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THE EXISTENCE AND APPROXIMATION FOR SOLUTIONS OF VARIATIONAL INCLUSION PROBLEMS

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Abstract. In this paper, we investigate the existence of solutions of a variational inclusion problem. An iterative algorithm is investigated with the aid of Michael's selection theorem and Nadler's theorem. A strong convergence theorem is established in the framework of real Banach spaces.
Keywords. Variational inclusion; Variational inequality; m-accretive mapping; Nonexpansive mapping.

1. INTRODUCTION

In the current research trends, nonlinear optimization problems with set-valued operators have attracted much attention since they find a number of real applications in the real world, such as, machine learning, medical imaging, signal processing, traffic and transportation; see, e.g., [1, 2, 3, 4] and the references therein.

Let H be a real Hilbert space, and let $C(H)$ denote the family of all nonempty compact subsets of H. Let A be a mapping on H. Throughout this paper, we denote the domain and range of A by $D(A)$ and $R(A)$, respectively.

Recall the following set-valued variational inclusion problems in H. For a given maximal monotone mapping $A : H \to H$, a nonlinear mapping $N : H \times H \to H$, set-valued mappings $T, F : H \to C(H)$, and a single-valued mapping $g : H \to H$, a fixed $f \in H$, find $u \in H$, $w \in T(u)$, and $y \in F(u)$ such that

$$f \in N(w, y) + A(g(u)), \tag{1.1}$$

If $f = 0$, it is reduced to the set-valued variational inclusion studied by Noor [5].

Using the resolvent operator technique, some existence theorems and convergence theorems were established for the variations of set-valued variational inclusion (1.1) recently; see, e.g., [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and the references therein.

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Let $CB(E)$ be the family of all nonempty closed and bounded subsets of E. The *Hausdorff metric* Θ on $CB(E)$ is defined by

$$\Theta(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\right\},\ A,B \in CB(E).$$

A set-valued mapping $T : E \to CB(E)$ is said to be a $\phi-$contractive mapping (with respect to the Hausdoff metric) if there exists a s continuous increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that, for any $x,y \in D(T)$,

$$\Theta(Tx,Ty) \le \phi(\|x-y\|).$$

The mapping T is called a Lipschitz continuous mapping (with respect to the Hausdorff metric) if there exists a constant $L > 0$ such that $x,y \in D(T)$,

$$\Theta(Tx,Ty) \le L\|x-y\|.$$

It is easy to see that every Lipschitz continuous set-valued mapping is $\phi-$contractive with respect to the function $\phi : [0,+\infty) \to [0,+\infty)$ given for all $t \in [0,+\infty)$ by $\phi(t) = Lt$.

If $T : E \to E$ is single-valued mapping, then T is $\phi-$contractive if there exists a continuous increasing function $\phi : [0,+\infty) \to [0,+\infty)$ with $\phi(0) = 0$ such that, for any $x,y \in D(T)$,

$$\|Tx-Ty\| \le \phi(\|x-y\|).$$

The single-valued mapping T is called a Lipschitz continuous mapping if there exists a constant $L > 0$ such that $x,y \in D(T)$,

$$\|Tx-Ty\| \le L\|x-y\|.$$

It is well known that every single-valued Lipschitz mapping is uniformly continuous (where a mapping $T : E \to E$ is uniformly continuous if, for all $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that, for all $x,y \in D(T)$ satisfying $\|x-y\| < \delta(\varepsilon)$, $\|Tx-Ty\| \le \varepsilon$). It is also known (see, e.g., Alber *et al.* [21]) that if $T : E \to E$ is a uniformly continuous mapping, then there exists a continuous increasing mapping $\pi : \mathbb{R} \to \mathbb{R}$ with $\pi(0) = 0$ (which is called the modulus of uniform continuity) such that, for any $x,y \in D(T)$, $\|Tx-Ty\| \le \pi(\|x-y\|)$. Thus, in the single-valued case, the class of $\phi-$contractive mappings includes the class of uniformly continuous mappings. So, the typical example of class of $\phi-$contractive single-valued mappings is the class of uniformly continuous mappings which is a proper supper class of single-valued Lipschitz mappings. In view of this, it is natural to extend the definition of $\phi-$contractive mappings to the set-valued case.

In 2002, Chang, Kim, and Kim [6] obtained the following result.

Theorem 1.1. *Let E be a real uniformly smooth Banach space. Let $T,F : E\times E \to CB(E)$, $A : D(A) \subseteq E \to 2^E$ be three set-valued mappings. Let $N : E\times E \to E$ be a single-valued continuous mapping, and let $g : D(A) \to D(A)$ be a single-valued continuous mapping satisfying the following conditions,*

(i) $A\circ g : E \to 2^E$ is m-accretive;
(ii) $T : E \to CB(E)$ is μ- Lipschitzian continuous;
(iii) $F : E \to CB(E)$ is ξ-Lipschitzian continuous, where μ and ξ are positive constants;
(iv) the mapping $x \mapsto N(x,y)$ is φ-strongly accretive with respect to the mapping T;
(v) the mapping $y \mapsto N(x,y)$ is accretive with respect to the set-valued mapping F.

Then, for any given $f \in E$, $\lambda > 0$, *there exists* $q \in D(A), w \in T(q)$, $k \in F(q)$ *such that* (q,w,k) *is a solution of the set-valued variational inclusion* (1.1). *Furthermore, let* $D(A)$ *is a closed set in E, and suppose that* $\{\alpha_n\}$, $\{\beta_n\}$ *are two sequences in [0, 1] satisfying (i)* $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$, *(ii)* $\sum_{n=0}^{\infty}\alpha_n = \infty$, *(iii)* $\sum_{n=0}^{\infty}||u_n|| < \infty$, *and* $\lim_{n\to\infty}||v_n|| = 0$. *If the ranges* $R(I - N(T(.),F(.)))$ *and* $R(A\circ g)$ *are both bounded and there exists a nonexpansive retraction* Q *of E onto D(A), then, for any given* $x_0 \in D(A)$, $h_0 \in T(x_0)$, $z_0 \in F(x_0)$, *the iterative sequences* $\{x_n\}$, $\{\omega_n\}$, $\{k_n\}$, $\{v_n\}$, $\{h_n\}$, $\{z_n\}$ *and* $\{p_n\}$ *defined by*

$$\begin{aligned}
x_{n+1} &= Qp_n,\\
p_n &\in (1-\alpha_n)x_n + \alpha_n(f + Qy_n - N(\omega_n,k_n) - \lambda A(g(Qy_n))) + u_n,\\
y_n &\in (1-\beta_n)x_n + \beta_n(f + x_n - N(h_n,z_n) - \lambda A(g(x_n))) + v_n,\\
h_n &\in T(x_n) : ||h_n - h_{n+1}|| \le \left(1 + \frac{1}{n+1}\right)\Theta(Tx_n, Tx_{n+1}),\\
z_n &\in F(x_n) : ||z_n - z_{n+1}|| \le \left(1 + \frac{1}{n+1}\right)\Theta(Fx_n, Fx_{n+1}),\\
\omega_n &\in T(Qy_n) : ||\omega_n - \omega_{n+1}|| \le \left(1 + \frac{1}{n+1}\right)\Theta((QTy)_n, T(Qy_{n+1})),\\
k_n &\in F(Qy_n) : ||k_n - k_{n+1}|| \le \left(1 + \frac{1}{n+1}\right)\Theta(F(Qy_n), F(Qy_{n+1})),
\end{aligned}$$

