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# ON A VISCOSITY ITERATIVE ALGORITHM FOR VARIATIONAL INCLUSION PROBLEMS AND THE FIXED POINT PROBLEM OF COUNTABLY MANY NONEXPANSIVE MAPPINGS

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**Abstract.** In this paper, we propose a viscosity iterative algorithm for finding a common solution of a common fixed point problem of a countable family of nonexpansive mappings and two variational inclusion problems. We investigate the convergence of the proposed algorithm and a strong convergence theorem is established in the setting of Banach spaces.

**Keywords.** Accretive operator; Nonexpansive mapping; Variational inclusion; Viscosity iterative algorithm.

## 1. INTRODUCTION-PRELIMINARIES

Let $E$ be a real Banach space with the dual $E^*$, and let $\emptyset \neq C \subset E$ be a closed convex set. Let $T$ be a mapping on $C$. We use the symbol $\mathrm{Fix}(T)$ to denote the set of fixed point of $T$. Recall that $T$ is said to be contractive if there exists a constant $\rho \in (0,1)$ such that $\|Tx - Ty\| \leq \rho\|x-y\|$, $\forall x,y \in C$. $T$ is said to be nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$, $\forall x,y \in C$. The class of nonexpansive mappings has many real applications in various optimization problems; see, e.g., [1, 2, 3, 4, 5, 6] and the references therein.

Recall that the generalized duality mapping, $J_q$ ($q>1$), from $E$ into $2^{E^*}$ is defined by

$$J_q(x) := \{x^* \in E^* : \langle x^*, x\rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing between $E$ and $E^*$. In particular, $J_2$ is called the normalized duality mapping and it is usually denoted by $J$. One knows that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \neq 0$.

The modulus of convexity of $E$ is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : x,y \in E,\ \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\}.$$

The modulus of smoothness of $E$ is the function $\rho_E : R_+ := [0,\infty) \to R_+$ defined by

$$\rho_E(\tau) = \sup\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : x,y \in E,\ \|x\| = \|y\| = 1\}.$$

A Banach space $E$ is said to be uniformly convex if $\delta_E(\varepsilon) > 0$, $\forall \varepsilon \in (0,2]$. It is said to be uniformly smooth if $\lim_{\tau\to 0^+} \frac{\rho_E(\tau)}{\tau} = 0$. Also, it is said to be $q$-uniformly smooth with $q > 1$

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if $\exists c > 0$ such that $\rho_E(t) \leq ct^q$, $\forall t > 0$. If $E$ is $q$-uniformly smooth, then $q \leq 2$. It is known that sequence space $\ell_p$ and Lebesgue space $L_p$ are $\min\{p,2\}$-uniformly smooth for every $p > 1$ [7, 8, 9].

Let $B : C \to 2^E$ be a set-valued operator with $Bx \neq \emptyset$, $\forall x \in C$. Let $q > 1$. An operator $B$ is said to be accretive if, for each $x,y \in C$, $\exists j_q(x-y) \in J_q(x-y)$ such that $\langle u-v, j_q(x-y)\rangle \geq 0$, $\forall u \in Bx, v \in By$. Further, $B$ is said to be $m$-accretive if $(I+\lambda B)C = E$ for all $\lambda > 0$. For an $m$-accretive operator $B$, we can define the mapping $J_\lambda^B : (I+\lambda B)C \to C$ by $J_\lambda^B = (I+\lambda B)^{-1}$ for each $\lambda > 0$. Such $J_\lambda^B$ is called the resolvent of $B$ for $\lambda > 0$. In the sequel, we will use the notation $T_\lambda := J_\lambda^B(I-\lambda A) = (I+\lambda B)^{-1}(I-\lambda A)$, $\forall \lambda > 0$. From Barbu [10], one has the following resolvent identity: $J_\lambda^B x = J_\mu^B(\frac{\mu}{\lambda}x + (1-\frac{\mu}{\lambda})J_\lambda^B x)$, $\forall \lambda, \mu > 0$, $x \in E$. An accretive operator $B$ is said to be $\alpha$-inverse-strongly accretive of order $q$ if, for each $x,y \in C$, there exists $j_q(x-y) \in J_q(x-y)$ such that $\langle u-v, j_q(x-y)\rangle \geq \alpha\|u-v\|^q$, $\forall u \in Bx, v \in By$ for some $\alpha > 0$. If $E = H$ a Hilbert space, then $B$ is called $\alpha$-inverse-strongly monotone.

In 2017, Chang, Wen and Yao [11] suggested a generalized viscosity implicit rule for finding a point in $(A+B)^{-1}0$, where $A : E \to E$ is an $\alpha$-inverse-strongly accretive mapping of order $q$ and $B : E \to 2^E$ is an $m$-accretive operator:

$$x_{j+1} = \alpha_j f(x_j) + (1-\alpha_j)J_\lambda^B(I-\lambda A)(t_j x_j + (1-t_j)x_{j+1}), \quad \forall j \geq 1,$$

where $f$ is a contractive mapping, $J_\lambda^B = (I+\lambda B)^{-1}$, $\lambda \in (0,\infty)$ and the sequences $\{t_j\}, \{\alpha_j\} \subset (0,1)$ are such that (i) $\lim_{j\to\infty}\alpha_j = 0$, $\sum_{j=1}^\infty \alpha_j = \infty$; (ii) $\sum_{j=1}^\infty |\alpha_{j+1}-\alpha_j| < \infty$; (iii) $0 < \varepsilon \leq t_j \leq t_{j+1} < 1$; (iv) $0 < \lambda \leq (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$. They proved the strong convergence of $\{x_j\}$ to a point of $(A+B)^{-1}0$, which solves some generalized variational inequality.

