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CONVERGENCE THEOREMS OF COMMON SOLUTIONS OF VARIATIONAL INEQUALITY AND f-FIXED POINT PROBLEMS IN BANACH SPACES

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Abstract. The purpose of this paper is to introduce the concept of a Bregman relatively f-nonexpansive mappings and study common solutions of the variational inequality problem involving Lipschitz monotone mappings and the fixed point problem of Bregman relatively f-nonexpansive mappings. Under some mild conditions, we obtained strong convergence theorems in real reflexive Banach spaces.

Keywords. Bregman relatively f-nonexpansive; f-fixed point; Monotone mapping; Variational inequality; Zero point.

1. INTRODUCTION

Let E be a real Banach space with its dual E^*. Let C be a nonempty subset of E. A mapping $A : C \to E^*$ is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

$A : C \to E^*$ is said to be *maximal monotone* if its graph $Gph(A)$, $Gph(A) = \{(x,Ax) \in E \times E^* : x \in C\}$, is not properly contained in the graph of any other monotone mapping. $A : C \to E^*$ is said to be α*-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha ||Ax - Ay||^2, \quad \forall x, y \in C.$$

It is known that α-inverse strongly monotone mappings are monotone and Lipschitz continuous. Let $T : C \to E$ be a mapping. The set of fixed points of T is defined by $F(T) := \{x \in C : Tx = x\}$ in this paper. Recall that T is said to be *contractive* and if there exists a constant $L \in (0,1)$ such that $||Tx - Ty|| \leq L||x - y||$, $\forall x, y \in C$. Furthermore, T is said to be *nonexpansive* if $L = 1$, that is, $||Tx - Ty|| \leq ||x - y||$, $\forall x, y \in C$. T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $||Tx - Tp|| \leq ||x - p||$ for all $x \in C$ and $p \in F(T)$. Recall that the mapping $I - T$, where I denotes the identity mapping, is said to be *demiclosed* at 0 if for any sequence $\{x_n\}$ in C converging weakly to x and $||(I - T)x_n|| \to 0$, then $(I - T)x = 0$.

Let C be nonempty, closed and convex subset of space E and let A be a monotone mapping. The variational inequality problem (VIP) is formulated as finding a point

$$x^* \in C \text{ such that } \langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

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We denote the solution set of (1.1) by $VI(C,A)$. This problem was first introduced by Hartman and Stampacchia [1] in 1966 and has been studied by several authors; see, e.g., [2, 3, 4, 5, 6, 7, 8] and there references therein.

In 2006, Nadezhkina and Takahashi [2] studied the following algorithm in a finite dimension Hilbert space setting. For any $x_0 \in C$, $\{x_n\}$ is defined by

$$\begin{cases} y_n = P_C[x_n - \gamma_n A x_n]; \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) P_C[x_n - \gamma_n A y_n], \ n \geq 0, \end{cases} \tag{1.2}$$

where A is L-Lipschitz continuous monotone mapping and P_C is the metric projection from H onto C. They proved that the sequence generated by (1.2) converges weakly to a solution of Problem (1.1) under certain conditions imposed on the control sequences $\{\gamma_n\}$ and $\{\alpha_n\}$.

In 2017, Thong and Hieu [3] introduced the following algorithm in a Hilbert space setting. For arbitrary $x_0 \in H$, $\{x_n\}$ is defined by

$$\begin{cases} y_n = P_C[x_n - \gamma_n A x_n]; \\ z_n = y_n - \gamma_n (A y_n - A x_n); \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) z_n, \ n \geq 0, \end{cases} \tag{1.3}$$

where A is L-Lipschitz continuous on H and $f : H \to H$ is a contractive mapping. They proved that the sequence generated by (1.3) converges strongly to $x^* = P_{VI(C,A)} f(x^*)$ provided that the control sequences $\{\gamma_n\}$ and $\{\alpha_n\}$ satisfy appropriate conditions.

Recently, several authors studied various algorithms for finding a common element of the set of fixed points of nonexpansive mapping and the set of solutions of variational inequality problems with Lipschitz monotone mappings; see, e.g., [2, 4, 9, 10, 11] and the references therein.

In 2003, Takahashi and Toyoda [9] introduced the following scheme for finding a point in $VI(C,A) \cap F(T)$ in a Hilbert space setting. For arbitrary $x_0 \in C$, $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T P_C(x_n - \gamma_n A x_n), \ n \geq 0, \tag{1.4}$$

where $A : C \to H$ is α-strongly monotone mapping, $T : C \to H$ is nonexpansive, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\gamma_n\}$ is a sequence in $(0, 2\alpha)$. They proved that if $VI(C,A) \cap F(T)$ nonempty, then the sequence generated by (1.4) converges weakly to some $x^* \in VI(C,A) \cap F(T)$.

In 2005, Iiduka and Takahashi [4] investigated the following algorithm. For arbitrary $x_0, x \in C$, $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x + (1-\alpha_n) T P_C(x_n - \gamma_n A x_n), \ n \geq 0, \tag{1.5}$$

where $A : C \to H$ is an α-strongly monotone mapping and $T : C \to H$ is a nonexpansive mapping. They proved that the sequence generated by (1.5) converges strongly to $x^* = P_{VI(C,A) \cap F(T)} x$ provided that the control sequences $\{\gamma_n\}$ and $\{\alpha_n\}$ satisfy appropriate conditions.

Recently, Zhang and Yuan [12] investigated an iterative process for approximating a common point of fixed points of nonexpansive mapping and solutions of the variational inequality problems with a finite family of α-inverse strongly monotone mappings in the setting of Hilbert spaces. For arbitrary $x_1, \in C$, $\{x_n\}$ is defined by

$$\begin{cases} y_{n,i} = P_C[x_n - \gamma_i A x_n]; \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T \sum_{i=1}^{n} \eta_i y_{n,i}, \ n \geq 1, \end{cases} \tag{1.6}$$

where $||y_{n,i} - P_C(x_n - \gamma_n A x_n)|| \leq e_{n,i}$, and $\lim_{n\to\infty} e_{n,i} = 0$, $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = \sum_{i=1}^{n} \eta_i = 1$, $\{\gamma_i\}$ is a real sequence in $(0, 2\mu_i)$, $f: C \to C$ is fixed α-contractive mapping, $T: C \to C$ is nonexpansive and $A_i: C \to H$ is μ_i-inverse strongly monotone mapping for $1 \leq i \leq n$. They proved that the sequence generated by (1.6) converges strongly to $x^* \in \cap_{i=1}^{n}(VI, A_i) \cap F(T)$ provided that the control sequences satisfying certain conditions.

Let E be a real Banach space with its dual E^*. For every $1 < p < \infty$, the *generalized duality mapping* $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{y^* \in E^* : \langle x, y^* \rangle = ||x||^p, ||y^*|| = ||x||^{p-1}\}, \tag{1.7}$$

for all $x \in E$. If $J = J_2$, then J_2 is called the *normalized duality mapping*. If $E = H$, a real Hilbert space, then $J = I$, where I is the identity mapping on E.

Beyond Hilbert spaces, Tufa and Zegeye [13] investigated an algorithm for finding a common point of the set of fixed points of a relatively nonexpansive mapping $T : C \to E$ and the set of solutions of variational inequality problems with Lipschitz monotone mappings $A_1, A_2 : C \to E^*$ in 2-uniformly convex and uniformly smooth real Banach spaces. They proved that the sequence generated by the following algorithm

$$\begin{cases} z_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_2 x_n]; \\ y_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_1 x_n]; \\ w_n = a_n J x_n + b_n J T x_n + c_n J(u_n) + d_n J(v_n); \\ x_{n+1} = \Pi_C J^{-1}[\alpha_n J u + (1 - \alpha_n) w_n], \ n \geq 0, \end{cases} \tag{1.8}$$

where J is the normalized duality mapping, $u_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_1 y_n]$, $v_n = \Pi_C J^{-1}[Jx_n - \gamma_n A_2 z_n]$, $\{\gamma_n\} \subset [a,b] \subset (0, \frac{1}{\mu L})$, for $L = \max\{L_1, L_2\}$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \subset [e, 1) \subset (0,1)$ such that $a_n + b_n + c_n + d_n = 1$, converges strongly to $x^* \in VI(C, A_1) \cap VI(C, A_2) \cap F(T)$.