$n = 0,1,2,3,\ldots$, *converge strongly to the solution* x, ω, k *of the set-valued variational inclusion.*

From Theorem 1.1, we have the following observations:

(1) the conditions that the set-valued operators $T : E \to CB(E)$, and $F : E \to CB(E)$ are Lipschitz continuous can be extended to the more general case of ϕ-contractive operators;
(2) the assumption that the operator $x \mapsto N(x,y)$ is $\phi-$strongly accretive with respect to the mapping T can be extended to the more general case of the mapping $x \mapsto N(x,y)$ being $\psi-$uniformly accretive with respect to mapping T;
(3) the assumptions on the errors that $\sum_{n=1}^{\infty}\|u_n\| < \infty$ and $\lim_{n\to\infty}\|v_n\| = 0$ are too strong. Borrowing the words of Ofoedu [22], these conditions lead to the definitions of Liu [23]. The most reasonable assumptions on the errors is the one placed by Xu [24], in which the imposition of the boundedness condition on the errors was made.

2. Preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is a topological dual space of E, $\langle .,. \rangle$ is the duality pairing between E and E^*, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|.\|f\|, \|f\| = \|x\|\}, \quad \forall x \in E.$$

Let $A : D(A) \subset E \to 2^E$ be a set-valued mapping. A is said to be accretive if, for any $x,y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \ge 0$$

for all $u \in Ax,\ v \in Ay$. A is said to be φ-strongly accretive if there exists a strictly increasing continuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ such that, for any $x,y \in D(A)$, $j(x-y) \in J(x-y)$, and for any $u \in Ax,\ v \in Ay$,

$$\langle u - v, j(x-y)\rangle \geq \varphi(\|x-y\|)\|x-y\|.$$

A is said to be Ψ-uniformly accretive if there exists a strictly increasing continuous function $\Psi : [0,\infty) \to [0,\infty)$ with $\Psi(0) = 0$ such that, for any $x,y \in D(A)$, $j(x-y) \in J(x-y)$ and for any $u \in Ax,\ v \in Ay$,

$$\langle u - v, j(x-y)\rangle \geq \Psi(\|x-y\|).$$

A is said to be m-accretive if A is accretive and $R(I+\rho A) = E$ for all $\rho > 0$, where I is the identity mapping on E. A is said to be ϕ-expansive if, for any $x,y \in D(A)$ and for any $u \in Ax,\ v \in Ay$,

$$\|u - v\| \geq \phi(\|x-y\|).$$

It is easy to see that if A is φ-strongly accretive, then A is φ-expansive, and every φ-strongly accretive operator is Ψ-uniformly accretive with $\Psi(t) = t\varphi(t)$, $\forall t \in [0,+\infty)$.

Let $T,F : E \to CB(E)$ be two set-valued mappings. Let $A : E \to 2^E$ be an m-accretive mapping, $g : E \to E$ be a single-valued mapping, and let $N : E \times E \to E$ be a nonlinear mapping. For any $f \in E$, $\lambda > 0$, we consider the following problem, which consists of finding $q \in E, w \in T(q), and\ k \in F(q)$ such that

$$f \in N(w,k) + \lambda A(g(q)). \tag{2.1}$$

This problem is called the set-valued variational inclusion problem in Banach spaces. A number of problems arising in pure and applied sciences can be reduced to the study of this kind of variational inclusion problem (see, e.g., [5, 25, 26]).

Remark 2.1. As enumerated in [6], we reconsider and throw more light on some special cases of problem (2.1)

(1) Observe that (1.1) is a special case of (2.1) in the sense that if $E = H$ is a Hilbert space, $A : D(A) = H \to H$ is a maximal monotone mapping, $T,F : H \to C(H)$ are two set-valued mappings, and $g : H \to H$ is a single-valued mapping and $\lambda = 1$. Problem (2.1) is equivalent to the problem of finding $q \in H,\ w \in T(q),\ and\ k \in F(q)$ such that

$$f \in N(w,k) + A(g(q)).$$

Problem (1.1) is the set-valued variational inclusion problem, which was introduced and studied by Noor [5] and Noor et al. [11] under some additional conditions.

(2) If $g = I$, $F = 0$, $T = I$, $S : E \to E$ is a single-valued mapping, and $N(x,y) = Sx$ for all $(x,y) \in E \times E$, then problem (2.1) is equivalent to the problem of finding $q \in D(A)$ such that $f \in Sq + \lambda Aq$. This problem was introduced and studied by Jung and Morales [17] and Bruck and Reich [27].

(3) If $E = H$, a real Hilbert space, $\lambda = 1$, and $A = \partial\phi$, the subdifferential of a proper convex lower semi-continuous function $\phi : H \to \mathbb{R} \cup \{+\infty\}$, then problem (2.1) is equivalent to the problem of finding $q \in H, w \in T(q)$, and $k \in F(q)$ such that

$$\langle N(w,k) - f, x - g(q)\rangle \geq \phi(g(q)) - \phi(x), \quad \forall x \in H. \tag{2.2}$$

This problem is called the generalized set-valued mixed variational inequality, which was introduced and studied by Noor *et al.* [11]. Some special cases were also considered in Noor [12]. This problem with N being some special cases was also considered in the setting of Banach spaces (see Chang [7]). Observe that if $g = I$, $F = 0$, $T = I$, and $S : D(S) \subseteq H \to H$ a single-valued mapping, $N(x,y) = S(x)$ for all $x \in D(S), y \in H$, and $\phi = 0$, then (2.2) is reduced to the problem of finding $q \in D(S)$ such that

$$\left\langle S(q), x-q \right\rangle \geq 0, \quad \forall x \in H. \tag{2.3}$$

Problem (2.3) is the classical variational inequality problem studied by many authors under different conditions on operator S.

(4) If $E = H$ is a Hilbert space, $T,F,M : H \to 2^H$ are three set-valued mappings, and $m,S,G : H \to H$ are three single-valued mappings, then $\Gamma_K(z) = m(z) + K$, where K is a closed convex subset of H, $N(x,y) = Sx + Gy$, and

$$\phi(x) = I_{\Gamma_{K(z)}}(x) = \begin{cases} 0, & x \in \Gamma_K(z), \\ +\infty, & x \notin \Gamma_K(z). \end{cases}$$

Then, problem (2.1) is equivalent to the problem of finding $q \in H, w \in T(q)$, *and* $k \in F(q)$, *and* $z \in M(q)$ such that

$$g(q) \in K(q), \langle Sw + Gk - f, x - g(q)\rangle \geq 0, \quad x \in K(z).$$

This problem is called the generalized strongly nonlinear implicit quasi-variational inequality studied by Huang [16].

