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APPROXIMATION OF COMMON SOLUTIONS FOR A FINITE FAMILY OF GENERALIZED DEMIMETRIC MAPPINGS AND MONOTONE INCLUSION PROBLEMS IN CAT(0) SPACES

G.C. UGWUNNADI[1,2*], C.C. OKEKE[2]

[1]*Department of Mathematics, University of Eswatini, Kwaluseni, Eswatini*
[2]*Department of Mathematics and Applied Mathematics,*
Sefako Makgatho Health Sciences University, Pretoria, South Africa

Abstract. In this paper, we introduce a modified Halpern-Mann algorithm and study the strong convergence of the algorithm for approximating common solution of a finite family of monotone inclusion problems and a finite family of generalized demimetric mappings in complete CAT(0) spaces. Some applications are also considered.
Keywords. Monotone operator; Fixed point; Zero point; Strong Convergence; Hadamard Space.

1. INTRODUCTION

The inclusion problem (IP) with a set-valued operator A in a Hilbert space H is consists of finding

$$x \in H \text{ such that } 0 \in Ax. \tag{1.1}$$

The solution set of problem (1.1) is denoted by $A^{-1}(0)$. This problem is closely related to many real-world problems, such as signal processing, medical imaging, and machine learning [1, 2, 3, 4, 5] and the references therein.

In 1970, Martinet [6] first studied solutions of problem (1.1) in Hilbert spaces. Later, Rockafellar [7] further studied the inclusion problem by introducing the following iterative algorithm in a Hilbert space H

$$x_1 \in H, \quad x_n = J_{\lambda_n}(x_{n-1}), \quad \forall n \geq 1, \tag{1.2}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers and J_λ is the resolvent of A defined by $J_\lambda = (I+\lambda A)^{-1}$ for $\lambda > 0$, and A is a maximal monotone operator in H. The algorithm is called the Proximal Point Algorithm (PPA). Rockafellar proved that the sequence $\{x_n\}$ generated by (1.2) converges weakly to a solution of (1.1) provided $\lambda_n \geq \lambda > 0$ for each $n \geq 1$. The generalizations and modified versions of the proximal point algorithm in Hilbert were studied by many authors recently; see, e.g., [8, 9, 10, 11, 12, 13] and the references therein.

*Corresponding author.
E-mail addresses: ugwunnadi4u@yahoo.com (G.C. Ugwunnadi), chibueze.okeke87@yahoo.com (C.C. Okeke)
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On the other hand, by using the duality mapping theory introduced by Kakavandi and Amini [14], Khatibzadeh and Ranjbar [15] introduced and study solutions of problem (1.1) via the proximal point algorithm in complete CAT(0) space X

$$x_1 \in X, \quad x_n = J^A_{\lambda_n} x_{n-1}, \quad \forall n \geq 1, \tag{1.3}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Recently, Ranjbar and Khatibzadeh [16] proposed the following Mann-type and Halpern-type proximal point algorithms in complete CAT(0) spaces for finding a solution of problem (1.1)

$$x_1 \in X, \quad x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) J^A_{\lambda_n} x_n, \quad \forall n \geq 1, \tag{1.4}$$

and

$$u, x_1 \in X, \quad x_{n+1} = \alpha_n u \oplus (1-\alpha_n) J^A_{\lambda_n} x_n, \quad \forall n \geq 1, \tag{1.5}$$

where $\{\lambda_n\} \subset (0,\infty)$ and $\{\alpha_n\} \subset [0,1]$. They obtained a Δ-convergence result using the Mann-type proximal point algorithm and they also obtained a strong convergence result using the Halpern-type proximal point algorithm.

Let X be a metric space, and let C be a nonempty closed and convex subset of X. A point $x \in C$ is called a fixed point of a mapping $T : C \to X$ provided $Tx = x$. We denote by $F(T) := \{x \in C : Tx = x\}$ the set of fixed points of T. Recently, many authors studied fixed points of nonlinear operators in convex metric spaces; see, e.g., [17, 18, 19, 20] and the references therein.

Recently, Aremu et al. [21] and Ugwunnadi et al. [22] used the concept of quasilinearization to define new operators in CAT(0) spaces as follows.

Definition 1.1. Let X be a complete CAT(0) space, and let C be a nonempty closed and convex subset of X. The mapping T from C into X is said to be

(i) k-demimetric (see [21]) if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$ such that

$$\langle \overrightarrow{xp}, \overrightarrow{xTx} \rangle \geq \frac{1-k}{2} d^2(x, Tx), \text{ for all } x \in X \text{ and } p \in F(T). \tag{1.6}$$

(ii) θ-generalized demimetric (see [22]) if $F(T) \neq \emptyset$ and there exists $\theta \in \mathbb{R}$ such that

$$d^2(x, Tx) \leq \theta \langle \overrightarrow{xu}, \overrightarrow{xTx} \rangle \tag{1.7}$$

for all $x \in C$ and $u \in F(T)$.

Remark 1.1. It is clear in Definition 1.1 that, for any $k \in (-\infty, 1)$, a k-demimetric mapping is $\frac{2}{1-k}$-generalized demimetric. Also, for $\theta > 0$, a θ-generalized demimetric is $\left(1 - \frac{2}{\theta}\right)$-demimetric.

Motivated by the above results, in this paper, we study a modified Halpern-Mann type algorithm for approximating common solution of a finite family of monotone inclusion problems and a finite family of generalized demimetric mappings. We also obtain a strong convergence theorem in Hadamard spaces. Our results unify and compliments many results in the current literature.

2. Preliminaries

A geodesic path joining two elements x,y in a metric space X is an isometry $c:[0,l]\to X$, where $d(x,y)=l$ such that $c(0)=x$ and $c(l)=y$. The image of a geodesic path is called a geodesic segment. A metric space for which every two points can be joined by a geodesic segment is called a geodesic space. We say that a metric space X is uniquely geodesic if every two points of X are joined by only one geodesic segment (i.e., CAT(0) space). The examples of CAT(0) spaces are Euclidean spaces $\mathbb{R}^n$ and Hilbert spaces. For more details, please see [23, 24, 25, 26]. Complete CAT(0) spaces are often called Hadamard spaces.

Let $(1-t)x\oplus ty$ denote the unique point z in the geodesic segment joining x to y for each x,y in a CAT(0) space such that $d(z,x)=td(x,y)$ and $d(z,y)=(1-t)d(x,y)$, where $t\in[0,1]$. Let $[x,y]:=\{(1-t)x\oplus ty : t\in[0,1]\}$. Then, a subset C of X is convex if $[x,y]\subseteq C$ for all $x,y\in C$.

