, sequence $\{x_n\}$ is bounded. So, we deduce from (3.3)-(3.5) that sequences $\{y_n\}$, $\{z_n\}$, and $\{t_n\}$ are also bounded.

Next, we show $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From Lemma 2.1, we have

$$\begin{aligned}\|y_{n+1}-y_n\| &= \left\|J^A_{\gamma^A_{n+1}}x_{n+1}-J^A_{\gamma^A_{n+1}}\left(\frac{\gamma^A_{n+1}}{\gamma^A_n}x_n+\big(1-\frac{\gamma^A_{n+1}}{\gamma^A_n}\big)J^A_{\gamma^A_n}x_n\right)\right\|\\ &\le \|x_{n+1}-x_n\|+\frac{|\gamma^A_{n+1}-\gamma^A_n|}{\gamma^A_n}\|x_n-J^A_{\gamma^A_n}x_n\|\\ &\le \|x_{n+1}-x_n\|+K_1|\gamma^A_{n+1}-\gamma^A_n|, \end{aligned}\tag{3.7}$$

where

$$K_1=\frac{\sup_n\{\|x_n-J^A_{\gamma^A_n}x_n\|\}}{r}<\infty.$$

Similarly, we also have

$$\|z_{n+1}-z_n\|\le\|Ty_{n+1}-Ty_n\|+K_2|\gamma^B_{n+1}-\gamma^B_n\|, \tag{3.8}$$

where

$$K_2=\frac{\sup_n\{\|Ty_n-J^B_{\gamma^B_n}(Ty_n)\|\}}{r}<\infty.$$

From Lemma 2.2 iii), we have

$$\begin{aligned}\|t_{n+1}-t_n\|^2 &= \|y_{n+1}-y_n\|^2+\delta^2\|T^*[(z_{n+1}-Ty_{n+1})-(z_n-Ty_n)]\|^2\\ &\quad +2\delta\langle y_{n+1}-y_n,T^*[(z_{n+1}-Ty_{n+1})-(z_n-Ty_n)]\rangle\\ &\le \|y_{n+1}-y_n\|^2+\delta^2\|T\|^2\|(z_{n+1}-Ty_{n+1})-(z_n-Ty_n)\|^2\\ &\quad +2\delta\langle Ty_{n+1}-Ty_n,(z_{n+1}-Ty_{n+1})-(z_n-Ty_n)\rangle\\ &\le \|y_{n+1}-y_n\|^2-\delta(2-\delta\|T\|^2)\|(z_{n+1}-Ty_{n+1})-(z_n-Ty_n)\|^2. \end{aligned}\tag{3.9}$$

It follows from $d_n=\alpha_nu+(1-\alpha_n)St_n$ that

$$\begin{aligned}\|d_{n+1}-d_n\| &\le |\alpha_{n+1}-\alpha_n|\|u\|+\|(1-\alpha_{n+1})St_{n+1}-(1-\alpha_n)St_n\|\\ &\le |\alpha_{n+1}-\alpha_n|\|u\|+|\alpha_{n+1}-\alpha_n|\|St_{n+1}\|+(1-\alpha_n)\|t_{n+1}-t_n\|\\ &\le \|t_{n+1}-t_n\|+K_3|\alpha_{n+1}-\alpha_n|, \end{aligned}\tag{3.10}$$

where $K_3=\|u\|+\sup_n\{\|St_n\|\}<\infty$. From (3.7), (3.9) and (3.10), we obtain

$$\|d_{n+1}-d_n\|\le\|x_{n+1}-x_n\|+K_1|\gamma^A_{n+1}-\gamma^A_n|+K_3|\alpha_{n+1}-\alpha_n|.$$

Thus, from the conditions C1) and C3), we have

$$\limsup_{n\to\infty}(\|d_{n+1}-d_n\|-\|x_{n+1}-x_n\|)\le 0.$$

So, it follows from Lemma 2.4 that

$$\lim_{n\to\infty}\|x_n-d_n\|=0. \tag{3.11}$$

Hence, we have

$$\|x_{n+1}-d_n\|=\beta_n\|x_n-d_n\|\to 0,$$

which together with (3.11) yields that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.12}$$

Next, we prove that the set of weak cluster points of the sequence $\{x_n\}$ is contained in Ω. Indeed, we denote by $\omega(x_n)$ the set of weak cluster points of the sequence $\{x_n\}$ and suppose that x^* is an arbitrarily in $\omega(x_n)$. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$. From the convexity of the function $\|.\|^2$ on H_1 and (3.5), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|\alpha_n u+(1-\alpha_n)St_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)[\alpha_n\|u-p\|^2+(1-\alpha_n)\|t_n-p\|^2]\\ &\le \|x_n-p\|^2+\alpha_n\|u-p\|^2-\|x_n-J^A_{\gamma^A_n}x_n\|^2-\delta(2-\delta\|T\|^2)\|z_n-Ty_n\|^2.\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}&\|x_n-J^A_{\gamma^A_n}x_n\|^2+\delta(2-\delta\|T\|^2)\|z_n-Ty_n\|^2\\ &\le(\|x_n-p\|^2-\|x_{n+1}-p\|^2)+\alpha_n\|u-p\|^2\\ &\le(\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|+\alpha_n\|u-p\|^2.\end{aligned}$$