Recently, Sunthrayuth and Cholamjiak [12] proposed the following modified viscosity-type extragradient method for a fixed point problem of a nonexpansive mapping and solution of the inclusion problem of an $\alpha$-inverse-strongly accretive mapping $A$ and an $m$-accretive operator $B$

$$\begin{cases} y_j = J_{\lambda_j}^B(x_j - \lambda_j Ax_j), \\ z_j = J_{\lambda_j}^B(x_j - \lambda_j Ay_j + r_j(y_j - x_j)), \\ x_{j+1} = \alpha_j f(x_j) + \beta_j x_j + \gamma_j Sz_j, \quad \forall j \geq 0, \end{cases}$$

where $S$ is a nonexpansive mapping on $C$, $J_{\lambda_j}^B = (I+\lambda_j B)^{-1}$, $\{r_j\}, \{\alpha_j\}, \{\beta_j\}, \{\gamma_j\} \subset (0,1)$ and $\{\lambda_j\} \subset (0,\infty)$ are such that: (i) $\alpha_j + \beta_j + \gamma_j = 1$; (ii) $\lim_{j\to\infty}\alpha_j = 0$, $\sum_{j=1}^\infty \alpha_j = \infty$; (iii) $\{\beta_j\} \subset [a,b] \subset (0,1)$; and (iv) $0 < \lambda \leq \lambda_j < \lambda_j/r_j \leq \mu < (\alpha q/\kappa_q)^{1/(q-1)}$, $0 < r \leq r_j < 1$. They proved the strong convergence of $\{x_j\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$, which solves some generalized variational inequality.

In this paper, we consider the variational inclusion with two accretive operators and a common fixed point problem of a countable family of nonexpansive mappings. A viscosity method is introduced and the strong convergence of the suggested method is obtained in a uniformly convex and $q$-uniformly smooth Banach space with $q \in (1,2]$. Our results improve and extend the corresponding results in [11, 12, 13, 14, 15, 16, 17] to a certain extent.

For obtain our main convergence theorem, we need the following lemmas.

**Lemma 1.1.** [13] *Let $E$ be a $q$-uniformly smooth Banach space with $q \in (1,2]$. Let $A : C \to E$ be an $\alpha$-inverse-strongly accretive mapping of order $q$. Then, for any given $\lambda \geq 0$,*

$$\|(I-\lambda A)x - (I-\lambda A)y\|^q \leq \|x-y\|^q - \lambda(\alpha q - \kappa_q \lambda^{q-1})\|Ax-Ay\|^q, \quad \forall x,y \in C,$$

*where* $\kappa_q > 0$ *is the q-uniform smoothness coefficient of E. In particular, if* $0 \le \lambda \le (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$, *then* $I - \lambda A$ *is nonexpansive.*

**Lemma 1.2.** [9] *Let* $q \in (1,2]$ *a fixed real number and let E be q-uniformly smooth. Then* $\|x + y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + \kappa_q\|y\|^q$, $\forall x, y \in E$, *where* $\kappa_q$ *is the q-uniform smoothness coefficient of E.*

**Lemma 1.3.** [9, 16] *Let* $q > 1$ *and* $r > 0$ *be two fixed real numbers and let E be uniformly convex. Let* $B_r := \{x \in E : \|x\| \le r\}$. *Then there exist strictly increasing, continuous and convex functions* $g, h : R_+ \to R_+$ *with* $g(0) = 0$ *and* $h(0) = 0$ *such that*

*(a)* $\|\mu x + (1-\mu)y\|^q \le \mu\|x\|^q + (1-\mu)\|y\|^q - \mu(1-\mu)g(\|x-y\|)$ *for all* $x, y \in B_r$ *and* $\mu \in [0,1]$*;*

*(b)* $\|\lambda x + \mu y + \nu z\|^q \le \lambda\|x\|^q + \mu\|y\|^q + \nu\|z\|^q - \lambda\mu g(\|x-y\|)$ *for all* $x, y, z \in B_r$ *and* $\lambda, \mu, \nu \in [0,1]$ *with* $\lambda + \mu + \nu = 1$.

*(c)* $h(\|x-y\|) \le \|x\|^q - q\langle x, j_q(y)\rangle + (q-1)\|y\|^q$ *for all* $x, y \in B_r$ *and* $j_q(y) \in J_q(y)$.

**Lemma 1.4.** [18] *Let* $\{S_n\}_{n=0}^{\infty}$ *be a sequence of self-mappings on C such that* $\sum_{n=1}^{\infty} \sup_{x\in C} \|S_n x - S_{n-1}x\| < \infty$. *Then for each* $y \in C$, $\{S_n y\}$ *converges strongly to some point of C. Moreover, let S be a self-mapping on C defined by* $Sy = \lim_{n\to\infty} S_n y$, $\forall y \in C$. *Then* $\lim_{n\to\infty} \sup_{x\in C} \|S_n x - Sx\| = 0$.

**Lemma 1.5.** [13, 19] *The following statements hold:*

*(i)* $\mathrm{Fix}(T_\lambda) = (A+B)^{-1}0$, $\forall \lambda > 0$*;*

*(ii)* $\|y - T_\lambda y\| \le 2\|y - T_r y\|$ *for* $0 < \lambda \le r$ *and* $y \in C$.

**Lemma 1.6.** [20] *Let E be strictly convex, and let* $\{S_n\}_{n=0}^{\infty}$ *be a sequence of nonexpansive mappings on C. Suppose that* $\bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$ *is nonempty. Let* $\{\lambda_n\}$ *be a sequence of positive numbers with* $\sum_{n=0}^{\infty} \lambda_n = 1$. *Then a mapping S on C defined by* $Sx = \sum_{n=0}^{\infty} \lambda_n S_n x$, $\forall x \in C$ *is defined well, nonexpansive and* $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$ *holds.*

**Lemma 1.7.** [21] *Let E be uniformly smooth,* $T : C \to C$ *a nonexpansive mapping with* $\mathrm{Fix}(T) \ne \emptyset$ *and* $f : C \to C$ *a fixed contraction. For each* $t \in (0,1)$, *let* $z_t \in C$ *be the unique fixed point of the contraction* $C \ni z \mapsto tf(z) + (1-t)Tz$ *on C, i.e.,* $z_t = tf(z_t) + (1-t)Tz_t$. *Then* $\{z_t\}$ *converges strongly to a fixed point* $x^* \in \mathrm{Fix}(T)$, *which solves* $\langle (I-f)x^*, J(x^*-x)\rangle \le 0$, $\forall x \in \mathrm{Fix}(T)$.