This now leads to the following important question.

Question. Can we obtain a strong convergence result for approximating a common element of the set of fixed points of a mapping, which is more general that the Bregman relatively nonexpansive mapping and the set of solutions of variational inequality problem with Lipschitz monotone mappings in spaces, which is beyond 2-uniformly convex and uniformly smooth real Banach spaces?

Motivated and inspired by the above results, it is our purpose in this paper to introduce the concept of a *Bregman relatively f-nonexpansive mapping* and construct an algorithm for approximating a common element of the set of *f-fixed points* of Bregman relatively f-nonexpansive mappings and the set of solutions of variational inequality problems with Lipschitz monotone mappings in the setting of real reflexive Banach spaces. Our results provide an affirmative answer to the question above. Our results improve, extend, and generalize many corresponding results in the literature.

2. Preliminaries

In this section, we recall some known and useful results which will be used in the sequel.

Let E be a real Banach space. E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$, whenever $x, y \in S_E = \{x \in E : ||x|| = 1\}$ and $x \neq y$. The norm of E is said to be *Gâteaux differentiable or smooth* if

$$\lim_{t\to 0} \frac{||x+ty|| - ||x||}{t},$$

exists for each $x, y \in S_E$. For the rest of this section, let E be a real reflexive Banach space with its dual E^*, and let $f : E \to (-\infty, +\infty]$ be a proper, convex and lower-semicontinuous function. We denote the domain of f by $dom\ f = \{x \in E : f(x) < \infty\}$. For any $x \in int\ dom\ f$ and any $y \in E$, we denote by $f'(x, y)$ the *right-hand derivative* of f at x in the direction of y, that is,

$$f'(x,y) = \lim_{t\to 0+} \frac{f(x+ty) - f(x)}{t}.$$

The function f is said to be *Gâteaux differentiable* at x if

$$\lim_{t\to 0} \frac{f(x+ty) - f(x)}{t}, \tag{2.1}$$

exists for any $y \in E$. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle y, \nabla f(x)\rangle := f'(x, y)$ for all $y \in E$. The function f is said to be *Gâteaux differentiable* if it is a Gâteaux differentiable at every point $x \in int\ dom f$. When the subdifferential of f is single-valued, it coincides with the gradient of f, ∂f, that is, $\partial f = \nabla f$ (see, e.g., [14]). Furthermore, f is said to be *uniformly Fréchet differentiable* on a subset C of E, if the limit in (2.1) is attained uniformly for $x \in C$ and $||y|| = 1$ and f is called *strongly coercive* if $\lim_{||x||\to\infty} \frac{f(x)}{||x||} = \infty$.

The *Fenchel conjugate* of f is a function $f^* : E^* \to (-\infty, +\infty]$, defined by $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$ for $x^* \in E^*$. The function f is said to be *Legendre* if it satisfies the following conditions:

(i) the interior of the domain of f, *int dom* f, is nonempty, f is Gâteaux differentiable and $dom\ \nabla f = int\ dom\ f$;

(ii) the interior of the domain of f^*, *int dom* f^*, is nonempty, f^* is Gâteaux differentiable and $dom\ \nabla f^* = int\ dom\ f^*$.

One of the important and interesting Legendre function is $f(x) = \frac{1}{p}||x||^p$ $(1 < p < \infty)$ with the conjugate functions $f^*(x^*) = \frac{1}{q}||x^*||^q$ $(1 < q < \infty)$, (see, e.g., [15, 16]), where $\frac{1}{p} + \frac{1}{q} = 1$. In this case, the gradient ∇f coincides with the *generalized duality mapping*, J_p, of E, that is, $\nabla f = J_p$.

Definition 2.1. Let $f : E \to \mathbb{R}$, be a Gâteaux differentiable convex function. The function $D_f : dom\ f \times int\ dom f \to [0, \infty)$ defined by

$$D_f(y,x) = f(y) - f(x) - \langle \nabla f(x), y - x\rangle,\ x, y \in E. \tag{2.2}$$

is called the *Bregman distance with respect to* f [17].

We note that the Bregman distance has the following two important properties (see, e.g., [18, 19]), the three point identity, for any $x \in dom\ f$ and $y, z \in int\ dom\ f$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle, \tag{2.3}$$

and the four point identity

$$D_f(y,x) - D_f(y,z) - D_f(w,x) + D_f(w,z) = \langle \nabla f(z) - \nabla f(x), y - w \rangle.$$

Lemma 2.1. [20, 21] *If $f : E \to \mathbb{R}$ is a strongly coercive Legendre function, then*

(i) $\nabla f : E \to E^*$ is one-to-one, onto and norm-to-weak* continuous;
(ii) $\{x \in E : D_f(x,y) \leq r\}$ is bounded for all $y \in E$ and $r > 0$;
(iii) dom $f^* = E^*$, f^* is Gâteaux differentiable and $\nabla f^* = (\nabla f)^{-1}$.

Lemma 2.2. [14] *If $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function. Thus, for all $z \in E$, we have*

$$D_f\Big(z, \nabla f^*\Big(\sum_{i=1}^{N} t_i \nabla f(x_i)\Big)\Big) \leq \sum_{i=1}^{N} t_i D_f(z, x_i),$$

where $\{x_i\} \subseteq E$ and $\{t_i\} \subseteq (0,1)$ with $\sum_{i=1}^{N} t_i = 1$.

Lemma 2.3. [22] *If $f : E \to \mathbb{R}$ is a uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is norm-to-norm uniformly continuous on bounded subsets of E and hence both f and ∇f are bounded on bounded subset of E.*

Definition 2.2. Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $\nu_f(x,t) : E \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\nu_f(x,t) = \inf_{\{y \in E : ||x-y|| = t\}} D_f(y,x),$$

is called the Modulus of total convexity of f at $x \in$ *int dom* f. f is said to be totally convex if $\nu_f(x,t) > 0$, for all $x \in E$ and $t > 0$.

We remark that f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets of E (see, e.g., [23, Theorem 2.10]). Note that if $f : E \to (-\infty, \infty]$ is a Legendre function and E is a reflexive Banach space, then $\nabla f^* = (\nabla f)^{-1}$ (see, e.g., [24]). The *Bregman projection* with respect to f on $x \in$ *int dom* f onto a nonempty, closed and convex set $C \subseteq$ *int dom* f is denoted by $P_C^f(x) \in C$, which satisfies

$$D_f(P_C^f(x), x) = \inf\{D_f(y,x) : y \in C\}. \tag{2.4}$$

Remark 2.1. Some of the special cases of the Bregman projection are the following:

(i) If, in (2.4), C is a closed and convex subset of a real reflexive Banach space E and $f(x) = ||x||^2$, then the Bregman projection $P_C^f(x)$ reduces to the *generalized projection* $\Pi_C(x)$, which is defined by

$$\phi(\Pi_C(x), x) = \min_{y \in C} \phi(y,x),$$

where $\phi(y,x) = ||y||^2 - 2\langle y, Jx \rangle + ||x||^2$, and J is the normalized duality mapping from E into 2^{E^*}.

(ii) If, in (2.4), $E = H$, a real Hilbert space and $f(x) = ||x||^2$, then the Bregman projection $P_C^f(x)$ reduces to the metric projection of x onto C.

