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AN INERTIAL ALGORITHM FOR APPROXIMATING SOLUTIONS OF SPLIT FEASIBILITY PROBLEMS IN REAL BANACH SPACES

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Abstract. In this paper, we construct and study an inertial algorithm for solving a split feasibility problem in real Banach spaces. The sequence generated via the algorithm is proved to be convergent strongly to a solution of the split feasibility problem.

Keywords. Monotone operators; Split feasibility problem; Relatively nonexpansive mapping; Resolvent operator.

1. INTRODUCTION

Let H_1 and H_2 be two real Hilbert spaces, and let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. The split feasibility problem is consists of finding a point $q \in H_1$ such that

$$q \in C \text{ and } Aq \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator.

The split feasibility problem was introduced in 1994 by Censor and Elfving [1] in finite dimensional Hilbert spaces. It is now known that the split feasibility problem is applicable in many disciplines such as image restoration, computer tomograph and radiation therapy treatment planning; see, e.g., [2, 3, 4, 5] and the references therein.

If problem (1.1) has a solution, it is known that $x \in C$ solves (1.1) if and only if it solves the following fixed point equation:

$$x = P_C((I - \gamma A^*(I - P_Q)A)x), x \in C, \tag{1.2}$$

where P_C and P_Q are the metric projections onto C and Q respectively, γ is a positive constant and A^* denotes the adjoint of A. Consequently, split feasibility problem (1.1) can be solved via fixed point methods. Recently, the solutions of split feasibility problem (1.1) has been extensively studied in Hilbert spaces by many authors; see, e.g., [6, 7, 8, 9] and the references therein.

The iterative approximation of fixed points of nonlinear mappings is important in the field of nonlinear analysis, and many convergence theorems of fixed points have been obtained in

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Hilbert and Banach spaces; see, e.g., [10, 11, 12, 13] and the references therein. For fixed points of nonexpansive mappings, Mann [14] in 1953 introduced the following iteration process:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \tag{1.3}$$

where the initial guess $x_1 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $(0,1)$. It is known that under appropriate conditions, the sequence $\{x_n\}$ generated by (1.3) converges weakly to a fixed point of T. However, even in a Hilbert space, Mann iteration may fail to converge strongly (see [15]). In general, the convergence rate of the Mann iteration is slow. Recently, fast iterative algorithms is now under the spotlight due to their applications.

In particular, the inertial extrapolation was first proposed by Polyak [16] as an acceleration process is popular. In recent years, some authors constructed various fast convergent iterative algorithms via inertial extrapolation techniques such as, inertial Mann algorithms, inertial forward-backward splitting algorithm, etc; see, e.g., [17, 18, 19, 20] and the references therein.

In this paper, we construct an inertial-type algorithm for finding a common solution of a split feasibility problem in certain real Banach spaces. We prove that the sequence generated by our new algorithm converges strongly to solution of the feasibility problem. Finally, we apply the obtained result to hierarchical variational inequality problems (see in Section 4). This paper is organized as follows. In Section 2, we present preliminary results and some important definitions needed for the main results of this paper. Section 3 is devoted to the main convergence theorem and its proof. The last section, Section 4, is devoted to theoretical application of the main result.

2. PRELIMINARIES

Throughout this paper, we assume that the Banach spaces are real. Recall that a Banach space E is said to be strictly convex if $\frac{||x+y||}{2} < 1$ for all $x, y \in S(E) := \{u \in E : ||u|| = 1\}$ with $x \neq y$. The modulus of convexity of E is defined by

$$\delta_E(\varepsilon) = \inf\{1 - \frac{1}{2}||x+y|| : ||x|| \leq 1, ||y|| \leq 1, ||x-y|| \geq \varepsilon\} \tag{2.1}$$

for all $\varepsilon \in [0,2]$. The space E is said to have the Kadec-Klee property if whenever $\{x_n\}$ is a sequence in E that converges weakly to $x_0 \in E$ and $||x_n|| \to ||x_0||$, as $n \to \infty$, then $\{x_n\}$ converges strongly to x_0. E is said to be uniformly convex if $\delta_E(0) = 0$, and $\delta_E(\varepsilon) > 0$, $\forall \varepsilon \in (0,2]$. The modulus of smoothness of E is a function: $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(||x+y|| + ||x-y||) - 1 : x \in S(E), ||y|| \leq t\}. \tag{2.2}$$

E is called uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. The spaces of Lebesgue integrable functions L_p, $p > 1$ are uniformly smooth (see, e.g., [21]).

Let E^* be the topological dual of E. For all $x \in E$ and $x^* \in E^*$, we denote by $\langle x^*, x\rangle$ the value of x^* at x. A mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{x^* \in E^* : \langle x^*, x\rangle = ||x||^2 = ||x^*||^2\}, x \in E \tag{2.3}$$

is called the normalized duality mapping. The following properties of J are very well known (see, e.g., [21, 22, 23]).

(1) If E is uniformly smooth, then J is uniformly continuous on each bounded subset of E.
(2) $J(x) \neq \emptyset$.

(3) If E is reflexive, then J is a surjective map from E to E^*.
(4) If E is strictly convex, then J is one to one and J is single valued if E is smooth.

Let C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be nonexpansive if

$$||Tx - Ty|| \leq ||x - y||, \quad \forall x, y \in C.$$

A point $x \in C$ is called a fixed point of T if $Tx = x$. The set of fixed points of T is defined as

$$F(T) := \{x \in C : Tx = x\}.$$

A mapping $T : C \to C$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$||Tx - x^*|| \leq ||x - x^*||, \quad \forall\, x \in C, x^* \in F(T).$$

It is clear that every nonexpansive mapping with nonempty set of fixed points is quasi- nonexpansive.

Recently, Chidume, Ikechukwu and Adam [12] introduced an inertial algorithm and proved that their algorithm is strongly convergent to a common fixed point of a countable family of relatively nonexpansive maps in a uniformly convex and uniformly smooth real Banach space. To be more precise, they proved the following Theorem.