In 2008, Breg and Nikolaev [27] introduced the concept of quailinearization mappings in CAT(0) spaces. They denoted a pair $(a,b)\in X\times X$ by $\overrightarrow{ab}$, which they called a vector and defined a mapping $\langle .,.\rangle : (X\times X)\times(X\times X)\to\mathbb{R}$ by

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\frac{1}{2}\left(d^2(a,d)+d^2(b,c)-d^2(a,c)-d^2(b,d)\right),\ (a,b,c,d\in X), \tag{2.1}$$

which is called the quasilinearization mapping. It is easy to verify that $\langle\overrightarrow{ab},\overrightarrow{ab}\rangle=d^2(a,b)$, $\langle\overrightarrow{ba},\overrightarrow{cd}\rangle=-\langle\overrightarrow{ab},\overrightarrow{cd}\rangle$, $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\langle\overrightarrow{ae},\overrightarrow{cd}\rangle+\langle\overrightarrow{eb},\overrightarrow{cd}\rangle$ and $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\langle\overrightarrow{cd},\overrightarrow{ab}\rangle$ for all $a,b,c,d,e\in X$. It has been established that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality (see [27]). Recall that the space X is said to satisfy the Cauchy-Swartz inequality if $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle\le d(a,b)d(c,d)\ \forall a,b,c,d\in X$. Let X be a complete CAT(0) space, and let X^* be its dual space. A multivalued operator $A:X\to 2^{X^*}$ with domain $\mathbb{D}(A):=\{x\in X : Ax\neq\emptyset\}$ is monotone if and only if, for all $x,y\in\mathbb{D}(A)$, $x^*\in Ax$, $y^*\in Ay$,

$$\langle x^*-y^*,\overrightarrow{yx}\rangle\ge 0\ \ \text{(see [15])}.$$

The resolvent of the operator A of order $\lambda>0$ is the multivalued mapping $J_\lambda^A : X\to 2^X$ defined in [15] as

$$J_\lambda^A(x):=\{z\in X\mid [\frac{1}{\lambda}\overrightarrow{zx}]\in Az\}.$$

The operator A satisfies the range condition if for every $\lambda>0$, $\mathbb{D}(J_\lambda^A)=X$ (see [15]). For simplicity, we shall write J_λ for the resolvent of a monotone operator A. Since our main contribution in this paper is on Hadamard spaces for monotone inclusion problems, it is worthwhile to provide a detailed proof of example of a monotone mapping in Hadamard spaces.

Example 2.1. [28] Let $X=\mathbb{R}^2$ be an $\mathbb{R}$-tree with the radical metric d_r, where $d_r(x,y)=d(x,y)$ if x and y are situated on the euclidean straight line passing through the origin and

$$d_r(x,y)=d(x,0)+d(y,0):=\|x\|+\|y\|,$$

otherwise let $p=(1,0)$ and $X=B\cup C$, where

$$B=\{(h,0):h\in[0,1]\}\ \text{ and }\ C=\{(h,k):h+k=1,\ h\in[0,1)\}.$$

Then, (X,d_r) is an Hadamard space and X^*, which is a space of element $[\overrightarrow{tab}]$ such that

$$[\overrightarrow{tab}] = \begin{cases} \{\overrightarrow{scd} : c,d \in B,\ s \in \mathbb{R}, t(\|b\|-\|a\|) = S(\|d\|-\|c|)\}\ a,b,\in B, \\ \{\overrightarrow{scd} : c,d \in C \in \{0\},\ s \in \mathbb{R}, t(\|b\|-\|a\|) = s(\|d\|-\|c\|),\}\ a,b \in C\cup\{0\}, \\ \{\overrightarrow{tab}\} \end{cases} \tag{2.2}$$

is the dual space of X (see [29]). Now, defined $A : X \to 2^{X^*}$ by

$$Ax := \begin{cases} \{[\overrightarrow{0p}]\}, & x \in B, \\ \{[\overrightarrow{0p}],[\overrightarrow{0x}]\}, & x \in C. \end{cases} \tag{2.3}$$

Then A is a multivalued monotone operator. To see this we consider the cases:

(I) If $x,y \in B$, then $Ax = Ay = \{[\overrightarrow{0p}]\}$ and $x^* = y^* = [\overrightarrow{0p}]$. So, $\langle x^*-y^*, \overrightarrow{yx}\rangle = 0 \geq 0$.

(II) If $x,y \in C$, then $Ax = \{[\overrightarrow{0p}],[\overrightarrow{0x}]\}$ and $Ay = \{[\overrightarrow{0p}],[\overrightarrow{0y}]\}$.

(i) If $x^* = y^* = [\overrightarrow{0p}]$; then $\langle x^*-y^*, \overrightarrow{yx}\rangle = 0 \geq 0$.

(ii) If $x^* = [\overrightarrow{0x}]$ and $y^* = [\overrightarrow{0y}]$, then

$$\begin{aligned} \langle x^*-y^*, \overrightarrow{yx}\rangle &= \langle \overrightarrow{yp}, \overrightarrow{yx}\rangle \\ &= \frac{1}{2}(d_r^2(y,x)+d_r^2(p,y)-d_r^2(p,x)) \\ &= \frac{1}{2}((\|y\|+\|x\|)^2+(1+\|y\|)^2-(1+\|x\|)^2) \\ &\geq 0 \text{ (since } 1/\sqrt{2} \leq \|x\|.\|y\| \leq 1). \end{aligned}$$

(iv) If $x^* = [\overrightarrow{0x}]$ and $y^* = [\overrightarrow{0p}]$, then $\langle x^*-y^*, \overrightarrow{yx}\rangle = \langle \overrightarrow{px}, \overrightarrow{yx}\rangle$, which is similar to (iii).

(III) If $x \in B$, $y \in C$. Then $Ax = \{[\overrightarrow{0p}]\}$, $Ay = \{[\overrightarrow{0p}],[\overrightarrow{0y}]\}$.

(i) If $x^* = y^* = [\overrightarrow{0p}]$, then $\langle x^*-y^*, \overrightarrow{yx}\rangle = 0 \geq 0$.

(ii) If $x^* = [\overrightarrow{0p}]$ and $y^* = [\overrightarrow{oy}]$, then

$$\begin{aligned} \langle x^*-y^*, \overrightarrow{yx}\rangle &= \langle \overrightarrow{yp}, \overrightarrow{yx}\rangle \\ &= \frac{1}{2}(d_r^2(y,x)+d_r^2(p,y)-d_r^2(p,x)) \\ &\geq 0 \end{aligned}$$

due to $d(p,x) \leq 1 \leq d(p,y)$. Thus, A is monotone.

We state some known and useful results which will be needed in the proof of our main theorem.

Lemma 2.1. [30] *Let X be a CAT(0) space, $x,y,z \in X$ and $\lambda \in [0,1]$. Then*

(i) $d(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d(x,z) + (1-\lambda)d(y,z)$.

(ii) $d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y)$.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X. For $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x,\{x_n\}) = r(\{x_n\})\}.$$

It is well known that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point ([31]). A sequence $\{x_n\}$ in X is said to be $\triangle$-convergent to $x \in X$, denoted by $\triangle - \lim_n x_n = x$ if x is the unique asymptotic center of $\{u_n\}$, for every subsequence $\{u_n\}$ of $\{x_n\}$.

Lemma 2.2. [32] *If $\{x_n\}$ is a bounded sequence in a closed and convex subset C of a complete CAT(0) space, then the asymptotic center of $\{x_n\}$ is in C.*

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space X, and let C be a closed and convex subset of X, which contains $\{x_n\}$. We employ the notation

$$\{x_n\} \rightharpoonup w \Leftrightarrow \limsup_{n\to\infty} d(x_n,w) = \inf_{x\in C}(\limsup_{n\to\infty} d(x_n,x)).$$

We note that $\{x_n\} \rightharpoonup w$ if and only if $A(\{x_n\}) = \{w\}$ (see [33]).

