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A SHRINKING PROJECTION ALGORITHM FOR VARIATIONAL INEQUALITY PROBLEMS WITH A LIPSCHITZ MONOTONE MAPPING AND FIXED POINT PROBLEMS OF RELATIVELY WEAK NONEXPANSIVE MAPPINGS

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Abstract. In this paper, a shrinking projection algorithm is investigated for variational inequality problems with Lipschitz monotone mappings and fixed point problems of relatively weak nonexpansive mappings. Strong convergence of the algorithm is established in a 2-uniformly convex and uniformly smooth real Banach space. As an application, zero problems of a Lipschitz monotone mapping is presented.
Keywords. Monotone and Lipschitz mapping; Generalized projection; Variational inequality problem; Fixed point problem; Relatively weak nonexpansive mapping.

1. Introduction

Variational inequality problems, which are connected with convex minimization problems, zeros of monotone-type mappings, complementarity problems, fixed point problems and so on, find many important applications in the real world; see, e.g., [1, 2, 3, 4, 5] and the references therein. Let E be a real Banach space with dual space E^*. Let C be a nonempty, closed, and convex subset of E, and let $A : C \to E^*$ be a monotone mapping. Recall that the classical variational inequality problem, which consists of finding $u \in C$ such that

$$\left\langle y-u, Au \right\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

Various iterative methods for solutions of problem (1.1) has been extensively studied by many authors; see, e.g., [6, 7, 8, 9, 10, 11, 12] and the references therein. We denote the set of solutions of variational inequality problem (1.1) by $VI(C,A) = \{u \in C : \langle y-u, Au\rangle \geq 0, \forall y \in C\}$. Recall that a mapping $A : C \to E^*$ is said to be monotone if $\left\langle x-y, Ax-Ay \right\rangle \geq 0$, $\forall x,y \in C$. $A : C \to E^*$ is said to be α-inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\left\langle x-y, Ax-Ay \right\rangle \geq \alpha \|Ax-Ay\|^2, \quad \forall x,y \in C.$$

If A is an α-inverse-strongly monotone mapping, then it is monotone and Lipschitz with constant $\frac{1}{\alpha}$. In 2000, Antipin studied solutions of variational inequality problem (1.1) in Euclidean spaces. In the framework of Hilbert spaces, Takahashi and Toyoda [13], in 2003, obtained a weak convergence theorem on common elements of the fixed point set of a nonexpansive mapping and the solution set of a variational inequality problem with δ-inverse-strongly monotone mapping. Based on a modified Halpern-like algorithm, Iiduka and Takahashi [14], in 2006,

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proved a strong convergence theorem on common elements of the fixed point set of a nonexpansive mapping and the solution set of a variational inequality problem with a δ-inverse strongly monotone mapping. In 2007, Ceng and Yao [15] obtained a strong convergence theorem on common elements of the fixed point set of a relatively nonexpansive mapping and the solution set of a variational inequality problem with a monotone and k-Lipschitz mapping. In fact, they proved the following theorem.

Theorem 1.1 (Ceng and Yao [15]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $f : C \to C$ be a contractive mapping with a contractive constant $L \in (0,1)$. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping, and let $S : C \to C$ be a nonexpansive mapping such that $F(T) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 \in C, \\ y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(y_n) + \beta_n S P_C(x_n - \lambda A x_n), \end{cases} \tag{1.2}$$

where $\{\lambda_n\} \subset (0,1)$ with $\sum_{i=1}^{\infty} \lambda_n < \infty$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

(1) $\alpha_n + \beta_n \le 1$, $\forall\, n \ge 1$*;* (2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{i=1}^{\infty} \alpha_n = \infty$*;* (3) $0 < \liminf \beta_n \le \limsup \beta_n < 1$.

Then, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to the some point $q = P_{F(T)\cap VI(C,A)} f(q)$ if and only if $\{Ax_n\}$ is bounded and $\liminf \langle y - x_n, Ax_n \rangle \ge 0$, $\forall\, y \in C$.

To solve variational inequality problems with a k-Lipschitz and monotone mapping and fixed point problems of a nonexpansive mapping, Nadezhkina and Takahashi [16] proved the following strong convergence theorem.

Theorem 1.2 (Nadezhkina and Takahashi [16]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping, and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C,A) \neq \emptyset$. Define inductively the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ by*

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = (1-\alpha_n)x_n + \alpha_n S P_C(x_n - \lambda_n A x_n), \\ C_n = \{v \in C : \|z_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \ \forall\ n \ge 0, \end{cases} \tag{1.3}$$

where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,\frac{1}{k})$ and $\{\alpha_n\} \subset [0,c]$ for some $c \in [0,1)$. Then, the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ converge strongly to the some point $q = P_{F(S)\cap VI(C,A)} x_0$.

In 2010, Ceng and Yao [17] proved the following strong convergence theorem in a real Hilbert space using a hybrid extragradient-like approximation method.

Theorem 1.3 (Ceng amd Yao [17]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping, and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C,A) \neq \emptyset$. Define inductively the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ by*

$$\begin{cases} x_0 \in C, \\ y_n = (1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n Ax_n), \\ z_n = (1-\alpha_n-\beta_n)x_n + \alpha_n y_n + \beta_n SP_C(x_n - \lambda_n Ax_n), \\ C_n = \{v \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + (3-3\gamma_n+\alpha_n)b^2\|Ax_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \ \forall \ n \geq 0, \end{cases} \tag{1.4}$$

where $\{\lambda_n\} \subset [a,b]$ *with* $a > 0$ *and* $b < \frac{1}{2k}$, $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*
(1) $\alpha_n + \beta_n \leq 1$, $\forall\, n \geq 0$, (2) $\lim_{n\to\infty} \alpha_n = 0$, (3) $\liminf \beta_n > 0$, *and* (4) $\lim_{n\to\infty} \gamma_n = 1$ *and* $\gamma_n > \frac{3}{4}$, $\forall\, n \geq 0$. *Then, the sequences* $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *converge strongly to some point* $q = P_{F(T)\cap VI(C,A)}x_0$.