Theorem 2.1. [12] *Let E be a uniformly convex and uniformly smooth real Banach space. Let $T_i : E \to E$, where $i = 1, 2, 3, \cdots$, be a relatively nonexpansive mapping such that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\} \subset (0,1)$ and $\{\beta_i\} \subset (0,1)$ are sequences such that $\sum_{i=1}^{\infty} \alpha_i = 1$ and $T : E \to E$ is defined by $Tx = J^{-1}\big(\sum_{i=1}^{\infty} \alpha_i[\beta_i Jx + (1-\beta_i)JT_i x]\big)$ for each $x \in E$. Let $\{u^n\}$ be a sequence generated by the following algorithm: $u^0, u^1 \in E$ and*

$$\begin{cases} C_0 = E, \\ w^n = u^n + \alpha_n(u^n - u^{n-1}), \\ v^n = J^{-1}\big((1-\beta)Jw^n + \beta JTw^n\big), \\ C_{n+1} = \{z \in C_n : \psi(z, v^n) \leq \psi(z, w^n)\}, \\ u^{n+1} = \Pi_{C_{n+1}} u^o, \ \forall\, n \geq 0, \end{cases} \tag{2.4}$$

where $\alpha_n \in (0,1)$ and $\beta \in (0,1)$. Then, $\{u^n\}$ converges strongly to a point $p = \Pi_{F(T)} u^0$.

Let H be a Hilbert space, and let $B : H \to 2^H$ be a set-valued operator. The variational inclusion problem is defined as follows

$$\text{find } x \in H : x \in B^{-1}(0). \tag{2.5}$$

Problem (2.5) is a unified framework for many real problems in finance, economics, transportation, etc. When B is a maximal monotone operator, Martinet [24] introduced the following proximal point algorithm

$$x_{n+1} = J_{\lambda_n}^B x_n, \ n \geq 1, \tag{2.6}$$

where $J_{\lambda_n}^B$ is the resolvent of B associated with $\{\lambda_n\} \subset (0, \infty)$. The proximal point algorithm was further developed by Rockafellar and many others; see, e.g., [25, 26, 27, 28].

For solving split feasibility problems and the fixed point problem of nonexpansive mappings, Takahashi, Xu and Yao [9] introduced the following problem, which consists of finding $x \in H$ such that

$$x \in B^{-1}(0) \text{ and } Ax \in F(T), \tag{2.7}$$

where $B : H_1 \to 2^{H_1}$ is a maximal monotone operator, $A : H_1 \to H_2$ is a bounded linear operator and $T : H_2 \to H_2$ is a nonexpansive mapping. They considered the following iterative algorithm:

$$x_1 \in H, \quad x_{n+1} = J_{\lambda_n}(I - \gamma_n A^*(I-T)Ax_n), \quad \forall n \geq 1, \tag{2.8}$$

where $\{\lambda_n\}$ and $\{\gamma_n\}$ satisfy suitable conditions. They showed that the sequence $\{x_n\}$ generated by (2.8) converges weakly to a point $p \in B^{-1}(0) \cap F(T)$.

Let $S : C \to C$ and $T : Q \to Q$ be two mappings. The so-called split common fixed point problem (SCFP) for mappings S and T is to find a point

$$q \in C \text{ such that } q \in F(S) \text{ and } Aq \in F(T), \tag{2.9}$$

where $F(S)$ and $F(T)$ denote the sets of fixed points of S and T, respectively. If $F(S)$ and $F(T)$ stand for the zero sets of monotone mappings, the split common fixed point problem (SCFP) is called split common null point problem (SCNPP). The split common fixed point problem in Hilbert spaces was introduced by Moudafi [29] in 2010 and has been studied extensively recently. In 2015, Takahashi [30] introduced and studied the split feasibility problem and the split common null point problem in the setting of Banach spaces, which is more general than the setting of Hilbert spaces. Motivated by the results of Takahashi [30], Tang *et al.* [31] proved weak and strong convergence theorems for the split common fixed point problem involving a quasi-strictly pseudo-contractive mapping and an asymptotical nonexpansive mapping in two Banach spaces. Precisely, they proved the following theorem.

Theorem 2.2. [31] *Assume that*

(1) E_1 is a real 2- uniformly convex and 2-uniformly smooth Banach space with the Opial's property satisfying $0 < k < \frac{1}{\sqrt{2}}$, k is the best smoothness constant;
(2) E_2 is a real Banach space;
(3) $A : E_1 \to E_2$ is a bounded linear operator and A^* is the adjoint of A;
(4) $S : E_1 \to E_2$ is an $\{\ell_n\}-$ asymptotically nonexpansive mapping with $\{\ell_n\} \subset (1,\infty)$ and $\ell_n \to 1$ and $T : E_2 \to E_2$ is a $\tau-$ quasi-strictly pseudocontractive mapping with $F(S) \neq \emptyset$ and $F(T) \neq \emptyset$ and T is demiclosed.

Let E_1, E_2, T, S, A and $\{\ell_n\}$ be as stated in the assumptions above. For each $x_1 \in E_1$, let $\{x_n\}$ be the sequence generated by

$$\begin{cases} z_n = x_n + \gamma J_1^{-1} A^* J_2 (T-I)Ax_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n S^n z_n, \forall\, n \geq 1, \end{cases} \tag{2.10}$$

where $\{\alpha_n\} \subset (0,1)$ satisfying $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, γ is a positive constant satisfying $0 < \gamma < \min\{\frac{1-2k^2}{||A||^2}, \frac{1-\tau}{||A||^2}\}$, with $L = \sup_{n\geq 1} \ell_n$ and $\sum_{n=1}^{\infty}(\ell_n - 1) < \infty$.
(I) If $\Gamma := \{v \in F(S) : Av \in F(T)\} \neq \emptyset$, then $\{x_n\}$ converges weakly to $x^* \in \Gamma$.
(II) If, in addition, $\Gamma \neq \emptyset$ and S is semicompact, then $\{x_n\}$ converges strongly to $x^* \in \Gamma$.