Lemma 2.3. [34] *Let X be a CAT(0) space. For any $u,v,\in X$ and $t \in (0,1)$, let $u_t = tu \oplus (1-t)v$. Then, for all $x,y \in X$,*

(i) $\langle \overrightarrow{u_tx}, \overrightarrow{u_ty}\rangle \le t\langle \overrightarrow{ux}, \overrightarrow{u_ty}\rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_ty}\rangle$;

(ii) $\langle \overrightarrow{u_tx}, \overrightarrow{uy}\rangle \le t\langle \overrightarrow{ux}, \overrightarrow{uy}\rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy}\rangle$ *and* $\langle \overrightarrow{u_tx}, \overrightarrow{vy}\rangle \le t\langle \overrightarrow{ux}, \overrightarrow{vy}\rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy}\rangle$.

Lemma 2.4. [33] *If $\{x_n\}$ is a bounded sequence in a closed and convex subset C of a complete CAT(0) space, then $\triangle - \lim_{n\to\infty} x_n = p$ implies that $\{x_n\} \rightharpoonup p$.*

Theorem 2.1. [15] *Let X be a CAT(0) space and let J_λ^A be the resolvent of the operator A of order λ. We have*

(i) For any $\lambda > 0$, $\mathbb{R}(J_\lambda^A) \subset \mathbb{D}(A)$, $F(J_\lambda^A) = A^{-1}(0)$.

(ii) If A is monotone then J_λ^A is a single-valued and firmly nonexpansive mapping.

The following remark is a consequence of Theorem 2.1.

Remark 2.1. (see [35]) If X is a CAT(0) space and J_λ^A is the resolvent of a monotone operator $A : X \to 2^{X^*}$ of order $\lambda > 0$, then

$$d^2(u,J_\lambda^A x) + d^2(J_\lambda^A x,x) \le d^2(u,x),$$

for all $u \in A^{-1}(0)$ and $x \in \mathbb{D}(J_\lambda^A)$.

Proof. Indeed, for any $u \in A^{-1}(0)$, $x \in \mathbb{D}(J_\lambda^A)$ and $\lambda > 0$, we obtain from Theorem 2.1 (i) and (ii) that

$$\begin{aligned} d^2(J_\lambda^A x,u) &\le \langle \overrightarrow{J_\lambda^A x\, u}, \overrightarrow{xu}\rangle \\ &= \frac{1}{2}\left(d^2(J_\lambda^A x,u) + d^2(u,x) - d^2(J_\lambda^A x,x)\right), \end{aligned}$$

which implies

$$d^2(u,J_\lambda^A x) + d^2(J_\lambda^A x,x) \le d^2(u,x).$$

□

Lemma 2.5. [36] *Let $\{x_n\}$ be a sequence in a complete CAT(0) space X, and $x \in X$. Then $\{x_n\}$ is $\triangle-$convergent to x if and only if $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$ for all $y \in X$.*

Lemma 2.6. [19] *Every bounded sequence in a complete CAT(0) spaces always has a convergent subsequence.*

Lemma 2.7. [17] *Let C be a nonempty, closed and convex subset of CAT(0) space X. Let $\{x_i : i = 1,2,...,N\}$ be in C, and $\alpha_1, \alpha_2, ..., \alpha_N \in (0,1)$ such that $\sum_{i=1}^N \alpha_i = 1$. Then the following inequality hold:*

(i) $d\left(z, \bigoplus_{i=1}^N \alpha_i x_i\right) \leq \sum_{i=1}^N \alpha_i d(z,x_i)$ for all $z \in C$.
(ii) $d^2\left(z, \bigoplus_{i=1}^N \alpha_i x_i\right) \leq \sum_{i=1}^N \alpha_i d^2(z,x_i) - \sum_{i,j=1,\ i\neq j}^N \alpha_i\alpha_j d^2(x_i,x_j)$ for all $z \in C$.

Lemma 2.8. [37] *Let C be a nonempty, convex subset of CAT(0) space X. Let $\{u_i : i = 1,2,...,N\} \subset C$, and $\alpha_1, \alpha_2, ..., \alpha_N \in (0,1)$ such that $\sum_{i=1}^N \alpha_i = 1$. Then the following inequalities hold:*

$$\begin{aligned}\left\langle \overrightarrow{\bigoplus_{i=1}^N \alpha_i u_i x}, \overrightarrow{xy}\right\rangle &\leq \sum_{i=1}^N \alpha_i \langle \overrightarrow{u_i x}, \overrightarrow{xy}\rangle + \frac{1}{2}\left(\sum_{i=1}^N \alpha_i d^2(u_i,x) - d^2\left(\bigoplus_{i=1}^N \alpha_i u_i, x\right)\right) \\ &\leq \sum_{i=1}^N \alpha_i \langle \overrightarrow{u_i x}, \overrightarrow{xy}\rangle + \frac{1}{2}\sum_{i=1}^N \alpha_i d^2(u_i,x). \end{aligned} \tag{2.4}$$

Lemma 2.9. [37] *Let X be a CAT(0) space and let C a nonempty convex subset of X. Assume that $\{S_i\}_{i=1}^N : C \to X$ is a finite family of k_i -demimetric mapping with $k_i \in (-\infty, 1)$ for each $i \in \{1,2,...,N\}$ such that $\bigcap_{i=1}^N F(S_i) \neq \emptyset$. Let $\{\alpha_i\}_{i=1}^N$ be a positive sequence with $\sum_{i=1}^N \alpha_i = 1$. Then $\bigoplus_{i=1}^N \alpha_i S_i : C \to X$ is a k-demimetric mapping if $k := \max\{k_i : i = 1,2,...,N,\} \leq 0$ and $F\left(\bigoplus_{i=1}^N \alpha_i S_i\right) = \bigcap_{i=1}^N F(S_i)$.*

Definition 2.1. Let C be a nonempty closed and convex subset of a complete CAT(0) space X. The metric projection $P_C : X \to C$ is defined by

$$u = P_C(x) \ \Leftrightarrow\ d(u,x) = \inf\{d(y,x) : y \in C\}, \text{ for all } x \in X.$$

Lemma 2.10. [27] *Let C be a nonempty closed and convex subset of complete CAT(0) space X. For any $x \in X$ and $u \in C$, $u = P_C x$ if and only if*

$$\langle \overrightarrow{yu}, \overrightarrow{ux}\rangle \geq 0.$$

Lemma 2.11. [22] *Let C be a nonempty closed and convex subset of a CAT(0) space X and let $T : C \to X$ be a θ-generalized demimetric mapping with $\theta \in \mathbb{R}$. Then, it is closed and convex.*

Lemma 2.12. [22] *Let C be a nonempty closed and convex subset of a CAT(0) space X and let $T : C \to X$ be a θ-generalized demimetric mapping. Then, for any $\theta \in [0,\infty)$ and $k \in (0,1]$, $(1-k)I \oplus kT$ is θk-generalized demimetric from C into X.*

Lemma 2.13. [21] *Let X be a CAT(0) space, $T : X \to X$ a k-demimetric mapping with $k \in (-\infty, \lambda)$ with $\lambda \in (0,1)$ and $F(T) \neq \theta$. Suppose that $T_\lambda x := (1-\lambda) \oplus \lambda T x$. Then T_λ is quasi-nonexpansive mapping and $F(T_\lambda) = F(T)$.*