Lemma 2.4. [23] *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. The Bregman projection P_C^f from E onto C, where C is a nonempty, closed and convex subset of E, is a unique vector with the following properties:*

(i) $z = P_C^f(x)$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \forall y \in C$.
(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x)$, *for all* $y \in C$ *and* $x \in E$.

Let $f : E \to \mathbb{R}$ be a Legendre function. Let $V_f : E \times E^* \to [0, +\infty)$, associated with f, be defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*. \tag{2.5}$$

Then, V_f is nonnegative with the following properties (see, e.g., [25])

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \quad \forall x \in E, x^* \in E^*, \tag{2.6}$$

and

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*.$$

Let C be nonempty, closed and convex subset of E. Let $T : C \to E$ be a mapping. A point $x \in C$ is called an *asymptotic fixed point of* T if C contains a sequence $\{x_n\}$ which converges weakly to x and $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$. We denote the set of asymptotic fixed points of T by $\widehat{F}(T)$.

Definition 2.3. [26] Let C be a nonempty, closed and convex subset of E. A mapping $T : C \to E$ is called relatively nonexpansive if

(i) $F(T) \neq \emptyset$;
(ii) $\phi(p, Ty) \leq \phi(p, y)$ $\forall y \in C, p \in F(T)$;
(iii) $\widehat{F}(T) = F(T)$.

Definition 2.4. [27] Let C be a nonempty, closed and convex subset of E and let $T : C \to E^*$ be a map. A point $p \in C$ is called an *asymptotic J-fixed point of* T if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty} ||Tx_n - Jx_n|| = 0$.

We denote the set of asymptotic J-fixed points of T by $\widehat{F}_J(T)$.

Definition 2.5. [27] Let C be a nonempty, closed and convex subset of E. A mapping $T : C \to E^*$ is called relatively J-nonexpansive if

(i) $\widehat{F}_J(T) = F(T) \neq \emptyset$;
(ii) $\phi(p, J^{-1}Ty) \leq \phi(p, y)$ $\forall y \in C, p \in \widehat{F}_J(T)$.

Definition 2.6. [28] A Gâteaux differentiable function f is called strongly convex with constant $\mu > 0$ if the following inequality holds:

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \mu ||x - y||^2,$$

for all points x, y in its domain, or equivalently (see, e.g., [29])

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2} ||x - y||^2.$$

If E is a smooth and strictly convex Banach space, then $f(x) = ||x||^2$ is an example of a strongly convex function with parameter $0 < \mu \leq 2$.

Definition 2.7. [21] A function f is called uniformly convex function with modulus ϕ if, for each x, y in its domain and $t \in [0,1]$,

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)-t(1-t)\phi(||x-y||),$$

where ϕ is a function that is increasing and vanishes only at 0.

Lemma 2.5. *Let f be a convex and lower semi-continuous function on a Banach space E. The following assertions are equivalent (see, e.g., [21]):*

(i) f is uniformly convex;

(ii) There exists modulus ϕ, $\forall(x,x^*),(y,y^*) \in Gph(\partial f)$ such that

$$f(y) \geq f(x)+\langle y-x, x^*\rangle+\phi(||x-y||);$$

(iii) dom $f^* = E^*, f^*$ is Frèchet differentiable and ∇f^* is uniformly continuous.

Note that a strongly convex function is uniformly convex with $\phi(\alpha) = \frac{\mu}{2}\alpha^2$ and hence the class of uniformly convex functions contains the class of strongly convex functions.

Lemma 2.6. [30] *Let E be a Banach space and let $r > 0$ be a constant. Let $f : E \to \mathbb{R}$ be a continuous and convex function, which is uniformly convex on bounded subset of E. Then*

$$f\Big(\sum_{k=0}^{n} \alpha_k x_k\Big) \leq \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i\alpha_j\rho_r(||x_i - x_j||),$$

for all $0 \leq i, j \leq n$, $x_k \in B_r$, $\alpha_k \in (0,1)$ with $\sum_{k=0}^{n} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Lemma 2.7. [21] *Let $f : E \to \mathbb{R}$ be a continuous convex function on a reflexive Banach space E, which is strongly coercive. Then the following statements are equivalent:*

(i) f is bounded on bounded subset and uniformly smooth on bounded subset of E;

(ii) f^* is Fréchet differentiable and f^* is uniformly norm-to-norm continuous on bounded subset of E^*;

(iii) dom $f^* = E^*$, f^* is strongly coercive and uniformly convex on bounded subset of E^*.

Lemma 2.8. [30] *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable function, which is uniformly convex on bounded subset of E. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequence in E. Then the following statements are equivalent:*

(i) $\lim_{n\to\infty} D_f(x_n, y_n) = 0$;

(ii) $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 2.9. [31] *Let f be a strongly convex function with constant $\mu > 0$. Then, for all $y \in$ dom f and $x \in$ int dom f,*

$$D_f(y,x) \geq \frac{\mu}{2}||x-y||^2,$$

where $D_f(y,x)$ is a Bregman distance with respect to f.

Lemma 2.10. [32] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation: $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n$, $n \geq n_0$, where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.11. [33] *Let $\{s_k\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{k_j}\}$ of $\{s_k\}$such that $s_{k_j} < s_{k_j+1}$ for all $j \geq 0$. Define an integer sequence $\{m_k\}_{k\geq k_0}$ as $m_k = \max\{k_0 \leq l \leq k : s_l < s_{l+1}\}$. Then $m_k \to \infty$ as $k \to \infty$ and for all $k \geq k_0$ $\max\{s_{m_k}, s_k\} \leq s_{m_k+1}$.*

3. MAIN RESULTS

From now on, E is assumed to be a real reflexive Banach space with its dual E^*, C is a nonempty, closed and convex subset of E, and $f : E \to (-\infty, +\infty]$ is a proper, continuous and convex function on a real Banach space E. We denote the Fenchel conjugate of f by $f^* : E^* \to (-\infty, +\infty]$ and the family of such functions by $\mathscr{F}(E)$. Let $\{\alpha_n\} \subset (0,1)$ be a real sequence satisfying $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Definition 3.1. Let C be a nonempty, closed and convex subset of E. Let $T : C \to E^*$ be a mapping. A point $p \in C$ is said to be a *f-fixed point* of T if $Tp = \nabla f p$.

We denote the set of f-fixed points of T by $F_f(T)$, that is, $F_f(T) := \{p \in C : Tp = \nabla f p\}$.

Definition 3.2. Let $T : C \to E^*$ be a mapping. A point $p \in C$ is said to be an *f-asymptotic fixed point* of T if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty} ||\nabla f x_n - Tx_n|| = 0$.

We denote the set of f-asymptotic fixed points of T by $\widehat{F}_f(T)$.

Definition 3.3. A mapping $T : C \to E^*$ is said to be Bregman relatively f-nonexpansive if

(i) $F_f(T) \neq \emptyset$;
(ii) $D_f(p, \nabla f^* Ty) \leq D_f(p, y), \forall\, y \in C, p \in \widehat{F}_f(T)$;
(iii) $\widehat{F}_f(T) = F_f(T)$.

First, we prove the following lemmas.

Lemma 3.1. *If $T : C \to E^*$ is a Bregman relatively f-nonexpansive mapping, then $F_f(T)$ is closed and convex.*

Proof. We first prove that $F_f(T)$ is closed. Let $\{x_n\}$ be a sequence in $F_f(T)$ such that $x_n \to p \in C$. From the definition of T, we have $D_f(x_n, \nabla f^* Tp) \leq D_f(x_n, p)$, for each $n \geq 0$. Thus,

$$D_f(p, \nabla f^* Tp) = \lim_{n\to\infty} D_f(x_n, \nabla f^* Tp) \leq \lim_{n\to\infty} D_f(x_n, p) = D_f(p, p) = 0,$$

which implies that $p = \nabla f^* Tp$. Hence, $\nabla f p = Tp$. So, we have $p \in F_f(T)$, that is, $F_f(T)$ is closed.