In order to prove that $\{x_n\}$ is bounded, we observe that the authors used the following known Lemma.

Lemma 2.1. [32] *For a given $r > 0$, a real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $g(0) = 0$ such that*

$$||tx + (1-t)y||^2 \leq t||x||^2 + (1-t)||y||^2 - t(1-t)g(||x-y||)$$

for all $x, y \in E$ *with* $||x|| \leq r$, $||y|| \leq r$ *and* $t \in [0,1]$.

Lemma 2.1 requires that x, y belong to a ball. Consequently, to use this Lemma, the authors need to verify first that the sequences $\{z_n - p\}$ and $\{S^n z_n - p\}$ of their work are bounded. This was not done and so in our view, it makes the proof of their main result, Theorem 3.1 incomplete. As an appendage to the target of this manuscript,

We prove that the iterative sequence studied by Tang *et al.* [31] is bounded without applying Lemma 2.1. We achieve this by using an inequality proved by Chidume (see Lemma 2.2 below). Let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$, and let E_2 be a smooth, strictly convex and reflexive Banach space. Let $B : E_1 \to 2^{E_1^*}$ be a maximal monotone operator, and let $A : E_1 \to E_2$ be a bounded linear operator with adjoint $A^* : E_2^* \to E_1^*$. For $i = 1, 2, 3, \cdots$, let $S_i : E_1 \to E_1$ be a countable family of relatively nonexpansive mappings with $Sx = J^{-1}\left(\sum_{i=1}^{\infty} \delta_i(\sigma_i JS_i x + (1-\sigma_i)JS_i x)\right)$ for each $x \in E_1, \{\delta_i\} \subset (0,1)$ and $\{\sigma_i\} \subset (0,1)$ are such $\sum_{i=1}^{\infty} \delta_i = 1$, and let $T : E_2 \to E_2$ be a closed relatively quasi-nonexpansive mapping with $F(T) \neq \emptyset$.

Consider the following problem

$$\text{Find } x^* \in E_1 : x^* \in B^{-1}(0) \cap F(S) \text{ and } Ax^* \in F(T). \tag{2.11}$$

Observe that problem (2.11) includes problems (2.5), (2.7) and (2.9) respectively.

In this paper, we construct an inertial algorithm and prove that the sequence generated by the algorithm converges strongly to a solution of (2.11) provided that the solution set of (2.11) is nonempty. Our result generalizes many recent results in the literature, such as, [12, 9]

Lemma 2.2. [13] *Let E be a p-unifoprmly smooth Banach space iwth $p > 1$. Then there exists a constant $C > 0$ such that*

$$||tx + (1-t)y - z||^p \leq [1 - t(p-1)]||y - z||^p + tC||x - z||^p - t(1 - t^{(p-1)}C)||x - y||^p,$$

where t is a real number in $(0,1)$, for any $x, y, z \in E$

Proposition 2.1. *Assume that the hypotheses of Theorem 2.2 are satisfied with $\{\alpha_n\} \subset (0, \frac{1}{C})$ and $\liminf_{n\to\infty} \alpha_n(1 - C\alpha_n) > 0$, where C is the constant appearing in Lemma 2.2 and $\sum_{n=1}^{\infty}(C\ell_n - 1) < \infty$.*
(I) If $\Gamma := \{v \in F(S) : Av \in F(T)\} \neq \emptyset$, then $\{x_n\}$ converges weakly to $p \in \Gamma$.
(II) If, in addition, $\Gamma \neq \emptyset$ and S is semicompact, then $\{x_n\}$ converges strongly to $p \in \Gamma$.

Proof. Fix $p \in \Gamma$. Following the arguments that yield (3.4) in the proof of [31, Theorem 3.1], we obtain

$$||z_n - p|| \leq ||x_n - p||. \tag{2.12}$$

Now, using Lemma 2.2, we get

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||\alpha_n S^n z_n + (1-\alpha_n)z_n - p||^2 \\
&\leq (1-\alpha_n)||z_n - p||^2 + \alpha_n C||S^n z_n - p||^2 - \alpha_n(1 - \alpha_n C)||z_n - S^n z_n||^2 \\
&\leq (1-\alpha_n)||z_n - p||^2 + \alpha_n C\ell_n||z_n - p||^2 - \alpha_n(1 - \alpha_n C)||z_n - S^n z_n||^2 \\
&= [1 + \alpha_n(C\ell_n - 1)]||x_n - p||^2 - \alpha_n(1 - \alpha_n C)||z_n - S^n z_n||^2 \\
&\leq [1 + \alpha_n(C\ell_n - 1)]||x_n - p||^2.
\end{aligned}$$

From Lemma 2.3, we have that $\lim_{n\to\infty} ||x_n - p||$ exists. Hence $\{x_n\}$ is bounded. The remaining part of the convergence proof follows the same argument as that of Tang *et al.* [31]. □

Let C be a nonempty, closed, and convex subset of a strictly convex and reflexive Banach E Then the metric projection $P_C x = \arg\min_{y\in C} ||x-y||$, $\forall x \in E$, is the unique minimizer of the norm distance.

