Appl. Set-Valued Anal. Optim. 3 (2021), No. 2, pp. 193-202
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.3.2021.2.04

CONVERGENCE THEOREMS FOR A FIXED POINT OF η-DEMIMETRIC MAPPINGS IN BANACH SPACES

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Abstract. The purpose of this paper is to propose and investigate an algorithm for solving a fixed point of η-demimetric mappings. We establish the strong convergence of the proposed algorithm under some mild conditions in Banach spaces. We apply these results to obtain new strong convergence theorems which are connected with the η-demimetric fixed point problems in Hilber/Banach spaces.
Keywords. Fixed point; η-Demimetric mapping; Monotone mappings; Strong convergence; Uniformly convex spaces.

1. Introduction

Let C be a nonempty subset of a real smooth Banach space E with its dual E^*. Let $\eta \in (-\infty, 1)$ and $p \in (1, \infty)$. A mapping $T : C \to E$ with $F(T) \neq \emptyset$ is called η-*demimetric* (see, [1, 2]) if, for any $x \in C$ and $x^* \in F(T)$, there exists $j_E^p(x - Tx) \in J_E^p(x - Tx)$ such that

$$\langle x - x^*, j_E^p(x - Tx)\rangle \geq \frac{1-\eta}{2}||x - Tx||^p, \tag{1.1}$$

where J_E^p is the *generalized duality* mapping from E into 2^{E^*} defined by

$$J_E^p(x) := \{f \in E^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}. \tag{1.2}$$

We note that, in (1.2), if $p = 2$, then $J_E^2 = J_E$ is called the *normalized duality mapping*. It is well-known (see, for example, ([3, 4]) that if E is smooth, then J_E^p is one-to-one and single valued and it also satisfies

$$J_E^p(x) = \|x\|^{p-2}J_E(x), \quad x \neq 0.$$

Furthermore, if E is uniformly smooth , then J_E^p is uniformly continuous on bounded subsets of E; if E is reflexive, strictly convex and smooth Banach space, then $J_{E^*}^q : E^* \to 2^E$ is one-to-one, surjective, and it is the duality mapping from E^* into E. Thus, $J_E^p J_{E^*}^q = I_{E^*}$ and $J_{E^*}^q J_E^p = I_E$, where q is a real number satisfying $\frac{1}{p} + \frac{1}{q} = 1$ (see, e.g., [5, 6]). We also note that if $E := H$, a real Hilbert space, then $J_H = I$, where I is the identity mapping.

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Received April 6, 2020; Accepted June 19, 2020.

We remark that inequality (1.1) is equivalent to

$$\langle x-x^*, J_E(x-Tx)\rangle \geq \frac{1-\eta}{2}||x-Tx||^2,$$

for all $x\in C$ and $x^*\in F(T)$.

Let E be a reflexive, strictly convex, and smooth Banach space and let $A: E\to 2^{E^*}$ be a maximal monotone mapping with $A^{-1}(0)\neq\emptyset$. Then, for each $x\in E$ and $r>0$, consider the mapping defined by $J_r^A x := \{z\in E : 0\in J(z-x)+rAz\}$ (equivalently, $J_r^A=(I+(J_E^p)^{-1}A)^{-1}: E\to E^*$), which is point-to-point and nonexpansive mapping (see, for example, Proposition 57.5(b) of [7]). Then, the resolvent J_r^A of A with $A^{-1}(0)$ nonempty is (-1)-demimetric (see, for example, [8]). Let C be a nonempty, closed and convex subset of a strictly convex, reflexive and smooth real Banach space E. Let P_C be the metric projection of E onto C. Then P_C is (-1)-demimetric (see [9]). Other examples of η-demimetric mappings in Hilbert spaces are those mappings in a class of demicontractive mappings. A mapping $T: C\to H$, where C is a subset of a Hilbert space H, is called *k-demicontractive* if there exists $k\in[0,1)$ such that

$$||Tx-Tx^*||^2\leq ||x-x^*||^2+k||x-Tx||^2,$$

for each $x\in C$ and $x^*\in F(T)$. We note that the class of k-demicontractive mappings contains the class of quasi-nonexpansive mappings, that is, $F(T)\neq\emptyset$ and

$$||Tx-Tx^*||\leq||x-x^*||, \text{ for all } x\in C, x^*\in F(T).$$

Several authors have studied various methods for fixed points of k-demicontractive mappings (see, for example, [10, 11]). It was shown in [12] that k-demicontractive mappings are k-demimetric.