Next, we show that $F_f(T)$ is convex. Let $p, q \in F_f(T)$ and $w = tp + (1-t)q$, for $t \in (0,1)$. Then, from (2.2) and the definition of T, we have

$$\begin{aligned} D_f(w, \nabla f^* Tw) &= f(w) - \langle tp + (1-t)q - \nabla f^* Tw, Tw\rangle - f(\nabla f^* Tw) \\ &= f(w) - t\langle p - \nabla f^* Tw, Tw\rangle - (1-t)\langle q - \nabla f^* Tw, Tw\rangle - f(\nabla f^* Tw) \\ &= f(w) + tf(p) - tf(p) - t\langle p - \nabla f^* Tw, Tw\rangle + (1-t)f(q) - (1-t)f(q) \\ &\quad - (1-t)\langle q - \nabla f^* Tw, Tw\rangle - f(\nabla f^* Tw) \\ &= f(w) + tD_f(p, \nabla f^* Tw) + (1-t)D_f(q, \nabla f^* Tw) - tf(p) - (1-t)f(q) \\ &\leq f(w) + tD_f(p, w) + (1-t)D_f(q, w) - tf(p) - (1-t)f(q) \\ &= f(w) - \langle w - w, \nabla f w\rangle - f(w) = 0. \end{aligned}$$

This impels $w = \nabla f^* T w$. Hence $\nabla f w = T w$ and $w \in F_f(T)$, that is, $F_f(T)$ is convex. □

Theorem 3.1. *Let $f \in \mathscr{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function, which is strongly convex with constant $\mu > 0$ on bounded subsets of E. Let $A_i : C \to E^*$ be a L_i-Lipschitz monotone mapping with constant L_i, for $i = 1,2$ and let $T : C \to E^*$ be a Bregman relatively f-nonexpansive mapping such that $\Omega = VI(C,A_1) \cap VI(C,A_2) \cap F_f(T) \neq \emptyset$. For arbitrary $x_0, w \in C$, define an iterative sequence by*

$$\begin{cases} y_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_1 x_n), \\ z_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_2 x_n), \\ w_n = \nabla f^*(\beta_n \nabla f x_n + \delta_n T x_n + \eta_n \nabla f u_n + \theta_n \nabla f v_n), \\ x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f w + (1-\alpha_n) \nabla f w_n), \end{cases} \tag{3.1}$$

where $u_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)$, $v_n = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_2 z_n)$, $0 < \underline{\gamma} \leq \gamma_n \leq \overline{\gamma} < \frac{\mu}{L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\varepsilon, 1) \subset (0,1)$ such that $\beta_n + \delta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ is bounded.

Proof. Let $q \in \Omega$. Now, from (3.1), Lemma 2.4 (ii), (2.2), (1.1) and the monotonicity of A_1, we have

$$\begin{aligned} D_f(q,u_n) &\leq D_f(q, \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)) - D_f(u_n, \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)) \\ &= f(q) - \langle q - \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n), \nabla f x_n - \gamma_n A_1 y_n \rangle \\ &\quad - f(\nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)) - f(u_n) \\ &\quad + \langle u_n - \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n), \nabla f x_n - \gamma_n A_1 y_n \rangle \\ &\quad + f(\nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)) \\ &= f(q) + \langle u_n - q, \nabla f x_n \rangle - \langle u_n - q, \gamma_n A_1 y_n \rangle - f(u_n) \\ &= f(q) - \langle q - x_n, \nabla f x_n \rangle - f(x_n) + \langle q - x_n, \nabla f x_n \rangle + f(x_n) \\ &\quad + \langle u_n - q, \nabla f x_n \rangle - \langle u_n - q, \gamma_n A_1 y_n \rangle - f(u_n) \\ &= D_f(q,x_n) + \langle u_n - x_n, \nabla f x_n \rangle + f(x_n) - f(u_n) - \langle u_n - q, \gamma_n A_1 y_n \rangle \end{aligned}$$

and

$$\begin{aligned} D_f(q,u_n) &\leq D_f(q,x_n) - D_f(u_n,x_n) - \langle u_n - q, \gamma_n A_1 y_n \rangle \\ &= D_f(q,x_n) - D_f(u_n,x_n) + \gamma_n \langle q - y_n, A_1 y_n - A_1 q \rangle \\ &\quad + \gamma_n \langle q - y_n, A_1 q \rangle + \langle y_n - u_n, \gamma_n A_1 y_n \rangle \\ &\leq D_f(q,x_n) - D_f(u_n,x_n) + \langle y_n - u_n, \gamma_n A_1 y_n \rangle. \end{aligned} \tag{3.2}$$

In addition, we from (2.3) that

$$D_f(u_n,x_n) = D_f(u_n,y_n) + D_f(y_n,x_n) + \langle \nabla f x_n - \nabla f y_n, y_n - u_n \rangle. \tag{3.3}$$

Substituting (3.3) into (3.2), we obtain

$$\begin{aligned} D_f(q,u_n) &\leq D_f(q,x_n) - D_f(u_n,y_n) - D_f(y_n,x_n) \\ &\quad - \langle \nabla f x_n - \nabla f y_n, y_n - u_n \rangle + \langle y_n - u_n, \gamma_n A_1 y_n \rangle \\ &= D_f(q,x_n) - D_f(u_n,y_n) - D_f(y_n,x_n) \\ &\quad + \langle y_n - u_n, \gamma_n A_1 y_n + \nabla f y_n - \nabla f x_n \rangle. \end{aligned} \tag{3.4}$$

From Lemma 2.4 (i), we have

$$\begin{aligned}\langle y_n-u_n,\gamma_nA_1y_n+\nabla fy_n-\nabla fx_n\rangle &= \langle u_n-y_n,\gamma_nA_1x_n-\gamma_nA_1y_n\rangle \\ &\quad +\langle u_n-y_n,\nabla fx_n-\gamma_nA_1x_n-\nabla fy_n\rangle \\ &\leq \langle u_n-y_n,\gamma_nA_1x_n-\gamma_nA_1y_n\rangle. \qquad (3.5)\end{aligned}$$

Furthermore, (3.4), (3.5), Lemma 2.9 and the fact that A_1 is L_1-Lipschitz, we obtain

$$\begin{aligned}D_f(q,u_n) &\leq D_f(q,x_n)-D_f(u_n,y_n)-D_f(y_n,x_n)+\langle u_n-y_n,\gamma_nA_1x_n-\gamma_nA_1y_n\rangle \\ &\leq D_f(q,x_n)-D_f(u_n,y_n)-D_f(y_n,x_n)+\gamma_n||u_n-y_n||||A_1x_n-A_1y_n|| \\ &\leq D_f(q,x_n)-D_f(u_n,y_n)-D_f(y_n,x_n)+\gamma_nL||u_n-y_n||||x_n-y_n|| \\ &\leq D_f(q,x_n)-D_f(u_n,y_n)-D_f(y_n,x_n)+\frac{\gamma_nL}{2}[||u_n-y_n||^2+||x_n-y_n||^2] \\ &\leq D_f(q,x_n)-\frac{\mu}{2}||u_n-y_n||^2-\frac{\mu}{2}||y_n-x_n||^2 \\ &\quad +\frac{\gamma_nL}{2}[||u_n-y_n||^2+||x_n-y_n||^2] \\ &= D_f(q,x_n)-\left(\frac{\mu-\gamma_nL}{2}\right)\Big[||u_n-y_n||^2+||x_n-y_n||^2\Big]. \qquad (3.6)\end{aligned}$$