Consequently (borrowing the summary given in [6]), it follows that for a suitable choice of mappings T,F,A,g,N and space E, one can obtain a number of known and new classes of variational inequalities, variational inclusions, and the corresponding optimization problems from set-valued variational inclusion problem (2.1). Furthermore, these types of variational inclusions enable us to study many important problems arising in the mathematical, physical, and engineering sciences in a general and unified framework.

Let $T,F : E \to 2^E$ be two set-valued mappings, and let $N : E \times E \to E$ be a nonlinear mapping. Recall that the mapping $x \mapsto N(x,y)$ is said to be Ψ-uniformly accretive with respect to the mapping T if there exists a strictly increasing continuous function $\Psi : [0,\infty) \to [0,\infty)$ with $\Psi(0) = 0$ such that, for any $x_1, x_2 \in E$,, $j(x_1 - x_2) \in J(x_1 - x_2)$, and for any $u_1 \in Tx_1, u_2 \in Tx_2$, $y \in E$,

$$\langle N(u_1,y) - N(u_2,y), j(x_1 - x_2)\rangle \geq \Psi(\|x_1 - x_2\|).$$

The mapping $y \mapsto N(x,y)$ is said to be Ψ-uniformly accretive with respect to the mapping F, if there exists a strictly increasing continuous function $\Psi : [0,\infty) \to [0,\infty)$ with $\Psi(0) = 0$ such that, for any $y_1, y_2 \in E$,, $j(y_1 - y_2) \in J(y_1 - y_2)$, and for any $v_1 \in Ty_1, v_2 \in Ty_2$, $x \in E$,

$$\langle N(x,v_1) - N(x,v_2), j(y_1 - y_2)\rangle \geq \Psi(\|y_1 - y_2\|).$$

The mapping $y \mapsto N(x,y)$ is said to be accretive with respect to the mapping F, for any $y_1, y_2 \in E$,, $j(y_1 - y_2) \in J(y_1 - y_2)$, and for any $v_1 \in Ty_1, v_2 \in Ty_2$, $x \in E$,

$$\langle N(x,v_1) - N(x,v_2), j(y_1 - y_2)\rangle \geq 0.$$

In the remainder of this paper, the following lemmas play an important role.

Lemma 2.1. *Let E be a real Banach space, and let $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x,y \in E$, $j(x+y) \in J(x+y)$, the inequality $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$, holds.*

Lemma 2.2. *Let E be a real smooth Banach space. Let $T,F : E \to 2^E$ be two set-valued mappings, and let $N : E \times E \to E$ be a nonlinear mapping satisfying the following conditions:*
(1) the mapping $x \mapsto N(x,y)$ is Ψ-uniformly accretive with respect to the mapping T;
(2) the mapping $y \mapsto N(x,y)$ is accretive with respect to the mapping F.
Then, the mapping $S : E \to 2^E$, defined by $Sx = N(Tx,Fx)$, $\forall x \in E$, is Ψ-uniformly accretive.

Proof. Since E is smooth, one sees that the normalized duality mapping $J : E \to 2^{E^*}$ is single-valued. So, for any $x_1,x_2 \in E$ and $u_i \in Sx_i, i = 1,2$, there exist $w_i \in Tx_i$ and $v_i \in Fx_i$ such that $u_i = N(w_i,v_i)$, $i = 1,2$. By conditions (1) and (2), we have that

$$\begin{aligned} \langle u_1 - u_2, J(x_1 - x_2)\rangle &= \langle N(w_1,v_1) - N(w_2,v_2), J(x_1 - x_2)\rangle \\ &= \langle N(w_1,v_1) - N(w_2,v_1), J(x_1 - x_2)\rangle \\ &\quad + \langle N(w_2,v_1) - N(w_2,v_2), J(x_1 - x_2)\rangle \\ &\geq \Psi(\|x_1 - x_2\|). \end{aligned}$$

This implies that $S = N(T(.),F(.))$ is Ψ-uniformly accretive. □

Lemma 2.3. *(Michael's selection theorem [28]). Let X and Y be two Banach spaces, and let $T : X \to 2^Y$ be a lower semi-continuous mapping with nonempty closed convex values. Then, T admits a continuous selection, i.e., there exists a continuous mapping $h : X \to Y$ such that $h(x) \in T(x)$ for each $x \in X$.*

Lemma 2.4. *Let E be a real uniformly smooth Banach space, and let $T : E \to 2^E$ be a lower semi-continuous m-accretive mapping. Then, the following conclusions hold:*
(1) T admits a continuous and m-accretive selection.
(2) In addition, if T is also Ψ-uniformly accretive, then T admits a continuous, m-accretive, and Ψ-uniformly accretive selection.

Proof. (1) It is well known that if E is a uniformly smooth Banach space and $T : E \to 2^E$ is an m-accretive mapping, then, for each $x \in E, T(x)$ is nonempty closed and convex (see, for example, Deimling [25, p.293]). By Michael's theorem, T admits a continuous selection $h : E \to E$ such that, for all $x \in E$, $h(x) \in T(x)$.

Next, we prove that $h : E \to E$ is m-accretive. In fact, since $T : E \to 2^E$ is accretive, for any $x,y \in E$ and $u \in Tx, v \in Ty$, we have $\langle u - v, J(x-y)\rangle \geq 0$. In particular, letting $u = h(x) \in Tx$ *and* $v = h(y) \in Ty$, we have

$$\langle h(x) - h(y), J(x-y)\rangle \geq 0.$$

This implies that $h : E \to E$ is a continuous accretive mapping. By a well-known result due to Martin [29], we have that h is a continuous m-accretive mapping.

(2) In addition, if T is also Ψ-uniformly accretive, then the selection $h : E \to E$ given in (1) is also Ψ-uniformly accretive. In fact, for any $x,y \in E$ and $u \in Tx$, $v \in Ty$, we have

$$\langle u - v, J(x-y)\rangle \geq \Psi(\|x-y\|). \tag{2.4}$$

Letting $u = h(x) \in Tx$, *and* $v = h(y) \in Ty$, we have

$$\langle h(x) - h(y), J(x-y)\rangle \geq \Psi(\|x-y\|).$$

This implies that h is Ψ-uniformly accretive. This completes the proof of this Lemma. □

Lemma 2.5. *(Nadler's Theorem* [26]*). Let E be a complete metric space, and let $T : E \longrightarrow CB(E)$ be a set-valued mapping. Then, for any given $\varepsilon > 0$ and given $x, y \in E$ and $u \in Tx$, there exists $v \in Ty$ such that $d(u,v) \leq (1+\varepsilon)\Theta(Tx,Ty)$, where $\Theta(.,.)$ is the Hausdorff metric on $CB(E)$.*

Lemma 2.6. *(Kazmi* [18]*). Let E be a uniformly smooth Banach space, and let $A : D(A) \subseteq E \longrightarrow 2^E$ be an m-accretive and ϕ-expansive mapping. Then A is surjective.*

We now invoke Michael's selection theorem (Lemma 2.3) and Nadler's theorem (Lemma 2.5) to generate the following algorithm for the solution of set-valued variational inclusion (1.1).