Let E be a smooth, reflexive, and strictly convex Banach space. Consider the functional [33] defined by

$$\phi(x,y) = ||x||^2 - 2\langle x, J(y)\rangle + ||y||^2,\ \forall x,y \in E \tag{2.13}$$

where J is the normalized duality mapping. It is clear that, in a Hilbert space H, (2.13) reduces to $\phi(x,y) = ||x-y||^2$, $\forall x,y \in H$. It is obvious from the definition of ϕ that

$$(||x|| - ||y||)^2 \leq \phi(x,y) \leq (||x|| + ||y||)^2,\ \forall\, x,y \in E. \tag{2.14}$$

$$\phi(x, J^{-1}(\alpha Jy + (1-\alpha)Jz) \leq \alpha\phi(x,y) + (1-\alpha)\phi(x,z),\ \forall x,y \in E. \tag{2.15}$$

and

$$\phi(x,y) \leq ||x||||x - Jy|| + ||y||||x-y||. \tag{2.16}$$

Following Alber [33], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C x = \arg\min_{y\in C} \phi(y,x), \quad \forall\, x \in E, \tag{2.17}$$

that is, $\Pi_C(x) = x$, where x is the unique solution to the minimization problem $\phi(\bar{x};x) = \inf_{y\in C}\phi(y,x)$. The existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, e.g., [33]). In Hilbert space H, $\Pi_C = P_C$.

Definition 2.1. A point $p \in C$ is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $F(\hat{T})$. We say that a mapping T is relatively nonexpansive (see, e.g., [34]) if the following conditions are satisfied
(1) $F(T) \neq \emptyset$;
(2) $\phi(p,Tx) \leq \phi(p,x)$, $\forall\, x \in C, p \in F(T)$;
(3) $F(T) = F(\hat{T})$.
If T satisfies (1) and (2), then it is said to be relatively quasi-nonexpansive. It is easy to see that the class of relatively quasi-nonexpansive mappings contains the class of relatively nonexpansive mappings.

Recently, many authors studied various numerical methods for fixed points of relatively nonexpansive and relatively quasi-nonexpansive mappings; see, e.g., [35, 36, 37] and the references therein. Clearly, in a Hilbert space H, relatively nonexpansive and relatively quasi-nonexpansive mappings coincide with nonexpansive and quasi-nonexpansive mappings, respectively due to $\phi(x,y) = ||x-y||^2$, $\forall x,y \in H$. This implies

$$\phi(p,Tx) \leq \phi(p,x) \Leftrightarrow ||Tx - p|| \leq ||x-p||,\ \forall x \in C, p \in F(T).$$

For the examples of relatively quasi-nonexpansive mappings, we refer to [35].

In order to prove our main results, we also need the following lemmas.

Lemma 2.3. *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying $a_{n+1} \leq (1+\delta_n)a_n + b_n$, $\forall n \geq 1$. If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.4. [22] *Let E be a strictly convex, smooth, reflexive real Banach space. Let C be a nonempty, closed, convex subset of E. Let $x_1 \in E, z \in C$. Then the following holds*

(1) $z = P_C x_1$;
(2) $\langle z - y, J(x_1 - z)\rangle \geq 0,\ \forall\ y \in C$.

Lemma 2.5. [32] *Let E be a 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$. Then*

$$||x+y||^2 \leq ||x||^2 + \langle x, Jy\rangle + 2||ky||^2\ \forall x, y \in E. \tag{2.18}$$

Lemma 2.6. [33] *Let E be a smooth, strictly convex, and reflexive Banach space, and let C be nonempty, closed and convex subset of E. Then, the following conclusions hold*

(1) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y),\ x \in C, y \in E$;
(2) if $x \in E$ and $z \in C$, then $z = \Pi_C x$ iff $\langle z - y, Jx - Jz\rangle \geq 0,\ \forall y \in C$;
(3) for $x, y \in E$, $\phi(x, y) = 0$ iff $x = y$.

Lemma 2.7. [38] *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $||x_n - y_n|| \to 0$.*

Remark 2.1. Using (2.16), it is easy to see that the converse of Lemma 2.7 is also true whenever $\{x_n\}$ and $\{y_n\}$ are both bounded.

Lemma 2.8. [39] *Let C be a nonempty, closed convex subset of a uniformly convex and uniformly smooth real Banach space E. Let $T_i : C \to E,\ \ i = 1, 2, 3\cdots$ be countably infinite family of relatively nonexpansive mappings such that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose that $\{\alpha_i\} \subset (0,1)$ and $\{\beta_i\} \subset (0,1)$ are such $\sum_{i=1}^{\infty} \alpha_i = 1$ and $T : C \to E$ is defined by*

$$Tx = J^{-1}\big(\sum_{i=1}^{\infty} \alpha_i(\beta_i Jx + (1-\beta_i)JT_i x)\big)\ \text{for each } x \in C.$$

Then, T is relatively nonexpansive and $F(T) = \cap_{i=1}^{\infty} F(T_i)$

Lemma 2.9. [11] *Let E be a strictly convex and reflexive smooth Banach space. Let $B : E \to 2^{E^*}$ be a maximal monotone operator and J_λ^B be the resolvent of B for $\lambda > 0$. Then $\phi(u, J_\lambda^B x) \leq \phi(u, x),\ \forall u \in B^{-1}(0), x \in E$.*

Lemma 2.10. [11] *Let E be a smooth and strictly convex real Banach space, and let C be a nonempty, closed and convex subset of E. Let T be a mapping from C into itself such that $F(T) \neq \emptyset$ and $\phi(y, Tx) \leq \phi(y, x),\ \forall (y, x) \in F(T) \times C$. Then $F(T)$ is closed and convex.*

Lemma 2.11. [35] *Let C be a nonempty closed convex subset of a smooth, uniformly convex Banach space E. Let T be a closed relatively quasi-nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

3. Main results

Now, we are ready to give our main results.