In 2016, Hojo and Takahashi [13] used the shrinking projection method to approximate fixed points of η-demimetric mappings in Banach spaces. Indeed, they proved the following result.

Theorem 1.1. *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space E. Let η be in $(-\infty,1)$. Let $T: C\to C$ be an η-demimetric and demiclosed mapping with with $F(T)\neq\emptyset$. Let $x_1\in C$, $C_1=C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n=\alpha_n x_n+(1-\alpha_n)Tx_n,\\ C_n=\{z\in C: 2\langle x_n-z, J_E(x_n-z_n)\rangle\geq(1-\eta)||x_n-z_n||^2\},\\ Q_n=\{z\in C:\langle x_n-z, J_E(x_1-x_n)\rangle\geq 0\},\\ x_{n+1}=P_{C_n\cap Q_n}x_1, n\geq 1,\end{cases} \tag{1.3}$$

where $0\leq\alpha_n\leq a<1$ for some $a\in\mathbb{R}$. Then, the sequence $\{x_n\}$ converges strongly to a point $\widehat{z_0}\in F(T)$, where $\widehat{z_0}=P_{F(T)}x_1$.

We observe that Algorithms (1.3) requires to calculate C_n and Q_n, which are not easy at each iterative step. This leads us to the following question.

Question: Can one obtain an iterative scheme which converges strongly to a fixed point of an η-demimetric mapping and does not involve the calculation of C_n and Q_n for each iterative step in Banach spaces?

Motivated and inspired by Hojo and Takahashi [13], we study a new iterative scheme for fixed points of η-demimetric mappings. We obtain strong convergence of the scheme under some mild conditions in the setting of Banach spaces. Our results provide an affirmative answers to the above question.

2. PRELIMINARIES

A real Banach space E is said to be *smooth* if $\lim_{t\to 0}\frac{||x+ty||-||x||}{t}$ exists for each $x,y\in S(E):=\{x\in E:||x||=1\}$. A space E is called *q-uniformly smooth* if there exist a constant $c>0$ and a real number $q\in(1,\infty)$ such that $\rho_E(\tau)\le c\tau^q$, where $\rho_E:[0,\infty)\to[0,\infty)$ is defined by

$$\rho_E(\tau):=\sup\left\{\frac{\|x+y\|+\|x-y\|}{2}-1:\|x\|=1,\|y\|=\tau\right\}.$$

E is called *uniformly smooth* if $\lim_{\tau\to 0}\frac{\rho_E(\tau)}{\tau}=0$. A Banach space E is called *uniformly convex* if and only if $\sigma(\varepsilon)>0$, for every $\varepsilon\in(0,2]$, where $\sigma:(0,2]\to[0,1]$ is defined by

$$\sigma(\varepsilon)=\inf\left\{1-||\frac{x+y}{2}||:||x||=||y||=1;\varepsilon=||x-y||\right\}.$$

Let $p>0$. Then, E is said to be $p-$uniformly convex if there exists a constant $c>0$ such that $\sigma(\varepsilon)\ge c\varepsilon^p$, for all $\varepsilon\in(0,2]$.

We remark that E is uniformly smooth if and only if E^* is uniformly convex (see [14]). We also know that E is q-uniformly smooth if and only if its dual E^* is p-uniformly convex. The examples of such spaces are the L_p, l_p and W_m^p spaces for $1<p<\infty$ (see, for example, [15]), where

$$L_p\ (l_p)\ or\ W_m^p \text{ is } \begin{cases} \text{2-uniformly smooth and } p\text{-uniformly convex if } 2\le p<\infty; \\ \text{2-uniformly convex and } p\text{-uniformly smooth if } 1<p<2. \end{cases}$$

The following result was proved by Xu [15] in q-uniformly smooth spaces.