It follows from $\|x_{n+1}-x_n\|\to 0$, the conditions C3) and C4) that

$$\lim_{n\to\infty}\|x_n-J^A_{\gamma^A_n}x_n\|^2=\lim_{n\to\infty}\|z_n-Ty_n\|^2=0.$$

So, we have

$$\lim_{n\to\infty}\|x_n-J^A_{\gamma^A_n}x_n\|=\lim_{n\to\infty}\|J^B_{\gamma^B_n}(Ty_n)-Ty_n\|=\lim_{n\to\infty}\|t_n-y_n\|=0. \tag{3.13}$$

From Lemma 2.2 i), we get

$$\lim_{n\to\infty}\|x_n-J^A_r x_n\|=\lim_{n\to\infty}\|J^B_r(Ty_n)-Ty_n\|=0.$$

Since $x_{n_k}\rightharpoonup x^*$ and $\lim_{n\to\infty}\|x_n-y_n\|=0$, one has $y_{n_k}\rightharpoonup x^*$. Since T is bounded linear operator, $Ty_{n_k}\rightharpoonup Tx^*$. By use of Lemma 2.4, we obtain $x^*\in A^{-1}0$, and $Tx^*\in B^{-1}0$, that is, $x^*\in A^{-1}0\cap T^{-1}(B^{-1}0)$.

Now, we show that $x^*\in \mathrm{Fix}(S)$. Note that

$$\|x_n-t_n\|\to 0. \tag{3.14}$$

From the condition C3), the boundedness of $\{St_n\}$ and $\|d_n-St_n\|=\alpha_n\|u-St_n\|$, we get

$$\lim_{n\to\infty}\|d_n-St_n\|=0, \tag{3.15}$$

which together with (3.11) yields

$$\lim_{n\to\infty}\|x_n - St_n\| = 0. \tag{3.16}$$

Thus, it follows from (3.14) and (3.16) that

$$\lim_{n\to\infty}\|t_n - St_n\| = 0. \tag{3.17}$$

Since $x_{n_k} \rightharpoonup x^*$ and (3.14), we have $t_{n_k} \rightharpoonup x^*$. So, from Lemma 2.4 and (3.17), we obtain $x^* \in \mathrm{Fix}(S)$. Hence, $x^* \in \Omega$. Consequently, $\omega(x_n) \subseteq \Omega$.

Finally, we show that $x_n \to x^\dagger = P_\Omega^{H_1} u$. Putting $x^\dagger = P_\Omega^{H_1} u$, we have from (3.4) that

$$\begin{aligned}
\|x_{n+1} - x^\dagger\|^2 &= \beta_n\langle x_n - x^\dagger, x_{n+1} - x^\dagger\rangle + (1-\beta_n)\langle \alpha_n u + (1-\alpha_n)St_n - x^\dagger, x_{n+1} - x^\dagger\rangle \\
&\le \beta_n \frac{\|x_n - x^\dagger\|^2 + \|x_{n+1} - x^\dagger\|^2}{2} \\
&\quad + (1-\beta_n)(1-\alpha_n)\frac{\|t_n - x^\dagger\|^2 + \|x_{n+1} - x^\dagger\|^2}{2} \\
&\quad + \alpha_n(1-\beta_n)\langle u - x^\dagger, x_{n+1} - x^\dagger\rangle \\
&\le \beta_n \frac{\|x_n - x^\dagger\|^2 + \|x_{n+1} - x^\dagger\|^2}{2} \\
&\quad + (1-\beta_n)(1-\alpha_n)\frac{\|x_n - x^\dagger\|^2 + \|x_{n+1} - x^\dagger\|^2}{2} \\
&\quad + \alpha_n(1-\beta_n)\langle u - x^\dagger, x_{n+1} - x^\dagger\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
[1+\alpha_n(1-\beta_n)]\|x_{n+1} - x^\dagger\|^2 \le [1-\alpha_n(1-\beta_n)]\|x_n - x^\dagger\|^2 \\
+ 2(1-\beta_n)\alpha_n\langle u - x^\dagger, x_{n+1} - x^\dagger\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - x^\dagger\|^2 &\le [1-\alpha_n(1-\beta_n)]\|x_n - x^\dagger\|^2 \\
&\quad + \alpha_n(1-\beta_n)\frac{2}{1+\alpha_n(1-\beta_n)}\langle u - x^\dagger, x_{n+1} - x^\dagger\rangle.
\end{aligned} \tag{3.18}$$