Motivated by the results of Ceng and Yao [17], Chidume *et al.* [6], in 2018, introduced a hybrid extragradient-like algorithm in a uniformly smooth and 2-uniformly convex real Banach space. They proved a strong convergence theorem of common element in the solution set of a variational inequality problem with a monotone and k-Lipschitz mapping and the fixed point set of a countable family of relatively nonexpansive mappings. In fact, they proved the following theorem.

Theorem 1.4 (Chidume *et al.* [6])**.** *Let* C *be a nonempty, closed, and convex subset of a uniformly smooth and 2-uniformly convex real Banach space* E *such that* $J(C)$ *is convex. Let* $A : C \to E^*$ *be a monotone and k-Lipschitz map. Let* $\{S_i\}_{i=1}^{\infty}$ *be a countable family of relatively nonexpansive maps such that* $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$, *where* $S_i : C \to C$, $\forall\, i$. *Let* $\{\eta_i\}_{i=1}^{\infty} \subset (0,1)$ *and* $\{\mu_i\}_{i=1}^{\infty} \subset (0,1)$ *be sequences such that* $\sum_{i=1}^{\infty} \eta_i = 1$. *Assume that* $\cap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset$. *Define inductively the sequence* $\{x_n\}$ *by*

$$\begin{cases} x_0 \in C := C_0, \\ y_n = \Pi_C J^{-1}(Jx_n - \gamma_n \lambda Ax_n), \\ z_n = J^{-1}((1-\alpha_n-\beta_n)Jx_n + \alpha_n Jy_n + \beta_n JS\Pi_C(J^{-1}(Jx_n - \lambda Ax_n)), \\ C_{n+1} = \{v \in C_n : \phi(z,z_n) \leq \phi(z,x_n) + (3-3\gamma_n)b^2\|Ax_n\|^2 + b\alpha_n\tau_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall \ n \geq 0, \end{cases} \tag{1.5}$$

where $Sx = J^{-1}\Big(\sum_{i=1}^{\infty} \eta_i(\mu_i Jx + (1-\mu_i)JS_i x)\Big)$, *for each* $x \in C$, $\lambda \in (0,b]$ *with* $b < \frac{\alpha}{2k}$, $\tau_n = \max\{\|x_n\|, 1\}\|Ax_n\|\sigma(\|\gamma_n \lambda Ax_n\|)$, *as well as* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*
(1) $\alpha_n + \beta_n \leq 1$, $\forall\, n \geq 0$,
(2) $\lim_{n\to\infty} \alpha_n = 0$,
(3) $\liminf_{n\to\infty} \beta_n > 0$,
(4) $\lim_{n\to\infty} \gamma_n = 1$, *and* $\gamma_n > 1 - \frac{\alpha}{4}$, $\forall\, n \geq 0$.
Then, the sequences $\{x_n\}$, $\{y_n\}$, *and* $\{z_n\}$ *are well defined and converge strongly to some point* $p = \Pi_{F(T)\cap VI(C,A)}x_0$.

It is our purpose in this paper to prove a strong convergence theorem of common solutions for a variational inequality problem with a monotone and k-Lipschitz mapping and a fixed point problem of a relatively weak nonexpansive mapping in a 2-uniformly convex and uniformly smooth real Banach space. Furthermore, we extend our theorem to a countable family of relatively weak nonexpansive mappings. Finally, an application of our theorem to zeros of a monotone and k-Lipschitz monotone mapping is presented.

2. Preliminaries

Recall that a mapping $A : E \to 2^{E^*}$ is said to be monotone if, for each $x,\ y \in E$, $\langle x-y, x^*-y^*\rangle \geq 0$, $\forall x^* \in Ax, y^* \in Ay$. Additionally, it is called maximally monotone if the graph of A, $G(A) = \{(x,y) : y \in Ax\}$, is not properly contained in that of any other monotone operator. It is well known that A is maximally monotone if and only if, for $(x,x^*) \in E \times E^*$, $\langle x-y, x^*-y^*\rangle \geq 0$, $\forall (y,y^*) \in G(A)$ implies that $x^* \in A$; see, e.g., Zegeye and Shahzad [18].

A mapping $J : E \to E^*$ defined by

$$J(x) := \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2,\ \|x^*\| = \|x\|,\ \forall\, x \in E\},$$

is called the normalized duality mapping on E, where $\langle \cdot,\cdot\rangle$ denotes the duality pairing between the elements of E and E^*. The following properties of the normalized duality map are needed in the sequel (see, e.g., Ibaraki and Takahashi [19]).

- If E is uniformly convex, then J is one-to-one and onto.
- If E is uniformly smooth, then J is single-valued.
- In particular, if a Banach space E is uniformly smooth and uniformly convex, the dual space is also uniformly smooth and uniformly convex. Hence, the normalized duality map J on E and the normalized duality map J_* on its dual space E^* are both uniformly continuous on bounded sets, and $J_* = J^{-1}$.

The modulus of convexity of a space E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1, \varepsilon = \|x-y\|\right\}.$$

The space E is uniformly convex if $\delta_E(\varepsilon) > 0$, for every $\varepsilon \in (0,2]$. If there exist a constant $c > 0$ and a real number $p > 1$ such that $\delta_E(\varepsilon) \geq c\varepsilon^p$, then E is said to be p-uniformly convex. Moreover, typical examples of such spaces are L_p, ℓ_p, and W_p^m (Sobolev spaces), for $1 < p < \infty$, where

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} p-\text{uniformly convex}, & \text{if } 2 \leq p < \infty; \\ 2-\text{uniformly convex}, & \text{if } 1 < p \leq 2. \end{cases}$$

Let E be a real normed space with dimension greater than 2. The modulus of smoothness of E, $\rho_E : [0,\infty) \to [0,\infty)$, is defined by

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau,\ \tau > 0\right\}.$$

The space E is said to be smooth if $\rho_E(\tau) > 0$, $\forall \tau > 0$. Furthermore, it is said to be uniformly smooth if $\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$.

In the sequel, we shall need the following definitions and results. Let E be a smooth real Banach space with dual space E^*. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,\ \forall\, x,y \in E, \tag{2.1}$$

where J is the normalized duality mapping from E into E^*, was introduced by Alber and has been studied by Alber [20], Kamimura and Takahashi [21], and a host of other authors. It will play a central role.