Lemma 2.1. *Let $x,y\in E$. If E is q-uniformly smooth, then there is a $c_q>0$ such that*

$$||x+y||^q\le||x||^q+q\langle y,J_E^q(x)\rangle+c_q||y||^q. \tag{2.1}$$

Let $1<q,p<\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. The *Bregman distance* with power p is defined by

$$\phi_p(x,y) \;=\; \frac{1}{q}||x||^p-\langle J_E^p x,y\rangle+\frac{1}{p}||y||^p. \tag{2.2}$$

Lemma 2.2. [16] *Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi_p(x_n,y_n)\to 0$ as $n\to\infty$, then $x_n-y_n\to 0$ as $n\to\infty$.*

Let C be a nonempty, closed and convex subset of a smooth and strictly convex real Banach space E. The *Bregman projection* is the unique minimizer of the Bregman distance [17] given by

$$\Pi_C x=\arg\min_{y\in C}\phi_p(x,y), x\in E.$$

If $E=H$, a Hilbert space, then the Bregman projection Π_C reduces to the metric projection P_C from H onto C.

We remark that the Bregman projection has the following properties (see [3]):

$$\langle J_E^p(x)-J_E^p(\Pi_C x),z-\Pi_C x\rangle\le 0,\ \forall z,\in C, \tag{2.3}$$

and

$$\phi_p(\Pi_C x,z)\le\phi_p(x,z)-\phi_p(x,\Pi_C x),\ \forall z\in C. \tag{2.4}$$

Let $V_p : E^* \times E \to [0, +\infty)$ be defined by

$$V_p(\bar{x}, x) = \frac{1}{q}||\bar{x}||^q - \langle \bar{x}, x\rangle + \frac{1}{p}||x||^p, \forall x \in E, \bar{x} \in E^*.$$

Then, we observe that V_p is characterized by

$$V_p(\bar{x}, x) = \phi_p((J_E^p)^{-1}(\bar{x}), x), \tag{2.5}$$

for all $x \in E$ and $\bar{x} \in E^*$. V_p is convex in the first variable, that is, for all $z \in E$,

$$\phi_p\Big((J_E^p)^{-1}\Big(\sum_{i=1}^{N} t_i J_E^p(x_i)\Big), z\Big) \le \sum_{i=1}^{N} t_i \phi_p(x_i, z), \tag{2.6}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum\limits_{i=1}^{N} t_i = 1$ (see, e.g., [18]). Moreover, by the subdifferential inequality, we have

$$V_p(\bar{x}, x) \le V_p(\bar{x} + \bar{y}, x) - \langle \bar{y}, (J_E^p)^{-1}(\bar{x}) - x\rangle, \tag{2.7}$$

for all $x \in E$ and $\bar{x}, \bar{y} \in E^*$ (see also [19, 20]).

We also need the following lemmas.

Lemma 2.3. [21] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$: $a_{m_k} \le a_{m_k+1}$ and $a_k \le a_{m_k+1}$. In fact, m_k is the largest number n in the set $\{1, 2, ..., k\}$ such that the condition $a_n \le a_{n+1}$ holds.*

Lemma 2.4. [22] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation: $a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\delta_n$, $n \ge n_0$, where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \delta_n \le 0$. Then, $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.5. [2] *Let C be a nonempty, closed and convex subset of a smooth and strictly convex Banach space E. Let η be in $(-\infty, 1)$. Let T be an η-demimetric mapping of C into E. Then $F(T)$ is closed and convex*

3. Main results

Let C be a subset of a real Banach space E, and let $T : C \to E$ be a mapping. The mapping $(I - T)$ is called *demiclosed at zero* if for a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x = Tx$. Throughout this section, unless otherwise specified, we assume that $p > 1$ and $q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and $\{\alpha_n\} \subset (0, e) \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$.

We now in a position to prove our main theorem.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a p-uniformly convex and smooth real Banach space E. Let $T : C \to E$ be an η-demimetric mapping with $F(T) \neq \emptyset$. For arbitrary $x_0, u \in C$, define an iterative sequence by*

$$\begin{cases} y_n = (J_E^p)^{-1}\big[J_E^p x_n - t_n J_E^p(x_n - Tx_n)\big], \\ x_{n+1} = \Pi_C (J_E^p)^{-1}\big[\alpha_n J_E^p u + (1-\alpha_n) J_E^p y_n\big], \end{cases} \tag{3.1}$$

where $0 < \delta \le t_n \le \gamma < \big(\frac{(1-\eta)q}{2c_q}\big)^{\frac{1}{q-1}}$, for all $n \ge 0$ and c_q is the constant in (2.1). Then $\{x_n\}$ is bounded.

