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A MODIFIED INERTIAL PROJECTION AND CONTRACTION METHOD FOR SOLVING BILEVEL SPLIT VARIATIONAL INEQUALITY PROBLEMS

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Abstract. The main purpose of this paper is to study a bilevel split variational inequality problem in two real Hilbert spaces. We propose a new modified inertial projection and contraction method for solving this problem when one of the cost operators is pseudomonotone and Lipschitz continuous, but not sequentially weakly continuous. Strong convergence of the proposed method is established and numerical examples are given to support our theoretical findings.
Keywords. Bilevel split variational inequality problem; Generalized demimetric mapping; Pseudomonotone operator; Projection and contraction method.

1. INTRODUCTION

Let H_1 and H_2 be real Hilbert spaces endowed with $\langle .,. \rangle$ and norm $||\cdot||$. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $B : H_1 \to H_2$ be a bounded linear operators, and let $A, F_1 : H_1 \to H_1$ and $F_2 : H_2 \to H_2$ be nonlinear mappings. The bilevel split variational inequality problem (BSVIP) is defined as follows:

$$\text{Finding } z^* \in \Gamma \text{ such that } \langle F_1(z^*), z - z^* \rangle \geq 0,\ \forall z \in \Gamma, \tag{1.1}$$

where $\Gamma := \{z^* \in VI(C,A) : Bz^* \in VI(Q,F_2)\}$ is the solution set of the following split variational inequality problem (SVIP) introduced by Censor et al. [1]:

$$\text{Finding } z^* \in C \text{ solves } \langle A(z^*), u - z^* \rangle \geq 0,\ \forall u \in C \tag{1.2}$$

such that

$$x^* = Bz^* \in Q \text{ solves } \langle F_2(x^*), y - x^* \rangle \geq 0,\ \forall y \in Q, \tag{1.3}$$

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and $VI(C,A)$ and $VI(Q,F_1)$ denote the solution sets of variational inequalities (1.2) and (1.3), respectively. Censor et al. [1] proposed and studied the following method for solving the SVIP (1.2)-(1.3) as follows: For $x_1 \in H_1$,

$$x_{n+1} = P_C(I-\lambda A)(x_n + \tau B^*(P_Q(I-\lambda F_2)-I)Bx_n), \; n \geq 1, \tag{1.4}$$

where A and F_2 are β_1 and β_2-inverse strongly monotone, and λ and τ satisfy some conditions. They proved weak convergence of the sequence generated by (1.4) to a solution of problem (1.2)-(1.3). The SVIP (1.2)-(1.3) is a very interesting union of classical variational inequality problem (1.2), which was first introduced by Stampacchia [2] and have been used as a strong methodology in studying traffic equilibrium control problems, machine learning, medical imaging, see, e.g., [3, 4, 5] for more details. A special case of the SVIP, when $A = F_2 = 0$, is the split feasibility problem (SFP) introduced and studied by Censor and Elfving [6], which has been studied and applied in many fields such as phase retrieval, medical image reconstruction, signal processing, radiation therapy treatment planning, and so on; see, e.g, [7, 8, 9] for more details. It is also known that problem (1.2) is equivalent to the fixed point problem:

$$\text{Finding } z^* \in C \text{ such that } z^* = P_C(z^* - \lambda A z^*), \tag{1.5}$$

where $\lambda > 0$ and P_C is the metric projection of H_1 onto C. Many methods have been developed for solving the VIP (1.2). One of the methods is the extragradient method which was proposed and studied by Korpelevich [10] and Antipin [11] in finite dimensional Euclidean spaces as follows:

$$x_1 \in C, \; y_n = P_C(x_n - \lambda A x_n), \; x_{n+1} = P_C(x_n - \lambda A y_n), \; \forall n \geq 1, \tag{1.6}$$

where A is a monotone and $L-$Lipschitz continuous operator. Under the assumption that $VI(C,A)$ is nonempty, the sequence generated by (1.6) converges to a point in $VI(C,A)$. This extragradient method may be costly in computation, since it requires two orthogonal projections at each iteration. There are some methods to overcome this difficulty, one of which is the subgradient extragradient method introduced by Censor et al. [12], where the second projection onto the constrained set is replaced with a projection onto a specific constructible half-space. Another method, which was proposed by Tseng [13], is called the Tseng's extragradient method. The method only requires one projection onto the feasible set. Since the subgradient extragradient method and Tseng's extragradient methods requires only one projection onto the constrained set in each iteration, they have attracted the attention of many researchers working in this field; see, e.g., [14, 15, 16, 17] and the references therein. However, there is also a setback in terms of the applicability of subgradient extragradient method and Tseng's extragradient method which is the fact that the stepsize depends on the knowledge of the Lipschitz constant. The third method is the projection and contraction method (PCM), proposed and studied by He [18] (see also Sun [19]):