If $E = H$, a real Hilbert space, equation (2.1) reduces $\phi(x,y) = \|x-y\|^2$, $\forall, x, y \in H$. The following properties are known (see, e.g., Nilsrakoo and Saejung [22]): for all $x, y, z \in E$ and $\tau \in (0,1)$,

$$(A_1)\ \ (\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2, \tag{2.2}$$

$$(A_2)\ \ \phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle z-x, Jy - Jz\rangle, \tag{2.3}$$

$$(A_3)\ \ \phi(\tau x + (1-\tau)y, z) \leq \tau\phi(x,z) + (1-\tau)\phi(y,z). \tag{2.4}$$

If E is smooth and strictly convex (see, e.g., Honda *et al.* [23]), then

$$\phi(x,y) = 0 \text{ if and only if } x = y. \tag{2.5}$$

Define a mapping $V : E \times E^* \to \mathbb{R}$ by $V(x,x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2$. Then, it is easy to see that

$$V(x,x^*) = \phi(x, J^{-1}(x^*)),\ \forall\ x \in E,\ x^* \in E^*. \tag{2.6}$$

Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive real Banach space E. The generalized projection mapping introduced by Alber [24] is a mapping $\Pi_C : E \to C$ such that, for any $x \in E$, there exists a unique element $x_0 := \Pi_C(x) \in C$ such that $\phi(x_0, x) = \min_{y\in C} \phi(y,x)$. If $E = H$ is a real Hilbert space, we remark that the generalized projection Π_C coincides with the metric projection from H onto C.

Definition 2.1. Let C be a nonempty, closed, and convex subset of E, and let $T : C \to C$ be a mapping. A point $x^* \in C$ is called a *fixed point* of T if $T(x^*) = x^*$. The set of fixed points of T is denoted by $F(T)$. A point $p \in C$ is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$.

Definition 2.2. A mapping $T : C \to C$ is said to be relatively nonexpansive if the following conditions hold (see, e.g., Matsushita and Takahashi [25]):

(1) $F(T) \neq \emptyset$,
(2) $\phi(p, Tx) \leq \phi(p,x)$, $\forall\, x \in C$ and $p \in F(T)$,
(3) $\hat{F}(T) = F(T)$.

Definition 2.3. A point $p \in C$ is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}_{n=1}^{\infty}$, which converges strongly to p and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ (see, e.g., Reich [26] as well as Matsushita and Takahashi [25]). The set of strong asymptotic fixed points of T is denoted by $\tilde{F}(T)$.

Definition 2.4. A mapping $T : C \to C$ is said to be relatively weak nonexpansive if the following conditions hold (see, e.g., Zegeye and Shahzad [18]):

(1) $F(T) \neq \emptyset$,
(2) $\phi(p, Tx) \leq \phi(p,x)$, $\forall\, x \in C$ and $p \in F(T)$,
(3) $\tilde{F}(T) = F(T)$.

If E is strictly convex and reflexive real Banach space and $A : E \to E^*$ is a continuous monotone mapping with $A^{-1}(0) \neq \varnothing$, it is known that $J_r := (J + rA)^{-1}J$, for $r > 0$, is relatively weak nonexpansive (see, e.g., Kohasaka [27]). Clearly, every relatively nonexpansive mapping is relatively weak nonexpansive. Let $T : C \to C$ be a mapping. We have that $F(T) \subset \tilde{F}(T) \subset \hat{F}(T)$. It follows that, for any relatively nonexpansive map, $F(T) = \tilde{F}(T) = \hat{F}(T)$.

An example of a relatively weak nonexpansive mapping, which is not relatively nonexpansive was given in Zhang *et al.* [12]. Let E be a real Banach space with dual space E^*. A mapping $A : C \to E^*$ is said to be hemicontinuous if, for each $x, y \in C$, a mapping $F : [0,1] \to E^*$ defined by $F(t) := A(tx + (1-t)y)$ is continuous with respect to the weak topology of E^*. Let $N_C(v)$ denote the normal cone for C at a point $v \in C$, that is,

$$N_C(v) := \{w^* \in E^* : \langle v - z, w^* \rangle \geq 0, \ \forall\, z \in C\}.$$

The following lemmas are needed in the sequel.

Lemma 2.1 (Alber [24])**.** *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive real Banach space E. Then,*

$$\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \leq \phi(y, x), \quad \forall x \in E, y \in C.$$

Lemma 2.2 (Xu [28])**.** *Let E be a 2-uniformly convex and smooth real Banach space. Then, there exists a positive constant α such that $\alpha\|x - y\|^2 \leq \phi(x, y)$, $\forall\, x, y \in E$. Without loss of generality, we may assume that $\alpha \in (0,1)$.*

Lemma 2.3 (Xu [28])**.** *Let E be a 2-uniformly convex and smooth real Banach space. Then, there exists a positive constant c_2 such that $\forall x, y \in E$, $f_x \in J_2(x), f_y \in J_2(y)$, the following inequality holds: $\langle x - y, f_x - f_y \rangle \geq c_2\|x - y\|^2$.*

Lemma 2.4 (Nilsrakoo and Saejung [22])**.** *Let E be a smooth real Banach space. Then, $\phi(u, J^{-1}[\beta Jx + (1-\beta)Jy]) \leq \beta\phi(u, x) + (1-\beta)\phi(u, y)$, $\forall \beta \in [0,1]$, $u, x, y \in E$.*

Lemma 2.5 (Reich [26])**.** *Let E be a uniformly smooth real Banach space. Then, there exists a nondecreasing function $\sigma : [0,\infty) \to [0,\infty)$ satisfying the following conditions:*
(i) $\sigma(ct) \leq c\sigma(t)$, $c \geq 1$;
(ii) $\lim_{t \to 0^+} \sigma(t) = 0$;
(iii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + \max\{\|x\|, 1\}\|y\|\sigma(\|y\|)$, $\forall x, y \in E$.