Let $\{\alpha_n\}, \{\hat{\alpha}_n\}\{\beta_n\}$ and $\{\hat{\beta}_n\}$ be four sequences in $[0,1]$. Let $f \in E$ be any given point, and let $\lambda > 0$ be any given positive number. Let $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in E. Let $x_0 \in E$ be arbitrarily given. Let $h, z : E \to E$ be continuous mappings such that $h(x_0) \in Tx_0$ *and* $z(x_0) \in Fx_0$. Fix

$$y_0 \quad \in \quad \hat{\alpha}_0 x_0 + \hat{\beta}_0(f + x_0 - N(h(x_0), z(x_0)) - \lambda A(g(x_0))) + \hat{\gamma}_0 v_0.$$

Let $w, k : E \to E$ be continuous mappings such that $w(y_0) \in T(y_0)$ *and* $k(y_0) \in Fy_0$. Take

$$x_1 \quad \in \quad \alpha_0 x_0 + \beta_0(f + y_0 - N(w(y_0), k(y_0)) - \lambda A(g(y_0))) + \gamma_0 u_0.$$

From $h(x_0) \in Tx_0, z(x_0) \in Fx_0$ and Naddler's theorem (Lemma 2.5), we have that there exists $h_0 \in Tx_1$ and $z_0 \in Fx_1$ such that

$$\|h(x_0) - h_0\| \leq (1+1)\Theta(Tx_0, Tx_1),$$
$$\|z(x_0) - z_0\| \leq (1+1)\Theta(Fx_0, Fx_1).$$

Also, by Michael's seletion theorem, we have $h(x_1) \in Tx_1$ and $z(x_1) \in Fx_1$. Fix

$$y_1 \quad \in \quad \hat{\alpha}_1 x_1 + \hat{\beta}_1(f + x_1 - N(h(x_1), z(x_1)) - \lambda A(g(x_1))) + \hat{\gamma}_1 v_1.$$

Again applying Michael's selection theorem, and letting $w, k : E \to E$ be continuous mappings such that $w(y_0) \in T(y_0)$ and $k(y_0) \in F(y_0)$, we obtain by Nadler's theorem that there exists $w_0 \in T(y_1)$ and $k_0 \in F(y_1)$ such that

$$\|w(y_0) - w_0\| \leq (1+1)\Theta(T(y_0), T(y_1)),$$
$$\|k(y_0) - k_0\| \leq (1+1)\Theta(F(y_0), F(y_1)).$$

Choose

$$x_2 \quad \in \quad \alpha_1 x_1 + \beta_1(f + y_1 - N(w(y_1), k(y_1)) - \lambda A(g(y_1))) + \gamma_1 u_1.$$

Since $h(x_1) \in Tx_1, z(x_1) \in Fx_1$, then by Naddler's theorem there exists $h_1 \in Tx_2$ *and* $z_1 \in Fx_2$ such that

$$\|h(x_1) - h_1\| \leq \left(1 + \frac{1}{2}\right)\Theta(Tx_1, Tx_2),$$

$$\|z(x_1) - z_1\| \leq \left(1 + \frac{1}{2}\right)\Theta(Fx_1, Fx_2).$$

Thus, by Michael's seletion theorem, we have $h(x_2) \in Tx_2$ *and* $z(x_2) \in Fx_2$. Fix

$$y_2 \in \hat{\alpha}_2 x_2 + \hat{\beta}_2(f + x_2 - N(h(x_2), z(x_2)) - \lambda A(g(x_2))) + \hat{\gamma}_2 v_2.$$

Again applying Michael's selection theorem, and letting $w, k : E \to E$ be continuous mappings such that $w(y_1) \in T(y_1)$ and $k(y_1) \in F(y_1)$, we obtain by Nadler's theorem that there exists $w_1 \in T(y_2)$ and $k_1 \in F(y_2)$ such that

$$\|w(y_1) - w_1\| \leq \left(1 + \frac{1}{2}\right) \Theta(T(y_1), T(y_2)),$$
$$\|k(y_1) - k_1\| \leq \left(1 + \frac{1}{2}\right) \Theta(F(y_1), F(y_2)).$$

Choose

$$x_3 \in \alpha_2 x_2 + \beta_2(f + y_2 - N(w(y_2), k(y_2)) - \lambda A(g(y_2))) + \gamma_2 u_2.$$

From $h(x_2) \in Tx_2, z(x_2) \in Fx_2$ and Naddler's theorem, there exists $h_2 \in Tx_3$ *and* $z_2 \in Fx_3$ such that

$$\|h(x_2) - h_2\| \leq \left(1 + \frac{1}{3}\right) \Theta(Tx_2, Tx_3),$$
$$\|z(x_2) - z_2\| \leq \left(1 + \frac{1}{3}\right) \Theta(Fx_2, Fx_3).$$

Thus, by Michael's seletion theorem, we have $h(x_3) \in Tx_3$ *and* $z(x_3) \in Fx_3$. Fix

$$y_3 \in \hat{\alpha}_3 x_3 + \hat{\beta}_3(f + x_3 - N(h(x_3), z(x_3)) - \lambda A(g(x_3))) + \hat{\gamma}_3 v_3.$$

Applying Michael's selection theorem, and letting $w, k : E \to E$ be continuous mappings such that $w(y_2) \in T(y_2)$ and $k(y_2) \in F(y_2)$, we obtain by Nadler's theorem that there exists $w_2 \in T(y_3)$ and $k_2 \in F(y_3)$ such that

$$\|w(y_2) - w_2\| \leq \left(1 + \frac{1}{3}\right) \Theta(T(y_2), T(y_3)),$$
$$\|k(y_2) - k_2\| \leq \left(1 + \frac{1}{3}\right) \Theta(F(y_2), F(y_3)).$$

Choose $x_4 \in \alpha_3 x_3 + \beta_3(f + y_3 - N(w(y_3), k(y_3)) - \lambda A(g(y_3))) + \gamma_3 u_3$.

Continuing in this way, we obtain the following algorithm.