Lemma 2.14. [34] *Let X be a complete CAT(0) space. Then, for all $u,x,y \in X$, the following inequality holds:*

$$d^2(x,u) \leq d^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu}\rangle.$$

Lemma 2.15. [38] *Let X be a complete CAT(0) space. For all $u,x,y \in X$ and $\alpha \in [0,1]$, let $z_1 = \alpha x \oplus (1-\alpha)u$ and $z_2 = \alpha y \oplus (1-\alpha)u$. Then*

$$\langle \overrightarrow{z_1z_2}, \overrightarrow{xz_2} \rangle \leq \alpha \langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Lemma 2.16. [39] *If $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the following inequality:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \geq 0,$$

where, (i) $\{\alpha_n\} \subset [0,1]$, $\sum \alpha_n = \infty$; (ii) $\lim sup\ \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$ and $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

Lemma 2.17. [40] *If $\{a_n\}$ is a sequence of real numbers and there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$, then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied: $a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$. for all sufficiently large numbers $k \in \mathbb{N}$. In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

3. Main results

Theorem 3.1. *Let X be a complete $CAT(0)$ space with dual X^* and let C be a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^N : C \to X$ be a finite family of θ_i-generalized demimetric mapping and Δ-demiclosd at 0 with $\theta_i \in (0,\infty)$ for each $i \in \{1,2,...,N\}$. Let $A_i : X \to 2^{X^*}$ $(i = 1,2,...,N)$ be multivalued monotone mappings which satisfy the range condition. Assume that $\Upsilon := \bigcap_{i=1}^N F(T_i) \cap \left(\bigcap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$. Let $\{u_n\}$ be a sequence in X such that $u_n \to u \in X$. Assume for $k \in (0,\gamma)$ with $\gamma \in (0,1)$ and $\theta_i k > 0$. For any $x_1 \in X$, let $\{x_n\}$ in X be a sequence generated by*

$$\begin{cases} y_n = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 x_n, \\ z_n = (1-\gamma)y_n \oplus \gamma \left[\bigoplus_{i=1}^N \xi_i((1-k) \oplus kT_i)y_n\right], \\ x_{n+1} = \alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n, \end{cases} \tag{3.1}$$

where $\lambda \in (0,\infty)$, $\{\alpha_n\}$, $\{\sigma\}$, $\{\beta_n\}$ and $\{\xi_i\}_{i=1}^N$ are sequences in $(0,1)$ satisfying the following conditions

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{i=1}^\infty \alpha_n = \infty$;

(ii) $\alpha_n + \beta_n + \sigma_n = 1$.

Then $\{x_n\}$ converges strongly to $x^* \in \Upsilon$.

Proof. Let $S_i = (1-k) \oplus kT_i$ and $W_N = \bigoplus_{i=1}^N \xi_i S_i$. Then we can rewrite algorithm (3.1) as:

$$\begin{cases} y_n = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 x_n, \\ z_n = (1-\gamma)y_n \oplus \gamma W_N y_n, \\ x_{n+1} = \alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n, \end{cases} \tag{3.2}$$

since $T_i : C \to X$ is θ_i-generalized demimetric, by Lemma 2.11, we have that $F(T_i)$ is closed and convex for each $i \in \{1,2,\cdots,N\}$. Also, J_λ^i is firmly nonexpansive by Theorem 2.1 and hence nonexpansive for each $i = 1,2,...,N$. Therefore $F(J_\lambda^i)$ is closed and convex for each $i = 1,2,...,N$. Hence, $\bigcap_{i=1}^N F(T_i) \cap \left(\bigcap_{i=1}^N A_i^{-1}(0)\right)$ is nonempty closed and convex. Therefore, $P_{\bigcap_{i=1}^N F(T_i) \cap \left(\bigcap_{i=1}^N A_i^{-1}(0)\right)}$ is well defined. Furthermore, T_i is θ_i-generalized demimetric with $\theta_i \in (0,\infty)$ for each $i \in \{1,2,...,N\}$. So, for any $k \in (0,\gamma)$, with $\gamma \in (0,1)$, we find from

Lemma 2.12 that S_i is $\theta_i k$-generalized demimetric for each i. By Remark 1.1, we have that S_i is $\left(1-\frac{2}{\theta_i k}\right)$-demimetric. We obtain from Lemma 2.9 that $W_N = \bigoplus_{i=1}^N \xi_i S_i$ is demimetric. It follows by Lemma 2.13 that $V_N := (1-\gamma)\oplus \gamma W_N$ is quasi-nonexpansive and $F(V_N) = F(W_N) = \bigcap_{i=1}^N F(S_i) = \bigcap_{i=1}^N F(T_i)$. Let $p\in\Upsilon$, $\Psi_\lambda^N := J_\lambda^N \circ J_\lambda^{N-1}\circ\cdots\circ J_\lambda^2\circ J_\lambda^1$, where $\Psi_\lambda^0 = I$. Then by the definition of (y_n) in (3.2) and Remark 2.1, we obtain

$$\begin{aligned}
d^2(y_n,p) &\le d^2(\Psi_\lambda^{N-1}x_n,p) - d^2(\Psi_\lambda^{N-1}x_n,y_n)\\
&\le d^2(\Psi_\lambda^{N-2}x_n,p) - d^2(\Psi_\lambda^{N-2}x_n,\Psi_\lambda^{N-1}x_n) - d^2(\Psi_\lambda^{N-1}x_n,y_n)\\
&\le d^2(\Psi_\lambda^{N-3}x_n,p) - d^2(\Psi_\lambda^{N-3}x_n,\Psi_\lambda^{N-2}x_n) - d^2(\Psi_\lambda^{N-2}x_n,\Psi_\lambda^{N-1}x_n)\\
&\quad - d^2(\Psi_\lambda^{N-1}x_n,y_n)\\
&\le d^2(x_n,p) - \sum_{i=1}^N d^2(\Psi_\lambda^{i-1}x_n,\Psi_\lambda^{i}x_n). \qquad (3.3)
\end{aligned}$$

Using (z_n) in (3.2), we get

$$d(z_n,p) \le d(V_N y_n,p) \le d(y_n,p) \le d(x_n,p),$$

which together the definition of (x_{n+1}) implies that

$$\begin{aligned}
d(x_{n+1},p) &= \alpha_n d(u_n,p) + \beta_n d(x_n,p) + \sigma_n d(z_n,p)\\
&\le \alpha_n d(u_n,p) + (\alpha_n+\sigma_n) d(x_n,p)\\
&= (1-\alpha_n) d(x_n,p) + \alpha_n d(u_n,p).
\end{aligned}$$

Since $\{u_n\}$ is bounded, there exists $M>0$ such that $\sup d(u_n,p)\le M$. Letting $M^* = \max\{d(x_1,p), M\}$ for all $n\in\mathbb{N}$ implies that $d(x_1,p)\le M^*$. Suppose that, for some $t\in\mathbb{N}$, $d(x_t,p)\le M^*$, then

$$\begin{aligned}
d(x_{n+1},p) &\le (1-\alpha_t) d(x_t,p) + \alpha_t d(x_t,p)\\
&= (1-\alpha_t)M^* + \alpha_t M^* = M^*.
\end{aligned}$$