Theorem 3.1. *Let E_1 be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$, and let E_2 be a smooth, strictly convex and reflexive Banach space. Let $B : E_1 \to 2^{E_1^*}$ be a maximal monotone operator, and let $A : E_1 \to E_2$ be a* *bounded linear operator with the adjoint $A^*: E_2^* \to E_1^*$. For $i = 1,2,3,\cdots$, let $S_i : E_1 \to E_1$ be a countable family of relatively nonexpansive mappings with $Sx = J_1^{-1}\big(\sum_{i=1}^{\infty}\delta_i(\sigma_i J_1 x + (1-\sigma_i)J_1 S_i x)\big)$ for each $x \in E_1, \{\delta_i\} \subset (0,1)$ and $\{\sigma_i\} \subset (0,1)$ are such that $\sum_{i=1}^{\infty}\delta_i = 1$, and let $T : E_2 \to E_2$ be a closed relatively quasi-nonexpansive mapping. For arbitrary $x_0, x_1 \in E_1$, let the sequence $\{x_n\}$ be generated as follows*

$$\begin{cases} C_0 = E_1, \\ w_n = x_n + (x_n - x_{n-1}), \\ u_n = J_1^{-1}\big(J_1 w_n + \gamma A^* J_2(P_{F(T)} - I)Aw_n\big), \\ v_n = J_1^{-1}\big((1-\alpha_n)J_1 u_n + \alpha_n J_1 J_\lambda^B u_n\big), \\ y_n = J_1^{-1}\big((1-\beta)J_1 v_n + \beta J_1 S v_n\big), \\ C_{n+1} = \{w \in C_n : \phi(w,y_n) \leq \phi(w,v_n) \leq \phi(w,u_n) \leq \phi(w,w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall\, n \geq 1, \end{cases} \tag{3.1}$$

where $J_\lambda^A = (J_1 + \lambda A)^{-1}J_1$, J_1 is the normalized duality mapping of E_1, and J_2 is the normalized duality mapping of E_2. Suppose that $\Gamma := \{x^* \in B^{-1}(0) \cap F(S) : Ax^* \in F(T)\} \neq \emptyset$ and the following conditions are satisfied:
(1) $\alpha_n \in [a,1), a > 0,\ \beta \in (0,1)$;
(2) $0 < \gamma \leq \frac{1}{k||A||^2}$.
Then $\{x_n\}$ converges strongly to $x^* = \Pi_\Gamma x_1$.

Proof. We divide the proof into six steps. First, we note that by Lemma 2.8 that S is relatively nonexpansive. In view of Lemma2.11, we have that $F(T)$ is closed and convex.

Step 1. Show that C_n is a closed and convex subset of E_1 for each $n \geqslant 1$.
Set

$$A_n = \{w \in C_n : \phi(w,y_n) \leq \phi(w,v_n)\},$$

$$B_n = \{w \in C_n : \phi(w,v_n) \leq \phi(w,u_n)\},$$

and

$$D_n = \{w \in C_n : \phi(w,u_n) \leq \phi(w,w_n)\}.$$

Then, $C_{n+1} = A_n \cap B_n \cap D_n$, $\forall n \geq 1$. Note that

$$\begin{aligned} \phi(w,y_n) &\leq \phi(w,v_n) \Leftrightarrow 2\langle w, Jy_n - Jv_n\rangle \leq \|y_n\|^2 - \|v_n\|^2, \\ \phi(w,v_n) &\leq \phi(w,u_n) \Leftrightarrow 2\langle w, Jv_n - Ju_n\rangle \leq \|v_n\|^2 - \|u_n\|^2, \\ \phi(w,u_n) &\leq \phi(w,w_n) \Leftrightarrow 2\langle w, Ju_n - Jw_n\rangle \leq \|u_n\|^2 - \|w_n\|^2. \end{aligned}$$

Consequently, A_n, B_n, D_n are closed and convex and so, C_{n+1} is closed and convex for all $n \geq 1$.

Step 2. Show that $\Gamma \subseteq C_n$, for each $n \geqslant 1$.
Fix $p \in \Gamma$. Using inequality (2.15) and Lemma 2.5, we obtain

$$\begin{aligned} \phi(p,y_n) &= \phi\big(p, J_1^{-1}\big((1-\beta)J_1 v_n + \beta J_1 S v_n\big)\big) \\ &\leq (1-\beta)\phi(p,v_n) + \beta\phi(p,Sv_n) \\ &\leq (1-\beta)\phi(p,v_n) + \beta\phi(p,v_n) = \phi(p,v_n), \end{aligned}$$

$$\begin{aligned}\phi(p,v_n) &= \phi\big(p, J_1^{-1}\big((1-\alpha_n)J_1u_n + \alpha_n J_1 J_\lambda^B u_n\big)\big)\\ &\le (1-\alpha_n)\phi(p,u_n) + \alpha_n\phi(p, J_1J_\lambda^B u_n)\\ &\le \phi(p,u_n),\end{aligned}$$

and

$$\begin{aligned}&\phi(p,u_n)\\ &= ||p||^2 - 2\langle p, J_1w_n + \gamma A^*J_2(P_{F(T)}-I)Aw_n\rangle + ||J_1w_n + \gamma A^*J_2(P_{F(T)}-I)Aw_n||^2\\ &= ||p||^2 - 2\langle p, J_1w_n\rangle - 2\gamma\langle p, A^*J_2(P_{F(T)}-I)Aw_n\rangle + ||J_1w_n + \gamma A^*J_2(P_{F(T)}-I)Aw_n||^2\\ &\le ||p||^2 - 2\langle p, J_1w_n\rangle - 2\gamma\langle p, A^*J_2(P_{F(T)}-I)Aw_n\rangle + ||w_n||^2\\ &\quad + 2\gamma\langle w_n, A^*J_2(P_{F(T)}-I)Aw_n\rangle + 2k^2||\gamma A^*J_2(P_{F(T)}-I)Aw_n||^2\\ &= ||p||^2 + ||w_n||^2 - 2\langle p, J_1w_n\rangle - 2\gamma\langle Ap - Aw_n, J_2(P_{F(T)}-I)Aw_n\rangle\\ &\quad + 2k^2\gamma^2||A||^2||(P_{F(T)}-I)Aw_n||^2.\end{aligned}$$

It follows from Lemma 2.4 that

$$\begin{aligned}&\langle Ap - Aw_n, J_2(P_{F(T)}-I)Aw_n\rangle\\ &= \langle Ap - P_{F(T)}Aw_n + P_{F(T)}Aw_n - Aw_n, J_2(P_{F(T)}-I)Aw_n\rangle\\ &= \langle Ap - P_{F(T)}Aw_n, J_2(P_{F(T)}-I)Aw_n\rangle + ||(P_{F(T)}-I)Aw_n||^2\\ &\ge ||(P_{F(T)}-I)Aw_n||^2.\end{aligned}$$

So,

$$\phi(p,u_n) \le \phi(p,w_n) - 2\gamma(1-k^2\gamma||A||^2)||(P_{F(T)}-I)Aw_n||^2,$$

that is, $\phi(p,u_n)\le\phi(p,w_n)$. Hence $p\in C_{n+1}$ and $\Gamma\in C_n$, $\forall n\ge 1$. Thus, $\{x_{n+1}\}$ is well defined.