Similarly, we obtain that

$$D_f(q,v_n)\leq D_f(q,x_n)-\left(\frac{\mu-\gamma_nL}{2}\right)\Big[||v_n-z_n||^2+||x_n-z_n||^2\Big]. \qquad (3.7)$$

Again, from (2.5), (3.1), (2.6) and Lemma 2.6, we get

$$\begin{aligned}D_f(q,w_n) &= V_f(q,\beta_n\nabla fx_n+\delta_nTx_n+\eta_n\nabla fu_n+\theta_n\nabla fv_n) \\ &= f(q)-\langle q,\beta_n\nabla fx_n+\delta_nTx_n+\eta_n\nabla fu_n+\theta_n\nabla fv_n\rangle \\ &\quad +f^*(\beta_n\nabla fx_n+\delta_nTx_n+\eta_n\nabla fu_n+\theta_n\nabla fv_n) \\ &\leq \beta_nf(q)+\delta_nf(q)+\eta_nf(q)+\theta_nf(q) \\ &\quad -\langle q,\beta_n\nabla fx_n\rangle-\langle q,\delta_nTx_n\rangle-\langle q,\eta_n\nabla fu_n\rangle-\langle q,\theta_n\nabla fv_n\rangle \\ &\quad +\beta_nf^*(\nabla fx_n)+\delta_nf^*(Tx_n)+\eta_nf^*(\nabla fu_n)+\theta_nf^*(\nabla fv_n) \\ &\quad -\beta_n\delta_n\rho_r^*(||\nabla fx_n-Tx_n||) \\ &= \beta_nV_f(q,\nabla fx_n)+\delta_nV_f(q,Tx_n)+\eta_nV_f(q,\nabla fu_n) \\ &\quad +\theta_nV_f(q,\nabla fv_n)-\beta_n\delta_n\rho_r^*(||\nabla fx_n-Tx_n||) \\ &= \beta_nD_f(q,x_n)+\delta_nD_f(q,\nabla f^*Tx_n)+\eta_nD_f(q,u_n)+\theta_nD_f(q,v_n) \\ &\quad -\beta_n\delta_n\rho_r^*(||\nabla fx_n-Tx_n||).\end{aligned}$$

Using the assumption on T, inequalities (3.6) and (3.7), we obtain

$$\begin{aligned}
D_f(q,w_n) &\leq \beta_n D_f(q,x_n)+\delta_n D_f(q,x_n)+\eta_n D_f(q,u_n)+\theta_n D_f(q,v_n) \\
&\quad -\beta_n\delta_n\rho_r^*(||\nabla f x_n - T x_n||) \\
&\leq \beta_n D_f(q,x_n)+\delta_n D_f(q,x_n) \\
&\quad +\eta_n\Big[D_f(q,x_n)-\Big(\frac{\mu-\gamma_n L}{2}\Big)(||u_n-y_n||^2+||x_n-y_n||^2)\Big] \\
&\quad +\theta_n\Big[D_f(q,x_n)-\Big(\frac{\mu-\gamma_n L}{2}\Big)(||v_n-z_n||^2+||x_n-z_n||^2)\Big] \\
&\quad -\beta_n\delta_n\rho_r^*(||\nabla f x_n - T x_n||) \\
&= D_f(q,x_n)-\eta_n\Big[\Big(\frac{\mu-\gamma_n L}{2}\Big)(||u_n-y_n||^2+||x_n-y_n||^2)\Big] \\
&\quad -\theta_n\Big[\Big(\frac{\mu-\gamma_n L}{2}\Big)(||v_n-z_n||^2+||x_n-z_n||^2)\Big] \\
&\quad -\beta_n\delta_n\rho_r^*(||\nabla f x_n - T x_n||) \\
&\leq D_f(q,x_n). \qquad (3.8)
\end{aligned}$$

Now, from (3.1), (3.8), Lemma 2.4 (ii) and Lemma 2.2, we obtain

$$\begin{aligned}
D_f(q,x_{n+1}) &= D_f(q,\nabla f^*(\alpha_n\nabla f w+(1-\alpha_n)\nabla f w_n)) \\
&\leq \alpha_n D_f(q,w)+(1-\alpha_n)D_f(q,w_n) \\
&\leq \alpha_n D_f(q,w)+(1-\alpha_n)D_f(q,x_n) \\
&\leq \max\{D_f(q,w),D_f(q,x_n)\}.
\end{aligned}$$

It follows that $D_f(q,x_n)\leq\max\{D_f(q,w),D_f(q,x_0)\}$. Thus, $\{D_f(q,x_n)\}$ is bounded. Therefore, by Lemma 2.1, we have that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{z_n\}$, $\{u_n\}$, $\{w_n\}$ and $\{v_n\}$. □

Next, we prove the main theorem of this section.

Theorem 3.2. *Let $f\in\mathscr{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function, which is strongly convex with constant $\mu>0$ on bounded subsets of E. Let $A_i: C\to E^*$ be a L_i-Lipschitz monotone mapping with constant L_i, for $i=1,2$ and let $T: C\to E^*$ be a Bregman relatively f-nonexpansive mapping such that $\Omega=VI(C,A_1)\cap VI(C,A_2)\cap F_f(T)\neq\emptyset$. For arbitrary $x_0,w\in C$, define an iterative sequence by*

$$\begin{cases}
y_n=P_C^f\nabla f^*(\nabla f x_n-\gamma_n A_1 x_n),\\
z_n=P_C^f\nabla f^*(\nabla f x_n-\gamma_n A_2 x_n),\\
w_n=\nabla f^*(\beta_n\nabla f x_n+\delta_n T x_n+\eta_n\nabla f u_n+\theta_n\nabla f v_n),\\
x_{n+1}=P_C^f\nabla f^*(\alpha_n\nabla f w+(1-\alpha_n)\nabla f w_n),
\end{cases} \qquad (3.9)$$

where $u_n=P_C^f\nabla f^*(\nabla f x_n-\gamma_n A_1 y_n)$, $v_n=P_C^f\nabla f^*(\nabla f x_n-\gamma_n A_2 z_n)$, $0<\underline{\gamma}\leq\gamma_n\leq\overline{\gamma}<\frac{\mu}{L}$, for $L:=\max\{L_1,L_2\}$ and $\{\beta_n\},\{\delta_n\},\{\eta_n\},\{\theta_n\}\subset[\varepsilon,1)\subset(0,1)$ such that $\beta_n+\delta_n+\eta_n+\theta_n=1$, for all $n\geq 0$. Then, $\{x_n\}$ converges strongly to an element $x^*=P_\Omega^f(w)$.

Proof. From Theorem 3.1, we know that $\{x_n\}$ is bounded. Let $q=P_\Omega^f(w)$ and $r_n=\nabla f^*(\alpha_n\nabla f w+(1-\alpha_n)\nabla f w_n)$. In view of Lemma 2.4 (i), we get $\langle u-q,\nabla f w-\nabla f q\rangle\leq 0$, $\forall u\in\Omega$. Now, from

(2.6), (3.9) and Lemma 2.4, we get

$$\begin{aligned} D_f(q,x_{n+1}) &\leq D_f(q,\nabla f^*(\alpha_n\nabla fw+(1-\alpha_n)\nabla fw_n)) \\ &= V_f(q,\alpha_n\nabla fw+(1-\alpha_n)\nabla fw_n) \\ &\leq V_f(q,\alpha_n\nabla fw+(1-\alpha_n)\nabla fw_n-\alpha_n(\nabla fw-\nabla fq)) \\ &\quad +\langle\alpha_n(\nabla fw-\nabla fq),r_n-q\rangle \\ &= D_f(q,\nabla f^*(\alpha_n\nabla fq+(1-\alpha_n)\nabla fw_n))+\alpha_n\langle\nabla fw-\nabla fq,r_n-q\rangle. \end{aligned}$$