**Lemma 1.8.** [22] *Let* $\{a_n\}$ *be a sequence in* $[0,\infty)$ *such that* $a_{n+1} \le (1-s_n)a_n + s_n\nu_n$, $\forall n \ge 0$, *where* $\{s_n\}$ *and* $\{\nu_n\}$ *satisfy the conditions: (i)* $\{s_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} s_n = \infty$*; (ii)* $\limsup_{n\to\infty} \nu_n \le 0$ *or* $\sum_{n=0}^{\infty} |s_n\nu_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

**Lemma 1.9.** [23] *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* $\{\Gamma_{n_i}\}$ *of* $\{\Gamma_n\}$ *which satisfies* $\Gamma_{n_i} < \Gamma_{n_i+1}$ *for each integer* $i \ge 1$. *Define the sequence* $\{\tau(n)\}_{n\ge n_0}$ *of integers as follows:*

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\},$$

*where integer* $n_0 \ge 1$ *such that* $\{k \le n_0 : \Gamma_k < \Gamma_{k+1}\} \ne \emptyset$. *Then, the following hold:*

*(i)* $\tau(n_0) \le \tau(n_0+1) \le \cdots$ *and* $\tau(n) \to \infty$*;*

*(ii)* $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \le \Gamma_{\tau(n)+1}$ $\forall n \ge n_0$.

## 2. Main Results

From now on, we always suppose that $C$ is a nonempty closed convex subset of a uniformly convex and $q$-uniformly smooth Banach space $E$ with $q\in(1,2]$. Let $B,M:C\to 2^E$ be $m$-accretive operators, and let $A,F:C\to E$ be $\alpha$-inverse-strongly accretive mapping of order $q$ and $\beta$-inverse-strongly accretive mapping of order $q$, respectively. Let $f:C\to C$ be a $\rho$-contraction with constant $\rho\in[0,\frac{1}{q})$, and let $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self-mappings on $C$. Assume that $\Omega:=\bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap(A+B)^{-1}0\cap(F+M)^{-1}0\neq\emptyset$.

**Algorithm 2.1.** Initial Step. Give $\zeta\in(0,1)$ and $x_0\in C$.

Iteration Steps. Given the current iterate $x_n$, compute $x_{n+1}$ as follows:

Step 1. Calculate $v_n=J^B_{\lambda_n}(x_n-\lambda_nAx_n)$;

Step 2. Calculate $u_n=J^B_{\lambda_n}(x_n-\lambda_nAv_n+r_n(v_n-x_n))$;

Step 3. Calculate $x_{n+1}=(1-\zeta)S_nu_n+\zeta J^M_{\mu_n}(\alpha_nf(u_n)+(1-\alpha_n)u_n-\mu_nFu_n)$, where $\{r_n\}$, $\{\alpha_n\}\subset(0,1)$ and $\{\lambda_n\},\{\mu_n\}\subset(0,\infty)$.

Set $n:=n+1$ and go to Step 1.

**Lemma 2.1.** *If $\{x_n\}$ is the sequence generated by Algorithm 2.1, then it is bounded.*

*Proof.* Fix $p\in\Omega$. Then

$$\begin{aligned}p=S_np\ &=J^B_{\lambda_n}(p-\lambda_nAp)=J^B_{\lambda_n}((1-r_n)p+r_n(p-\tfrac{\lambda_n}{r_n}Ap))\\&=J^M_{\mu_n}(p-\mu_nFp)=J^M_{\mu_n}(\alpha_np+(1-\alpha_n)(p-\tfrac{\mu_n}{1-\alpha_n}Fp)).\end{aligned}$$

Using Lemmas 1.1, we have

$$\begin{aligned}\|v_n-p\|^q\ &\le\|(I-\lambda_nA)x_n-(I-\lambda_nA)p\|^q\\&\le\|x_n-p\|^q-\lambda_n(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Ap\|^q,\end{aligned}$$

which hence leads to $\|v_n-p\|\le\|x_n-p\|$. By the convexity of $\|\cdot\|^q$, we deduce that

$$\begin{aligned}&\|u_n-p\|^q\\&\le\|((1-r_n)x_n+r_n(v_n-\tfrac{\lambda_n}{r_n}Av_n))-((1-r_n)p+r_n(p-\tfrac{\lambda_n}{r_n}Ap))\|^q\\&\le(1-r_n)\|x_n-p\|^q+r_n\|(I-\tfrac{\lambda_n}{r_n}A)v_n-(I-\tfrac{\lambda_n}{r_n}A)p\|^q\\&\le(1-r_n)\|x_n-p\|^q+r_n[\|v_n-p\|^q-\tfrac{\lambda_n}{r_n}(\alpha q-\tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Av_n-Ap\|^q]\\&\le(1-r_n)\|x_n-p\|^q+r_n[\|x_n-p\|^q-\lambda_n(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Ap\|^q\\&\quad-\tfrac{\lambda_n}{r_n}(\alpha q-\tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Av_n-Ap\|^q]\\&=\|x_n-p\|^q-r_n\lambda_n(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Ap\|^q-\lambda_n(\alpha q-\tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Av_n-Ap\|^q.\end{aligned}$$

This ensures that $\|u_n-p\|\le\|x_n-p\|$. We now put

$$y_n:=J^M_{\mu_n}(\alpha_nf(u_n)+(1-\alpha_n)u_n-\mu_nFu_n)$$

for all $n \geq 0$. Since $J^M_{\mu_n}$ and $I - \frac{\mu_n}{1-\alpha_n}F$ are nonexpansive for all $n \geq 0$, we obtain that

$$\begin{aligned}\|y_n - p\| &\leq \|(\alpha_n f(u_n) + (1-\alpha_n)(u_n - \tfrac{\mu_n}{1-\alpha_n}Fu_n)) - (\alpha_n p + (1-\alpha_n)(p - \tfrac{\mu_n}{1-\alpha_n}Fp))\| \\ &\leq (1-\alpha_n)\|u_n - p\| + \alpha_n\|f(u_n) - f(p)\| + \alpha_n\|f(p) - p\| \\ &\leq (1-\alpha_n(1-\rho))\|u_n - p\| + \alpha_n\|f(p) - p\| \\ &\leq (1-\alpha_n(1-\rho))\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &\leq \max\{\|x_n - p\|, \tfrac{\|f(p)-p\|}{1-\rho}\}.\end{aligned}$$

Hence, from the nonexpansivity of $S_n$, we get

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1-\zeta)\|S_n u_n - p\| + \zeta\|y_n - p\| \\ &\leq (1-\zeta)\|x_n - p\| + \zeta \max\{\|x_n - p\|, \tfrac{\|f(p)-p\|}{1-\rho}\} \\ &\leq \max\{\|x_n - p\|, \tfrac{\|f(p)-p\|}{1-\rho}\}.\end{aligned}$$

By induction, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|f(p)-p\|}{1-\rho}\}, \quad \forall n \geq 0.$$