Step 3. Show that $\{x_n\}$ is a Cauchy sequence.

Fix $v\in\Gamma$. It follows from the definition of C_n that $x_n = \Pi_{C_n}x_1$ for all $n\geqslant 1$. In view of Lemma 2.6 (1), we have

$$\phi(x_n,x_1) = \phi(\Pi_{C_n}x_1, x_1) \leqslant \phi(v,x_1) - \phi(v,\Pi_{C_n}x_1) \leqslant \phi(v,x_1), \forall n\geqslant 1. \tag{3.2}$$

This shows that $\{\phi(x_n,x_1)\}$ is bounded. Consequently, $\{w_n\},\{u_n\},\{v_n\},\{y_n\}$ are all bounded. Since $x_n = \Pi_{C_n}x_1$ and $x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1}\subseteq C_n$, we have

$$\phi(x_n,x_1) \leqslant \phi(x_{n+1},x_1), \quad \forall n\geqslant 1, \tag{3.3}$$

which implies that $\{\phi(x_n,x_1)\}$ is nondecreasing and bounded. So $\lim_{n\to\infty}\phi(x_n,x_1)$ exists. Again by Lemma 2.6 (1), we have

$$\begin{aligned}\phi(x_{n+1},x_n) &= \phi(x_{n+1},\Pi_{C_n}) \leqslant \phi(x_{n+1},x_1) - \phi(\Pi_{C_n}x_1,x_1)\\ &= \phi(x_{n+1},x_1) - \phi(x_n,x_1),\end{aligned} \tag{3.4}$$

which implies that

$$\lim_{n\to\infty}\phi(x_{n+1},x_n) = 0, \tag{3.5}$$

which together with Lemma 2.7 shows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.6}$$

For arbitrary positive integers m,n with $m \leqslant n$, it follows from $x_n = \Pi_{C_n} x_1 \subseteq C_m$ and Lemma 2.6 (1) that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_1) \leqslant \phi(x_m, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1). \end{aligned} \tag{3.7}$$

Since $\lim_{n\to\infty} \phi(x_n, x_1)$ exists, it follows from (3.7) and Lemma 2.7 that

$$\lim_{m,n\to\infty} \|x_n - x_m\| = 0.$$

Hence, $\{x_n\}$ is a cauchy sequence. Consequently, there exists $x^* \in E_1$ such that $x_n \to x^*$ as $n \to \infty$.

Step 4. Show that $\lim_{n\to\infty} \|u_n - J_\lambda{}^B u_n\| = 0$, $\lim_{n\to\infty} \|(P_{F(T)} - I)Aw_n\| = 0$ and $\lim_{n\to\infty} \|v_n - Sv_n\| = 0$.

Since $\{x_n\}$ is a Cauchy sequence, we obtain that $||x_{n+1} - x_n|| \to 0, n \to \infty$. Now, $||x_n - w_n|| = ||x_n - x_n + (x_n - x_{n-1})|| = ||x_n - x_{n-1}|| \to 0$ as $n \to \infty$. It follows from Remark 2.1 that $||x_{n+1} - w_n|| \leq ||x_{n+1} - x_n|| + ||x_n - w_n|| \to 0$ as $n \to \infty$. Hence, $\phi(x_{n+1}, w_n) \to 0$ as $n \to \infty$. Furthermore, since $x_{n+1} \in C_{n+1}$, we obtain

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, v_n) \leq \phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, w_n) \to 0, \ n \to \infty. \tag{3.8}$$

Therefore,

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = \lim_{n\to\infty} \phi(x_{n+1}, v_n) = \lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0, \tag{3.9}$$

and

$$\lim_{n\to\infty} ||x_{n+1} - y_n|| = \lim_{n\to\infty} ||x_{n+1} - v_n|| = \lim_{n\to\infty} ||x_{n+1} - u_n|| = 0. \tag{3.10}$$

Observe that

$$\begin{aligned} ||x_n - y_n|| &\leq ||x_{n+1} - x_n|| + ||x_{n+1} - y_n|| \to 0, \\ ||x_n - v_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - v_n|| \to 0, \\ ||x_n - u_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - u_n|| \to 0, \end{aligned} \tag{3.11}$$

and hence

$$\begin{aligned} ||y_n - v_n|| &\leq ||y_n - x_{n+1}|| + ||x_{n+1} - v_n|| \to 0, \\ ||y_n - u_n|| &\leq ||y_n - x_{n+1}|| + ||x_{n+1} - u_n|| \to 0, \\ ||u_n - v_n|| &\leq ||u_n - x_n|| + ||x_n - v_n|| \to 0, \\ ||u_n - w_n|| &\leq ||u_n - x_n|| + ||x_n - w_n|| \to 0. \end{aligned} \tag{3.12}$$

It follows that

$$\begin{aligned} & 2\gamma(1 - k^2\gamma||A||^2)||(P_{F(T)} - I)Aw_n||^2 \\ \leq\ & \phi(p, w_n) - \phi(p, u_n) \\ =\ & ||p||^2 + 2\langle p, J_1 w_n\rangle + ||w_n||^2 - ||p||^2 - 2\langle p, J_1 u_n\rangle - ||u_n||^2 \\ =\ & 2\langle p, J_1 w_n - J_1 u_n\rangle + ||w_n||^2 - ||u_n||^2 \\ \leq\ & 2||p||||J_1 w_n - J_1 u_n|| + ||w_n - u_n||\big(||w_n|| + ||u_n||\big). \end{aligned} \tag{3.13}$$