By induction, we obtain that $d(x_n,p)\le M^*$ for all $n\in\mathbb{N}$. Hence $\{x_n\}$ is bounded. From Lemma 2.1(ii), we obtain

$$\begin{aligned}
d^2(x_{n+1},p) &= d^2(\alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n, p)\\
&\le d^2\left[(1-\sigma_n)\left(\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n\right)\oplus \sigma_n z_n, p\right]\\
&\le (1-\sigma_n) d^2\left(\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n, p\right) + \sigma_n d^2(z_n,p)\\
&\le \alpha_n d^2(u_n,p) + \beta_n d^2(x_n,p) - \frac{\alpha_n\beta_n}{1-\sigma_n} d^2(u_n,z_n) + \sigma_n d^2(y_n,p)\\
&\le \alpha_n d^2(u_n,p) + \beta_n d^2(x_n,p) + \sigma_n d^2(x_n,p) - \sigma_n \sum_{i=1}^N d^2(\Psi_\lambda^{i-1}x_n,\Psi_\lambda^{i}x_n)\\
&\le (1-\alpha_n) d^2(x_n,p) + \alpha_n d^2(u_n,p) - \sigma_n \sum_{i=1}^N d^2(\Psi_\lambda^{i-1}x_n,\Psi_\lambda^{i}x_n). \qquad (3.4)
\end{aligned}$$

We divide the remaining proof in two cases.

Case 1. Assume that $\{d(x_n,p)\}_{n=1}^{\infty}$ is a non-increasing sequence of a real numbers. Since $\{d(x_n.p)\}_{n=1}^{\infty}$ is bounded, then its limit exists. With the fact that $\alpha_n \to 0$ as $n \to \infty$ and $\sigma_n > 0$, (3.4) gives

$$\sum_{i=1}^{N} d^2(\Psi_{\lambda}^{i=1}x_n, \Psi_{\lambda}^{i}x_n) \leq d^2(x_n,p) - d^2(x_{n+1},p) + \alpha_n d^2(u_n,p),$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{N} d^2(\Psi_{\lambda}^{i=1}x_n, \Psi_{\lambda}^{i}x_n) = 0.$$

Note that $d^2(\Psi_{\lambda}^{i-1}x_n, \Psi_{\lambda}^{i}x_n)$ is nonnegative for each $i = 1,2,\cdots,N$. Hence, for each $i = 1,2,...,N$, we obtain

$$\lim_{n\to\infty} d(\Psi_{\lambda}^{i-1}x_n, \Psi_{\lambda}^{i}x_n) = 0. \tag{3.5}$$

Using (y_n) in (3.2) and (3.5), we get

$$d(y_n,x_n) \leq \sum_{i=1}^{N} d(\Psi_{\lambda}^{i-1}x_n, \Psi_{\lambda}^{i}x_n) \to 0, \text{ as } n \to \infty. \tag{3.6}$$

It follows from (3.2) that

$$\begin{aligned}
d^2(x_{n+1},p) &= d^2(\alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n, p) \\
&\leq d^2\left[(1-\sigma_n)\left(\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n\right) \oplus \sigma_n z_n, p\right] \\
&\leq (1-\sigma_n)d^2\left[\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n\right] + \sigma_n d^2(z_n,p) \\
&\leq \alpha_n d^2(u_n,p) + \beta_n d^2(x_n,p) - \frac{\alpha_n\beta_n}{1-\sigma_n}d^2(u_n,x_n) + \sigma_n d^2(z_n,p) \\
&\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n d^2(u_n,p) - \frac{\alpha_n\beta_n}{1-\sigma_n}d^2(u_n,x_n),
\end{aligned}$$

which implies that

$$\frac{\alpha_n\beta_n}{1-\sigma_n}d^2(u_n,x_n) \leq d^2(x_n,p) - d^2(x_{n+1},p) + \alpha_n d^2(x_n,p).$$

Hence,

$$\lim_{n\to\infty} d(u_n,x_n) = 0. \tag{3.7}$$

Also, using Lemma 2.1(ii), we can get that

$$\begin{aligned}
d^2(x_{n+1},p) &\leq d^2((1-\sigma_n)\left(\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n\right) \oplus \sigma_n z_n, p) \\
&\leq (1-\sigma_n)d^2\left(\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n, p\right) + \sigma_n d^2(z_n,p) \\
&\quad -\sigma_n(1-\sigma_n)d^2\left(\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n, z_n\right) \\
&\leq \alpha_n d^2(u_n,p) + \beta_n d^2(x_n,p) + \sigma_n d^2(z_n,p) \\
&\quad -\sigma_n(1-\sigma_n)d^2\left(\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n, z_n\right) \\
&\leq d^2(x_n,p) + \alpha_n d^2(u_n,p) - \sigma_n(1-\sigma_n)d^2\left(\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n, z_n\right).
\end{aligned}$$

Therefore,

$$\sigma_n(1-\sigma_n)d^2\left(\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n, z_n\right) \leq d^2(x_n,p) - d^2(x_{n+1},p) + \alpha_n d^2(u_n,p)$$

and

$$\lim_{n\to\infty} d\left(\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n, z_n\right) = 0. \tag{3.8}$$

On the other hand, we obtain from (3.7) and (3.8) that

$$\begin{aligned}
d(z_n,x_n) &\leq d\left(z_n, \frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n\right) + d\left(\frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n, x_n\right) \\
&\leq d\left(z_n, \frac{\alpha_n}{1-\sigma_n}u_n \oplus \frac{\beta_n}{1-\sigma_n}x_n\right) + \frac{\alpha_n}{1-\sigma_n}d(u_n,x_n).
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} d(z_n,x_n) = 0. \tag{3.9}$$

We obtain from (3.6) and (3.9) that

$$d(y_n,z_n) \leq d(y_n,x_n) + d(x_n,z_n) \to 0, \text{ as } n\to\infty. \tag{3.10}$$

From (3.2), (3.8) and (3.9), we get

$$d(x_{n+1},x_n) \leq \alpha_n d(u_n,x_n) + \beta_n d(x_n,x_n) + \sigma_n d(z_n,x_n) \to 0, \ n\to\infty. \tag{3.11}$$