Lemma 2.6 (Alber [20])**.** *Let E be a reflexive strictly, convex, and smooth Banach space with E^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*), \tag{2.7}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 2.7 (Alber [20])**.** *Let C be a nonempty, closed, and convex subset of a smooth real Banach space E. Then, for $x \in E$ and $x_0 \in C$, $x_0 := \Pi_C x$ if and only if*

$$\langle y - x_0, Jx_0 - Jx \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.8 (Zegeye and Shahzad [30])**.** *Let C be a nonempty, closed, and convex subset of a real reflexive, strictly convex, and smooth Banach space E. If $A : C \to E^*$ is a continuous monotone mapping, then $VI(C, A)$ is closed and convex.*

Lemma 2.9 (Chidume *et al.* [8]). *Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth real Banach space E, and let $T_i : C \to E$, $i = 1, 2, \cdots$, be a countable family of relatively weak nonexpansive maps. Assume that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and $\{\alpha_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$. Let the map $T : C \to E$ be defined by*

$$Tx = J^{-1}\Big(\sum_{i=1}^{\infty} \alpha_i J T_i x\Big)$$

for each $x \in C$. Then, T is relatively weak nonexpansive and $F(T) = \cap_{i=1}^{\infty} F(T_i)$.

3. Main results

We now prove the following theorem.

Theorem 3.1. *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let C be a nonempty, closed, and convex subset of E such that $J(C)$ is convex, where J is the normalized duality map. Let $A : C \to E^*$ be a monotone and k-Lipschitz mapping, and let $T : C \to C$ be a relatively weak nonexpansive mapping. Assume that $W := F(T) \cap VI(C,A) \neq \emptyset$. For arbitrary $x_0 \in C$, let the sequence $\{x_n\}_{n=0}^{\infty}$ be iteratively defined by*

$$\begin{cases} x_0 \in C := C_0, \\ y_n = \Pi_C J^{-1}(Jx_n - \gamma_n \lambda A x_n), \\ z_n = J^{-1}((1-\alpha_n-\beta_n)Jx_n + \alpha_n J y_n + \beta_n J T \Pi_C(J^{-1}(Jx_n - \lambda A y_n))), \\ C_{n+1} = \{v \in C_n : \phi(v,z_n) \leq \phi(v,x_n) + (3-3\gamma_n)b^2\|Ax_n\|^2 + b\alpha_n\tau_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall \ n \geq 0, \end{cases} \tag{3.1}$$

where Π_C denotes the generalized projection of E onto C, $\lambda \in (0,b]$ with $b < \frac{\alpha}{2k}$, and $\tau_n = \max\{\|x_n\|, 1\}\|Ax_n\|\sigma(\|\gamma_n\lambda A x_n\|)$ ($\sigma : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying the conditions in Lemma 2.5), as well as $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:
(1) $\alpha_n + \beta_n \leq 1$, $\forall\, n \geq 0$,
(2) $\lim_{n\to\infty} \alpha_n = 0$,
(3) $\liminf_{n\to\infty} \beta_n > 0$,
(4) $\lim_{n\to\infty} \gamma_n = 1$, *and* $\gamma_n > 1 - \frac{\alpha}{4}$, $\forall\, n \geq 0$.
Then, the sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $p = \Pi_{F(T)\cap VI(C,A)} x_0$.

Proof. We follow the ideas employed in Chidume, Chinwendu and Adamu [6]. The proof is divided into five steps.

Step 1. Prove $F(T) \cap VI(C,A) \subset C_n$, $\forall n \geq 0$.

Let us assume that $\{x_n\}_{n=1}^{\infty}$ is well defined. This clearly implies that $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are also well defined. Let $u \in F(T) \cap VI(C,A)$ be arbitrary and set $t_n = \Pi_C J^{-1}(Jx_n - \lambda A y_n)$. Using

Lemma 2.1 and (2.1), we obtain that

$$\begin{aligned}
&\phi(u,t_n)\\
&\leq \phi(u,J^{-1}(Jx_n-\lambda Ay_n))-\phi(t_n,J^{-1}(Jx_n-\lambda Ay_n))\\
&=\|u\|^2-2\langle u,Jx_n-\lambda Ay_n\rangle+\|J^{-1}(Jx_n-\lambda Ay_n)\|^2-\|t_n\|^2+2\langle t_n,Jx_n-\lambda Ay_n\rangle\\
&\quad -\|J^{-1}(Jx_n-\lambda Ay_n)\|^2\\
&=\|u\|^2-2\langle u,Jx_n\rangle+\|x_n\|^2-(\|t_n\|^2-2\langle t_n,Jx_n\rangle+\|x_n\|^2)+2\langle u-t_n,\lambda Ay_n\rangle\\
&=\phi(u,x_n)-\phi(t_n,x_n)+2\langle u-y_n,\lambda Ay_n\rangle+2\langle y_n-t_n,\lambda Ay_n\rangle.
\end{aligned}$$

Since $A: C\to E^*$ is Lipschitz monotone, then (2.3) gives that

$$\begin{aligned}
\phi(u,t_n) &\leq \phi(u,x_n)-\phi(t_n,x_n)+2\langle y_n-t_n,\lambda Ay_n\rangle\\
&= \phi(u,x_n)-\phi(t_n,y_n)-\phi(y_n,x_n)+2\langle t_n-y_n,Jx_n-\lambda Ay_n-Jy_n\rangle. \qquad (3.2)
\end{aligned}$$

The estimation of $\langle t_n-y_n,Jx_n-\lambda Ay_n-Jy_n\rangle$ yields

$$\begin{aligned}
&\langle t_n-y_n,Jx_n-\lambda Ay_n-Jy_n\rangle\\
&=\langle t_n-y_n,Jx_n-Jy_n-\lambda Ax_n\rangle+\langle t_n-y_n,\lambda(Ax_n-Ay_n)\rangle\\
&=\langle t_n-y_n,Jx_n-Jy_n-\lambda\gamma_nAx_n+\lambda\gamma_nAx_n-\lambda Ax_n\rangle+\langle t_n-y_n,\lambda(Ax_n-Ay_n)\rangle\\
&=\langle t_n-y_n,Jx_n-Jy_n-\lambda\gamma_nAx_n\rangle+\langle t_n-y_n,\lambda\gamma_nAx_n-\lambda Ax_n\rangle\\
&\quad+\langle t_n-y_n,\lambda(Ax_n-Ay_n)\rangle\\
&=\langle t_n-y_n,Jx_n-Jy_n-\lambda\gamma_nAx_n\rangle-(1-\gamma_n)\langle t_n-y_n,\lambda Ax_n\rangle\\
&\quad+\langle t_n-y_n,\lambda(Ax_n-Ay_n)\rangle.
\end{aligned}$$