From (3.12) and the fact that J_1 is uniformly continuous on bounded subsets of E_1, we get

$$\lim_{n\to\infty} ||(P_{F(T)} - I)Aw_n|| = 0 \tag{3.14}$$

and then

$$||J_1 v_n - J_1 u_n|| = \alpha_n ||J_1 u_n - J_1 J_\lambda^B u_n||.$$

By the third conclusion in (3.12) and the uniform continuity of J_1 on bounded subsets of E_1, we get $\lim_{n\to\infty} ||J_1 v_n - J_1 u_n|| = 0$. Hence

$$\lim_{n\to\infty} ||J_1 u_n - J_1 J_\lambda^B u_n|| = \lim_{n\to\infty} ||u_n - J_\lambda^B u_n|| = 0. \tag{3.15}$$

Similarly, $||J_1 y_n - J_1 v_n|| = \alpha_n ||J_1 v_n - J_1 S v_n||$. Using condition (1) , the first conclusion in (3.12) and the uniform continuity of J_1, we have $\lim_{n\to\infty} ||J_1 y_n - J_1 v_n|| = 0$. Hence,

$$\lim_{n\to\infty} ||J_1 v_n - J_1 S v_n|| = \lim_{n\to\infty} ||v_n - S v_n|| = 0. \tag{3.16}$$

Step 5. Show that $x_n \to \Pi_\Gamma x_1$.

Since $\{x_n\}$ is bounded, there exists $\{x_{n_k}\}$ a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. By the third conclusion of (3.11), we have that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup x^*$. From (3.15), we get

$$\lim_{n\to\infty} ||u_{n_k} - J_\lambda^B u_{n_k}|| = 0. \tag{3.17}$$

From Lemma 2.9, we have $x^* \in F(J_\lambda^B) = B^{-1}(0)$. Again, from the second conclusion of (3.11), we have that there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \rightharpoonup x^*$. It follows from (3.16) that

$$\lim_{n\to\infty} ||v_{n_k} - S v_{n_k}|| = 0. \tag{3.18}$$

Since S is relatively nonexpansive, we have $x^* \in F(S)$. Next, we show that $Ax^* \in F(T)$. From Lemma 2.4, we have

$$\begin{aligned}
\|(1 - P_{F(T)})Ax^*\|^2 &= \langle J_2\big(Ax^* - P_{F(T)})(Ax^*)\big), Ax^* - P_{F(T)})(Ax^*)\rangle \\
&= \langle J_2\big(Ax^* - P_{F(T)})(Ax^*)\big), Ax^* - Aw_n + Aw_n - P_{F(T)})(Aw_n) \\
&\quad + P_{F(T)})(Aw_n) - P_{F(T)})(Ax^*)\rangle \\
&= \langle J_2\big(Ax^* - P_{F(T)})(Ax^*)\big), Ax^* - Aw_n\rangle + \langle Ax^* - P_{F(T)})(Ax^*), \\
&\quad Aw_n - P_{F(T)})(Aw_n)\rangle + \langle Ax^* - P_{F(T)})(Ax^*), P_{F(T)})(Aw_n) \\
&\quad - P_{F(T)})(Ax^*)\rangle \\
&\leqslant \langle Ax^* - P_{F(T)})(Ax^*), Ax^* - Aw_n\rangle + \langle Ax^* - P_{F(T)})(Ax^*), \\
&\quad Aw_n - P_{F(T)})(Aw_n)\rangle.
\end{aligned}$$

Since A is a bounded linear operator, we have that

$$\lim_{n\to\infty} \|Aw_n - Ax^*\| = 0.$$

Hence, it follows from (3.14) that $\|(I - P_{F(T)}Ax^*)\| = 0$. This implies that $Ax^* \in F(T)$. Therefore, $x^* \in \Gamma$.

We now show that $x^* = P_{F(T)}x_1$. Let $z = P_\Gamma x_1$. Then, $z \in \Gamma$. Since $x_n = P_{C_n}x_1$ and $\Gamma \subseteq C_n$, we have $\phi(x_n,x_1) \leq \phi(z,x_1)$. On the other hand, from lower semicontinuity of norm, we have

$$\begin{aligned}
\phi(x^*,x_1) &= ||x^*||^2 - 2\langle x^*, Jx_1\rangle + ||x_1||^2 \\
&\leq \liminf_{k\to\infty}\left(||x_{n_k}||^2 - 2\langle x_{n_k}, Jx_1\rangle + ||x_1||^2\right) \\
&\leq \liminf_{k\to\infty}\left(\phi(x_{n_k},x_1)\right) \leq \limsup_{k\to\infty}\left(\phi(x_{n_k},x_1)\right) \leq \phi(z,x_1).
\end{aligned} \tag{3.19}$$

From the fact that $z = P_\Gamma x_1$, we get $\phi(z,x_1) \leq \phi(p,x_1)$, $\forall p \in \Gamma$, which shows that $\phi(z,x_1) \leq \phi(x^*,x_1)$. From (3.19), we have $\phi(z,x_1) = \phi(x^*,x_1)$. Uniqueness of $P_\Gamma x_1$ gives us $z = x^*$. Next we show that $x_{n_k} \to x^*$. Using (3.19) again, we obtain

$$\begin{aligned}
\phi(x^*,x_1) &\leq \liminf_{k\to\infty}\left(\phi(x_{n_k},x_1)\right) \leq \limsup_{k\to\infty}\left(\phi(x_{n_k},x_1)\right) \leq \phi(z,x_1.) \\
&= \phi(x^*,x_1).
\end{aligned}$$

Thus $\lim_{k\to\infty}\phi(x_{n_k},x_1) = \phi(x^*,x_1)$. Therefore, $\lim_{k\to\infty}||x_{n_k}|| = ||x^*||$. By the Kadec Klee property of E_1, we get $x_{n_k} \to x^*$, as $k \to \infty$. Since $\{x_n\}$ is Cauchy, we conclude that $x_n \to x^* = P_\Gamma x_1$, $n \to \infty$. This completes the proof. □

When T is nonexpansive, we have the following result.