Let $s_n = \|x_n - x^\dagger\|^2$ and $c_n = \dfrac{2}{1+\alpha_n(1-\beta_n)}\langle u - x^\dagger, x_{n+1} - x^\dagger\rangle$. Then the inequality (3.18) can be rewritten in the following form

$$s_{n+1} \le [1-\alpha_n(1-\beta_n)]s_n + \alpha_n(1-\beta_n)c_n. \tag{3.19}$$

Now, we will show that $\limsup_{n\to\infty} c_n \le 0$. Indeed, suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - x^\dagger, x_n - x^\dagger\rangle = \lim_{k\to\infty}\langle u - x^\dagger, x_{n_k} - x^\dagger\rangle.$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_l}} \rightharpoonup x^*$. We may assume without loss of generality that $x_{n_k} \rightharpoonup x^*$. It follows from $\omega(x_n) \subseteq \Omega$ that $x^* \in \Omega$. So, from $x^\dagger = P_\Omega^{H_1} u$ and (2.2), we deduce that

$$\limsup_{n\to\infty}\langle u - x^\dagger, x_n - x^\dagger\rangle = \langle u - x^\dagger, x^* - x^\dagger\rangle \le 0,$$

which together with (3.12), conditions C2), and C3) gets $\limsup_{n\to\infty} c_n \leq 0$. Since $\sum_{n=1}^{\infty} \alpha_n = \infty$ and condition C2), we have $\sum_{n=1}^{\infty} \alpha_n(1-\beta_n) = \infty$. Hence, all conditions of Lemma 2.5 are satisfied. Therefore, we immediately deduce that $s_n \to 0$, that is, $x_n \to x^{\dagger} = P_{\Omega}^{H_1} u$. This completes the proof. □

The viscosity approximation method, which was firstly introduced and investigated by Moudafi [15], acts as a bridge linking fixed point and variational inequality problem. Indeed, the fixed point of the potential nonlinear operator is also a unique solution to some monotone variational inequality.

Next, we give the following viscosity method

$$\begin{aligned} u_n &= J_{\gamma_n^A}^A e_n, \\ v_n &= J_{\gamma_n^B}^B (Tu_n), \\ w_n &= u_n + \delta T^*(v_n - Tu_n), \\ e_{n+1} &= \beta_n e_n + (1-\beta_n)[\alpha_n f(e_n) + (1-\alpha_n) S w_n],\ n \geq 0, \end{aligned} \tag{3.20}$$

where $f:\ H_1 \to H_1$ is a contractive mapping from H_1 into itself with the contraction coefficient $c \in (0,1)$.

The convergence analysis of the above viscosity approximation method is easy to obtain with the aid the Theorem 3.1. However, we here still give the proof for the sake of the completeness.

Theorem 3.2. *If conditions* C1)–C4) *are satisfied, then the sequence* $\{e_n\}$ *generated by* (3.20) *converges strongly to* $x^* \in \Omega$ *which is the unique solution to the variational inequality*

$$\langle (I-f)x^*, y - x^* \rangle \geq 0,\ \forall y \in \Omega.$$

Proof. Since $P_S^{H_1} f$ is contractive mapping, Banach contraction mapping principle guarantees that $P_S^{H_1} f$ has a unique fixed point x^* which is the unique solution of the variational inequality

$$\langle (I-f)x^*, y - x^* \rangle \geq 0,\ \forall y \in \Omega.$$

From Theorem 3.1, replacing u by $f(x^*)$ in (3.2), we have the sequence $\{x_n\}$ converging strongly to $P_{\Omega}^{H_1} f(x^*) = x^*$.

Now we only need to prove that $\|e_n - x_n\| \to 0$, as $n \to \infty$. Note that

$$\|e_{n+1} - x_{n+1}\| \leq \beta_n \|e_n - x_n\| + (1-\beta_n)[\alpha_n c\|e_n - x^*\| + (1-\alpha_n)\|w_n - t_n\|]. \tag{3.21}$$

By use of Lemma 2.2 iii), we have

$$\begin{aligned} \|w_n - t_n\|^2 &= \|u_n - y_n\|^2 + \delta^2 \|T^*[(v_n - Tu_n) - (z_n - Ty_n)]\|^2 \\ &\quad + 2\delta \langle u_n - y_n, T^*[(v_n - Tu_n) - (z_n - Ty_n)] \rangle \\ &\leq \|u_n - y_n\|^2 - \delta(2 - \delta\|T\|^2)\|(v_n - Tu_n) - (z_n - Ty_n)\|^2 \\ &\leq \|u_n - y_n\|^2. \end{aligned} \tag{3.22}$$

From the nonexpansiveness of $J_{\gamma_n^A}^A$, we have

$$\|u_n - y_n\| \leq \|e_n - x_n\|. \tag{3.23}$$

Thanks to (3.21)–(3.23), we obtain

$$\begin{aligned}\|e_{n+1}-x_{n+1}\| &\le \beta_n\|e_n-x_n\|+(1-\beta_n)[\alpha_n c\|e_n-x^*\|+(1-\alpha_n)\|e_n-x_n\|]\\ &\le [1-\alpha_n(1-\beta_n)]\|e_n-x_n\|+\alpha_n(1-\beta_n)c(\|e_n-x_n\|+\|x_n-x^*\|)\\ &= [1-\alpha_n(1-\beta_n)(1-c)]\|e_n-x_n\|+\alpha_n(1-\beta_n)c\|x_n-x^*\|.\end{aligned}$$

From Lemma 2.5, we get $\lim_{n\to\infty}\|e_n-x_n\|=0$. Thus, $\lim_{n\to\infty}\|e_n-x^*\|=0$, we obtain $\lim_{n\to\infty}\|e_n-x^*\|=0$. This completes the proof. □

If the mapping S in Theorem 3.2 is an identity mapping, then we have the following result.