Furthermore, since S_i is k_i-demimetric mapping for each $i \in \{1,2,...,N\}$ with $k = \max\{k_i\} \leq 0$, then

$$\begin{aligned}
\langle \overrightarrow{y_n z_n}, \overrightarrow{y_n p} \rangle &= -\langle \overrightarrow{z_n y_n}, \overrightarrow{y_n p} \rangle \\
&= -\langle \overrightarrow{((1-\gamma)y_n \oplus \gamma W_N y_n) y_n}, \overrightarrow{y_n p} \rangle \\
&\geq -(1-\gamma)\langle \overrightarrow{y_n y_n}, \overrightarrow{y_n p} \rangle - \gamma \langle \overrightarrow{W_N y_n y_n}, \overrightarrow{y_n p} \rangle \\
&\geq -\gamma \langle \overrightarrow{W_N y_n y_n}, \overrightarrow{y_n p} \rangle \\
&\geq -\gamma \langle \overrightarrow{\bigoplus_{i=1}^{N} \xi_i S_i y_n y_n}, \overrightarrow{y_n p} \rangle \\
&\geq -\gamma \sum_{i=1}^{N} \xi_i \langle \overrightarrow{S_i y_n y_n}, \overrightarrow{y_n p} \rangle - \frac{1}{2}\gamma \sum_{i=1}^{N} \alpha_i d^2(S_i y_n, y_n) \\
&\geq \gamma \sum_{i=1}^{N} \frac{1-k_i}{2} \xi_i d^2(S_i y_n, y_n) - \frac{1}{2}\gamma \sum_{i=1}^{N} \xi_i d^2(S_i y_n, y_n) \\
&= \gamma \sum_{i=1}^{N} \frac{-k_i}{2} \xi_i d^2(S_i y_n, y_n) \\
&\geq \frac{-k}{2}\gamma \sum_{i=1}^{N} \xi_i d^2(S_i y_n, y_n).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{-k}{2}\gamma \sum_{i=1}^{N} \xi_i d^2(S_i y_n, y_n) &\leq \langle \overrightarrow{y_n z_n}, \overrightarrow{y_n p} \rangle \\
&\leq d(y_n, z_n) d(y_n, p). \qquad (3.12)
\end{aligned}$$

Since $\{y_n\}$ is bounded, $k \leq 0$, and $\gamma, \xi_i \in (0,1)$ for all $n \geq 1$ and $i \in \{1,2,...,N\}$, then we find from (3.10) and (3.12) that

$$\lim_{n\to\infty} d(S_i y_n, y_n) = 0, \quad \text{for } i \in \{1,2,...,N\}. \qquad (3.13)$$

Now, since $\{x_n\}$ is bounded and X is complete CAT(0) spaces, we conclude from Lemma 2.6 that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim x_{n_j} = v \in X$. By (3.6), we get $\Delta\text{-}\lim y_{n_j} = v$. With (3.13) and the fact that S_i is Δ-demiclosed at 0, for each $i \in \{1,2...,N\}$, we obtain that $v \in \bigcap_{i=1}^{N} F(S_i) = \bigcap_{i=1}^{N} F(T_i)$. Furthermore, Ψ_λ^i is firmly nonexpansive, in particular, it is nonexpansive for each $i = 1,2,...,N$. Hence by (3.6), we obtain that $v \in \bigcap_{i=1}^{N} A_i^{-1}(0)$. Therefore, $v \in \bigcap_{i=1}^{N} F(T_i) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(0)\right) = \Upsilon$. Thus, from Lemma 2.5, we get

$$\limsup_{n\to\infty} \langle \overrightarrow{uv}, \overrightarrow{x_n v} \rangle \leq 0. \qquad (3.14)$$

Letting $w_n := \frac{\beta_n}{1-\alpha_n} x_n \oplus \frac{\sigma_n}{1-\alpha_n} z_n$, we have

$$\begin{aligned}
\langle \overrightarrow{u_n v}, \overrightarrow{w_n v} \rangle &= \langle \overrightarrow{u_n v}, \overrightarrow{w_n x_n} \rangle + \langle \overrightarrow{u_n v}, \overrightarrow{x_n v} \rangle \\
&\leq d(u_n, v) d(w_n, x_n) + \langle \overrightarrow{u_n u}, \overrightarrow{x_n v} \rangle + \langle \overrightarrow{uv}, \overrightarrow{x_n v} \rangle \\
&\leq \frac{\beta_n}{1-\alpha_n} d(u_n, v) d(x_n, z_n) + d(u_n, u) d(x_n, v) + \langle \overrightarrow{uv}, \overrightarrow{x_n v} \rangle.
\end{aligned}$$

Therefore, in view of the fact that $u_n \to u$ as $n \to \infty$ with (3.9) and (3.14), we obtain

$$\limsup_{n\to\infty} \langle \overrightarrow{u_n v}, \overrightarrow{w_n v} \rangle \leq 0. \tag{3.15}$$

Also,

$$\begin{aligned} d(w_n, v) &= d\Big(\frac{\beta_n}{1-\alpha_n}x_n \oplus \frac{\sigma_n}{1-\alpha_n}z_n, v\Big) \\ &\leq \frac{\beta_n}{1-\alpha_n}d(x_x, v) + \frac{\sigma_n}{1-\alpha_n}d(z_{n,v}) \\ &\leq \frac{\beta_n}{1-\alpha_n}d(x_n, v) + \frac{\sigma_n}{1-\alpha_n}d(x_{n,v}) \\ &= d(x_n, v). \end{aligned}$$

Finally, we show that $x_n \to v$. Using(3.2), and letting $\vartheta_n := \alpha_n v \oplus (1-\alpha_n)z_n$ and $x_{n+1} = \alpha_n u_n \oplus (1-\alpha_n)w_n$, we conclude from Lemma 2.14 and Lemma 2.15 that

$$\begin{aligned} d^2(x_{n+1}, v) &\leq d^2(\vartheta_n, v) + 2\langle \overrightarrow{x_{n+1}\vartheta_n}, \overrightarrow{x_{n+1}v} \rangle \\ &\leq (1-\alpha_n)d^2(w_n, v) + 2\langle \overrightarrow{\vartheta_n x_{n+1}}, \overrightarrow{v x_{n+1}} \rangle \\ &\leq (1-\alpha_n)d^2(x_n, v) + 2\alpha_n \langle \overrightarrow{u_n v}, \overrightarrow{w_n v} \rangle. \end{aligned}$$

Therefore

$$d^2(x_{n+1}, v) \leq (1-\alpha_n)d^2(x_n, v) + 2\alpha_n \langle \overrightarrow{u_n v}, \overrightarrow{w_n v} \rangle. \tag{3.16}$$

From (3.15), (3.16) and Lemma 2.16, we obtain $d(x_n, v) \to 0$ as $n \to \infty$, that is, $x_n \to v$ as $n \to \infty$.

Case 2. Suppose that $\{d(x_n, p)\}_{n=1}^{\infty}$ is a not monotone decreasing real sequence. Set $\Upsilon_n := d(x_n, x^*)$ for all $n \geq 1$. Then, there exists a subsequence Υ_{n_s} of Υ_n such that $\Upsilon_{n_s} < \Upsilon_{n_s+1}$ for all $k \geq 1$. Now, define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Upsilon_k < \Upsilon_{k+1}\}.$$

It follows from Lemma 2.17 that $\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}$. Using (3.4), we get

$$\sigma_{\tau(n)} \sum_{i=1}^{N} d^2\left(\Psi_\lambda^{i-1}x_{\tau(n)}, \Psi_\lambda^i x_{\tau(n)}\right) \leq d^2(x_{\tau(n)}, v) - d^2(x_{\tau(n)+1}, v) + \alpha_{\tau(n)}d^2(u_{\tau(n)}, v).$$

Now, $\alpha_{\tau(n)} \to 0$ as $n \to \infty$ gives

$$\lim_{n\to\infty} \sum_{i=1}^{N} d^2\left(\Psi_\lambda^{i-1}x_{\tau(n)}, \Psi_\lambda^i x_{\tau(n)}\right) = 0.$$