From Lemma 2.2 and (3.8), we arrive at

$$\begin{aligned} D_f(q,x_{n+1}) &\leq \alpha_n D_f(q,q)+(1-\alpha_n)D_f(q,w_n) \\ &\quad +\alpha_n\langle\nabla fw-\nabla fq,r_n-q\rangle \\ &= (1-\alpha_n)D_f(q,w_n)+\alpha_n\langle\nabla fw-\nabla fq,r_n-q\rangle \\ &\leq (1-\alpha_n)D_f(q,x_n)+\alpha_n\langle\nabla fw-\nabla fq,r_n-q\rangle \\ &\leq (1-\alpha_n)D_f(q,x_n)+\alpha_n||\nabla fw-\nabla fq||||r_n-x_n|| \\ &\quad +\alpha_n\langle\nabla fw-\nabla fq,x_n-q\rangle. \end{aligned} \tag{3.10}$$

On the other hand, (3.9), Lemma 2.4 and Lemma 2.2 yield that

$$\begin{aligned} D_f(q,x_{n+1}) &= D_f(q,P_C^f\nabla f^*(\alpha_n\nabla fw+(1-\alpha_n)\nabla fw_n)) \\ &= D_f(q,\nabla f^*(\alpha_n\nabla fw+(1-\alpha_n)\nabla fw_n)) \\ &\leq \alpha_n D_f(q,w)+(1-\alpha_n)D_f(q,w_n). \end{aligned}$$

By using (3.8), we obtain

$$\begin{aligned} D_f(q,x_{n+1}) \leq{}& \alpha_n D_f(q,w)+(1-\alpha_n)D_f(q,x_n) \\ &-(1-\alpha_n)\eta_n\Big[\Big(\frac{\mu-\gamma_n L}{2}\Big)(||u_n-y_n||^2+||x_n-y_n||^2)\Big] \\ &-(1-\alpha_n)\theta_n\Big[\Big(\frac{\mu-\gamma_n L}{2}\Big)(||v_n-z_n||^2+||x_n-z_n||^2)\Big] \\ &-\beta_n\delta_n\rho_r^*(||\nabla fx_n-Tx_n||), \end{aligned}$$

which implies that

$$\begin{aligned} &(1-\alpha_n)\eta_n\Big[\Big(\frac{\mu-\gamma_n L}{2}\Big)(||u_n-y_n||^2+||x_n-y_n||^2)\Big] \\ &\quad +(1-\alpha_n)\theta_n\Big[\Big(\frac{\mu-\gamma_n L}{2}\Big)(||v_n-z_n||^2+||x_n-z_n||^2)\Big] \\ &\quad +\beta_n\delta_n\rho_r^*(||\nabla fx_n-Tx_n||) \\ &\leq \alpha_n(D_f(q,w)-D_f(q,x_n))+D_f(q,x_n)-D_f(q,x_{n+1}). \end{aligned} \tag{3.11}$$

Now, we divide the rest of the proof into two parts as follows.

Case 1. Assume that there exists $n_0\in\mathbb{N}$ such that $D_f(q,x_n)$ is decreasing for all $n\geq n_0$. It then follow that $D_f(q,x_n)$ is convergent and hence $D_f(q,x_n)-D_f(q,x_{n+1})\to 0$ as $n\to\infty$. Thus, from (3.11) and the conditions on $\alpha_n,\beta_n,\delta_n,\eta_n$ and θ_n, we get

$$\lim_{n\to\infty}[||u_n-y_n||^2+||x_n-y_n||^2]=\lim_{n\to\infty}[||v_n-z_n||^2+||x_n-z_n||^2]=0,$$

and $\lim_{n\to\infty}\rho_r^*(||\nabla f x_n - Tx_n||) = 0$. Hence, $\lim_{n\to\infty}||u_n - y_n|| = \lim_{n\to\infty}||x_n - y_n|| = 0$,

$$\lim_{n\to\infty}||v_n - z_n|| = \lim_{n\to\infty}||x_n - z_n|| = 0, \tag{3.12}$$

and

$$\lim_{n\to\infty}||\nabla f x_n - Tx_n|| = 0. \tag{3.13}$$

From (3.13) and the fact that ∇f^* is uniformly continuous on bounded subsets of E^*, we get

$$\lim_{n\to\infty}||x_n - \nabla f^* T x_n|| = 0. \tag{3.14}$$

Moreover, from (3.9) and Lemma 2.2, we obtain

$$\begin{aligned} D_f(x_n, r_n) &= D_f(x_n, \nabla f^*[\alpha_n \nabla f w + (1-\alpha_n)\nabla f w_n]) \\ &\leq \alpha_n D_f(x_n, w) + (1-\alpha_n) D_f(q, w_n) \\ &= \alpha_n D_f(x_n, w) + (1-\alpha_n)[\beta_n D_f(x_n, x_n) + \delta_n D_f(x_n, \nabla f^* T x_n)] \\ &\quad +(1-\alpha_n)[\eta_n D_f(x_n, u_n) + \theta_n D_f(x_n, v_n)]. \end{aligned} \tag{3.15}$$

Thus, from Lemma 2.8, (3.12), (3.14) and (3.15), we get $\lim_{n\to\infty} D_f(x_n, r_n) = 0$, which implies that $\lim_{n\to\infty}||x_n - r_n|| = 0$. Now, since $\{x_n\}$ is bounded in E, we have that there exists $u \in E$ and a subsequence $\{x_{n_k}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup u$ and

$$\limsup_{n\to\infty}\langle x_n - q, \nabla f w - \nabla f q\rangle = \lim_{k\to\infty}\langle x_{n_k} - q, \nabla f w - \nabla f q\rangle. \tag{3.16}$$

From (3.13) and the fact that T is relatively f-nonexpansive, we have $u \in F_f(T)$.

Next, we show that $u \in VI(C, A_1) \cap VI(C, A_2)$. Let

$$F_1 x = \begin{cases} A_1 x + N_C x, & x \in C, \\ \emptyset, & x \notin C, \end{cases}$$

where $N_C(x) = \{v \in E^* : \langle x - z, v\rangle \geq 0, \forall z \in C\}$ is the normal cone to C. Then, F_1 is a maximal monotone mapping, and $0 \in F_1 x$ if and only if $x \in VI(C, A_1)$ (see, e.g., [34]). Let $(x,y) \in Gph(F_1)$. Then $y \in A_1 x + N_C(x)$ and $y - A_1 x \in N_C(x)$. Thus, we obtain

$$\langle x - v, y - A_1 x\rangle \geq 0, \quad \forall v \in C. \tag{3.17}$$

Note that x and u_{n_k} are in C. We have $\langle \nabla f x_{n_k} - \gamma_{n_k} A_1 y_{n_k} - \nabla f u_{n_k}, u_{n_k} - x\rangle \geq 0$, which implies that

$$\langle x - u_{n_k}, \frac{\nabla f u_{n_k} - \nabla f x_{n_k}}{\gamma_{n_k}} + A_1 y_{n_k}\rangle \geq 0.$$

From (3.17), we find that

$$\begin{aligned} \langle x - u_{n_k}, y\rangle &\geq \langle x - u_{n_k}, A_1 x\rangle \\ &\geq \langle x - u_{n_k}, A_1 x\rangle - \langle x - u_{n_k}, \frac{\nabla f u_{n_k} - \nabla f x_{n_k}}{\gamma_{n_k}} + A_1 y_{n_k}\rangle \\ &= \langle x - u_{n_k}, A_1 x - A_1 u_{n_k}\rangle + \langle x - u_{n_k}, A_1 u_{n_k} - A_1 y_{n_k}\rangle \\ &\quad - \frac{1}{\gamma_{n_k}}\langle x - u_{n_k}, \nabla f u_{n_k} - \nabla f x_{n_k}\rangle \\ &\geq \langle x - u_{n_k}, A_1 u_{n_k} - A_1 y_{n_k}\rangle - \frac{1}{\gamma_{n_k}}\langle x - u_{n_k}, \nabla f u_{n_k} - \nabla f x_{n_k}\rangle. \end{aligned} \tag{3.18}$$