Algorithm 2.1. Let $h, z, w, k : E \to E$ be continuous mappings such that for $x_0 \in E$ *arbitary*, $h(x_0) \in Tx_0,\ z(x_0) \in F(x_0)$. As illustrated above, compute the sequences $\{y_n\}, \{x_n\}, \{h_n\}, \{k_n\}, \{w_n\}$, and $\{z_n\}$ iteratively such that

$$
\begin{aligned}
(i)\quad & x_{n+1} \in \alpha_n x_n + \beta_n(f + y_n - N(w(y_n), k(y_n)) - \lambda A(g(y_n))) + \gamma_n u_n, \\
(ii)\quad & y_n \in \hat{\alpha}_n x_n + \hat{\beta}_n(f + x_n - N(h(x_n), z(x_n)) - \lambda A(g(x_n))) + \hat{\gamma}_n v_n, \\
(iii)\quad & h(x_n) \in Tx_n, \|h(x_n) - h_n\| \leq \left(1 + \frac{1}{n+1}\right)\Theta(Tx_n, Tx_{n+1}), \\
(iv)\quad & z(x_n) \in Fx_n, \|z(x_n) - z_n\| \leq \left(1 + \frac{1}{n+1}\right)\Theta(Fx_n, Fx_{n+1}), \\
(v)\quad & w(y_n) \in T(y_n), \|w(y_n) - w_n\| \leq \left(1 + \frac{1}{n+1}\right)\Theta(T(y_n), T(y_{n+1})), \\
(vi)\quad & k(y_n) \in F(y_n), \|k(y_n) - k_n\| \leq \left(1 + \frac{1}{n+1}\right)\Theta(F(y_n), F(y_{n+1})), n = 0, 1, 2, \ldots
\end{aligned}
$$

The sequence $\{x_n\}$ defined by Algorithm 2.1 is called the Ishikawa-type iterative sequence with bounded errors. In Algorithm 2.1, if $\hat{\beta}_n = \hat{\gamma}_n = 0$ for all $n \geq 0$, then $x_n = y_n$. Taking $w = h$ and $z = k$, we obtain the following algorithm.

Algorithm 2.2.

$$
\begin{aligned}
& x_{n+1} \in \alpha_n x_n + \beta_n(f + x_n - N(w(y_n), k(y_n)) - \lambda A(g(x_n))) + \gamma_n u_n, \\
& h(x_n) \in T(x_n), \|h(x_n) - h_n\| \leq \left(1 + \frac{1}{n+1}\right) M(T(x_n), T(x_{n+1})), \\
& z(x_n) \in F(x_n), \|z(x_n) - z_n\| \leq \left(1 + \frac{1}{n+1}\right) M(F(x_n), F(x_{n+1})), n = 0, 1, 2, \ldots
\end{aligned}
$$

The sequence $\{x_n\}$ defined by Algorithm 2.2 is called the Mann-type iterative sequence with bounded errors.

3. Main Results

3.1. **Existence theorem for solutions of set-valued variational inclusion (2.1).** In this subsection, we establish an existence theorem for solutions of set-valued variational inclusion (2.1).

Theorem 3.1. *Let E be a real uniformly smooth Banach space. Let $T, F : E \to CB(E)$ and $A : E \to 2^E$ be three set-valued mappings such that T and F are continuous. Let $g : E \to E$ be a single-valued mapping, and let $N : E \times E \to E$ be a single-valued continuous mapping. Suppose that the following conditions are satisfied:*

- (i) $A \circ g : E \longrightarrow 2^E$ *is* $m-$accretive;
- (ii) for $\lambda > 0, \lambda A \circ g : E \to 2^E$ is m $-$ accretive and $\phi-$expansive;
- (iii) the mapping $x \mapsto N(x, y)$ is $\Psi-$uniformly accretive with respect to the mapping T;
- (iv) the mapping $y \mapsto N(x, y)$ is accretive with respect to the mapping F.

Then, there exist $q \in E$, $w \in T(q)$, and $k \in F(q)$ such that (q, w, k) is a solution of set-valued variational inclusion (2.1).

Proof. It follows from conditions (iii) and (iv) and Lemma 2.2 that the mapping $S : E \to 2^E$ defined by $Sx = N(Tx, Fx)$, $\forall x \in E$ is Ψ-uniformly accretive. Since N is continuous, and both T and F are continuous, then S is continuous. Hence, S is a continuous and accretive mapping. So, by Morales [30], we have that S is m-accretive. From Lemma 2.4(2), S admits a continuous selection $h : E \to E$, which is Ψ-uniformly accretive and m-accretive such that $h(x) \in Sx = N(Tx, Fx)$, $\forall x \in E$.

Next, we consider the variational inclusion

$$f \in h(x) + \lambda A(g(x)), \tag{3.1}$$

where λ is some positive constant. Since $\lambda A \circ g : E \to 2^E$ is m-accretive, $h : E \to E$ is continuous and φ-uniformly accretive, then $h + \lambda A \circ g : E \to 2^E$ is m-accretive and Ψ-uniformly accretive. Moreover, $h + \lambda A \circ g : E \to 2^E$ is also m-accretive and ϕ-expansive. By Lemma 2.6, we have that $h + \lambda A \circ g : E \to 2^E$ is surjective. Therefore, for any given $f \in E$ *and* $\lambda > 0$, there exists a unique $q \in E$ such that $f \in h(q) + \lambda A(g(q))$. The uniqueness of $q \in E$ is a direct consequence of the Ψ-uniform accretivity of $h + \lambda A \circ g$. Since $h(q) + \lambda A(g(q)) \subset N(T(q), F(q)) + \lambda A(g(q))$, there exist $w \in T(q)$, $k \in F(q)$ such that $f \in N(w, k) + \lambda A(g(q))$. This completes the proof. ☐

Remark 3.1. Theorem 3.1 generalizes and improves of the works of Chang, Kim, and Kim [6, Theorem 3.1]. The Lipschitz continuity assumed on T and F in [6] is replaced with the continuity assumption on T and F in our own case. Moreover, the $\phi-$strongly accretive assumption on the operator N with respect to the mapping T in [6, Theorem 3.1] is extended to the $\Psi-$uniformly accretive assumption on the operator N with respect to the mapping T in our own case.

3.2. **Approximate solutions for set-valued variational inclusion (2.1).** In Theorem 3.1, we proved that, under mild conditions, there exists unique $q \in E$, and there exist $w \in T(q)$, *and* $k \in F(q)$, such that (q, w, k) is a solution of set-valued variational inclusion (2.1). In this subsection, we consider solutions of variational inclusion (2.1). In fact, we shall prove the following theorem.

Theorem 3.2. *Let E be a real uniformly smooth Banach space. Let $T, F : E \longrightarrow CB(E)$ and $A : E \subset E \longrightarrow 2^E$ be three set-valued mappings, let $g : E \longrightarrow E$ be a single-valued mapping, and let $N : E \times E \longrightarrow E$ be a single-valued continuous mapping satisfying the following properties:*

- (i) $A \circ g : E \longrightarrow 2^E$ *is* $m-accretive$;
- (ii) T, F are two continuous mappings and are also $\varphi-$contractive mapping with respect to the Hausdorff metric Θ;
- (iii) the mapping $x \mapsto N(x, y)$ is $\Psi-$uniformly accretive with respect to the mapping T;
- (iv) the mapping $y \mapsto N(x, y)$ is accretive with respect to the mapping F.

Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}$, and $\{\hat{\gamma}_n\}$ be sequences in (0,1) satisfying the following conditions: (v) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1$; (vi) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = +\infty$; (vii) $\lim_{n\to\infty} \hat{\beta}_n = 0$; and (viii) $\lim_{n\to\infty} \hat{\gamma}_n = 0$, $\sum_{n=0}^{\infty} \gamma_n < \infty$. If ranges $R(I - N(T(.), F(.)))$ and $R(A \circ g)$ are both bounded, and the sequences $\{u_n\}, \{v_n\}$ are bounded sequences in E, then the iterative sequences defined by Algorithm 2.1 converge strongly to a solution of set-valued variational inclusion problem (2.1).

Proof. First, in (i) and (ii) Algorithm 2.1, we choose $d_n \in A(g(x_n))$ and $e_n \in A(g(y_n))$ such that

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n(f + y_n - N(w(y_n), k(y_n)) - \lambda e_n) + \gamma_n u_n, \\ y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n(f + x_n - N(h(x_n), z(x_n)) - \lambda d_n) + \hat{\gamma}_n v_n. \end{aligned} \tag{3.2}$$

Let

$$b_n := f + x_n - N(h(x_n), z(x_n)) - \lambda d_n,$$

and

$$c_n := f + y_n - N(w(y_n), k(y_n)) - \lambda e_n.$$

Then, (3.2) can be written as

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n c_n + \gamma_n u_n, \\ y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n b_n + \hat{\gamma}_n v_n, \quad n \geq 0. \end{aligned} \tag{3.3}$$

Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences and ranges of $R(I - N(T(.), F(.)))$ *and* $R(A \circ g)$ are bounded subsets of E, then, for the unique $q \in E$, and for some $w \in T(q)$, *and* $k \in F(q)$ such that (q, w, k) is a solution of the set-valued variational inclusion (2.1), we let d be given by

$$d = \max\left\{ \sup_{n\geq 0}\{\|u_n - q\|\}, \sup_{n\geq 0}\{\|v_n - q\|\}, \sup_{w\in\Omega_x}\{\|w - q\|\}, \|x_1 - q\|, \|x_0 - q\| \right\},$$

where $\Omega_x = \{u : u \in f + x - N(T(x), F(x)) - \lambda A(g(x))\}, \forall x \in E$. Thus,

$$\begin{aligned} &\|c_n - q\| \leq d, \text{ for all n} \geq 0, \\ &\|b_n - q\| \leq d, \text{ for all n} \geq 0. \end{aligned} \tag{3.4}$$

Next, we prove that

$$\|x_n - q\| \leq d \text{ and} \|y_n - q\| \leq d, \text{ for all n} \geq 0. \tag{3.5}$$

In fact, by the defintion of d, we know that $\|x_0 - q\| \leq d$ *and* $\|x_1 - q\| \leq d$. Again by (3.3), we have that

$$\begin{aligned} \|x_2 - q\| &= \|\alpha_1(x_1 - q) + \beta_1(c_1 - q) + \gamma_1(u_1 - q)\| \\ &\leq \alpha_1 d + \beta_1 d + \gamma_1 d \\ &= d. \end{aligned}$$

By induction, we can prove that $\|x_n - q\| \leq d$. To see this, we observe from above that $\|x_0 - q\| \leq d$. Assume that it is true for some $n = k \geq 0$, that is, $\|x_k - q\| \leq d$. We show that it is true for $n = k+1$. Now,

$$\begin{aligned} \|x_{k+1} - q\| &= \|\alpha_k(x_k - q) + \beta_k(c_k - q) + \gamma_k(u_k - q)\| \\ &\leq \alpha_k d + \beta_k d + \gamma_k d \\ &= d. \end{aligned}$$

Therefore, we have that $\|x_n - q\| \le d$, $\forall n \ge 0$. Moreover, we also have

$$\begin{aligned}\|y_n - q\| &= \|\hat{\alpha}_n(x_n - q) + \hat{\beta}_n(b_n - q) + \hat{\gamma}_n(v_n - q)\| \\ &\le \hat{\alpha}_n\|x_n - q\| + \hat{\beta}_n\|b_n - q\| + \hat{\gamma}_n\|v_n - q\| \\ &\le d.\end{aligned}$$

So, (3.5) is proved.

Next, using (3.3)-(3.5) and Lemma 2.1, we obtain that

$$\begin{aligned}\|x_{n+1} - q\|^2 \le\ & \alpha_n^2\|x_n - q\|^2 + 2\beta_n \langle c_n - q, j(y_n - q)\rangle \\ & +2\beta_n \langle c_n - q, j(x_{n+1} - q) - j(y_n - q)\rangle \\ & +2\gamma_n \langle u_n - q, j(x_{n+1} - q)\rangle. \end{aligned} \tag{3.6}$$

Observe that

$$\begin{aligned} & \|x_{n+1} - q - (y_n - q)\| \\ \le\ & \alpha_n\|x_n - y_n\| + \beta_n\|c_n - y_n\| + \gamma_n\|u_n - y_n\| \\ \le\ & \alpha_n\|x_n - (\hat{\alpha}_n x_n + \hat{\beta}_n b_n + \hat{\gamma}_n v_n)\| + \beta_n(\|c_n - q\| + \|y_n - q\|) \\ & +\gamma_n(\|u_n - q\| + \|y_n - q\|) \\ \le\ & \alpha_n[\hat{\beta}_n\|x_n - b_n\| + \hat{\gamma}_n\|x_n - v_n\|] + \beta_n\|c_n - q\| + \gamma_n\|u_n - q\| \\ & +(\beta_n + \gamma_n)\|y_n - q\| \\ \le\ & \alpha_n\hat{\beta}_n\|x_n - q\| + \alpha_n\hat{\beta}_n\|b_n - q\| + \alpha_n\hat{\gamma}_n\|x_n - q\| + \alpha_n\hat{\gamma}_n\|v_n - q\| \\ & +\beta_n\|c_n - q\| + \gamma_n\|u_n - q\| + (\beta_n + \gamma_n)\|y_n - q\| \\ =\ & (\alpha_n\hat{\beta}_n + \alpha_n\hat{\gamma}_n)\|x_n - q\| + \alpha_n\hat{\beta}_n\|b_n - q\| + \alpha_n\hat{\gamma}_n\|v_n - q\| \\ & +\beta_n\|c_n - q\| + \gamma_n\|u_n - q\| + (\beta_n + \gamma_n)\|y_n - q\| \\ \le\ & \alpha_n(\hat{\beta}_n + \hat{\gamma}_n)d + \alpha_n\hat{\beta}_n d + \alpha_n\hat{\gamma}_n d + \beta_n d + \gamma_n d + (\beta_n + \gamma_n)d. \end{aligned} \tag{3.7}$$