Following an argument similar to the one in Case 1, we obtain

$$\lim_{n\to\infty} d(y_{\tau(n)}, x_{\tau(n)}) = 0, \ \lim_{n\to\infty} d(y_{\tau(n)}, z_{\tau(n)}) = 0, \lim_{n\to\infty} d(u_{\tau(n)}, x_{\tau(n)}) = 0$$

and

$$\lim_{n\to\infty} d(x_{\tau(n)+1}, x_{\tau(n)}) = 0 = \lim_{n\to\infty} d(S_i y_{\tau(n)}, y_{\tau(n)}). \tag{3.17}$$

Following the similar argument of proof in Case 1, we get

$$\langle \overrightarrow{u_{\tau n} p}, \overrightarrow{w_{\tau(n)} p} \rangle \leq 0.$$

From (3.16), we have

$$d^2(x_{\tau(n)+1},p) \leq (1-\alpha_{\tau(n)})d^2(x_{\tau(n)},p) + 2\alpha_{\tau(n)}\langle \overrightarrow{u_{\tau(n)}p}, \overrightarrow{w_{\tau(n)}p}\rangle.$$

Since $d^2(x_{\tau(n)},p) < d^2(x_{\tau(n)+1},p)$, then

$$\begin{aligned} \alpha_{\tau(n)}d^2(x_{\tau(n)},p) &\leq d^2(x_{\tau(n)},p) - d^2(x_{\tau(n)+1},p) \\ &\quad +2\alpha_{\tau(n)}\langle \overrightarrow{u_{\tau(n)}p}, \overrightarrow{w_{\tau(n)}p}\rangle \\ &< 2\alpha_{\tau(n)}\langle \overrightarrow{u_{\tau n}p}, \overrightarrow{w_{\tau(n)}p}\rangle. \end{aligned}$$

Using the fact that $\alpha_{\tau(n)} > 0$, we obtain

$$d^2(x_{\tau(n)},p) < 2\langle \overrightarrow{u_{\tau n}p}, \overrightarrow{w_{\tau(n)}p}\rangle.$$

Since $\limsup\limits_{n\to\infty}\langle \overrightarrow{u_{\tau n}p}, \overrightarrow{w_{\tau(n)}p}\rangle \leq 0$, then

$$\limsup_{n\to\infty} d^2(x_{\tau(n)},p) \leq 0.$$

Hence, $\lim\limits_{n\to\infty} d(x_{\tau(n)},p) = 0$. Since $\lim\limits_{n\to\infty} d(x_{\tau(n)+1},x_{\tau(n)}) = 0$, then

$$\lim_{n\to\infty} d(x_{\tau(n)},p) = \lim_{n\to\infty} d(x_{\tau(n)+1},p) = 0.$$

Therefore, by Lemma 2.17, we obtain $d(x_n,p) \leq d(x_{\tau(n)+1},p) \to 0$ as $n\to\infty$. Hence $x_n \to p$ as $n\to\infty$. □

Let X be complete CAT(0) space and let X^* be its dual space. Let $f : X \to (-\infty,\infty]$ be a proper lower semicontinuous and convex function with domain $\mathbb{D}(f) := \{x \in X : f(x) < +\infty\}$. Then the subdifferential of f is a set-valued function $\partial f : X \to 2^{X^*}$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz}\rangle, \ (z \in X)\}, & if\ x \in \mathbb{D}(f), \\ \emptyset & \text{otherwise} \end{cases}$$

It has been shown in [14] that

(1) ∂f is a monotone operator;
(2) ∂f satisfies the range condition. That is, $\mathbb{D}(J_\lambda^{\partial f}) = X$ for all $\lambda > 0$;
(3) f attains its minimum at $x \in X$ if and only if $0 \in \partial f(x)$.

Now, we consider the following Minimization Problem (MP), which consists of finding $x \in X$ such that

$$f(x) = \min_{y\in X} f(y). \tag{3.18}$$

From Theorem 3.1, we obtain the following result.

Corollary 3.1. *Let X be a complete CAT(0) space with dual X^* and let C be a nonempty closed and convex subset of X. Let $\{T_i\}_{i=1}^N : C \to X$ be a finite family of θ_i-generalized demimetric mappings and Δ-demiclosd at 0 with $\theta_i \in (0,\infty)$ for each $i \in \{1,2,\dots,N\}$. Let $f_i : X \to (-\infty,\infty]$ $(i = 1,2,\dots,N)$ be a finite family of proper, lower semicontinuous and convex functions. Assume that $\Upsilon := \bigcap_{i=1}^N F(T_i) \cap \left(\bigcap_{i=1}^N \partial f_i^{-1}(0)\right) \neq \emptyset$. Let $\{u_n\}$ be a sequence in X such that $u_n \to u \in X$.*

Assume $k \in (0,\gamma)$ with $\gamma \in (0,1)$ and $\theta_i k > 0$. For any $x_1 \in X$, let $\{x_n\}$ in X be a sequence generated by

$$\begin{cases} y_n = J_\lambda^{\partial f_N} \circ J_\lambda^{\partial f_{N-1}} \circ \cdots \circ J_\lambda^{\partial f_2} \circ J_\lambda^{\partial f_1} x_n, \\ z_n = (1-\gamma)y_n \oplus \gamma \left[\bigoplus_{i=1}^N \xi_i((1-k)\oplus kT_i)y_n\right], \\ x_{n+1} = \alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n, \end{cases} \tag{3.19}$$

where $\lambda \in (0,\infty)$, $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\beta_n\}$ and $\{\xi_i\}_{i=1}^N$ are sequences in $(0,1)$ satisfying the following conditions

- *(i)* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{i=1}^\infty \alpha_n = \infty$;
- *(ii)* $\alpha_n + \beta_n + \sigma_n = 1$.

Then $\{x_n\}$ converges strongly to $x^* \in \Upsilon$.

4. APPLICATIONS

In this section, using Theorem 3.1, we obtain new strong convergence theorems in complete CAT(0) space.

Definition 4.1. [41] Let C be a nonempty subset of a CAT(0) space X. A mapping $T : C \to X$ is called a strict pseudo-contraction if there exists a constant $0 \leq \delta < 1$ such that

$$d^2(Tx,Ty) \leq d^2(x,y) + 4\delta\left(\frac{1}{2}x \oplus \frac{1}{2}Ty, \frac{1}{2}Tx \oplus \frac{1}{2}y\right) \tag{4.1}$$

for all $x,y \in C$. If (4.1) holds, we also say that T is a δ-strict pseudo-contraction.

The definition of pseudo-contractions finds its origin in Hilbert spaces. Note that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings. That is, T is nonexpansive if and only if T is a 0-strict pseudo-contraction.