Furthermore, from the fact that A_1 and ∇f are uniformly continuous, we obtain $\lim_{n\to\infty}||A_1u_n - A_1y_n|| = 0$ and $\lim_{n\to\infty}||\nabla fu_n - \nabla fx_n|| = 0$. Consequently, inequality (3.18) provides that $\langle x-u, y\rangle \geq 0$. Hence, by the maximality of F_1, we obtain $u \in F_1^{-1}(0)$. Therefore, $u \in VI(C,A_1)$. Similarly, we get $u \in VI(C,A_2)$. Thus $u \in VI(C,A_1)\cap VI(C,A_2)\cap F_f(T)$. It follows from Lemma 2.4 (i) that

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n - q, \nabla fw - \nabla fq\rangle &= \lim_{k\to\infty}\langle x_{n_k} - q, \nabla fw - \nabla fq\rangle \\ &= \langle u-q, \nabla fw - \nabla fq\rangle \leq 0. \end{aligned} \tag{3.19}$$

Therefore, from (3.10), (3.19) and Lemma 2.10, we conclude that $D_f(q,x_n) \to 0$ as $n\to\infty$. By using Lemma 2.8, we conclude that $x_n \to q$ as $n\to\infty$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $D_f(q,x_{n_i}) < D_f(q,x_{n_i+1})$, $\forall i\in\mathbb{N}$. It follows from Lemma 2.11 that there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that $m_k\to\infty$ as $k\to\infty$ and $\max\{D_f(q,x_{m_k}), D_f(q,x_k)\} \leq D_f(q,x_{m_k+1})$, for all $k\in\mathbb{N}$. Thus, from (3.11) and the conditions on $\alpha_n, \beta_n, \delta_n, \eta_n$ and θ_n, we get

$$\lim_{k\to\infty}||u_{m_k} - y_{m_k}|| = \lim_{k\to\infty}||x_{m_k} - y_{m_k}|| = 0,$$

$$\lim_{k\to\infty}||v_{m_k} - z_{m_k}|| = \lim_{k\to\infty}||x_{m_k} - z_{m_k}|| = 0,$$

and

$$\lim_{k\to\infty}||\nabla fx_{m_k} - Tx_{m_k}|| = 0.$$

Moreover, following the methods in Case 1, we get $\lim_{k\to\infty}||x_{m_k} - r_{m_k}|| = 0$, and

$$\limsup_{k\to\infty}\langle x_{m_k} - q, \nabla fw - \nabla fq\rangle \leq 0. \tag{3.20}$$

Now, from (3.10), we obtain

$$\begin{aligned}\alpha_{m_k}(D_f(q,x_{m_k}) &\leq D_f(q,x_{m_k}) - D_f(q,x_{m_k+1}) + \alpha_{m_k}||x_{m_k} - r_{m_k}||||\nabla fw - \nabla fq|| \\ &\quad + \alpha_{m_k}\langle x_{m_k} - q, \nabla fw - \nabla fq\rangle \\ &\leq \alpha_{m_k}||x_{m_k} - r_{m_k}||||\nabla fw - \nabla fq|| + \alpha_{m_k}\langle x_{m_k} - q, \nabla fw - \nabla fq\rangle.\end{aligned}$$

Consequently, since $\alpha_{m_k} > 0$, we get

$$D_f(q,x_{m_k}) \leq ||x_{m_k} - r_{m_k}||||\nabla fw - \nabla fq|| + \langle x_{m_k} - q, \nabla fw - \nabla fq\rangle. \tag{3.21}$$

Therefore, from (3.20) and (3.21), we obtain $D_f(q,x_{m_k}) \to 0$ as $k\to\infty$. But, we have $D_f(q,x_{m_k+1}) \to 0$ as $k\to\infty$. Hence, the fact $D_f(q,x_k) \leq D_f(q,x_{m_k+1})$ implies $\lim_{k\to\infty} D_f(q,x_k) = 0$. Using Lemma 2.8, we conclude that $x_k \to q$ as $k\to\infty$. □

We remark that the proof of Theorem 3.2 provides the following result for a common point in the f-fixed point set of a Bregman relatively f-nonexpansive mapping and in the solution set of variational inequality problems with a finite family of Lipschitz monotone mappings in Banach spaces.

Theorem 3.3. *Let $f \in \mathscr{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function, which is strongly convex with constant $\mu > 0$ on bounded subsets of E. Let $A_i : C \to E^*$ be a L_i-Lipschitz monotone mapping with constant L_i, for $i =$*

$1,2,\cdots,N$, and let $T : C \to E^*$ be a Bregman relatively f-nonexpansive mapping such that $\Omega = \cap_{i=1}^{N} VI(C,A_i) \cap F_f(T) \neq \emptyset$. For arbitrary $x_0, w \in C$, define an iterative sequence by

$$\begin{cases} y_{n,i} = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_i x_n), \\ u_{n,i} = P_C^f \nabla f^*(\nabla f x_n - \gamma_n A_i y_{n,i}), \\ w_n = \nabla f^*(\beta_n \nabla f x_n + \delta_n T x_n + \sum_{i=1}^{n} \eta_{n,i} \nabla f u_{n,i}), \\ x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f w + (1-\alpha_n) \nabla f w_n), \end{cases}$$

where $0 < \underline{\gamma} \leq \gamma_n \leq \overline{\gamma} < \frac{\mu}{L}$, for $L := \max\{L_i : i = 1,2,\cdots,N\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_{n,i}\} \subset [\varepsilon, 1) \subset (0,1)$ such that $\beta_n + \delta_n + \sum_{i=1}^{N} \eta_{n,i} = 1$, for all $n \geq 0$. Then, $\{u_n\}$ converges strongly to $x^* = P_\Omega^f(w)$.

If, in Theorem 3.2, $C = E$, then the Bregman projection P_C^f is reduced to the identity mapping on E. Hence, $VI(C,A_1) = A_1^{-1}(0)$ and $VI(C,A_2) = A_2^{-1}(0)$. Thus, we obtain the following corollary for a common point in the f-fixed point set of Bregman relatively f-nonexpansive mappings and the zero point set of Lipschitz monotone mappings in Banach spaces.

Corollary 3.1. *Let $f \in \mathscr{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E. Let $A_i : E \to E^*$ be a L_i-Lipschitz monotone mapping with constant L_i, for $i = 1,2$ and $T : E \to E^*$ be a Bregman relatively f-nonexpansive mapping such that $\Omega = F_f(T) \cap A_1^{-1}(0) \cap A_2^{-1}(0) \neq \emptyset$. For arbitrary $x_0, w \in E$, define an iterative sequence by*

$$\begin{cases} y_n = \nabla f^*(\nabla f x_n - \gamma_n A_1 x_n), \\ z_n = \nabla f^*(\nabla f x_n - \gamma_n A_2 x_n), \\ w_n = \nabla f^*(\beta_n \nabla f x_n + \delta_n T x_n + \eta_n \nabla f u_n + \theta_n \nabla f v_n), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f w + (1-\alpha_n) \nabla f w_n), \end{cases} \tag{3.22}$$

where $u_n = \nabla f^*(\nabla f x_n - \gamma_n A_1 y_n)$, $v_n = \nabla f^*(\nabla f x_n - \gamma_n A_2 z_n)$, $0 < \underline{\gamma} \leq \gamma_n \leq \overline{\gamma} < \frac{\mu}{L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\varepsilon, 1) \subset (0,1)$ such that $\beta_n + \delta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to $x^* = P_\Omega^f(w)$.