By employing $y_n=\Pi_CJ^{-1}(Jx_n-\lambda Ax_n)$, Lemmas 2.2 and 2.7, $\gamma_n\in[0,1]$, $\lambda\leq b$, and the Lipschitz property of A, we see that

$$\begin{aligned}
&\langle t_n-y_n,Jx_n-\lambda Ay_n-Jy_n\rangle\\
&\leq-(1-\gamma_n)\langle t_n-y_n,\lambda Ax_n\rangle+\langle t_n-y_n,\lambda(Ax_n-Ay_n)\rangle\\
&\leq(1-\gamma_n)\|Ax_n\|(\|t_n-y_n\|+\|x_n-y_n\|)+(1-\gamma_n)\|t_n-y_n\|\|Ax_n\|\\
&\quad+\lambda k\|t_n-y_n\|\|x_n-y_n\|\\
&\leq\frac{1}{2}(1-\gamma_n)(2b^2\|Ax_n\|^2+\|t_n-y_n\|^2+\|x_n-y_n\|^2)\\
&\quad+\frac{1}{2}(1-\gamma_n)(b^2\|Ax_n\|^2+\|t_n-y_n\|^2)+\frac{1}{2}bk(\|t_n-y_n\|^2+\|x_n-y_n\|^2)\\
&\leq\frac{1}{2}(1-\gamma_n)(2b^2\|Ax_n\|^2+\frac{1}{\alpha}\phi(t_n,y_n)+\frac{1}{\alpha}\phi(y_n,x_n))\\
&\quad+\frac{1}{2}(1-\gamma_n)(b^2\|Ax_n\|^2+\frac{1}{\alpha}\phi(t_n,y_n))+\frac{1}{2}bk(\frac{1}{\alpha}\phi(y_n,x_n)+\frac{1}{\alpha}\phi(t_n,y_n))\\
&\leq\frac{1}{2}\Big(\frac{1-\gamma_n}{\alpha}+\frac{bk}{\alpha}\Big)\phi(y_n,x_n)\\
&\quad+\frac{1}{2}\Big(2\frac{(1-\gamma_n)}{\alpha}+\frac{bk}{\alpha}\Big)\phi(t_n,y_n)+\frac{3}{2}(1-\gamma_n)b^2\|Ax_n\|^2,
\end{aligned}$$

which together with (3.2) and the assumptions that $b < \frac{\alpha}{2k}$ and $\gamma_n > 1 - \frac{\alpha}{4}$ implies that

$$\begin{aligned}
\phi(u,t_n) &\leq \phi(u,x_n) - \phi(t_n,y_n) - \phi(y_n,x_n) + 2\langle t_n - y_n, Jx_n - \lambda Ay_n - Jy_n\rangle \\
&\leq \phi(u,x_n) - \phi(t_n,y_n) - \phi(y_n,x_n) + \Big(\frac{1-\gamma_n}{\alpha} + \frac{bk}{\alpha}\Big)\phi(y_n,x_n) \\
&+ \Big(2\frac{(1-\gamma_n)}{\alpha} + \frac{bk}{\alpha}\Big)\phi(t_n,y_n) + 3(1-\gamma_n)b^2\|Ax_n\|^2 \\
&\leq \phi(u,x_n) - \Big(1 - \frac{(1-\gamma_n)}{\alpha} - \frac{bk}{\alpha}\Big)\phi(y_n,x_n) \\
&- \Big(1 - \frac{2(1-\gamma_n)}{\alpha} - \frac{bk}{\alpha}\Big)\phi(t_n,x_n) + 3(1-\gamma_n)b^2\|Ax_n\|^2 \\
&\leq \phi(u,x_n) - \frac{1}{4}\phi(y_n,x_n) + 3(1-\gamma_n)b^2\|Ax_n\|^2 \qquad (3.3)\\
&\leq \phi(u,x_n) + 3(1-\gamma_n)b^2\|Ax_n\|^2. \qquad (3.4)
\end{aligned}$$

Moreover, by utilizing Lemmas 2.1 and 2.5, $\gamma_n \leq 1$, and $\lambda \leq b$, we have

$$\begin{aligned}
\phi(u,y_n) &\leq \phi(u, J^{-1}(Jx_n - \lambda\gamma_n Ax_n)) \\
&= \|u\|^2 - 2\langle u, Jx_n - \lambda\gamma_n Ax_n\rangle + \|J^{-1}(Jx_n - \lambda\gamma_n Ax_n)\|^2 \\
&\leq \|u\|^2 - 2\langle u, Jx_n - \lambda\gamma_n Ax_n\rangle + \|x_n\|^2 - 2\langle x_n, \lambda\gamma_n Ax_n\rangle \\
&+ \max\{\|x_n\|, 1\}\|\lambda\gamma_n Ax_n\|\sigma(\|\lambda\gamma_n Ax_n\|) \\
&= \|u\|^2 - 2\langle u, Jx_n\rangle + \|x_n\|^2 + 2\langle u - x_n, \lambda\gamma_n Ax_n\rangle + b\tau_n \\
&= \phi(u,x_n) + b\tau_n. \qquad (3.5)
\end{aligned}$$

Further, by applying Lemma 2.4, the fact that T is relatively weak nonexpansive map, as well as inequalities (3.4) and (3.5), we conclude that

$$\begin{aligned}
&\phi(u,z_n) \\
= \;& \phi(u, J^{-1}((1-\alpha_n-\beta_n)Jx_n + \alpha_n Jy_n + \beta_n JTt_n)) \\
\leq \;& (1-\alpha_n-\beta_n)\phi(u,x_n) + \alpha_n\phi(u,y_n) + \beta_n\phi(u,Tt_n) \\
\leq \;& (1-\alpha_n-\beta_n)\phi(u,x_n) + \alpha_n(\phi(u,x_n) + b\tau_n) + \beta_n(\phi(u,x_n) + 3(1-\gamma_n)b^2\|Ax_n\|^2) \\
= \;& \phi(u,x_n) + 3(1-\gamma_n)b^2\|Ax_n\|^2 + b\alpha_n\tau_n. \qquad (3.6)
\end{aligned}$$

Therefore, $u \in C_{n+1}$, which follows that $F(T) \cap VI(C,A) \subset C_n$, $\forall n \geq 0$.