Thus, $\lim_{n\to\infty}\|x_{n+1} - q - (y_n - q)\| = 0$. Since j is uniformly continuous on any bounded subsets of E, we have that $\lim_{n\to\infty}\|j(x_{n+1} - q) - j(y_n - q)\| = 0$. Setting

$$\xi_n = |\langle c_n - q, j(x_{n+1} - q) - j(y_n - q)\rangle|,$$

we obtain that $\lim_{n\to\infty}\xi_n = 0$ due to

$$\begin{aligned}\zeta_n &= |\langle c_n - q, j(x_{n+1} - q) - j(y_n - q)\rangle| \\ &\le \|c_n - q\|\|j(x_{n+1} - q) - j(y_n - q)\| \\ &\le d\|j(x_{n+1} - q) - j(y_n - q)\| \to 0 \ as\ n \to \infty. \end{aligned} \tag{3.8}$$

Next, since $w(y_n) \in T(y_n), k(y_n) \in F(y_n),\ and\ e_n \in Ag(y_n)$, one has

$$N(w(y_n), k(y_n)) + \lambda e_n \in N(T(y_n), F(y_n)) + \lambda Ag(y_n).$$

Since q is the unique solution of the variational inclusion problem, one has $f \in h(q) + \lambda Ag(q)$. This shows that $f \in N(T(q), F(q)) + \lambda Ag(q)$. By the assumptions of Theorem 3.2, the mapping

$N(T(.),F(.))+\lambda Ag(.) : E \longrightarrow 2^E$ is Ψ-uniformly accretive. Thus,

$$\begin{aligned}\langle f-(N(w(y_n),k(y_n))+\lambda e_n), j(y_n-q)\rangle &= -\langle N(w(y_n),k(y_n))+\lambda e_n - f, j(y_n-q)\rangle \\ &\leq -\varphi(\|y_n-q\|).\end{aligned}$$

It follows that

$$\begin{aligned}& 2\beta_n\langle c_n-q, j(y_n-q)\rangle \\ =\ & 2\beta_n\langle f+y_n-N(w(y_n),k(y_n))-\lambda e_n-q, j(y_n-q)\rangle \\ =\ & 2\beta_n[\langle y_n-q, j(y_n-q)\rangle - \langle N(w(y_n),k(y_n))+\lambda e_n-f, j(y_n-q)\rangle] \\ \leq\ & 2\beta_n[\|y_n-q\|^2-\Psi(\|y_n-q\|)]. \end{aligned} \tag{3.9}$$

Substituting (3.9) and (3.8) into (3.6), we get

$$\|x_{n+1}-q\|^2 \leq \alpha_n^2\|x_n-q\|^2+2\beta_n[\|y_n-q\|^2-\Psi(\|y_n-q\|)]+2\beta_n\zeta_n+2\gamma_n d^2. \tag{3.10}$$

Furthermore, from (3.3), (3.5) and Lemma 2.1, we observe that

$$\begin{aligned}\|y_n-q\|^2 &\leq \hat{\alpha}_n\|x_n-q\|^2+2\hat{\beta}_n\langle b_n-q, j(y_n-q)\rangle+2\hat{\gamma}_n\langle v_n-q, j(y_n-q)\rangle \\ &\leq \|x_n-q\|^2+2\hat{\beta}_n\|b_n-q\|\|y_n-q\|+2\hat{\gamma}_n\|v_n-q\|\|y_n-q\| \\ &\leq \|x_n-q\|^2+2d^2(\hat{\beta}_n+\hat{\gamma}_n)\end{aligned} \tag{3.11}$$

Thus, it follows from (3.11) that

$$\|y_n-q\|^2 \leq \|x_n-q\|^2+2d^2(\hat{\beta}_n+\hat{\gamma}_n). \tag{3.12}$$

Observe that $2d^2(\hat{\beta}_n+\hat{\gamma}_n)\to\infty$ as $n\to\infty$. Substituting (3.12) into (3.10), we have that

$$\begin{aligned}& \|x_{n+1}-q\|^2 \\ \leq\ & \alpha_n^2\|x_n-q\|^2+2\beta_n[\|y_n-q\|^2-\Psi(\|y_n-q\|)]+2\beta_n\zeta_n+2d^2\gamma_n \\ \leq\ & \alpha_n^2\|x_n-q\|^2+2\beta_n[\|x_n-q\|^2+2d^2(\hat{\beta}_n+\hat{\gamma}_n)]-2\beta_n\Psi(\|y_n-q\|)+2\beta_n\zeta_n+2d^2\gamma_n \\ =\ & (1-(\beta_n+\gamma_n))^2\|x_n-q\|^2+2\beta_n\|x_n-q\|^2+4d^2\beta_n(\hat{\beta}_n+\hat{\gamma}_n) \\ & -2\beta_n\Psi(\|y_n-q\|)+2\beta_n\zeta_n+2d^2\gamma_n \\ =\ & \|x_n-q\|^2-2\beta_n\|x_n-q\|^2-2\gamma_n\|x_n-q\|^2+(\beta_n+\gamma_n)^2\|x_n-q\|^2 \\ & +2\beta_n\|x_n-q\|^2+4\beta_n d^2(\hat{\beta}_n+\hat{\gamma}_n)-2\beta_n\Psi(\|y_n-q\|)+2\beta_n\zeta_n+2d^2\gamma_n \\ \leq\ & \|x_n-q\|^2+[-2\gamma_n+2\gamma_n+\beta_n^2+\gamma_n^2]\|x_n-q\|^2+4d^2\beta_n(\hat{\beta}_n+\hat{\gamma}_n) \\ & -2\beta_n\Psi(\|y_n-q\|)+2\beta_n\zeta_n+2d^2\gamma_n \\ \leq\ & \|x_n-q\|^2+d^2(\beta_n^2+\gamma_n^2)+4d^2\beta_n(\hat{\beta}_n+\hat{\gamma}_n)-2\beta_n\Psi(\|y_n-q\|)+2\beta_n\zeta_n+2d^2\gamma_n \\ \leq\ & \|x_n-q\|^2-\beta_n\Psi(\|y_n-q\|)-\beta_n\left[\Psi(\|y_n-q\|)-2\mu_n-d^2\beta_n-2\zeta_n\right]+3d^2\gamma_n\end{aligned} \tag{3.13}$$

Let $\sigma=\inf_{n\geq 0}\|y_n-q\|$, Next, we prove that $\sigma=0$. Suppose for contradiction that $\sigma>0$. Then $\|y_n-q\|\geq\sigma>0$, $\forall n\geq 0$. By the strictly increasing property of φ *with* $\varphi(0)=0$, we have $\varphi(\sigma)>0$. Hence, it follows from (3.13) that

$$\|x_{n+1}-q\|^2 \leq \|x_n-q\|^2-\beta_n\Psi(\sigma)-\beta_n\left[\Psi(\sigma)-2\mu_n-\beta_n d^2-2\zeta_n\right]+3\gamma_n d^2. \tag{3.14}$$