Corollary 3.1. *Let H_1 and H_2 be two real Hilbert spaces and let $f:\ H_1\to H_1$ be a contractive mapping on H_1. Let $A:\ H_1\to 2^{H_1}$ and $B:\ H_2\to 2^{H_2}$ be two maximal monotone operators on H_1 and H_2, respectively. Let $T:\ H_1\to H_2$ be bounded linear operator from H_1 onto H_2 such that $\Omega=A^{-1}0\cap T^{-1}(B^{-1}0)\neq\emptyset$. If conditions* C1)–C4) *are satisfied, then the sequence $\{e_n\}$ generated by the following iterative method: $e_0\in H_1$ and*

$$\begin{aligned}&u_n=J^A_{\gamma_n^A}e_n,\\ &v_n=J^B_{\gamma_n^B}(Tu_n),\\ &w_n=u_n+\delta T^*(v_n-Tu_n),\\ &e_{n+1}=\beta_n e_n+(1-\beta_n)[\alpha_n f(e_n)+(1-\alpha_n)w_n],\ n\ge 0,\end{aligned} \tag{3.24}$$

converges strongly to $x^*\in\Omega$, which is a unique solution of the variational inequality

$$\langle (I-f)x^*,y-x^*\rangle\ge 0,\ \forall y\in\Omega.$$

If $H_1=H_2$, $Ax=Bx=0$ for all $x\in H_1$ and $\gamma_n^A=\gamma_n^B=1$ for all $n\ge 1$ in Theorem 3.2, we obtain the following result.

Corollary 3.2. *Let H_1 be a real Hilbert space and let $f:\ H_1\to H_1$ be a contractive mapping on H_1. Let $S:\ H_1\to H_1$ be a nonexpansive mapping such that $\Omega=\mathrm{Fix}(S)\neq\emptyset$. If conditions* C2)–C3) *are satisfied, then the sequence $\{e_n\}$ generated by the following iterative method; $e_0\in H_1$,*

$$e_{n+1}=\beta_n e_n+(1-\beta_n)[\alpha_n f(e_n)+(1-\alpha_n)Se_n],\ n\ge 0, \tag{3.25}$$

converges strongly to $x^*\in\Omega$, which is a unique solution of the variational inequality

$$\langle (I-f)x^*,y-x^*\rangle\ge 0,\ \forall y\in\Omega.$$

4. Applications

4.1. **Split minimum point problem.** Let H be a real Hilbert space and let $g:H\to(-\infty,\infty]$ be a proper, lower semicontinuous and convex function. The subdifferential of g is the multi-valued mapping $\partial g:\ H\to 2^H$, which is defined by

$$\partial g(x)=\{z\in H:\ g(y)-g(x)\ge\langle y-x,z\rangle,\ \forall y\in H\}$$

for all $x\in H$. We know that ∂g is the maximal monotone operator [17] and $x_0\in\arg\min_{x\in H}g(x)$ if and only if $\partial g(x_0)\ni 0$.

We have the following result on the split minimum point problem in two real Hilbert spaces.

Theorem 4.1. *Let H_1 and H_2 be two real Hilbert spaces and let $f: H_1 \to H_1$ be a contractive mapping on H_1. Let $f_1: H_1 \to (-\infty,\infty]$, $f_2: H_2 \to (-\infty,\infty]$ be two proper, lower semicontinuous and convex functions and let $T: H_1 \to H_2$ be a bounded linear operator such that*

$$\Omega = \arg\min_{x\in H_1} f_1(x) \cap T^{-1}(\arg\min_{x\in H_2} f_2(x)) \neq \emptyset.$$

If conditions C1)–C4) *are satisfied, then the sequence $\{e_n\}$ generated by $e_0 \in H_1$ and*

$$\begin{aligned}
u_n &= \arg\min_{x\in H_1}\left\{f_1(x) + \frac{1}{2\gamma_n^A}\|x-e_n\|^2\right\},\\
v_n &= \arg\min_{y\in H_2}\left\{f_2(y) + \frac{1}{2\gamma_n^B}\|y-Tu_n\|^2\right\},\\
w_n &= u_n + \delta T^*(v_n - Tu_n),\\
e_{n+1} &= \beta_n e_n + (1-\beta_n)[\alpha_n f(e_n) + (1-\alpha_n)w_n],\ n \geq 0,
\end{aligned} \tag{4.1}$$

converges strongly to $x^* \in \Omega$, which is a unique solution of the variational inequality $\langle (I-f)x^*, y-x^*\rangle \geq 0$, $\forall y \in \Omega$.