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ d(x_n, y_n) = (x_n - y_n) - \lambda(Ax_n - Ay_n), \\ x_{n+1} = x_n - \gamma \eta_n d(x_n, y_n), \; \forall n \geq 1, \end{cases} \tag{1.7}$$

where $\gamma \in (0,2), \lambda \in (0,1/L)$ and $\eta_n := \frac{\langle x_n - y_n, d(x_n,y_n)\rangle}{||d(x_n,y_n)||^2}$. He [18] established that the sequence $\{x_n\}$ generated by (1.7) converges weakly to a solution of (1.2). Since PCM requires only one projection onto the feasible set C, it reduces the computational cost per each iteration. This attracted the attention of many researchers who devoted their studies to improving the PCM in many different ways; see, e.g., [20, 21, 22] and the references therein. To speed up convergence rate of algorithms, Polyak [23] studied the heavy ball method, an inertial extrapolation process for minimizing a smooth convex function. Since then, many authors have introduced this technique in different methods for solving VIPs (see, e.g., [24, 25, 26, 27, 28] for details). In particular, Cholamjiak et al. [28] introduced the following inertial PCM for solving the VIP with pseudomonotone operator:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda A w_n), \\ z_n = w_n - \gamma\eta_n d(w_n, y_n), \\ d(w_n,y_n) = (w_n - y_n) - \lambda(Aw_n - Ay_n), \\ \eta_n := \frac{\langle w_n - y_n, d(w_n,y_n)\rangle}{||d(w_n,y_n)||^2}, \\ x_{n+1} = (1-\alpha_n-\delta_n)x_n + \alpha_n z_n, \ \forall n \geq 1, \end{cases} \tag{1.8}$$

where A is L-Lipschitz continuous, pseudomonotone, and sequentially weakly continuous, $\gamma \in (0,2), \lambda \in (0,1/L)$, $\{\tau_n\} \subset (0,\infty)$, $\tau_n = \circ(\alpha_n)$, $\{\alpha_n\} \subset (a, 1-\delta_n)$ for some $a > 0$, $\{\delta_n\} \subset (0,1), \theta > 0$ and θ_n is selected such that $0 \leq \theta_n \leq \bar{\theta}_n$ and $\bar{\theta}_n := \min\left\{\theta, \frac{\tau_n}{||x_n - x_{n-1}||}\right\}$ if $x_n \neq x_{n-1}$; otherwise, $\bar{\theta}_n = \theta$. They proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to the solution of the VIP. We note that Algorithm (1.8) requires to know the Lipschitz constant of operator A, which is often unknown or difficult to estimate in practice.

A special case of the BSVIP (1.1), when $H_1 = H_2$, $F_2 = 0$ and $B = I$(identity mapping), is known as a bilevel variational inequality problem (BVIP). The BVIP covers many aspects of mathematical programs with equilibrium constraints [29], bilevel convex programming models [30], and the minimum-norm problems with the solution set of variational inequalities [31]. Using the projection and contraction method, Thong et al. [32] studied the BVIP as follows:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = x_n - \alpha\eta_n d(x_n, y_n), \\ d(x_n,y_n) = (x_n - y_n) - \lambda_n(Ax_n - Ay_n), \\ \eta_n := \frac{\langle x_n - y_n, d(x_n,y_n)\rangle}{||d(x_n,y_n)||^2}, \\ x_{n+1} = z_n - \alpha_n\gamma F(z_n), \ \forall n \geq 1, \end{cases} \tag{1.9}$$

where A is L_1-Lipschitz continuous, pseudomonotone, and sequentially weakly continuous, F is $\beta-$strongly monotone and L_2-Lipschitz continuous, $\alpha \in (0,2), \mu \in (0,1)$, $\lambda_0 > 0$, $0 < \gamma < \frac{2\beta}{L_2^2}$, $\{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lambda_{n+1} := \min\left\{\lambda_n, \mu\frac{||x_n - y_n||}{||Ax_n - Ay_n||}\right\}$ if $Ax_n \neq Ay_n$; otherwise, $\lambda_{n+1} = \lambda_n$. They proved that the sequence $\{x_n\}$ generated by (1.9) converges strongly to the unique solution of the BVIP. Very recently, Tan et al. [33] proposed the following inertial subgradient extragradient method and inertial Tseng's method, respectively, for solving the BVIP:

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n) \\ T_n = \{x \in H : \langle w_n - \lambda_n A w_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = P_{T_n}(w_n - \lambda_n A y_n), \\ x_{n+1} = z_n - \alpha_n \gamma F(z_n), \ \forall n \geq 1, \end{cases} \tag{1.10}$$

and

$$\begin{cases} x_0, x_1 \in H, \\ w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ z_n = y_n - \lambda_n(A y_n - A w_n), \\ x_{n+1} = z_n - \alpha_n \gamma F(z_n), \ \forall n \geq 1. \end{cases} \tag{1.11}$$

Under some assumptions on the parameters, they proved that Algorithms (1.10) and (1.11) converge strongly to the unique solution of the BVIP when A is L_1-Lipschitz continuous, pseudomonotone and sequentially weakly continuous, and F is $\beta-$strongly monotone and L_2-Lipschitz-continuous.

Motivated by the above works and using the idea presented in (1.5), letting $S(x) := P_Q(x - \lambda F_2(x))$, $\lambda > 0$, we obtain that $VI(Q, F_2) = Fix(S)$ ($Fix(S)$ denotes the fixed points set of S). In this case, we redefine BSVIP (1.1) as follows: Suppose that $A : H_1 \to H_1$ is pseudomonotone and K-Lipschitz continuous (not sequentially weakly continuous), $F : H_1 \to H_1$ is $\beta-$strongly monotone and $L-$Lipschitz continuous, $B : H_1 \to H_2$ is a bounded linear operator with $B \neq 0$, and $S : H_2 \to H_2$ is a κ-generalized demimetric mapping with $\kappa > 0$. Our goal is to study the following BSVIP:

$$\text{Finding } z^* \in \Gamma \text{ such that } \langle F(z^*), z - z^* \rangle \geq 0, \ \forall z \in \Gamma, \tag{1.12}$$

where $\Gamma := \{z^* \in VI(C, A) : Bz^* \in Fix(S)\}$. We then propose a modified projection and contraction method with inertial extrapolation steps, and prove that the sequence generated by this algorithm converges strongly to a unique solution of BSVIP (1.12). We also give some numerical examples to show the efficiency of our proposed method.

2. Preliminaries

Let C be a closed convex subset of a real Hilbert space H. Then, for each $x \in H$, there exists a unique point $z = P_C(x)$ such that $||x - z|| = \inf_{y \in C} ||x - y||$. The operator $P_C : H \to C$ is called the metric projection from H onto C. We have that $z = P_C(x)$ if and only if the following relation holds $\langle x - z, y - z \rangle \leq 0$, $\forall y \in C$.

Recall that an operator $A : H \to H$ is said to be:

(a) $L-$Lipschitz continuous with $L > 0$ if

$$||Ax - Ay|| \leq L||x - y||, \ \forall x, y \in H;$$

(b) β-strongly monotone if there exists $\beta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \beta ||x - y||^2, \ \forall x, y \in H;$$

(c) monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \forall x, y \in H;$$

(d) pseudomonotone if

$$\langle Ax, y - x \rangle \geq 0 \Rightarrow \langle Ay, y - x \rangle \geq 0, \ \forall x, y \in H.$$

It is clear that $(b) \Rightarrow (c) \Rightarrow (d)$, but the converses are not true in general. Recall that a mapping $S : C \to X$ with $Fix(S) \neq \emptyset$ is said to be κ-generalized demimetric if there exists $\kappa \in \mathbb{R} \setminus \{0\}$ such that $||x - Sx||^2 \leq \kappa \langle x - u, x - Sx \rangle$, for all $x \in C$ and $u \in Fix(S)$. Let $S : C \to X$ be a θ-generalized demimetric mapping with $\theta \in \mathbb{R} \setminus \{0\}$. Then $F(T)$ is closed and convex; see, [34].

The following lemma is trivial.

Lemma 2.1. *Let H be a real Hilbert space. Then, for all $x, y \in H$ and $\alpha \in \mathbb{R}$, the following hold*

(i) $||x - y||^2 = ||x||^2 + ||y||^2 - 2\langle x, y \rangle$,
(ii) $||x + y||^2 \leq ||x||^2 + 2\langle y, x + y \rangle$,
(iii) $||\alpha x + (1 - \alpha) y||^2 = \alpha ||x||^2 + (1 - \alpha) ||y||^2 - \alpha(1 - \alpha)||x - y||^2$.