Lemma 4.1. [41] *Let C be a nonempty, closed and convex subset of a Hadamard space X and let $T : C \to X$ be δ-strict pseudocontraction. Define $T_\delta : C \to X$ by $T_\delta x = \delta x \oplus (1-\delta)Tx$. Then T_δ is nonexpansive mapping and $F(T) = F(T_\delta)$.*

Lemma 4.2. *Let C be a nonempty closed and convex subset of a Hadamard space X. Let $S : C \to C$ be a nonexpansive mapping and let $T : C \to C$ be a δ-strict pseudo-contraction such that $F(S)\cap F(T) \neq \emptyset$. Let $W_\alpha = ((1-\alpha)I \oplus \alpha T)Sx$, for any $x \in C$, where $0 < \alpha < \frac{1}{1+\delta}$ and $\delta \in (0,1)$. Then $F(W_\alpha) = F(S)\cap F(T)$. Furthermore, if $F(W_\alpha) \neq \emptyset$, then W_α is 2-generalized demimetric.*

Proof. First, we show that $F(W_\alpha) = F(S)\cap F(T)$. It is easy to prove that $F(S)\cap F(T) \subseteq F(W_\alpha)$. Next, we show that $F(W_\alpha) \subseteq F(S)\cap F(T)$ for any $x \in F(W_\alpha)$ and $y \in F(S)\cap F(T)$

$$\begin{aligned} d^2(x,y) &= d^2((1-\alpha)I \oplus \alpha T)Sx, y) \\ &= (1-\alpha)d^2(Sx,y) + \alpha d^2(TSx,y) - \alpha(1-\alpha)d^2(Sx,TSx) \\ &= d^2(Sx,y) - \alpha d^2(Sx,y) \\ &\quad + \alpha d^2(TSx,y) - \alpha(1-\alpha)d^2(Sx,TSx). \end{aligned} \tag{4.2}$$

Since T is δ-strictly pseudocontractive, we obtain

$$\begin{aligned} d^2(TSx,y) &\leq d^2(Sx,y)+4\delta d^2\left(\frac{1}{2}Sx\oplus\frac{1}{2}y,\frac{1}{2}TSx\oplus\frac{1}{2}y\right) \\ &\leq d^2(Sx,y)+\delta[d^2(Sx,y)+d^2(TSx,y)+d^2(Sx,TSx) \\ &\quad +d^2(y,y)-d^2(Sx,y)-d^2(y,TSx)] \\ &= d^2(Sx,y)+\delta d^2(Sx,y)+\delta d^2(TSx,y) \\ &\quad +\delta d^2(Sx,TSx)-\delta d^2(Sx,y)-\delta d^2(y,TSx) \\ &= d^2(Sx,y)+\delta d^2(Sx,TSx). \end{aligned} \tag{4.3}$$

By (4.2) and (4.3), we obtain

$$\begin{aligned} d(x,y) &\leq d^2(Sx,y)+\alpha\delta d^2(Sx,TSx)-\alpha(1-\alpha)d^2(Sx,TSx) \\ &\leq d^2(x,y)-\alpha(1-\alpha(1+\delta))d^2(Sx,TSx). \end{aligned}$$

Hence,

$$\alpha(1-\alpha(1+\delta))d^2(Sx,TSx)\leq 0.$$

By the virtue of $0<\alpha<\frac{1}{1+\delta}$, we have $\alpha(1-\alpha(1+\delta))>0$ and then

$$TSx=Sx. \tag{4.4}$$

By (4.4), we have

$$\begin{aligned} d(x,Sx) &= d(((1-\alpha)I\oplus\alpha T)Sx,Sx) \\ &\leq (1-\alpha)d(Sx,Sx)+\alpha d(TSx,Sx) \\ &\leq \alpha d(x,Sx), \end{aligned}$$

which implies that

$$(1-\alpha)d(x,Sx)\leq 0.$$

Since $\alpha\in(0,1)$, we have

$$d(x,Sx)=0, \quad \Longrightarrow \quad x=Sx. \tag{4.5}$$

Therefore, we obtain that $x\in F(S)$. From (4.4) and (4.5), we get $x=Sx=TSx=Tx$, so $x\in F(T)$. Hence $x\in F(S)\cap F(T)$. So, $F(W_\alpha)\subset F(S)\cap F(T)$ hold.

Next, we show that W_α is 2-generalized demimetric. Let $p\in F(W_\alpha)$. Then, $p\in F(S)\cap F(T)$. From Lemma 4.1, we obtain

$$\begin{aligned} d^2(W_\alpha x,p) &\leq d^2([(1-\alpha)I\oplus\alpha T]Sx,p) \\ &\leq d^2(Sx,p) \\ &\leq d^2(x,p). \end{aligned} \tag{4.6}$$

Then, it follows (2.1) and (4.6) that

$$2\langle\overrightarrow{xW_\alpha x},\overrightarrow{px}\rangle+d^2(x,p)+d^2(W_\alpha x,x)\leq d^2(x,p),$$

thus,

$$d^2(W_\alpha x,x)\leq 2\langle\overrightarrow{xW_\alpha x},\overrightarrow{xp}\rangle.$$

Hence, W_α is 2-generalized demimetric mapping. This complete the proof. □

Lemma 4.3. [19] *Let C be a closed and convex subset of a complete CAT(0) space X and let $T : C \to X$ be a nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = p$. Then $p = Tp$.*

With the help of Lemma 4.2, we obtain the following result.

Theorem 4.1. *Let X be a complete CAT(0) space with dual X^* and let C be a nonempty closed and convex subset of X. Let $S : C \to C$ be a nonexpansive mapping and let $T : C \to C$ be a δ-strict pseudo-contraction such that $F(S) \cap F(T) \neq \emptyset$. For $0 < \alpha < \frac{1}{1+\delta}$, let $W_\alpha := ((1-\alpha)I \oplus \alpha T)S$. Let $A_i : X \to 2^{X^*}$ $(i = 1, 2, ..., N)$ be multivalued monotone mappings, which satisfy the range condition. Assume that $\Upsilon := F(T) \cap F(S) \cap \left(\bigcap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$. Let $\{u_n\}$ be a sequence in X such that $u_n \to u \in X$. For any $x_1 \in X$, let $\{x_n\}$ in X be a sequence generated by*

$$\begin{cases} y_n = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 x_n, \\ z_n = (1-\gamma)y_n \oplus \gamma W_\alpha y_n, \\ x_{n+1} = \alpha_n u_n \oplus \beta_n x_n \oplus \sigma_n z_n, \end{cases} \tag{4.7}$$

where $\lambda \in (0, \infty)$, $\gamma \in (0, 1)$ and $\{\alpha_n\}$, and $\{\sigma_n\}$, $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{i=1}^\infty \alpha_n = \infty$;

(ii) $\alpha_n + \beta_n + \sigma_n = 1$.

Then $\{x_n\}$ converges strongly to $x^* \in \Upsilon$.

Proof. Since T is δ-strictly pseudo-contractive and S is nonexpansive. From Lemma 4.1, we have that $T_\alpha := (1-\alpha)I \oplus \alpha T$ is nonexpansive. From Lemma 4.3, $(1-\alpha)I \oplus \alpha T$ and S are Δ-demiclosed at zero. If $\lim_{n\to\infty} d(T_\alpha x_n, x_n) = 0 = \lim_{n\to\infty} d(Sx_n, x_n)$, then

$$\begin{aligned} d(W_\alpha x_n, x_n) &= d(((1-\alpha)I \oplus \alpha T)Sx_n, x_n) \\ &\leq d(T_\alpha Sx_n, T_\alpha x_n) + d(T_\alpha x_n, x_n) \\ &\leq d(Sx_n, x_n) + d(T_\alpha x_n, x_n). \end{aligned}$$

Hence $\lim_{n\to\infty} d(W_\alpha x_n, x_n) = 0$, Since $F(T) \cap F(S) \neq \emptyset$, we find from Lemma 4.2 that W_α is 2-generalized demimetric mapping. We obtain the desired Theorem 3.1. □

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