Proof. Fix $x^* \in F(T)$. From (2.2), (3.1) and Lemma 2.1, we obtain

$$\begin{aligned}
\phi_p(y_n,x^*) &= \phi_p\Big((J_E^p)^{-1}\big[J_E^p x_n - t_n J_E^p(x_n - Tx_n)\big], x^*\Big)\\
&= \frac{1}{q}||J_E^p x_n - t_n J_E^p(x_n - Tx_n)||^q - \langle J_E^p x_n, x^*\rangle + t_n\langle J_E^p(x_n - Tx_n), x^*\rangle + \frac{1}{p}||x^*||^p\\
&\le \frac{1}{q}\Big(||J_E^p x_n||^q - q t_n\langle J_E^p(x_n - Tx_n), x_n\rangle + t_n^q c_q ||J_E^p(x_n - Tx_n)||^q\Big) - \langle J_E^p x_n, x^*\rangle\\
&\quad + t_n\langle J_E^p(x_n - Tx_n), x^*\rangle + \frac{1}{p}||x^*||^p\\
&= \frac{1}{q}||x_n||^p - t_n\langle J_E^p(x_n - Tx_n), x_n - x^*\rangle + \frac{t_n^q c_q}{q}||J_E^p(x_n - Tx_n)||^q - \langle J_E^p x_n, x^*\rangle + \frac{1}{p}||x^*||^p\\
&\le \frac{1}{q}||x_n||^p - t_n\frac{(1-\eta)}{2}||x_n - Tx_n||^p + \frac{t_n^q c_q}{q}||x_n - Tx_n||^p - \langle J_E^p x_n, x^*\rangle + \frac{1}{p}||x^*||^p,
\end{aligned}$$

which implies that

$$\phi_p(y_n,x^*) \;\le\; \phi_p(x_n,x^*) - t_n\Big(\frac{(1-\eta)}{2} - t_n^{q-1}\frac{c_q}{q}\Big)||x_n - Tx_n||^p. \tag{3.2}$$

Now, from (3.1) and (3.2), we derive

$$\begin{aligned}
\phi_p(x_{n+1},x^*) &\le \alpha_n\phi_p(u,x^*) + (1-\alpha_n)\phi_p(y_n,x^*)\\
&\le \alpha_n\phi_p(u,x^*) + (1-\alpha_n)\phi_p(x_n,x^*)\\
&\quad -(1-\alpha_n)t_n\Big(\frac{(1-\eta)}{2} - t_n^{q-1}\frac{c_q}{q}\Big)||(I-T)x_n||^p.
\end{aligned}$$

Since

$$\frac{(1-\eta)}{2} - t_n^{q-1}\frac{c_q}{q} > 0,$$

it follows that

$$\phi_p(x_{n+1},y^*) \le +\alpha_n\phi_p(u,x^*) + (1-\alpha_n)\phi_p(x_n,x^*).$$

Now, we show that $\{\phi_p(x_n,x^*)\}$ is a bounded sequence. It suffices to show that $\phi_p(x_n,x^*) \le M$ for all $n \ge 1$, where $M := \max\{\phi_p(u,x^*), \phi_p(x_0,x^*)\}$. Note that if $n = 0$, then $\phi_p(x_0,x^*) \le M$. Assume that $\phi_p(x_n,x^*) \le M$, for $n \ge 1$. Then

$$\phi_p(x_{n+1},x^*) \le (1-\alpha_n)\phi_p(x_n,x^*) + \alpha_n\phi_p(u,x^*) \le M.$$

This shows that $\{\phi(x_n,x^*)\}$ is bounded and hence $\{x_n\}$ and $\{y_n\}$ are also bounded. □

Theorem 3.2. *Let C be a nonempty, closed and convex subset of a p-uniformly convex and smooth real Banach space E. Let η be in $(-\infty,1)$. Let $T : C \to E$ be an η-demimetric mapping. Assume that $I - T$ is demiclosed at zero and $F(T) \neq \emptyset$. For arbitrary $x_0, u \in C$, define an iterative sequence $\{x_n\}$ by (3.1). Then, $\{x_n\}$ converges strongly to $\widehat{x} = \Pi_{F(T)}u$.*