Lemma 2.2. [35] *Let H be a real Hilbert space, and let $F : H \to H$ be a β-strongly monotone and $L-$Lipschitz continuous mapping on H. If $\alpha \in (0, 1)$, $\eta \in [0, 1 - \alpha]$ and $\mu \in \left(0, \frac{2\beta}{L^2}\right)$, then, for all $x, y \in H$, $||[(1 - \eta)x - \alpha\mu F(x)] - [(1 - \eta)y - \alpha\mu F(y)]|| \leq (1 - \eta - \alpha\tau)||x - y||$, where $\tau = 1 - \sqrt{1 - \mu(2\beta - \mu L^2)} \in (0, 1)$.*

Lemma 2.3. [36] *Let C be a nonempty closed and convex subset of a real Hilbert space H, and let $A : C \to H$ be a pseudomonotone and continuous operator. Then, $x^* \in VI(C, A) \Leftrightarrow \langle A(z), z - x^* \rangle \geq 0$, $\forall z \in C$.*

Lemma 2.4. [37] *Let $\{a_n\}$ be a nonnegative sequence of real numbers, $\{r_n\}$, and $\{s_n\}$ be sequences of real numbers such that $\{r_n\}$ is in $(0, 1)$ with condition $\sum_{n=1}^{\infty} r_n = \infty$. Assume that $a_{n+1} \leq (1 - r_n)a_n + r_n s_n$, $\forall n \geq 1$. If $\limsup_{k\to\infty} s_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the following condition: $\liminf_{k\to\infty} \left(a_{n_k+1} - a_{n_k}\right) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

In this section, we introduce our method for solving the bilevel split variational inequality problem. We begin with the following assumptions.

Assumption 3.1. Let H_1 and H_2 be two real Hilbert spaces. Suppose the following conditions are satisfied:

(A1) the feasible set C is a nonempty closed, and convex subset of H_1, and the solution set $\Gamma \neq \emptyset$;
(A2) $A : H_1 \to H_1$ is pseudomonotone, and $K-$Lipschitz continuous on H_1 with $K > 0$;
(A3) $A : H_1 \to H_1$ satisfies the following condition

$$\text{whenever} \quad \{x_n\} \subset C,\ x_n \rightharpoonup z, \quad \text{one has} \quad \|Az\| \leq \liminf_{n\to\infty} \|Ax_n\|;$$

(A4) $F: H_1 \to H_1$ is $\beta-$strongly monotone and $L-$Lipschitz continuous on H_1 with $L > 0$ such that $\delta = 1 - \sqrt{1-\sigma(2\beta - \sigma L^2)}$, where $\sigma \in \left(0, \frac{2\beta}{L^2}\right)$;

(A5) $B: H_1 \to H_2$ is a bounded linear operator such that $B \neq 0$ and $S: H_2 \to H_2$ is κ-generalized demimetric with $\kappa \in (0,\infty)$ such that S is demiclosed at the origin;

(A6) $\{c_n\}$ is a positive sequence with $c_n = \circ(b_n)$, $\{a_n\} \subset (b, 1-b_n)$ for some $b > 0$, $\{\lambda_n\}$ satisfies $0 < c \leq \lambda \leq d < \frac{2}{\kappa||B||^2}$ and $\{b_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=1}^{\infty} b_n = \infty$.

We remark here that Assumption (A3) is strictly weaker than the sequentially weakly continuous assumption which has frequently been used i for solving pseudomonotone VIPs recently. An example of an operator satisfying condition (A3) but not sequentially weakly continuous is $A(x) = x\|x\|$ (see [38, Remark 3.2]). Let $x^* \in H_1$ denote the unique solution of the BSVIP (1.12). We present the following modified inertial projection and contraction method for solving BSVIP (1.12).

Algorithm 3.1. Initialization: Choose $\alpha \in (0,2)$, $\tau_1 > 0$, $\mu \in (0,1)$, and $\theta \in [0,1)$, and let $x_0, x_1 \in H_1$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \min\left\{\theta, \frac{c_n}{||x_n - x_{n-1}||}\right\}, \text{ if } x_n \neq x_{n-1}; \text{ otherwise, set } \bar{\theta}_n = \theta.$$

Step 2. Set

$$y_n = x_n + \theta_n(x_n - x_{n-1})$$

and compute

$$w_n = P_C(y_n - \tau_n A y_n).$$

Update

$$\tau_{n+1} = \begin{cases} \min\left\{\frac{\mu||y_n - w_n||}{||Ay_n - Aw_n||}, \tau_n\right\}, \text{ if } Ay_n \neq Aw_n, \\ \tau_n, \qquad\qquad\qquad \text{Otherwise.} \end{cases} \tag{3.1}$$