Step 2. Prove that $\{x_n\}_{n=0}^{\infty}$ is a well defined.

This can be obtained by using the fact that C_{n+1} is a closed and convex subset of C.

Step 3. Prove that $x_n \to p \in C$ as $n \to \infty$.

Let $u \in C_n$, for all $n \geq 0$. From $x_n = \Pi_{C_n}x_0$ and Lemma 2.1, we have that

$$\phi(x_n,x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(u,x_0),$$

which yields that $\{\phi(x_n,x_0)\}_{n=0}^{\infty}$ is bounded. It follows from inequality (2.2) that sequence $\{x_n\}_{n=0}^{\infty}$ is bounded. Furthermore, for each $n \in \mathbb{N}$, since $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we obtain the following inequality by utilizing lemma 2.1 again

$$\phi(x_n,x_0) \leq \phi(x_{n+1},x_n) + \phi(x_n,x_0) \leq \phi(x_{n+1},x_0). \tag{3.7}$$

Thus, $\{\phi(x_n,x_0)\}_{n=0}^{\infty}$ converges. Now, for arbitrary $n,m \in \mathbb{N}$ with $m > n$, we have $x_m = \Pi_{C_m}x_0 \in C_m \subset C_n$. The utilization of Lemma 2.1 gives that

$$\phi(x_m,x_n) = \phi(x_m,\Pi_{C_n}x_0) \leq \phi(x_m,x_0) - \phi(x_n,x_0) \to 0 \text{ as } n,m \to \infty. \tag{3.8}$$

From a result of Kamimura and Takahashi [21], we arrive at

$$\|x_m - x_n\| \to 0 \text{ as } n,m \to \infty. \tag{3.9}$$

So, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in C. Since C is closed, it follows that there exists $p \in C$ such that

$$x_n \to p \text{ as } n \to \infty. \tag{3.10}$$

Step 4. Prove $p \in F(T)$.

In view of $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we obtain from inequalities (3.4) and (3.6) that

$$\phi(x_{n+1},t_n) \leq \phi(x_{n+1},x_n) + 3(1-\gamma_n)b^2\|Ax_n\|^2$$

and

$$\phi(x_{n+1},z_n) \leq \phi(x_{n+1},x_n) + 3(1-\gamma_n)b^2\|Ax_n\|^2 + b\alpha_n\tau_n.$$

Using the facts that $\gamma_n \to 1$ and $\alpha_n \to 0$ as well as $\{x_n\}$, $\{Ax_n\}$, $\{\gamma_n\}$, and $\{\alpha_n\}$ are bounded, we obtain that $\lim_{n\to 0}\phi(x_{n+1},t_n) = 0$ and $\lim_{n\to 0}\phi(x_{n+1},z_n) = 0$. From a result of Kamimura and Takahashi [21], we obtain that

$$\|x_{n+1} - t_n\| \to 0 \text{ and } \|x_{n+1} - z_n\| \to 0 \text{ as } n \to \infty. \tag{3.11}$$

The utilization of conditions (3.9) and (3.11) gives that

$$\|t_n - x_n\| \leq \|t_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty \tag{3.12}$$

and

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.13}$$

Thus,

$$t_n \to p \text{ as } n \to \infty. \tag{3.14}$$

Since J is uniformly continuous on bounded sets, we see from conditions (3.10) and (3.14) that

$$\phi(u,x_n) - \phi(u,t_n) = 2\langle u, Jt_n - Jx_n\rangle + \|x_n\|^2 - \|t_n\|^2 \to 0 \text{ as } n \to \infty. \tag{3.15}$$

Inequality (3.3) gives that

$$\frac{1}{4}\phi(y_n,x_n) \quad \leq \quad \phi(u,x_n) - \phi(u,t_n) + 3(1-\gamma_n)b^2\|Ax_n\|^2. \tag{3.16}$$

Using the fact that $\gamma_n \to 1$ and $\{Ax_n\}$ is bounded, it follows that $\lim_{n\to\infty}\phi(y_n,x_n) = 0$. Consequently,

$$\|y_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.17}$$

By utilizing conditions (3.12) and (3.17), we have

$$\|t_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.18}$$

From recursion formula (3.1), we have $z_n = J^{-1}((1-\alpha_n-\beta_n)Jx_n + \alpha_n Jy_n + \beta_n JTt_n)$, and therefore,

$$\beta_n\|JTt_n - Jx_n\| \le \|Jx_n - Jz_n\| + \alpha_n\|Jy_n - Jx_n\| \to 0 \text{ as } n \to \infty. \tag{3.19}$$

By employing the fact that J and J^{-1} are uniformly continuous on bounded subsets of E and E^*, respectively, $\liminf \beta_n > 0$, conditions (3.13) and (3.17), we have that

$$\|Tt_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.20}$$

It follows from conditions (3.12) and (3.20) that

$$\|Tt_n - t_n\| \le \|Tt_n - x_n\| + \|x_n - t_n\| \to 0 \text{ as } n \to \infty. \tag{3.21}$$

Since T is relatively weak nonexpansive map, it follows from conditions (3.14) and (3.21) that $p \in F(T)$.

Step 5. Prove $x_n \to p \in VI(C,A)$.