Consequently, $\{x_n\}$ is bounded, and so are $\{u_n\}\{v_n\}, \{y_n\}, \{S_n u_n\}, \{Ax_n\}, \{Av_n\}$ and $\{Fu_n\}$. This completes the proof. □

**Theorem 2.1.** *Let $\{x_n\}$ be the sequence generalized by Algorithm 2.1. Suppose that the following conditions hold:*

*(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;*

*(C2) $0 < r \leq r_n < 1$ and $0 < \lambda \leq \lambda_n < \frac{\lambda_n}{r_n} \leq \overline{\lambda} < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}}$;*

*(C3) $0 < a \leq \frac{\mu_n}{1-\alpha_n} \leq b < (\frac{\beta q}{\kappa_q})^{\frac{1}{q-1}}$.*

*Assume that $\sum_{n=0}^{\infty} \sup_{x\in D}\|S_{n+1}x - S_n x\| < \infty$ for any bounded subset $D$ of $C$. Let $S : C \to C$ be a mapping defined by $Sx = \lim_{n\to\infty} S_n x$ $\forall x \in C$, and suppose that $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. Then $x_n \to x^* \in \Omega$.*

*Proof.* Fix $x^* \in \Omega$. Using Lemma 1.1 and Lemma 1.2, we get

$$\begin{aligned}&\|y_n - x^*\|^q \\ &\leq \|(1-\alpha_n)((u_n - \tfrac{\mu_n}{1-\alpha_n}Fu_n) - (x^* - \tfrac{\mu_n}{1-\alpha_n}Fx^*)) + \alpha_n(f(u_n) - x^*)\|^q \\ &\leq (1-\alpha_n)^q\|(u_n - \tfrac{\mu_n}{1-\alpha_n}Fu_n) - (x^* - \tfrac{\mu_n}{1-\alpha_n}Fx^*)\|^q \\ &\quad + q\alpha_n(1-\alpha_n)^{q-1}\langle f(u_n) - x^*, J_q(u_n - x^* - \tfrac{\mu_n}{1-\alpha_n}(Fu_n - Fx^*))\rangle + \kappa_q\alpha_n^q\|f(u_n) - x^*\|^q \\ &\leq (1-\alpha_n)[\|u_n - x^*\|^q - \tfrac{\mu_n}{1-\alpha_n}(\beta q - \kappa_q(\tfrac{\mu_n}{1-\alpha_n})^{q-1})\|Fu_n - Fx^*\|^q] \\ &\quad + q\alpha_n(1-\alpha_n)^{q-1}\langle f(u_n) - f(x^*), J_q(u_n - x^* - \tfrac{\mu_n}{1-\alpha_n}(Fu_n - Fx^*))\rangle \\ &\quad + q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*) - x^*, J_q(u_n - x^* - \tfrac{\mu_n}{1-\alpha_n}(Fu_n - Fx^*))\rangle + \kappa_q\alpha_n^q\|f(u_n) - x^*\|^q \\ &\leq (1-\alpha_n(1-q\rho))\|u_n - x^*\|^q - \mu_n(\beta q - \kappa_q(\tfrac{\mu_n}{1-\alpha_n})^{q-1})\|Fu_n - Fx^*\|^q \\ &\quad + q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*) - x^*, J_q(u_n - x^* - \tfrac{\mu_n}{1-\alpha_n}(Fu_n - Fx^*))\rangle + \kappa_q\alpha_n^q\|f(u_n) - x^*\|^q.\end{aligned}$$

Using Lemma 1.3 (a), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^q \\
&\le (1-\zeta)\|S_nu_n-x^*\|^q+\zeta\|y_n-x^*\|^q-\zeta(1-\zeta)g(\|S_nu_n-y_n\|) \\
&\le (1-\zeta)\|u_n-x^*\|^q+\zeta[(1-\alpha_n(1-q\rho))\|u_n-x^*\|^q-\mu_n(\beta q-\kappa_q(\frac{\mu_n}{1-\alpha_n})^{q-1})\|Fu_n-Fx^*\|^q \\
&+q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\mu_n}{1-\alpha_n}(Fu_n-Fx^*))\rangle+\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q] \\
&-\zeta(1-\zeta)g(\|S_nu_n-y_n\|) \\
&\le (1-\alpha_n\zeta(1-q\rho))\|x_n-x^*\|^q-(1-\alpha_n\zeta(1-q\rho))[r_n\lambda_n(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Ax^*\|^q \\
&+\lambda_n(\alpha q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Av_n-Ax^*\|^q]-\zeta\mu_n(\beta q-\kappa_q(\frac{\mu_n}{1-\alpha_n})^{q-1})\|Fu_n-Fx^*\|^q-\zeta(1-\zeta) \\
&\times g(\|S_nu_n-y_n\|)+\zeta q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\mu_n}{1-\alpha_n}(Fu_n-Fx^*))\rangle \\
&+\zeta\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q.
\end{aligned}$$

For each $n\ge 0$, we set

$$\begin{aligned}
&\Gamma_n=\|x_n-x^*\|^q, \\
&\varepsilon_n=\alpha_n\zeta(1-q\rho), \\
&\eta_n=(1-\alpha_n\zeta(1-q\rho))[r_n\lambda_n(\alpha q-\kappa_q\lambda_n^{q-1})\|Ax_n-Ax^*\|^q+\lambda_n(\alpha q-\tfrac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}})\|Av_n-Ax^*\|^q] \\
&\quad+\zeta\mu_n(\beta q-\kappa_q(\tfrac{\mu_n}{1-\alpha_n})^{q-1})\|Fu_n-Fx^*\|^q+\zeta(1-\zeta)g(\|S_nu_n-y_n\|), \\
&\vartheta_n=\zeta q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\tfrac{\mu_n}{1-\alpha_n}(Fu_n-Fx^*))\rangle+\zeta\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q.
\end{aligned}$$

It follows that

$$\Gamma_{n+1}\le(1-\varepsilon_n)\Gamma_n-\eta_n+\vartheta_n,\quad\forall n\ge 0,$$

and hence

$$\Gamma_{n+1}\le(1-\varepsilon_n)\Gamma_n+\vartheta_n,\quad\forall n\ge 0.$$

We next show the strong convergence of $\{\Gamma_n\}$ by the following two cases:

Case 1. Suppose that there exists an integer $n_0\ge 1$ such that $\{\Gamma_n\}$ is non-increasing. Then $\Gamma_n-\Gamma_{n+1}\to 0$. It follows that

$$0\le\eta_n\le\Gamma_n-\Gamma_{n+1}+\vartheta_n-\varepsilon_n\Gamma_n.$$

Since $\alpha_n\to 0$, $\varepsilon_n\to 0$ and $\vartheta_n\to 0$, we have $\eta_n\to 0$. This ensures that $\lim_{n\to\infty}g(\|S_nu_n-y_n\|)=0$,

$$\lim_{n\to\infty}\|Ax_n-Ax^*\|=\lim_{n\to\infty}\|Av_n-Ax^*\|=0, \tag{2.1}$$

and

$$\lim_{n\to\infty}\|Fu_n-Fx^*\|=0. \tag{2.2}$$

Note that $g$ is a strictly increasing, continuous and convex function with $g(0)=0$. So, it follows that

$$\lim_{n\to\infty}\|S_nu_n-y_n\|=0. \tag{2.3}$$

From $x_{n+1}=(1-\zeta)S_nu_n+\zeta y_n$, we get

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=(1-\zeta)\lim_{n\to\infty}\|S_nu_n-y_n\|=0. \tag{2.4}$$

On the other hand, noticing $v_n = J^B_{\lambda_n}(x_n - \lambda_n A x_n)$ and using Lemma 1.3, we get

$$\begin{aligned} \|v_n - x^*\|^q &\le \langle (x_n - \lambda_n A x_n) - (x^* - \lambda_n A x^*), J_q(v_n - x^*)\rangle \\ &\le \tfrac{1}{q}[\|(x_n - \lambda_n A x_n) - (x^* - \lambda_n A x^*)\|^q + (q-1)\|v_n - x^*\|^q \\ &\quad -h_1(\|x_n - \lambda_n(Ax_n - Ax^*) - v_n\|)], \end{aligned}$$

which together with Lemma 1.1 implies that

$$\begin{aligned} \|v_n - x^*\|^q &\le \|(x_n - \lambda_n A x_n) - (x^* - \lambda_n A x^*)\|^q - h_1(\|x_n - \lambda_n(Ax_n - Ax^*) - v_n\|) \\ &\le \|x_n - x^*\|^q - h_1(\|x_n - \lambda_n(Ax_n - Ax^*) - v_n\|). \end{aligned} \tag{2.5}$$

Since $J^M_{\mu_n}$ is firmly nonexpansive, we conclude from Lemma 1.3 (c) that

$$\begin{aligned} \|y_n - x^*\|^q &\le \langle (\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n F u_n) - (x^* - \mu_n F x^*), J_q(y_n - x^*)\rangle \\ &\le \tfrac{1}{q}[\|(\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n F u_n) - (x^* - \mu_n F x^*)\|^q + (q-1)\|y_n - x^*\|^q \\ &\quad -h(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n(Fu_n - Fx^*) - y_n\|)], \end{aligned}$$

which, together with the convexity of $\|\cdot\|^q$ and the nonexpansivity of $I - \frac{\mu_n}{1-\alpha_n}F$, implies that

$$\begin{aligned} \|y_n - x^*\|^q &\le \|(\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n F u_n) - (x^* - \mu_n F x^*)\|^q \\ &\quad -h(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n(Fu_n - Fx^*) - y_n\|) \\ &\le (1-\alpha_n)\|(u_n - \tfrac{\mu_n}{1-\alpha_n}Fu_n) - (x^* - \tfrac{\mu_n}{1-\alpha_n}Fx^*)\|^q + \alpha_n\|f(u_n) - x^*\|^q \\ &\quad -h(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n(Fu_n - Fx^*) - y_n\|) \\ &\le (1-\alpha_n)\|u_n - x^*\|^q + \alpha_n\|f(u_n) - x^*\|^q \\ &\quad -h(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n(Fu_n - Fx^*) - y_n\|). \end{aligned} \tag{2.6}$$

This together with (2.5) implies that

$$\begin{aligned} &\|x_{n+1} - x^*\|^q \le (1-\zeta)\|S_n u_n - x^*\|^q + \zeta\|y_n - x^*\|^q \\ &\le (1-\zeta)\|u_n - x^*\|^q + \zeta[(1-\alpha_n)\|u_n - x^*\|^q + \alpha_n\|f(u_n) - x^*\|^q \\ &\quad -h(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n(Fu_n - Fx^*) - y_n\|)] \\ &= (1-\zeta\alpha_n)\|u_n - x^*\|^q + \zeta\alpha_n\|f(u_n) - x^*\|^q \\ &\quad -\zeta h(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n(Fu_n - Fx^*) - y_n\|) \\ &\le (1-\zeta\alpha_n)[(1-r_n)\|x_n - x^*\|^q + r_n\|v_n - x^*\|^q] + \zeta\alpha_n\|f(u_n) - x^*\|^q \\ &\quad -\zeta h(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n(Fu_n - Fx^*) - y_n\|) \\ &\le (1-\zeta\alpha_n)\|x_n - x^*\|^q - (1-\zeta\alpha_n)r_n h_1(\|x_n - \lambda_n(Ax_n - Ax^*) - v_n\|) \\ &\quad +\zeta\alpha_n\|f(u_n) - x^*\|^q - \zeta h(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n(Fu_n - Fx^*) - y_n\|), \end{aligned}$$

which immediately yields

$$\begin{aligned} &(1-\zeta\alpha_n)r_n h_1(\|x_n - \lambda_n(Ax_n - Ax^*) - v_n\|) \\ &\quad +\zeta h(\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n(Fu_n - Fx^*) - y_n\|) \\ &\le \zeta\alpha_n\|f(u_n) - x^*\|^q + (1-\zeta\alpha_n)\|x_n - x^*\|^q - \|x_{n+1} - x^*\|^q \\ &\le \zeta\alpha_n\|f(u_n) - x^*\|^q + \Gamma_n - \Gamma_{n+1}. \end{aligned}$$

Since $h_1$ and $h$ are strictly increasing, continuous and convex functions with $h_1(0) = h(0) = 0$, we conclude from $\alpha_n \to 0$ and $\Gamma_n - \Gamma_{n+1} \to 0$ that

$$\lim_{n\to\infty}\|x_n - \lambda_n(Ax_n - Ax^*) - v_n\| = \lim_{n\to\infty}\|\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n(Fu_n - Fx^*) - y_n\| = 0. \tag{2.7}$$