Corollary 3.1. *Let H_1 and H_2 be real Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone operator, and let $A : H_1 \to H_2$ be a bounded linear operator with adjoint $A^* : H_2 \to H_1$. Let $S : H_1 \to H_1$ a nonexpansive mapping, and let $T : H_2 \to H_2$ be a closed relatively quasi-nonexpansive mapping. For arbitrary $x_0, x_1 \in H_1$, let the sequence $\{x_n\}$ be a sequence generated as follows*

$$\begin{cases}
C_0 = H_1, \\
w_n = x_n + (x_n - x_{n-1}), \\
u_n = w_n + \gamma A^*(P_{F(T)} - I)Aw_n, \\
v_n = (1-\alpha_n)u_n + \alpha_n J_\lambda^B u_n, \\
y_n = (1-\beta)v_n + \beta S v_n, \\
C_{n+1} = \{p \in C_n : ||p - y_n|| \leq ||p - v_n|| \leq ||p - u_n|| \leq ||p - w_n||\}, \\
x_{n+1} = P_{C_{n+1}}x_1, \ \forall\, n \geq 1
\end{cases} \tag{3.20}$$

where $J_\lambda^A = (I + \lambda A)^{-1}$. Suppose that $\Gamma := \{v \in B^{-1}(0) \cap F(S) : Av \in F(T)\} \neq \emptyset$, and the following conditions are satisfied: (1) $\alpha_n \in [a,1), a > 0,\ \beta \in (0,1)$; (2) $0 < \gamma \leq \frac{1}{||A||^2}$. Then $\lim_{n\to\infty} x_n = x^* = P_\Gamma x_1$.

4. THE APPLICATION

In this subsection, we consider the application to the hierarchical variational inequality problem.

Definition 4.1. Let E be a smooth, strictly convex and real reflexive Banach space, and let K be a nonempty, closed and convex subset of E. Let $S : K \to K$ be a nonlinear mapping with $F(S)$, a nonempty closed and convex subset of K, and let $V : K \to K$ be a nonlinear mapping.

The so-called hierarchical variational inequality problem for the mapping S with respect to the mapping V in Banach spaces is to find $x^* \in F(S)$ such that

$$\langle x^* - x, J(Vx^* - x^*)\rangle \geq 0, \ \forall x \in F(S). \tag{4.1}$$

From Lemma 2.4, hierarchical variational inequality problem (4.1) is equivalent to the following fixed point equation:

$$x^* = P_{F(T)}Vx^*. \tag{4.2}$$

Set $C = F(S)$ and $Q = F(P_F(S)oV)$ and $A = I$, where I denotes the identity mapping on E. Then, the hierarchical variational inequality problem (4.1) for a mapping S with respect to a mapping V is equivalent to the following split common fixed point problem, which consists of finding

$$x^* \in C \text{ such that } x^* \in Q. \tag{4.3}$$

Therefore, the set of solutions Γ of hierarchical variational inequality problem(4.1) is just the set of solutions of split common fixed point problem (4.3).

From Theorem 3.1, we have the following result.

Theorem 4.1. *Let E be a 2-uniformly convex and 2-uniformly smooth real Banach space with the best smoothness constant $k > 0$, and let $B : E \to 2^{E^*}$ be a maximal monotone operator. For $i = 1,2,\cdots$, let $S_i : E \to E$ be a countable family of relatively nonexpansive mappings with $Sx = J^{-1}\big(\sum_{i=1}^{\infty} \delta_i(\sigma_i Jx + (1-\sigma_i)JS_i x)\big)$ for each $x \in C$, $\{\delta_i\} \subset (0,1)$ and $\{\sigma_i\} \subset (0,1)$ are such that $\sum_{i=1}^{\infty} \delta_i = 1$. Let $T : E \to E$ be a closed relatively quasi-nonexpansive mapping. Suppose that $V : E \to E$ is a mapping such that $H := P_F(T)oV$ is a closed relatively quasi-nonexpansive mapping with $F(H) \neq \emptyset$. For arbitrary $x_0, x_1 \in E_1$, let the sequence $\{x_n\}$ be a sequence generated as follows*

$$\begin{cases} C_0 = E_1, \\ w_n = x_n + (x_n - x_{n-1}), \\ u_n = J^{-1}\big(Jw_n + \gamma J(P_{F(H)} - I)w_n\big), \\ v_n = J^{-1}\big((1-\alpha_n)Ju_n + \alpha_n JJ_\lambda^B u_n\big), \\ y_n = J^{-1}\big((1-\beta)Jv_n + \beta JSv_n\big), \\ C_{n+1} = \{p \in C_n : \phi(w, y_n) \leq \phi(p, v_n) \leq \phi(p, u_n) \leq \phi(p, w_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \ \forall\, n \geq 1, \end{cases} \tag{4.4}$$

where $J_\lambda^A = (J + \lambda A)^{-1}J$ with J being the normalized duality map of E. Assume that the following conditions are satisfied (1) $\alpha_n \in [a,1), a > 0$, $\beta \in (0,1)$; (2) $0 < \gamma \leq \frac{1}{k||A||^2}$. If Γ, the set of solutions of the hierarchical variational inequality problem (4.1), is nonempty, then $\{x_n\}$ converges strongly to $v = \Pi_\Gamma x_1$.

Proof. Taking $E_1 = E_2 = E, A = I, H = P_F(S)oV$ in Theorem (3.1), and noticing that $J_1 = J_2 = J$, we see that the conclusion of Theorem 4.1 follows from Theorem 3.1 immediately. □

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