Proof. From Theorem 3.1, we know that $\{x_n\}$ is bounded. Take $\widehat{x} = \Pi_{F(T)}u$. Using (2.3), we get

$$\langle J_E^p u - J_E^p\widehat{x}, z - \widehat{x}\rangle \le 0, \forall z \in F(T). \tag{3.3}$$

Now, from (2.5), (2.6), (2.7) and (3.1), we obtain

$$
\begin{aligned}
\phi_p(x_{n+1},\widehat{x}) &\le \phi_p\big((J_E^p)^{-1}(\alpha_n J_E^p u+(1-\alpha_n)J_E^p y_n),\widehat{x}\big)\\
&= V_p\big(\alpha_n J_E^p u+(1-\alpha_n)J_E^p y_n,\widehat{x}\big)\\
&= V_p\big(\alpha_n J_E^p u+(1-\alpha_n)J_E^p y_n-\alpha_n(J_E^p u-J_E^p\widehat{x}),\widehat{x}\big)\\
&\quad+\langle\alpha_n(J_E^p u-J_E^p\widehat{x}),x_{n+1}-\widehat{x}\rangle\\
&= V_p\big(\alpha_n J_E^p\widehat{x}+(1-\alpha_n)J_E^p y_n,\widehat{x}\big)+\alpha_n\langle J_E^p u-J_E^p\widehat{x},x_{n+1}-\widehat{x}\rangle\\
&= \phi_p\big((J_E^p)^{-1}(\alpha_n J_E^p\widehat{x}+(1-\alpha_n)J_E^p y_n),\widehat{x}\big)\\
&\quad+\alpha_n\langle J_E^p u-J_E^p\widehat{x},x_{n+1}-\widehat{x}\rangle\\
&\le \alpha_n\phi_p(\widehat{x},\widehat{x})+(1-\alpha_n)\phi_p(y_n,\widehat{x})+\alpha_n\langle J_E^p u-J_E^p\widehat{x},x_{n+1}-\widehat{x}\rangle\\
&= (1-\alpha_n)\phi_p(y_n,\widehat{x})+\alpha_n\langle J_E^p u-J_E^p\widehat{x},x_{n+1}-\widehat{x}\rangle,\\
&\le (1-\alpha_n)\phi_p(x_n,\widehat{x})+\alpha_n\langle J_E^p u-J_E^p\widehat{x},x_{n+1}-\widehat{x}\rangle\\
&\quad-(1-\alpha_n)t_n\Big(\frac{(1-\eta)}{2}-t_n^{q-1}\frac{c_q}{q}\Big)||x_n-Tx_n||^p,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\phi_p(x_{n+1},\widehat{x}) \;\le\;& (1-\alpha_n)\phi_p(x_n,\widehat{x})+\alpha_n\langle J_E^p u-J_E^p\widehat{x},x_n-\widehat{x}\rangle\\
&+\alpha_n||J_E^p u-J_E^p\widehat{x}||.||x_{n+1}-x_n||\\
&-(1-\alpha_n)t_n\Big(\frac{(1-\eta)}{2}-t_n^{q-1}\frac{c_q}{q}\Big)||(I-T)x_n||^p. \qquad (3.4)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\phi_p(x_{n+1},\widehat{x}) \;\le\;& (1-\alpha_n)\phi_p(x_n,\widehat{x})+\alpha_n\langle J_E^p u-J_E^p\widehat{x},x_n-\widehat{x}\rangle\\
&+\alpha_n||J_E^p u-J_E^p\widehat{x}||.||x_{n+1}-x_n||. \qquad (3.5)
\end{aligned}
$$

Next, we show that the sequence $\{\phi_p(x_n,\widehat{x})\}$ converges strongly to zero. For this, we consider two possible cases on $\{\phi_p(x_n,\widehat{x})\}$.