Note that

$$\|x_n - v_n\| \le \|x_n - \lambda_n(Ax_n - Ax^*) - v_n\| + \lambda_n\|Ax_n - Ax^*\|,$$

and

$$\begin{aligned}\|u_n - y_n\| &\le \|\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n(Fu_n - Fx^*) - y_n\| \\ &\quad + \alpha_n \|f(u_n) - u_n\| + \mu_n \|Fu_n - Fx^*\|.\end{aligned}$$

So it follows from (2.1), (2.2) and (2.7) that

$$\lim_{n\to\infty} \|x_n - v_n\| = \lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{2.8}$$

In a similar way, we have

$$\begin{aligned}\|u_n - x^*\|^q &\le \langle (x_n - \lambda_n A v_n + r_n(v_n - x_n)) - (x^* - \lambda_n A x^*), J_q(u_n - x^*)\rangle \\ &\le \tfrac{1}{q}[\|(x_n - \lambda_n A v_n + r_n(v_n - x_n)) - (x^* - \lambda_n A x^*)\|^q + (q-1)\|u_n - x^*\|^q \\ &\quad - h_2(\|x_n + r_n(v_n - x_n) - \lambda_n(Av_n - Ax^*) - u_n\|)],\end{aligned}$$

which implies that

$$\begin{aligned}\|u_n - x^*\|^q &\le \|(x_n - \lambda_n A v_n + r_n(v_n - x_n)) - (x^* - \lambda_n A x^*)\|^q \\ &\quad - h_2(\|x_n + r_n(v_n - x_n) - \lambda_n(Av_n - Ax^*) - u_n\|) \\ &\le \|x_n - x^*\|^q - h_2(\|x_n + r_n(v_n - x_n) - \lambda_n(Av_n - Ax^*) - u_n\|).\end{aligned}$$

This together with (2.6) ensures that

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\le (1-\zeta)\|S_n u_n - x^*\|^q + \zeta\|y_n - x^*\|^q \\ &\le (1-\zeta)\|u_n - x^*\|^q + \zeta[(1-\alpha_n)\|u_n - x^*\|^q + \alpha_n\|f(u_n) - x^*\|^q] \\ &\le (1-\zeta\alpha_n)[\|x_n - x^*\|^q - h_2(\|x_n + r_n(v_n - x_n) - \lambda_n(Av_n - Ax^*) - u_n\|)] \\ &\quad + \zeta\alpha_n\|f(u_n) - x^*\|^q \\ &\le \|x_n - x^*\|^q - (1-\zeta\alpha_n)h_2(\|x_n + r_n(v_n - x_n) - \lambda_n(Av_n - Ax^*) - u_n\|) \\ &\quad + \zeta\alpha_n\|f(u_n) - x^*\|^q.\end{aligned}$$

So, it follows that

$$(1-\zeta\alpha_n)h_2(\|x_n + r_n(v_n - x_n) - \lambda_n(Av_n - Ax^*) - u_n\|) \le \Gamma_n - \Gamma_{n+1} + \zeta\alpha_n\|f(u_n) - x^*\|^q.$$

Since $h_2$ is a strictly increasing, continuous and convex function with $h_2(0) = 0$, we find from $\alpha_n \to 0$ and $\Gamma_n - \Gamma_{n+1} \to 0$ that

$$\lim_{n\to\infty} \|x_n + r_n(v_n - x_n) - \lambda_n(Av_n - Ax^*) - u_n\| = 0. \tag{2.9}$$

Observe that

$$\begin{aligned}\|x_n - u_n\| &= \|x_n + r_n(v_n - x_n) - \lambda_n(Av_n - Ax^*) - u_n - r_n(v_n - x_n) + \lambda_n(Av_n - Ax^*)\| \\ &\le \|x_n + r_n(v_n - x_n) - \lambda_n(Av_n - Ax^*) - u_n\| + r_n\|v_n - x_n\| + \lambda_n\|Av_n - Ax^*\|.\end{aligned}$$

So it follows from (2.1), (2.8) and (2.9) that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.10}$$

This together with (2.4) and (2.8) leads to

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - y_n\| + \|y_n - u_n\| + \|u_n - x_n\| \to 0 \quad (n \to \infty).$$

On the other hand, taking into account

$$\begin{aligned}\|S_n x_n - x_n\| &\le \|S_n x_n - S_n u_n\| + \|S_n u_n - y_n\| + \|y_n - u_n\| + \|u_n - x_n\| \\ &\le 2\|x_n - u_n\| + \|S_n u_n - y_n\| + \|y_n - u_n\|,\end{aligned}$$

we deduce from (2.3), (2.8) and (2.10) that $\lim_{n\to\infty}\|S_nx_n - x_n\| = 0$. Using Lemma 1.4 and the assumption on $\{S_n\}_{n=0}^{\infty}$, we get $\lim_{n\to\infty}\|S_nx_n - Sx_n\| = 0$. Therefore, we conclude that

$$\|x_n - Sx_n\| \le \|x_n - S_nx_n\| + \|S_nx_n - Sx_n\| \to 0 \quad (n \to \infty). \tag{2.11}$$

For each $n \ge 0$, we put $T_{\lambda_n} := J^B_{\lambda_n}(I - \lambda_n A)$. It follows from (2.8) that

$$\lim_{n\to\infty}\|x_n - T_{\lambda_n}x_n\| = \lim_{n\to\infty}\|x_n - v_n\| = 0.$$

Noticing $0 < \lambda \le \lambda_n$ for all $n \ge 0$ and using Lemma 1.5 (ii), we obtain

$$\|T_\lambda x_n - x_n\| \le 2\|T_{\lambda_n}x_n - x_n\| \to 0 \quad (n \to \infty). \tag{2.12}$$

In addition, for each $n \ge 0$, we put $T_{\mu_n} := J^M_{\mu_n}(I - \mu_n F)$. From (2.8) and $\alpha_n \to 0$, we get

$$\begin{aligned}\|u_n - T_{\mu_n}u_n\| &\le \|u_n - J^M_{\mu_n}(\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n F u_n)\| \\ &\quad + \|J^M_{\mu_n}(\alpha_n f(u_n) + (1-\alpha_n)u_n - \mu_n F u_n) - J^M_{\mu_n}(u_n - \mu_n F u_n)\| \\ &\le \|u_n - y_n\| + \alpha_n\|f(u_n) - u_n\| \to 0 \quad (n \to \infty).\end{aligned}$$