Proof. First, we have $\Omega = (\partial f_1)^{-1}(0) \cap T^{-1}((\partial f_2)^{-1}(0))$. Note that

$$u_n = \arg\min_{x\in H_1}\left\{f_1(x) + \frac{1}{2\gamma_n^A}\|x-e_n\|^2\right\}$$

if and only if

$$\partial f_1(u_n) + \frac{1}{\gamma_n^A}(u_n - e_n) \ni 0,$$

which implies that $u_n = J^A_{\gamma_n^A} e_n$, where $A = \partial f_1$. Similarly, we also have $v_n = J^B_{\gamma_n^B}(Tu_n)$ with $B = \partial f_2$. Thus, by using Corollary 3.1, we get the desired conclusion immediately. □

4.2. **Split feasibility problem.** Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let i_C be the indicator function of C, that is,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C,\\ \infty, & \text{if } x \notin C. \end{cases}$$

It is easy to see that i_C is the proper, semicontinuous and convex function and its subdifferential ∂i_C is a maximal monotone operator. Observe taht

$$\partial i_C(u) = N(u,C) = \{v \in H:\ \langle u-y, f\rangle \geq 0\ \forall y \in C\},$$

where $N(u,C)$ is the normal cone of C at u.

We denote by J_r the resolvent operator of ∂i_C with $r > 0$. Suppose $u = J_r x$ for $x \in H$, i.e.,

$$\frac{x-u}{r} \in \partial i_C(u) = N(u,C).$$

Thus,

$$\langle x-u, u-y\rangle \geq 0, \quad \forall y \in C.$$

From (2.2), we get $u = P_C^H x$. In view of Corollary 3.1, we have the following result for solving the split feasibility problem in Hilbert spaces immediately.

Theorem 4.2. *Let H_1 and H_2 be two real Hilbert spaces and let $f: H_1 \to H_1$ be a contractive mapping on H_1. Let Q_i, $i = 1,2$ be two closed convex subsets of H_i, respectively. Let $T: H_1 \to H_2$ be a bounded linear operator such that $\Omega = Q_1 \cap T^{-1}(Q_2) \neq \emptyset$. If conditions* C2)–C4) *are satisfied, then the sequence $\{e_n\}$ generated by $e_0 \in H_1$ and*

$$\begin{aligned} u_n &= P_{Q_1}^{H_1} e_n, \\ v_n &= P_{Q_2}^{H_2}(Tu_n), \\ w_n &= u_n + \delta T^*(v_n - Tu_n), \\ e_{n+1} &= \beta_n e_n + (1-\beta_n)[\alpha_n f(e_n) + (1-\alpha_n) w_n],\ n \geq 0, \end{aligned} \tag{4.2}$$

converges strongly to $x^* \in \Omega$, which is a unique solution of the variational inequality $\langle (I - f)x^*, y - x^* \rangle \geq 0$, $\forall y \in \Omega$.

4.3. **Split equilibrium problem.** Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem:

$$\text{find} x \in C \text{ such that } F(x,y) \geq 0,\ \forall y \in C. \tag{4.3}$$

To study the equilibrium problem, we may assume that F satisfies the following restrictions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x,y \in C$;
(A3) for each $x,y,z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(A4) for each $x \in C$, $y \longmapsto F(x,y)$ is convex and lower semi-continuous.

We need the following lemma for solving the equilibrium problem.

Lemma 4.1. [21] *Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4)*, and let A_F be a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H:\ F(x,y) \geq \langle y - x, z \rangle,\ \forall y \in C\},\ x \in C \\ \emptyset,\ x \notin C. \end{cases}$$

Then A_F is a maximal monotone operator with the domain $D(A_F) \subset C$, $EP(F) = A_F^{-1} 0$, where $EP(F)$ stands for the solution set of (4.3)*, and the resolvent $T_r = (I + rA_F)^{-1}$ is defined by*

$$T_r^F x = \{z \in C:\ F(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0,\ \forall y \in C\},\ \forall x \in H.$$

Now, we have the following result.

Theorem 4.3. *Let H_1 and H_2 be two real Hilbert spaces and let $f: H_1 \to H_1$ be a contractive mapping on H_1. Let $F_i: H_i \times H_i \to \mathbb{R}$, $i = 1,2$ be two bifunctions which satisfy the conditions* (A1)-(A4) *and let $T: H_1 \to H_2$ be a bounded linear operator such that $\Omega =$*

$EP(F_1)\cap T^{-1}(EP(F_2))\neq\emptyset$. *If conditions* C1)–C4) *are satisfied, then the sequence* $\{e_n\}$ *generated by* $e_0\in H_1$ *and*

$$\begin{aligned} u_n &= T^{F_1}_{\gamma_n^A}(e_n),\\ v_n &= T^{F_2}_{\gamma_n^B}(Tu_n),\\ w_n &= u_n+\delta T^*(v_n-Tu_n),\\ e_{n+1} &= \beta_n e_n+(1-\beta_n)[\alpha_n f(e_n)+(1-\alpha_n)w_n],\ n\geq 0, \end{aligned} \tag{4.4}$$

converges strongly to $x^*\in\Omega$, *which is a unique solution of the variational inequality* $\langle (I-f)x^*, y-x^*\rangle\geq 0$, $\forall y\in\Omega$.

4.4. **Split variational inequality problem.** Let H be a real Hilbert space and let C be a closed convex subset of H. Let $A\ :C\to H$ be a single-valued monotone operator which is hemicontinuous. Then a point $u\in C$ is said to be a solution of the variational inequality involving operator A if $\langle y-u, Au\rangle\geq 0$ holds for all $y\in C$. We denote by $VI(C,A)$ the set of all solutions of the variational inequality.