Case 1. Assume that there exists $n_0\in\mathbb{N}$ such that the sequence of real numbers $\{\phi_p(x_n,\widehat{x})\}$ is decreasing for all $n\ge n_0$. It then follows that $\{\phi_p(x_n,\widehat{x})\}$ is convergent. Since the sequences $\{x_n\}$ is bounded, we conclude from (3.4) and the fact that $\alpha_n\to 0$ that

$$\lim_{n\to\infty}||x_n-Tx_n||=0.$$

It follows from (3.1) that

$$||J_E^p y_n-J_E^p x_n||=t_n||x_n-Tx_n||^{p-1}\to 0,$$

which together with the fact that $(J_E^p)^{-1}$ is uniformly continuous yields that

$$\lim_{n\to\infty}||y_n-x_n||=0. \qquad (3.6)$$

Furthermore, from (3.1) and the fact that $\alpha_n\to 0$ as $n\to\infty$, we have

$$
\begin{aligned}
\phi_p(x_{n+1},y_n) \;\le\;& \phi_p((J_E^p)^{-1}(\alpha_n J_E^p u+(1-\alpha_n)J_E^p y_n),y_n)\\
\le\;& \alpha_n\phi_p(u,y_n)+(1-\alpha_n)\phi(y_n,y_n)\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

By using Lemma 2.2, we have $x_{n+1} - y_n \to 0$ as $n \to \infty$. Using (3.6) yields

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.7}$$

Since E is uniformly convex, which implies that it is reflexive, and $\{x_n\}$ is bounded in E, we can find a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, which converges weakly to $\bar{x}$ and

$$\limsup_{n\to\infty} \langle J_E^p u - J_E^p \widehat{x}, x_n - \widehat{x}\rangle = \lim_{j\to\infty} \langle J_E^p u - J_E^p \widehat{x}, x_{n_j} - \widehat{x}\rangle. \tag{3.8}$$

Furthermore, the fact that $I - T$ is demiclosed at zero yields that $\bar{x} \in F(T)$. Therefore, from (3.3) and (3.8), we obtain

$$\begin{aligned} \limsup_{n\to\infty} \langle J_E^p u - J_E^p \widehat{x}, x_n - \widehat{x}\rangle &= \lim_{j\to\infty} \langle x_{n_j} - \widehat{x}, J_E^p u - J_E^p \widehat{x}\rangle \\ &= \langle J_E^p u - J_E^p \widehat{x}, \bar{x} - \widehat{x}\rangle \leq 0. \end{aligned} \tag{3.9}$$

In view of (3.5), (3.7), (3.9) and Lemma 2.4, we conclude that $\phi_p(x_n, \widehat{x})$ converges strongly to zero as $n \to \infty$. Therefore, $\{x_n\}$ converges strongly to $\widehat{x} = \Pi_{F(T)} u$.

Case 2. Assume that there exists a subsequence $\{\phi_p(x_{n_i}, \widehat{x})\}$ of $\{\phi_p(x_n, \widehat{x})\}$ such that $\phi_p(x_{n_i}, \widehat{x}) < \phi_p(x_{n_i+1}, \widehat{x})$ for all $i \geq 0$. In view of Lemma 2.3, we can define a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$ and $\phi(x_{m_k}, \widehat{x}) \leq \phi(x_{m_k+1}, \widehat{x})$, and $\phi(x_k, \widehat{x}) \leq \phi(x_{m_k+1}, \widehat{x})$ for all $k \in \mathbb{N}$. Since the sequences $\{x_{m_k}\}$ is bounded, it follows from (3.4) and the methods in Case 1 that $||x_{m_k} - Tx_{m_k}|| \to 0$, $\|x_{m_k+1} - x_{m_k}\| \to 0$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \langle J_E^p u - J_E^p \widehat{x}, x_{m_k} - \widehat{x}\rangle \leq 0.$$

Finally, making use of $\phi(x_{m_k}, \widehat{x}) \leq \phi_p(x_{m_k+1}, \widehat{x})$ for all $k \in \mathbb{N}$, and rearranging terms in (3.5), we derive

$$\begin{aligned} \alpha_{m_k} \phi_(x_{m_k}, \widehat{x}) \leq \phi_p(x_{m_k}, \widehat{x}) - \phi_p(x_{m_k+1}, \widehat{x}) + \alpha_{m_k} \langle J_E^p u - J_E^p \widehat{x}, x_{m_k} - \widehat{x})\rangle \\ + \alpha_{m_k} ||J_E^p u - J_E^p \widehat{x}|| \times ||x_{m_k+1} - x_{m_k}||. \end{aligned}$$