Since $\lim_{n\to\infty} a(1-\alpha_n) = a > 0$, without loss of generality, we may assume that $\exists \mu > 0$ such that $\mu \le a(1-\alpha_n) \le \mu_n\ \forall n \ge 0$. Using Lemma 1.5 (ii), we obtain from (2.10) that

$$\begin{aligned}\|T_\mu x_n - x_n\| &\le \|T_\mu x_n - T_\mu u_n\| + \|T_\mu u_n - u_n\| + \|u_n - x_n\| \\ &\le 2\|x_n - u_n\| + \|T_\mu u_n - u_n\| \\ &\le 2\|x_n - u_n\| + 2\|T_{\mu_n}u_n - u_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{2.13}$$

We now define a mapping $\Psi : C \to C$ by $\Psi x := \nu_1 Sx + \nu_2 T_\lambda x + (1 - \nu_1 - \nu_2)T_\mu x$, $\forall x \in C$ with $\nu_1 + \nu_2 < 1$ for constants $\nu_1, \nu_2 \in (0,1)$. From Lemma 1.5 (i) and Lemma 1.6, we know that $\Psi$ is nonexpansive and

$$\mathrm{Fix}(\Psi) = \mathrm{Fix}(S) \cap \mathrm{Fix}(T_\lambda) \cap \mathrm{Fix}(T_\mu) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap (A+B)^{-1}0 \cap (F+M)^{-1}0\ (=: \Omega).$$

Taking into account

$$\|\Psi x_n - x_n\| \le \nu_1\|Sx_n - x_n\| + \nu_2\|T_\lambda x_n - x_n\| + (1 - \nu_1 - \nu_2)\|T_\mu x_n - x_n\|,$$

we deduce from (2.11)-(2.13) that

$$\lim_{n\to\infty}\|\Psi x_n - x_n\| = 0. \tag{2.14}$$

Let $z_t = tf(z_t) + (1-t)\Psi z_t\ \forall t \in (0,1)$. Then it follows from Lemma 1.7 that $\{z_t\}$ converges strongly to a point $x^* \in \mathrm{Fix}(\Psi) = \Omega$, which solves

$$\langle (I - f)x^*, J(x^* - p)\rangle \le 0, \quad \forall p \in \Omega.$$

It follows that

$$\begin{aligned}\|z_t - x_n\|^q &\le (1-t)^q\|\Psi z_t - x_n\|^q + qt\langle f(z_t) - x_n, J_q(z_t - x_n)\rangle \\ &\le (1-t)^q(\|\Psi z_t - \Psi x_n\| + \|\Psi x_n - x_n\|)^q + qt\langle f(z_t) - z_t, J_q(z_t - x_n)\rangle + qt\|z_t - x_n\|^q \\ &\le (1-t)^q(\|z_t - x_n\| + \|\Psi x_n - x_n\|)^q + qt\langle f(z_t) - z_t, J_q(z_t - x_n)\rangle + qt\|z_t - x_n\|^q,\end{aligned}$$

which immediately attains

$$\langle f(z_t) - z_t, J_q(x_n - z_t)\rangle \le \frac{(1-t)^q}{qt}(\|z_t - x_n\| + \|\Psi x_n - x_n\|)^q + \frac{qt-1}{qt}\|z_t - x_n\|^q.$$

From (2.14), we have

$$\limsup_{n\to\infty}\langle f(z_t)-z_t,J_q(x_n-z_t)\rangle \le \frac{(1-t)^q}{qt}K+\frac{qt-1}{qt}K=(\frac{(1-t)^q+qt-1}{qt})K, \tag{2.15}$$

where $K$ is a constant such that $\|z_t-x_n\|^q\le K$ for all $n\ge 0$ and $t\in(0,1)$. It is clear that $((1-t)^q+qt-1)/qt\to 0$ as $t\to 0$. Since $J_q$ is norm-to-norm uniformly continuous on bounded subsets of $E$ and $z_t\to x^*$, we get $\|J_q(x_n-z_t)-J_q(x_n-x^*)\|\to 0$ as $t\to 0$. So, we obtain

$$\begin{aligned}&|\langle f(z_t)-z_t,J_q(x_n-z_t)\rangle-\langle f(x^*)-x^*,J_q(x_n-x^*)\rangle|\\ &\le|\langle f(x^*)-x^*,J_q(x_n-z_t)-J_q(x_n-x^*)\rangle|+|\langle f(z_t)-f(x^*),J_q(x_n-z_t)\rangle|\\ &\ +|\langle x^*-z_t,J_q(x_n-z_t)\rangle|\\ &\le\|f(x^*)-x^*\|\|J_q(x_n-z_t)-J_q(x_n-x^*)\|+(1+\rho)\|z_t-x^*\|\|x_n-z_t\|^{q-1}.\end{aligned}$$

Thus, for each $n\ge 0$, we have

$$\lim_{t\to 0}\langle f(z_t)-z_t,J_q(x_n-z_t)\rangle=\langle f(x^*)-x^*,J_q(x_n-x^*)\rangle.$$

From (2.15), as $t\to 0$, it follows that

$$\limsup_{n\to\infty}\langle f(x^*)-x^*,J_q(x_n-x^*)\rangle\le 0. \tag{2.16}$$

By (C3), (2.1) and (2.10), we get

$$\begin{aligned}\|u_n-x^*-\tfrac{\mu_n}{1-\alpha_n}(Fu_n-Fx^*)-(x_n-x^*)\| &\le\|u_n-x_n\|+\tfrac{\mu_n}{1-\alpha_n}\|Fu_n-Fx^*\|\\ &\le\|u_n-x_n\|+b\|Fu_n-Fx^*\|\to 0\quad(n\to\infty).\end{aligned} \tag{2.17}$$

Using (2.16) and (2.17), we have

$$\limsup_{n\to\infty}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\mu_n}{1-\alpha_n}(Fu_n-Fx^*))\rangle\le 0.$$

It follows from (2.8) that

$$\begin{aligned}&\|x_{n+1}-x^*\|^q\le(1-\alpha_n\zeta(1-q\rho))\|x_n-x^*\|^q\\ &+\zeta q\alpha_n(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\tfrac{\mu_n}{1-\alpha_n}(Fu_n-Fx^*))\rangle+\zeta\kappa_q\alpha_n^q\|f(u_n)-x^*\|^q\\ &=(1-\alpha_n\zeta(1-q\rho))\|x_n-x^*\|^q\\ &+\alpha_n\zeta(1-q\rho)[\tfrac{q(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle}{1-q\rho}+\tfrac{\kappa_q\alpha_n^{q-1}\|f(u_n)-x^*\|^q}{1-q\rho}].\end{aligned} \tag{2.18}$$

Note that $\{\alpha_n\zeta(1-q\rho)\}\subset[0,1]$, $\sum_{n=0}^\infty\alpha_n\zeta(1-q\rho)=\infty$ and

$$\limsup_{n\to\infty}[\frac{q(1-\alpha_n)^{q-1}\langle f(x^*)-x^*,J_q(u_n-x^*-\frac{\lambda_n}{1-\alpha_n}(Au_n-Ax^*))\rangle}{1-q\rho}+\frac{\kappa_q\alpha_n^{q-1}\|f(u_n)-x^*\|^q}{1-q\rho}]\le 0.$$

Applying Lemma 1.8 to (2.18), we deduce that $\Gamma_n\to 0$ as $n\to\infty$. Thus, $x_n\to x^*$ as $n\to\infty$.