Theorem 4.4. *Let* H_i, $i=1,2$, *be two real Hilbert spaces and let* C_i, $i=1,2$, *be two closed convex subsets of* H_i, *respectively. Let* $f:\ H_1\to H_1$ *be a contractive mapping on* H_1. *Let* $A_i:\ C_i\to H_i$, $i=1,2$, *be two single-valued monotone and hemicontinuous operators, and let* $T:\ H_1\to H_2$ *be a bounded linear operator such that* $\Omega=VI(C_1,A_1)\cap T^{-1}(VI(C_2,A_2))\neq\emptyset$. *If conditions* C1)–C4) *are satisfied, then the sequence* $\{e_n\}$ *generated by* $e_0\in H_1$ *and*

$$\begin{aligned} u_n &= VI(C_1,\gamma_n^{A_1}A_1+I^{H_1}-e_n),\\ v_n &= VI(C_2,\gamma_n^{A_2}A_2-I^{H_2}-Tu_n),\\ w_n &= u_n+\delta T^*(v_n-Tu_n),\\ e_{n+1} &= \beta_n e_n+(1-\beta_n)[\alpha_n f(e_n)+(1-\alpha_n)w_n],\ n\geq 0, \end{aligned} \tag{4.5}$$

converges strongly to $x^*\in\Omega$, *which is a unique solution of the variational inequality* $\langle (I-f)x^*, y-x^*\rangle\geq 0$, $\forall y\in\Omega$.

Proof. Define a mapping $T_{A_1}\subset H_1\times H_1$ by

$$T_{A_1}x=\begin{cases} A_1x+N_{C_1}(x), & x\in C_1\\ \emptyset, & x\notin C_1,\end{cases}$$

where $N_{C_1}(x)=\{z\in H_1:\ \langle y-x,z\rangle\leq 0 \text{ for all } y\in C_1\}$.

By Rockafellar [18], we know that the operator T_{A_1} is maximal monotone, and $T_{A_1}^{-1}(0)=VI(C_1,A_1)$. Note that

$$u_n=VI(C_1,\gamma_n^{A_1}A_1+I^{H_1}-e_n)$$

if and only if

$$\langle y-u_n,\gamma_n^{A_1}A_1(u_n)+u_n-e_n\rangle\geq 0$$

for all $y\in C_1$, that is, $-\gamma_n^{A_1}A_1(u_n)-u_n+e_n\in\gamma_n^{A_1}N_{C_1}(u_n)$. This implies that $u_n=J^{T_{A_1}}_{\gamma_n^{A_1}}e_n$. Similarly, if we define a mapping $T_{A_2}\subset H_2\times H_2$ by

$$T_{A_2}x=\begin{cases} A_1x+N_{C_2}(x), & x\in C_2\\ \emptyset, & x\notin C_2,\end{cases}$$

then T_{A_2} is maximal monotone, $T_{A_2}^{-1}(0) = VI(C_2, A_2)$ and $v_n = J_{\gamma_n^{A_2}}^{T_{A_2}}(Tu_n)$. From Corollary 3.1, we get the desired conclusion easily. □

5. Numerical experiments

The algorithms are implemented in MALAB 7.0 running on a HP Compaq 510, Core(TM) 2 Duo CPU. T5870 with 2.0 GHz and 2GB RAM.

Example 5.1. Consider the problem of finding an element $x^\dagger \in \mathbb{R}^4$ such that

$$x^\dagger \in \Omega = \arg\min_{x\in\mathbb{R}^4} f_1(x) \cap T^{-1}(\arg\min_{x\in\mathbb{R}^4} f_2(x)),$$

where f_1 and f_2 are defined by

$$f_i(x) = \langle A_i x, x\rangle + \langle B_i, x\rangle + C_i,\ i = 1,2$$

with

$$A_1 = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -4 & -4 & 4 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} -4 & -4 & 0 & 4 \end{pmatrix},\ C_1,\ C_2 \text{ are any constants,}$$

and $T:\ \mathbb{R}^4 \to \mathbb{R}^4$ is a bounded linear operator which is defined by

$$Tx = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 4 & 0 \\ 3 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

for all $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$.

Then, it is easy to show that f_1 and f_2 are two proper continuous convex functions on $\mathbb{R}^4$ and $x^\dagger \in S$ if and only if $2A_1 x^\dagger + B_1 = 0$ and $2A_2(Tx^\dagger) + B_2 = 0$. So, the set of solutions Ω in Example 5.1 is defined by

$$S = \left\{ \left(\frac{60}{23}, -\frac{266}{161}, -\frac{168}{161}, a \right) :\ a \in \mathbb{R} \right\},$$

and if $x_0 = (2, -1, -2, 4)$ then

$$x^\dagger = P_\Omega^{\mathbb{R}^4}(x_0) = \left(\frac{60}{23}, -\frac{266}{161}, -\frac{168}{161}, 4 \right) \approx (2.60869565, -1.65217391, -1.04347826, 4).$$

In Theorem 4.1, let $f(x) = x_0 = (2, -1, -2, 4)$ for all $x \in \mathbb{R}^4$, $\gamma_n^{A_1} = \gamma_n^{A_2} = 1$, $\alpha_n = 1/n$ and $\beta_n = 1/2$ for all $n \geq 0$. Then, we have the following table of numerical results (Table 1).