Let $S \subset E \times 2^{E^*}$ be a mapping defined by

$$Sv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

From a result of Rockafellar [29], we have that S is maximal monotone and $S^{-1}0 = VI(C,A)$. Let $(v,w) \in G(S)$. Therefore, $w \in Sv = Av + N_C(v)$. So, we obtain that $w - Av \in N_C(v)$. Since $t_n = \Pi_C J^{-1}(Jx_n - \lambda Ay_n) \in C$, we have that $\langle v - t_n, w - Av\rangle \ge 0$. Moreover, by applying Lemma 2.7 and the recursion formula again, it is easy to see that

$$\langle v - t_n, Jt_n - (Jx_n - \lambda Ay_n)\rangle \ge 0, \tag{3.22}$$

and thus $\langle v - t_n, \frac{Jx_n - Jt_n}{\lambda} - Ay_n\rangle \le 0$. Now,

$$\begin{aligned}
\langle v - t_n, w\rangle &\ge \langle v - t_n, Av\rangle \\
&\ge \langle v - t_n, Av\rangle + \left\langle v - t_n, \frac{Jx_n - Jt_n}{\lambda} - Ay_n\right\rangle \\
&\ge \langle v - t_n, Av - At_n\rangle + \langle v - t_n, At_n - Ay_n\rangle + \left\langle v - t_n, \frac{Jx_n - Jt_n}{\lambda}\right\rangle \\
&\ge -\|v - t_n\|\|At_n - Ay_n\| - \|v - t_n\|\frac{\|Jt_n - Jx_n\|}{\lambda} \\
&\ge -M\left(k\|t_n - y_n\| + \frac{\|Jt_n - Jx_n\|}{\lambda}\right),
\end{aligned}$$

where $M = \sup\{\|v - t_n\| : n \ge 0\}$. It follows from conditions (3.12), (3.14), and (3.18) that $\langle v - p, w\rangle \ge 0$. It is well known that S is maximally monotone if and only if for $(x,x^*) \in E \times E^*$, $\langle x - y, x^* - y^*\rangle \ge 0\ \forall (y,y^*) \in G(S)$ implies that $x^* \in Sx$ (see, e.g., Zegeye and Shahzad [18]). Since S is maximally monotone, we obtain that $p \in S^{-1}0 = VI(C,A)$.

Step 6. Prove $p = \Pi_{F(T)\cap VI(C,A)}x_0$.

Set $q = \Pi_{F(T)\cap VI(C,A)}x_0$ and $x_n = \Pi_{C_n}x_0$. Since $F(T)\cap VI(C,A) \subset C_n$, we have $\phi(x_n,x_0) \le \phi(q,x_0)$. By applying the fact that norm is lower semi-continuous and $p \in F(T)\cap VI(C,A) \subset C_n$, for all $n \ge 0$, we have that

$$\begin{aligned}
\phi(q,x_0) \le \phi(p,x_0) &= \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 \\
&\le \liminf\left(\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2\right) \\
&= \liminf \phi(x_n,x_0) \le \limsup \phi(x_n,x_0) \le \phi(q,x_0).
\end{aligned}$$

This yields that $\phi(q,x_0) \le \phi(p,x_0) \le \phi(q,x_0)$, and thus $\phi(q,x_0) = \phi(p,x_0)$. We obtain from Lemma 2.1 that $\phi(p,q) \le \phi(p,x_0) - \phi(q,x_0) = 0$. It follows that $p = q = \Pi_{F(T)\cap VI(C,A)}x_0$. □

Next, we give a strong convergence theorem for a countable family of relatively weak nonexpansive mapings.

Theorem 3.2. *Let E be a uniformly smooth and 2-uniformly convex real Banach space with dual space E^*. Let C be a nonempty, closed, and convex subset of E such that $J(C)$ is convex, where J is the normalized duality map. Let $A : C \to E^*$ be a monotone and k-Lipschitz map. Let $T_i : C \to C$, $i = 1,2,\cdots$, be a countable family of relatively weak nonexpansive maps. Assume that $W := \cap_{i=1}^{\infty}F(T_i)\cap VI(C,A) \ne \emptyset$. For arbitrary $x_0 \in C$, let the sequence $\{x_n\}_{n=0}^{\infty}$ be iteratively defined by*

$$\begin{cases}
x_0 \in C := C_0,\\
y_n = \Pi_C J^{-1}(Jx_n - \gamma_n\lambda Ax_n),\\
z_n = J^{-1}((1-\alpha_n-\beta_n)Jx_n + \alpha_n Jy_n + \beta_n JT\Pi_C(J^{-1}(Jx_n - \lambda Ay_n))),\\
C_{n+1} = \{v \in C_n : \phi(v,z_n) \le \phi(v,x_n) + (3-3\gamma_n)b^2\|Ax_n\|^2 + b\alpha_n\tau_n\},\\
x_{n+1} = \Pi_{C_{n+1}}x_0, \ \forall \ n \ge 0,
\end{cases}$$

where the map $T : C \to C$ is defined by $Tx = J^{-1}\Big(\sum_{i=1}^{\infty}\delta_i JT_i x\Big)$, for each $x \in C$, $\{\delta_i\}_{i=1}^{\infty}$ is a sequence in $(0,1)$ such that $\sum_{i=1}^{\infty}\delta_i = 1$, Π_C is the generalized projection of E onto C, $\lambda \in (0,b]$ with $b < \frac{\alpha}{2k}$, $\tau_n = \max\{\|x_n\|,1\}\|Ax_n\|\sigma(\|\gamma_n\lambda Ax_n\|)$ ($\sigma : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying the conditions in Lemma 2.5), as well as $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:
(1) $\alpha_n + \beta_n \le 1$, $\forall\, n \ge 0$,
(2) $\lim_{n\to\infty}\alpha_n = 0$,
(3) $\liminf_{n\to\infty}\beta_n > 0$,
(4) $\lim_{n\to\infty}\gamma_n = 1$, *and* $\gamma_n > 1 - \frac{\alpha}{4}$, $\forall\, n \ge 0$.
Then, $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $p = \Pi_{F(T)\cap VI(C,A)}x_0$.