If, in Theorem 3.2, $A_i = 0$, for $i = 1,2$, then we obtain a method of approximation for f-fixed points of a Bregman relatively f-nonexpansive mapping in Banach spaces.

Corollary 3.2. *Let $f \in \mathscr{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function, which is strongly convex with constant $\mu > 0$ on bounded subsets of E. Let $T : C \to E^*$ be a Bregman relatively f-nonexpansive mapping such that $F_f(T) \neq \emptyset$. For arbitrary $x_0, w \in C$, define an iterative sequence by*

$$\begin{cases} w_n = \nabla f^*(\beta_n \nabla f x_n + \delta_n T x_n), \\ x_{n+1} = P_C^f \nabla f^*(\alpha_n \nabla f w + (1-\alpha_n) \nabla f w_n), \end{cases} \tag{3.23}$$

where $\{\beta_n\}, \{\delta_n\} \subset [\varepsilon, 1) \subset (0,1)$ such that $\beta_n + \delta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to $x^* = P_{F_f(T)}(w)$.

If, in Theorem 3.3, $C = E$ and $T = \nabla f$, the identity mappong on E, then we get the following result for a common zero point of a finite family of Lipschitz monotone mappings in Banach spaces.

Corollary 3.3. *Let $f \in \mathscr{F}(E)$ be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function, which is strongly convex with constant $\mu > 0$ on bounded subsets of E. Let $A_i : E \to E^*$ be a L_i-Lipschitz monotone mapping with Lipschitz constant L_i, for $i = 1,2,\cdots,N$ such that $\Omega = \cap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset$. For arbitrary $x_0, w \in E$, define an iterative sequenc by*

$$\begin{cases} y_{n,i} = \nabla f^*(\nabla f x_n - \gamma_n A_i x_n), \\ u_{n,i} = \nabla f^*(\nabla f x_n - \gamma_n A_i y_{n,i}), \\ w_n = \nabla f^*(\beta_{n,0} \nabla f x_n + \sum_{i=1}^{n} \beta_{n,i} \nabla f u_{n,i}), \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f w + (1-\alpha_n) \nabla f w_n), \end{cases} \tag{3.24}$$

where $0 < \underline{\gamma} \leq \gamma_n \leq \overline{\gamma} < \frac{\mu}{L}$, for $L := \max\{L_i : i = 1,2,\cdots,N\}$ and $\{\beta_{n,i}\} \subset [\varepsilon, 1) \subset (0,1)$ such that $\sum_{i=0}^{N} \beta_{n,i} = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $x^* = P_{\Omega}^{f}(w)$.

If, in Corollary 3.1, E is a smooth, strictly convex and real reflexive Banach space, then $f(x) = \frac{1}{2}||x||^2$ is a strongly coercive, bounded and uniformly Frêchet differentiable Legendre function, which is strongly convex with constant $\mu = 2$ and conjugate $f^*(x^*) = \frac{1}{2}||x^*||^2$. This yields that $\nabla f = J_2 = J$ with $\nabla f^* = J_2^{-1} = J^{-1}$. Thus, we easily have the following results.

Corollary 3.4. *Let E be a smooth, strictly convex and real reflexive Banach space with its dual E^* Let $A_i : C \to E^*$ be a L_i-Lipschitz monotone mapping with Lipschitz constant L_i, for $i = 1,2$, and let $T : C \to E^*$ be a relatively J-nonexpansive mapping such that $\Omega = F_f(T) \cap A_1^{-1}(0) \cap A_2^{-1}(0) \neq \emptyset$. For arbitrary $x_0, w \in C$, define an iterative sequence by*

$$\begin{cases} y_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_1 x_n), \\ z_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_2 x_n), \\ w_n = J^{-1}(\beta_n J x_n + \delta_n T x_n + \eta_n J u_n + \theta_n J v_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J w + (1-\alpha_n) J w_n), \end{cases} \tag{3.25}$$

where $u_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_1 y_n)$, $v_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_2 z_n)$, $0 < \underline{\gamma} \leq \gamma_n \leq \overline{\gamma} < \frac{1}{L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\varepsilon, 1) \subset (0,1)$ such that $\beta_n + \delta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega$.

Corollary 3.5. *Let E be a smooth, strictly convex and real reflexive Banach space with its dual E^* Let $A_i : C \to E^*$ be a L_i-Lipschitz monotone mapping with Lipschitz constant L_i, for $i = 1,2$, and let $T : C \to E$ be a relatively nonexpansive mapping such that $\Omega = VI(C, A_1) \cap VI(C, A_2) \cap F(T) \neq \emptyset$. For arbitrary $x_0, w \in C$, define an iterative sequence by*

$$\begin{cases} y_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_1 x_n), \\ z_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_2 x_n), \\ w_n = J^{-1}(\beta_n J x_n + \delta_n J T x_n + \eta_n J u_n + \theta_n J v_n), \\ x_{n+1} = \Pi_C J^{-1}(\alpha_n J w + (1-\alpha_n) J w_n), \end{cases} \tag{3.26}$$

where $u_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_1 y_n)$, $v_n = \Pi_C J^{-1}(Jx_n - \gamma_n A_2 z_n)$, $0 < \underline{\gamma} \leq \gamma_n \leq \overline{\gamma} < \frac{1}{L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\varepsilon, 1) \subset (0,1)$ such that $\beta_n + \delta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega$.

If, in Corollary 3.4, we assume $E = H$, a real Hilbert space, and $f(x) = \frac{1}{2}||x||^2$, then $\nabla f = J_2 = I$ and $\nabla f^* = J_2^{-1} = I$, were I is identity mapping on H. Thus, we get the following results in Hilbert spaces.

Corollary 3.6. *Let H be a real Hilbert space. Let $A_i : C \to H$ be a L_i-Lipschitz monotone mapping with Lipschitz constant L_i, for $i = 1,2$, and let $T : C \to E$ be a quasi-nonexpansive mapping such that $(I - T)$ is demiclosed at 0 and $\Omega = A_1^{-1}(0) \cap A_2^{-1}(0) \cap F(T) \neq \emptyset$. For arbitrary $x_0, w \in C$, define an iterative sequence by*

$$\begin{cases} y_n = P_C(x_n - \gamma_n A_1 x_n), \\ z_n = P_C(x_n - \gamma_n A_2 x_n), \\ w_n = \beta_n x_n + \delta_n T x_n + \eta_n u_n + \theta_n v_n, \\ x_{n+1} = P_C(\alpha_n w + (1-\alpha_n) w_n), \end{cases} \tag{3.27}$$

where $u_n = P_C(x_n - \gamma_n A_1 y_n)$, $v_n = P_C(x_n - \gamma_n A_2 z_n)$, $0 < \underline{\gamma} \leq \gamma_n \leq \overline{\gamma} < \frac{1}{L}$, for $L := \max\{L_1, L_2\}$ and $\{\beta_n\}, \{\delta_n\}, \{\eta_n\}, \{\theta_n\} \subset [\varepsilon, 1) \subset (0,1)$ such that $\beta_n + \delta_n + \eta_n + \theta_n = 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to $x^* \in \Omega$.

4. Conclusion

In this paper, we gave the concepts of Bregman relatively f-nonexpansive mappings and investigate efficient iterative algorithms for the common solution problem of Bregman relatively f-nonexpansive mappings and variational inequality problems with Lipschitz monotone mappings in reflexive real Banach spaces. Theorem 3.2 extends the results in [2, 4, 9, 12, 35] from real Hilbert spaces to real reflexive Banach spaces. Corollary 3.5 extends of [13, Theorem 3.1] from 2-uniformly convex and uniformly smooth real Banach spaces to smooth, strictly convex and reflexive real Banach spaces.

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