The behavior of TOL_n in the case $\mathrm{TOL}_n < 10^{-4}$ is described in Figure 1.

In Theorem 4.1, let $f(x) = x/5$ for all $x \in \mathbb{R}^4$, $\gamma_n^{A_1} = \gamma_n^{A_2} = 1$, $\alpha_n = 1/n$ and $\beta_n = 1/2$ for all $n \geq 0$. Then, we have the following table of numerical results (Table 1).

Note that, in these cases, at the nth iteration step, we define the function TOL_n by

$$\mathrm{TOL}_n = \frac{1}{2}(\|2A_1 x_n + B_1\|^2 + \|2A_2(Tx_n) + B_2\|^2)$$

ε	n	TOL_n	x_n
10^{-2}	4460	9.99978481e-003	(2.60154635, -1.64518849, -1.04377354, 4)
10^{-3}	44450	9.99995419e-004	(2.60798073, -1.65147533, -1.04350784, 4)
10^{-4}	444351	9.99999830e-005	(2.60862416, -1.65210405, -1.04348121, 4)

TABLE 1. Table of numerical results with $\mathrm{TOL}_n = \|x_n - x^\dagger\|$

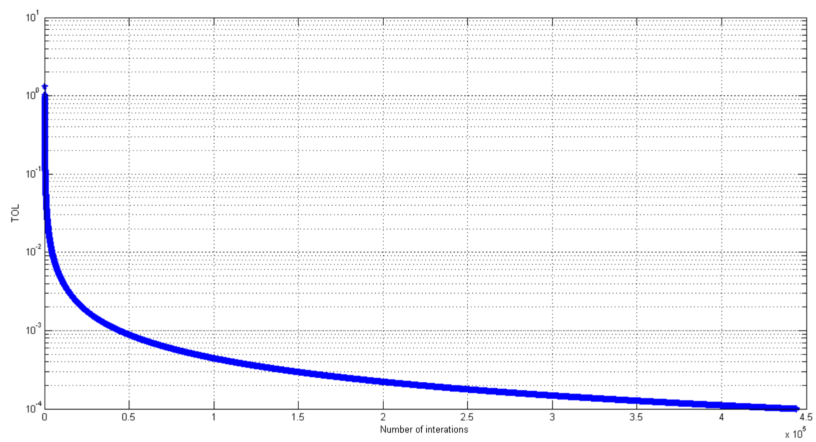

FIGURE 1. The behavior of TOL_n with the stop condition $\mathrm{TOL}_n < 10^{-4}$

ε	n	TOL_n	x_n
10^{-2}	3945	9.995938e-003	(2.586882, -1.631553, -1.043644, 1.254370e-002)
10^{-3}	12415	9.999292e-004	(2.601796, -1.645652, -1.043530, 6.305143e-003)
10^{-4}	39202	9.999527e-005	(2.606513, -1.650111, -1.043494, 3.162867e-003)
10^{-5}	123908	9.999958e-006	(2.608005, -1.651521, -1.043483, 1.585649e-003)
10^{-6}	391774	9.999991e-007	(2.608477, -1.651967, -1.043479, 7.947777e-004)

TABLE 2. Table of numerical results

and use the condition $\mathrm{TOL}_n < \varepsilon$ to stop the iteration process, where ε is an initial error.

The behavior of TOL_n in the case $\mathrm{TOL}_n < 10^{-6}$ is described in Figure 2:

Remark 5.1. If, in Theorem 4.1, $f(x) = x/5$ for all $x \in \mathbb{R}^3$, then

$$x^\dagger = P_\Omega 0 \approx (2.60869565, -1.65217391, -1.04347826, 0).$$

Example 5.2. Consider the following problem: Find an element $x^\dagger \in \mathbb{R}^{40}$ such that

$$x^\dagger \in \Omega = S_1 \cap T^{-1}(S_2) \neq \emptyset,$$

where $S_1 = \{x \in \mathbb{R}^{40} : \|x - a_1\|^2 \leq R_1^2\}$, $S_2 = \{x \in \mathbb{R}^{50} : \|x - a_2\|^2 \leq R_2^2\}$ and $T : \mathbb{R}^{40} \to \mathbb{R}^{50}$ is a bounded linear operator whose matrix has the elements are randomly generated in $[-10, 10]$.