Since $\beta_n \to 0$, $\mu_n \to 0$ *and* $\zeta_n \to 0$, there exists n_0 such that, for all $n \geq n_0$, $\varphi(\sigma) - 2\mu_n - \beta_n d^2 - 2\zeta_n > 0$. Therefore, for $n \geq n_0$, we have

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \beta_n \Psi(\sigma) + 3d^2\gamma_n,$$

that is, $\beta_n \Psi(\sigma) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + 3d^2\gamma_n$, $\forall n \geq n_0$. Hence, for any $m \geq n_0$, we have

$$\begin{aligned} \Psi(\sigma) \sum_{n=n_0}^{m} \beta_n &\leq \|x_{n_0} - q\|^2 - \|x_{m+1} - q\|^2 + 3d^2 \sum_{n=n_0}^{m} \gamma_n \\ &\leq \|x_{n_0} - q\|^2 + 3d^2 \sum_{n=n_0}^{m} \gamma_n. \end{aligned}$$

Letting $m \to \infty$ yields that $+\infty \leq \|x_{n_0} - q\|^2 + 3d^2 \sum_{n=n_0}^{\infty} \gamma_n < +\infty$, which is a contradiction. Hence, we have that $\sigma = 0$. Therefore, there exists a subsequence $\{y_{n_j}\} \subseteq \{y_n\}$ such that

$$y_{n_j} \to q \text{ as } j \to \infty \tag{3.15}$$

By (3.7) and (3.15), we see that

$$\|x_{n_j+1} - q\| \leq \|x_{n_j+1} - q - (y_{n_j} - q)\| + \|y_{n_j} - q\|.$$

Thus, $\lim_{j\to\infty} \|x_{n_j+1} - q\| = 0$. That is,

$$x_{n_j+1} = \alpha_{n_j} x_{n_j} + \beta_{n_j} c_{n_j} + \gamma_{n_j} \to q \quad as \ j \to \infty$$

Since $\beta_n \to 0$, $\gamma_n \to 0$, $\hat{\beta}_n \to 0$, $\{c_n\}$, $\{b_n\}$, $\{u_n\}$ *and* $\{v_n\}$ are bounded sequences, this implies that $x_{n_j} \to q \ \ as \ j \to \infty$. Therefore,

$$y_{n_j} = \hat{\alpha}_{n_j} x_{n_j} + \hat{\beta}_{n_j} b_{n_j} + \hat{\gamma}_{n_j} v_{n_j} \to q \quad as \ j \to \infty.$$

We can easily verify by induction that, for all $i \geq 0$, $x_{n_j+i} \to q$ and $y_{n_j+i} \to q$ as $j \to \infty$. Therefore, $x_n \to q$ and $y_n \to q$ as $n \to \infty$. Thus, by the continuity of h, z, w and k, $h(x_n) \to h(q), z(x_n) \to z(q), w(y_n) \to w(q)$ and $k(y_n) \to k(q)$. Furthermore, since $\Theta(Tx, Ty) \leq \varphi(\|x - y\|)$, it follows from the fact that φ is continuous that

$$\begin{aligned} \|h(x_n) - h_n\| &\leq \left(1 + \frac{1}{n}\right) \Theta(Tx_n, Tx_{n+1}) \\ &\leq \left(1 + \frac{1}{n}\right) \varphi(\|x_n - x_{n+1}\|) \to 0 \quad as \ n \to \infty, \ and \\ \|z(x_n) - z_n\| &\leq \left(1 + \frac{1}{n}\right) \Theta(Fx_n, Fx_{n+1}) \\ &\leq \left(1 + \frac{1}{n}\right) \varphi(\|x_n - x_{n+1}\|) \to 0 \quad as \ n \to \infty. \end{aligned}$$

This implies by the continuity of h and z that $\{h_n\}$ and $\{z_n\}$ converge to $h(q)=h^*\in T(q)$ *and* $z(q)=z^*\in F(q)$, respectively. Furthermore, we also have

$$\begin{aligned}
\|w(y_n)-w_n\| &\leq \left(1+\frac{1}{n}\right)\Theta(T(y_n),T(y_{n+1}))\\
&\leq \left(1+\frac{1}{n}\right)\varphi(\|y_n-y_{n+1}\|)\\
&\leq \left(1+\frac{1}{n}\right)\varphi(\|y_n-y_{n+1}\|)\ \to 0 \ as\ n\to\infty\ and\\
\|k(y_n)-k_n\| &\leq \left(1+\frac{1}{n}\right)\Theta(F(y_n),F(y_{n+1}))\\
&\leq \left(1+\frac{1}{n}\right)\varphi(\|y_n-y_{n+1}\|)\quad \to 0\ as\ n\to\infty.
\end{aligned}$$

From the continuity of w and k, we have that $\{w_n\}$ and $\{k_n\}$ converge to $w(q)=w^*\in T(q)$ and $k(q)=k^*\in F(q)$, respectively.

Next, we prove that $h^*=w^*=w$ and $z^*=k^*=k$, where (q,w,k), $q\in E$, $w\in T(q)$, *and* $k\in F(q)$ is the solution of the set-valued variational inclusion. In fact, since

$$\begin{aligned}
d(w^*,T(q)) &\leq d(w^*,w_n)+d(w_n,T(q))\\
&\leq d(w^*,w_n)+\Theta(T(y_n),T(q))\\
&\leq d(w^*,w_n)+\varphi(\|y_n-q\|)\to 0\quad as\ n\to\infty,
\end{aligned}$$

which implies that $w^*\in T(q)$. Similarly, we can prove that $h^*\in T(q)$. Observe that

$$\begin{aligned}
d(w^*,w) &\leq d(w^*,w_n)+d(w_n,w)\\
&\leq d(w^*,w_n)+\Theta(T(y_n),T(q))\\
&\leq d(w^*,w_n)+\varphi(\|y_n-q\|)\to 0\quad as\ n\to\infty
\end{aligned}$$

This implies that $w^*=w$. In view of

$$\begin{aligned}
d(h^*,w) &\leq d(h^*,h_n)+d(h_n,w)\\
&\leq d(h^*,h_n)+M(Tx_n,Tq)\\
&\leq d(h^*,h_n)+\varphi(\|x_n-q\|)\to 0\quad as\ n\to\infty,
\end{aligned}$$

we obtain that $h^*=w$. Following the same argument, we can prove that $z^*\in F(q)$, $k^*\in F(q)$, and $z^*=k^*=k$. Consequently, we have that the sequences $\{x_n\}$, $\{w_n\}$, and $\{k_n\}$ defined by (2.5) respectively converge strongly to q, w and k, which satisfy the set-valued variational inclusion. This completes the prove. □

Remark 3.2. All the observations in Section 1 are properly addressed. It also deserves mentioning that we can also easily solve set-valued variational inclusions 1.1 via Algorithm 2.2 following the prof of Theorem 3.2. Our theorems extend, generalize, improve, and unify the corresponding results of [5, 6, 7, 8, 10, 16, 11, 12, 13, 14, 15, 17, 18, 19, 20].

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