Dividing by α_{m_k} and passing to the limit as $k \to \infty$ in the resulting inequality, we obtain $\phi_p(x_{m_k}, \widehat{x}) \to 0$. Hence, $\phi_p(x_{m_k+1}, \widehat{x}) \to 0$ as $k \to \infty$. Since $\phi_p(x_k, \widehat{x}) \leq \phi_p(x_{m_k+1}, \widehat{x})$, we find that $\phi_p(x_k, \widehat{x}) \to 0$ as $k \to \infty$ and hence $x_k \to \widehat{x}$ as $k \to \infty$. Therefore, we have shown in both cases that the sequence $\{x_n\}$ generated by (3.1) converges strongly to $\widehat{x} \in F(T)$. This completes the proof of this theorem. □

If, in Theorem 3.2, $C = E$, then Π_C is reduced to the identity mapping on E. We get the following corollary.

Corollary 3.1. *Let E be a p-uniformly convex and smooth real Banach space. Let $T : E \to E$ be an η-demimetric mapping. Assume that $I - T$ is demiclosed at zero and $F(T) \neq \emptyset$. For arbitrary $x_0, u \in E$, define an iterative sequence by*

$$\begin{cases} y_n = (J_E^p)^{-1} \left[J_E^p x_n - t_n J_E^p (x_n - Tx_n)\right], \\ x_{n+1} = (J_E^p)^{-1} \left[\alpha_n J_E^p u + (1 - \alpha_n) J_E^p y_n\right], \end{cases}$$

where $0 < \delta \leq t_n \leq \gamma < \left(\frac{(1-\eta)q}{2c_q}\right)^{\frac{1}{q-1}}$, for all $n \geq 0$ and c_q is the constant in (2.1). Then, $\{x_n\}$ converges strongly to the minimum norm $\widehat{x}$ of $F(T)$.

If, in Theorem 3.2, $E = H$, a real Hilbert spaces, then $p = 2$. Hence J_E^p is the identity mapping and $c_q = 1$. Thus, we get the following corollary.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let η be in $(-\infty, 1)$. Let $T : C \to H$ be an η-demimetric mapping. Assume that $I-T$ is demiclosed at zero and $F(T) \neq \emptyset$. For arbitrary $x_0, u \in C$, define an iterative sequence $\{x_n\}$ by*

$$\begin{cases} y_n = x_n - t_n(x_n - Tx_n), \\ x_{n+1} = P_C\left[\alpha_n u + (1-\alpha_n)y_n\right], n \geq 1, \end{cases}$$

where $0 < \delta \leq t_n \leq \gamma < (1-\eta)$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $\widehat{x} = P_{F(T)}(u)$.

If, in Corollary 3.2, T is a self-mapping, then P_C is reduced to the identity mapping on C. Hence, we get the following corollary.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let η be in $(-\infty, 1)$. Let $T : C \to C$ be an η-demimetric mapping. Assume that $I-T$ is demiclosed at zero and $F(T) \neq \emptyset$. For arbitrary $x_0, u \in C$, define an iterative sequence $\{x_n\}$ by*

$$\begin{cases} y_n = x_n - t_n(x_n - Tx_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, n \geq 1, \end{cases}$$

where $0 < \delta \leq t_n \leq \gamma < (1-\eta)$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to an element $\widehat{x} = P_{F(T)}(u)$.

4. Applications

In this section, we present some applications of our results in the context of convex and nonlinear analysis problems.

The following lemmas are needed.

Lemma 4.1. [23] *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to H$ be a k-strictly pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$ as $n \to \infty$, then $z \in F(T)$, that is, $I-T$ is demiclosed at zero.*

Lemma 4.2. [24] *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be generalized hybrid mapping. If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$ as $n \to \infty$, then $z \in F(T)$.*

Lemma 4.3. [9] *Let C be a nonempty, closed and convex subsets of a uniformly smooth and strictly convex Banach space E. Let P_C be the metric projection from E onto C. If $x_n \rightharpoonup z$ and $x_n - P_C x_n \to 0$ as $n \to \infty$, then $z \in F(P_C)$, i.e., $I - P_C$ is demiclosed at zero.*

Theorem 4.1. *Let C be nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a k-strict pseudocontractive mapping with $F(T) \neq \emptyset$. For arbitrary $x_0, u \in C$, define an iterative sequence $\{x_n\}$ by*

$$\begin{cases} y_n = x_n - t_n(x_n - Tx_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, n \geq 1, \end{cases} \tag{4.1}$$

where $0 < \delta \leq t_n \leq \gamma < (1-k)$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to $\widehat{x} = P_{F(T)}(u)$.