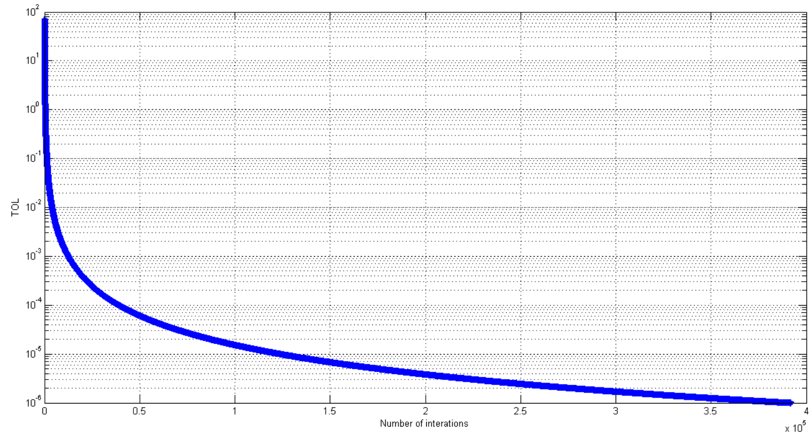

FIGURE 2. The behavior of TOL_n with the stop condition $\text{TOL}_n < 10^{-6}$

Now, we test the convergence of the sequence $\{x_n\}$ which is defined in Theorem 4.2 with the centers $a_1,\ a_2$ randomly generated in $[-1,1]$, the radius $R_1,\ R_2$ randomly generated in $[7,8]$ and $[9,10]$, respectively, and the initial x_0 randomly generated in $[-5,5]$.

It is easy to see that, in this case, $\Omega = S_1 \cap T^{-1}(S_2) \neq \emptyset$ because $0 \in \Omega$.

Remark 5.2. In this example, the function TOL is defined by

$$\text{TOL}_n = \frac{1}{2}(\|x_n - P_{S_1}^{\mathbb{R}^{40}}(x_n)\|^2 + \|Tx_n - P_{S_2}^{\mathbb{R}^{50}}(Tx_n)\|^2),$$

for all $n \geq 1$. Note that, at the nth step, if $\text{TOL}_n = 0$ then $x_n \in S$, that is, x_n is a solution of this problem.

After five attempts with the randomized data and $f(x) = x_0$ or $f(x) = x/2$ for all $x \in \mathbb{R}^{40}$, $\alpha_n = 1/n$, $\beta_n = 1/2$ for all $n \geq 1$ and $\delta = 1/\|T\|^2$, we obtain the following table of numerical results (Table 3).

The behavior of TOL_n in the Table 3 is described in Figure 3 and Figure 4.

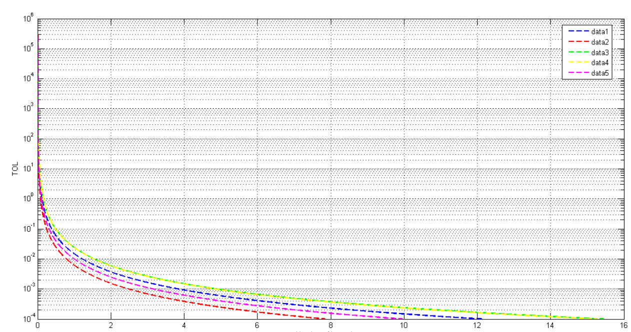

FIGURE 3. The behavior of TOL_n with the stop condition $\text{TOL}_n < 10^{-4}$ and $f(x) = x_0$

Stop condition: $\mathrm{TOL}_n < 10^{-3}$			Stop condition: $\mathrm{TOL}_n < 10^{-4}$		
No.	TOL_n	n	No.	TOL_n	n
$f(x) = x_0$					
1	$9.999930604862115e-004$	38381	1	$9.999978097851798e-005$	121258
2	$9.999759540680524e-004$	24794	2	$9.999922131723621e-005$	78401
3	$9.999842360618018e-004$	48879	3	$9.999944518918451e-005$	154551
4	$9.999952749402601e-004$	48303	4	$9.999993798300586e-005$	152641
5	$9.999536072386973e-004$	31470	5	$9.999836001196085e-005$	99529
$f(x) = x/2$					
1	$8.402767592710540e-004$	116	1	$5.484656397249216e-005$	55
2	$5.372343063055929e-004$	53	2	$7.095432510606713e-007$	97
3	$8.597287933614299e-004$	60	3	$3.859116638956402e-005$	77
4	$4.419986354387474e-004$	66	4	0	61
5	$2.435382620250198e-004$	82	5	$8.622270871144550e-006$	74

TABLE 3. Table of numerical results

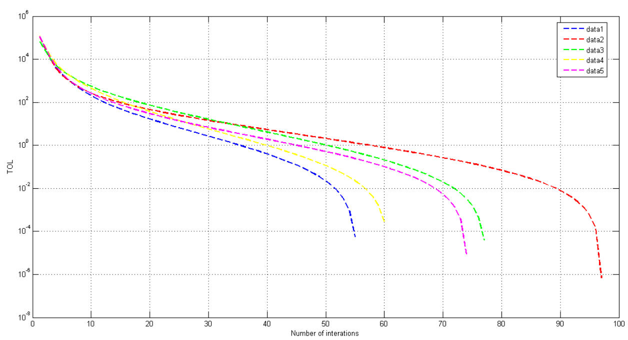

FIGURE 4. The behavior of TOL_n with the stop condition $\mathrm{TOL}_n < 10^{-4}$ and $f(x) = x/2$

Acknowledgments

The authors thank the referees for useful suggestions, which improved the presentation of this paper. The first author was supported by the Science and Technology Fund of Thai Nguyen University of Sciences.

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