Case 2. Suppose that there exits $\{\Gamma_{m_l}\}\subset\{\Gamma_m\}$ such that $\Gamma_{m_l}<\Gamma_{m_l+1}$, $\forall l\in\mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. Define a mapping $\tau:\mathbb{N}\to\mathbb{N}$ by

$$\tau(m):=\max\{l\le m:\Gamma_l<\Gamma_{l+1}\}.$$

Using Lemma 1.9, we have

$$\Gamma_{\tau(m)}\le\Gamma_{\tau(m)+1}\quad\text{and}\quad\Gamma_m\le\Gamma_{\tau(m)+1}.$$

Putting $\Gamma_m = \|x_m - x^*\|^q$, $\forall m \in \mathbb{N}$ and using the same inference as in Case 1, we can obtain

$$\lim_{m\to\infty} \|x_{\tau(m)+1} - x_{\tau(m)}\| = 0, \tag{2.19}$$

and

$$\limsup_{m\to\infty} \langle f(x^*) - x^*, J_q(u_{\tau(m)} - x^* - \frac{\mu_{\tau(m)}}{1-\alpha_{\tau(m)}}(Fu_{\tau(m)} - Fx^*))\rangle \le 0.$$

In view of $\Gamma_{\tau(m)} \le \Gamma_{\tau(m)+1}$ and $\alpha_{\tau(m)} > 0$, we conclude that

$$\begin{aligned}\|x_{\tau(m)} - x^*\|^q \quad &\le \tfrac{q(1-\alpha_{\tau(m)})^{q-1}}{1-q\rho} \langle f(x^*) - x^*, J_q(u_{\tau(m)} - x^* - \tfrac{\mu_{\tau(m)}}{1-\alpha_{\tau(m)}}(Fu_{\tau(m)} - Fx^*))\rangle \\ &\quad + \tfrac{\kappa_q \alpha_{\tau(m)}^{q-1}}{1-q\rho} \|f(u_{\tau(m)}) - x^*\|^q,\end{aligned}$$

and hence $\limsup_{m\to\infty} \|x_{\tau(m)} - x^*\|^q \le 0$. Thus, we have $\lim_{m\to\infty} \|x_{\tau(m)} - x^*\|^q = 0$. Using Lemma 1.2 and (2.19), we obtain

$$\begin{aligned}&\|x_{\tau(m)+1} - x^*\|^q - \|x_{\tau(m)} - x^*\|^q \\ &\le q\langle x_{\tau(m)+1} - x_{\tau(m)}, J_q(x_{\tau(m)} - x^*)\rangle + \kappa_q \|x_{\tau(m)+1} - x_{\tau(m)}\|^q \\ &\le q\|x_{\tau(m)+1} - x_{\tau(m)}\| \|x_{\tau(m)} - x^*\|^{q-1} + \kappa_q \|x_{\tau(m)+1} - x_{\tau(m)}\|^q \to 0 \quad (m \to \infty).\end{aligned}$$

Taking into account $\Gamma_m \le \Gamma_{\tau(m)+1}$, we have

$$\|x_m - x^*\|^q \le \|x_{\tau(m)} - x^*\|^q + q\|x_{\tau(m)+1} - x_{\tau(m)}\| \|x_{\tau(m)} - x^*\|^{q-1} + \kappa_q \|x_{\tau(m)+1} - x_{\tau(m)}\|^q.$$

It is easy to see from (2.19) that $x_m \to x^*$ as $m \to \infty$. This completes the proof. □

In the framework of Hilbert spaces, we have the following result immediately.

**Corollary 2.1.** *Let $C$ be a closed convex nonempty subset of a real Hilbert space $H$. Let $f : C \to C$ be a $\rho$-contraction with constant $\rho \in [0, \frac{1}{q})$, and let $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self-mappings on $C$. Suppose that $B, M : C \to 2^H$ are both maximal monotone operators, and $A, F : C \to E$ are $\alpha$-inverse-strongly monotone mapping and $\beta$-inverse-strongly monotone mapping, respectively. Assume that $\Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap (A+B)^{-1}0 \cap (F+M)^{-1}0 \ne \emptyset$. For any given $x_0 \in C$ and $\zeta \in (0,1)$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$\begin{cases} v_n = J_{\lambda_n}^B (x_n - \lambda_n A x_n), \\ u_n = J_{\lambda_n}^B (x_n - \lambda_n A v_n + r_n(v_n - x_n)), \\ x_{n+1} = (1-\zeta) S_n u_n + \zeta J_{\mu_n}^M (\alpha_n f(u_n) + (1-\alpha_n) u_n - \mu_n F u_n), \quad \forall n \ge 0, \end{cases}$$

*where the sequences $\{r_n\}, \{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\}, \{\mu_n\} \subset (0,\infty)$ are such that the following conditions hold: (C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$; (C2) $0 < r \le r_n < 1$ and $0 < \lambda \le \lambda_n < \frac{\lambda_n}{r_n} \le \overline{\lambda} < 2\alpha$; (C3) $0 < a \le \frac{\mu_n}{1-\alpha_n} \le b < 2\beta$. Assume that $\sum_{n=0}^{\infty} \sup_{x\in D} \|S_{n+1}x - S_n x\| < \infty$ for any bounded subset $D$ of $C$. Let $S : C \to C$ be a mapping defined by $Sx = \lim_{n\to\infty} S_n x$ $\forall x \in C$, and suppose that $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$. Then $x_n \to x^* \in \Omega$, which is the unique solution to $\langle (I-f)x^*, p - x^*\rangle \ge 0$, $\forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.*

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