Proof. We observe from Lemma 2.9 that $T : C \to C$ is relatively weak nonexpansive and $F(T) = \cap_{i=1}^{\infty}F(T_i)$. It follows by Theorem 3.1 that the sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $p = \Pi_{F(T)\cap VI(C,A)}x_0$. This completes the proof. □

4. APPLICATIONS

Theorem 4.1. *Let $E = L_p$, ℓ_p, or W_m^p, where $1 < p \le 2$. Let C be a nonempty, closed, and convex subset of E such that $J(C)$ is convex, where J is the normalized duality map. Let $A : C \to E^*$ be a monotone and k-Lipschitz map. Let $T_i : C \to C$, $i = 1,2,\cdots$, be a countable family of relatively* *weak nonexpansive maps. Assume that* $W := \cap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset$*. For arbitrary* $x_0 \in C$*, let the sequence* $\{x_n\}_{n=0}^{\infty}$ *be iteratively defined by*

$$\begin{cases} x_0 \in C := C_0, \\ y_n = \Pi_C J^{-1}(Jx_n - \gamma_n \lambda A x_n), \\ z_n = J^{-1}((1-\alpha_n-\beta_n)Jx_n + \alpha_n J y_n + \beta_n JT\Pi_C(J^{-1}(Jx_n - \lambda A y_n))), \\ C_{n+1} = \{v \in C_n : \phi(v,z_n) \leq \phi(v,x_n) + (3-3\gamma_n)b^2\|Ax_n\|^2 + b\alpha_n\tau_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall \ n \geq 0, \end{cases}$$

where the map $T : C \to C$ *is defined by* $Tx = J^{-1}\Big(\sum_{i=1}^{\infty} \delta_i J T_i x\Big)$*, for each* $x \in C$*,* $\{\delta_i\}_{i=1}^{\infty}$ *is a sequence in* $(0,1)$ *such that* $\sum_{i=1}^{\infty} \delta_i = 1$*,* Π_C *is the generalized projection of* E *onto* C*,* $\lambda \in (0,b]$ *with* $b < \frac{\alpha}{2k}$*,* $\tau_n = \max\{\|x_n\|,1\}\|Ax_n\|\sigma(\|\gamma_n\lambda A x_n\|)$ *(*$\sigma : [0,\infty) \to [0,\infty)$ *is a nondecreasing function satisfying the conditions in Lemma 2.5), as well as* $\{\alpha_n\}$*,* $\{\beta_n\}$*, and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*
(1) $\alpha_n + \beta_n \leq 1$, $\forall\, n \geq 0$,
(2) $\lim_{n\to\infty} \alpha_n = 0$,
(3) $\liminf_{n\to\infty} \beta_n > 0$,
(4) $\lim_{n\to\infty} \gamma_n = 1$, *and* $\gamma_n > 1 - \frac{\alpha}{4}$, $\forall\, n \geq 0$.
Then, $\{x_n\}_{n=1}^{\infty}$*,* $\{y_n\}_{n=1}^{\infty}$*, and* $\{z_n\}_{n=1}^{\infty}$ *converge strongly to* $p = \Pi_{F(T)\cap VI(C,A)} x_0$*.*

Proof. We observe that E is 2-uniformly convex and uniformly smooth. It follows by utilizing Theorem 3.2 that the sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $p = \Pi_{F(T)\cap VI(C,A)} x_0$. □

Next, we consider the problem of finding a zero of a k-Lipschitz monotone mapping.

Theorem 4.2. *Let* E *be a uniformly smooth and* 2*-uniformly convex real Banach space with dual space* E^* *such that* $J(E)$ *is convex, where* J *is the normalized duality mapping. Let* $A : E \to E^*$ *be a monotone and* k*-Lipschitz mapping. Let* $T_i : E \to E$*,* $i = 1,2,\cdots$*, be a countable family of relatively weak nonexpansive maps. Assume that* $W := \cap_{i=1}^{\infty} F(T_i) \cap A^{-1}0 \neq \emptyset$*, where* $A^{-1}0 = \{u \in E : Au = 0\} \neq \emptyset$*. For arbitrary* $x_0 \in E$*, let the sequence* $\{x_n\}_{n=0}^{\infty}$ *be iteratively defined by*

$$\begin{cases} x_0 \in E := C_0, \\ y_n = J^{-1}(Jx_n - \gamma_n \lambda A x_n), \\ z_n = J^{-1}((1-\alpha_n-\beta_n)Jx_n + \alpha_n J y_n + \beta_n JT(J^{-1}(Jx_n - \lambda A y_n))), \\ C_{n+1} = \{v \in C_n : \phi(v,z_n) \leq \phi(v,x_n) + (3-3\gamma_n)b^2\|Ax_n\|^2 + b\alpha_n\tau_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \ \forall \ n \geq 0, \end{cases}$$

where the map $T : C \to C$ *is defined by* $Tx = J^{-1}\Big(\sum_{i=1}^{\infty} \delta_i J T_i x\Big)$*, for each* $x \in E$*,* $\{\delta_i\}_{i=1}^{\infty}$ *is a sequence in* $(0,1)$ *such that* $\sum_{i=1}^{\infty} \delta_i = 1$*,* $\lambda \in (0,b]$ *with* $b < \frac{\alpha}{2k}$*,* $\tau_n = \max\{\|x_n\|,1\}\|Ax_n\|\sigma(\|\gamma_n\lambda A x_n\|)$ *(*$\sigma : [0,\infty) \to [0,\infty)$ *is a nondecreasing function satisfying the conditions in Lemma 2.5), as well as* $\{\alpha_n\}$*,* $\{\beta_n\}$*, and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying the following conditions:*
(1) $\alpha_n + \beta_n \leq 1$, $\forall\, n \geq 0$,
(2) $\lim_{n\to\infty} \alpha_n = 0$,
(3) $\liminf_{n\to\infty} \beta_n > 0$,
(4) $\lim_{n\to\infty} \gamma_n = 1$, *and* $\gamma_n > 1 - \frac{\alpha}{4}$, $\forall\, n \geq 0$.
Then, $\{x_n\}_{n=1}^{\infty}$*,* $\{y_n\}_{n=1}^{\infty}$*, and* $\{z_n\}_{n=1}^{\infty}$ *converge strongly to* $p = \Pi_{F(T)\cap A^{-1}0} x_0$*.*

Proof. Setting $C_0 = E$ and $\Pi_E = I$ in Theorem 3.2, we observe that $VI(E,A) = A^{-1}0$. It follows from Theorem 3.2 that $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ converge strongly to $p \in W := F(T) \cap A^{-1}0$. □

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