Proof. Note that T is k-demimetric and $I-T$ is demiclosed at zero. From Corollary 3.3, we get the desired conclusion immediately. □

If, in Corollary 3.3, T is an (α,β)-generalized hybrid mapping, then we have the following result.

Theorem 4.2. *Let C be nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be an (α,β) generalized hybrid mapping with $F(T) \neq \emptyset$. For arbitrary $x_0, u \in C$, define an iterative sequence $\{x_n\}$ by*

$$\begin{cases} y_n = x_n - t_n(x_n - Tx_n), \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, n \geq 1, \end{cases} \tag{4.2}$$

where $0 < \delta \leq t_n \leq \gamma < 1$, for all $n \geq 0$. Then, $\{x_n\}$ converges strongly to $\widehat{x} = P_{F(T)}(u)$.

Proof. Note that T is 0-demimetric. From Lemma 4.2, we have that $I-T$ is demiclosed. Using Corollary 3.3, we have the desired conclusion immediately. □

If, in Corollary 3.1, $T = P_C$, where C closed and convex subsets of a p-uniformly convex Banach space E, then we get the following result.

Theorem 4.3. *Let C be nonempty, closed and convex subset of a real p-uniformly convex Banach space E. Let $P_C : E \to C$ be the projection mapping. For arbitrary $x_0, u \in E$, define an iterative sequence by*

$$\begin{cases} y_n = (J_E^p)^{-1}\left[J_E^p x_n - t_n J_E^p(x_n - P_C x_n)\right], \\ x_{n+1} = (J_E^p)^{-1}\left[\alpha_n J_E^p u + (1-\alpha_n)J_E^p y_n\right], \end{cases} \tag{4.3}$$

where $0 < \delta \leq t_n \leq \gamma < \left(\frac{q}{c_q}\right)^{\frac{1}{q-1}}$, for all $n \geq 0$ and c_q is the constant in (2.1). Then, $\{x_n\}$ converges strongly to $\widehat{x} = P_C(u)$.

Proof. Take $T := P_C$. Then, T is (-1)-demimetric and by Lemma 4.3, we have that $(I-T)$ is demiclosed at zero. From Corollary 3.1, we have the desired conclusion easily. □

Furthermore, from Theorem 3.2, we also have the following strong convergence result for finding the zero of a maximal monotone operator in Banach spaces.

Theorem 4.4. *Let C be a nonempty, closed and convex subset of a p-uniformly convex and smooth real Banach space E. Let A be a maximal monotone mapping of E into 2^{E^*} and let $J_\lambda^A = (I + \lambda J_E^{-1}A)$ be the resolvent of A with $N(A) \neq \emptyset$. For arbitrary $x_0, u \in C$, define an iterative sequence by*

$$\begin{cases} y_n = (J_E^p)^{-1}\left[J_E^p x_n - t_n J_E^p(x_n - J_\lambda^A x_n)\right], \\ x_{n+1} = \Pi_C J_p^{-1}\left[\alpha_n J_p u + (1-\alpha_n)J_p y_n\right], \end{cases} \tag{4.4}$$

where $0 < \delta \leq t_n \leq \gamma < \left(\frac{q}{c_q}\right)^{\frac{1}{q-1}}$, for all $n \geq 0$ and c_q is the constant in (2.1). Then, $\{x_n\}$ converges strongly to $\widehat{x} \in N(A)$.

Proof. Note that $T := J_\lambda^A$ is nonexpansive and (-1)-demimetric. Furthermore, from [13, Theorem 4.6], we have that $(I - J_\lambda^A)$ is demiclosed at zero. Using Theorem 3.2, we obtain the desired conclusion immediately. □

Conclusions. Theorem 3.2 provides an algorithm which converge strongly to fixed points of η-demimetric mappings in the setting of Banach spaces. Theorem 3.2 improves the results announced by Hojo and Takahashi [13] in the sense that our algorithm does not require the involvement of computation of C_n and Q_n for each iterative setp. Our results provide affirmative answers to the question raised in Section 1.

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