Step 3. Compute

$$v_n = y_n - \alpha\eta_n d_n,$$

where

$$d_n := y_n - w_n - \tau_n(Ay_n - Aw_n), \; \eta_n = \frac{\langle y_n - w_n, d_n\rangle}{||d_n||^2}, \text{ if } d_n \neq 0; \text{otherwise } \eta_n = 0.$$

Step 4. Compute

$$u_n = v_n - \lambda_n B^*(I-S)Bv_n$$
$$x_{n+1} = a_n x_n + (1-a_n)u_n - b_n \sigma F u_n.$$

Set $n := n+1$ and return to **Step 1**.

Lemma 3.1. ([39, Lemma 3.1]) *The sequence $\{\tau_n\}$ generated by Algorithm 3.1 is non-increasing and*

$$\lim_{n\to\infty} \tau_n = \tau \geq \min\left\{\frac{\mu}{K}, \tau_1\right\}.$$

Lemma 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 under Assumption 3.1. Then $\{x_n\}$ is bounded.*

Proof. Let $x^* \in \Gamma$. Since $w_n \in C$, we have $\langle Ax^*, w_n - x^*\rangle \geq 0$, which implies by the pseudomonotonicity of A that $\langle Aw_n, w_n - x^*\rangle \geq 0$. Note that

$$\langle y_n - w_n - \tau_n Ay_n, w_n - x^*\rangle \geq 0. \tag{3.2}$$

It follows that $\langle y_n - w_n - \tau_n(Ay_n - Aw_n), w_n - x^*\rangle \geq 0$. Thus, $\langle w_n - x^*, d_n\rangle \geq 0$. From Step 3, (3.2), and Lemma 2.1(i), we have

$$\begin{aligned} ||v_n - x^*||^2 &= ||y_n - x^*||^2 - 2\alpha\eta_n\langle y_n - x^*, d_n\rangle + ||\alpha\eta_n d_n||^2 \\ &= ||y_n - x^*||^2 - 2\alpha\eta_n\langle y_n - w_n, d_n\rangle - 2\alpha\eta_n\langle w_n - x^*, d_n\rangle + ||\alpha\eta_n d_n||^2 \\ &\leq ||y_n - x^*||^2 - 2\alpha\eta_n\langle y_n - w_n, d_n\rangle + ||\alpha\eta_n d_n||^2 \\ &= ||y_n - x^*||^2 - \frac{2-\alpha}{\alpha}||y_n - v_n||^2. \end{aligned} \tag{3.3}$$

On the other hand, we have

$$||y_n - x^*|| \leq ||x_n - x^*|| + b_n \cdot \frac{\theta_n}{b_n}||x_n - x_{n-1}||.$$

From Step 1, we have $\frac{\theta_n}{b_n}||x_n - x_{n-1}|| \leq \frac{c_n}{b_n}$, which together with (A6) implies that $\lim_{n\to\infty}\frac{\theta_n}{b_n}||x_n - x_{n-1}|| = 0$, so $\{\frac{\theta_n}{b_n}||x_n - x_{n-1}||\}$ is bounded. Thus, there exists a constant $M > 0$ such that $\frac{\theta_n}{b_n}||x_n - x_{n-1}|| \leq M$ for all $n \geq 1$. Hence, $||y_n - x^*|| \leq ||x_n - x^*|| + b_n M$, which together with (3.3), and the fact that $\alpha \in (0,2)$, obtains

$$||v_n - x^*|| \leq ||y_n - x^*|| \leq ||x_n - x^*|| + b_n M. \tag{3.4}$$

Furthermore, from Step 4, the fact that S is κ-generalized demimetric with $\kappa > 0$, and $\lambda_n < \frac{2}{\kappa||B||^2}$, we have

$$\begin{aligned} ||u_n - x^*||^2 &= ||v_n - x^*||^2 - 2\lambda_n\langle Bv_n - Bx^*, (I-S)Bv_n\rangle + \lambda_n^2||B^*(I-S)Bv_n||^2 \\ &\leq ||v_n - x^*||^2 - \frac{2\lambda_n}{\kappa}||(I-S)Bv_n||^2 + \lambda_n^2||B^*(I-S)Bv_n||^2 \\ &\leq ||v_n - x^*||^2 + \lambda_n\left(\lambda_n||B||^2 - \frac{2}{\kappa}\right)||(I-S)Bv_n||^2 && (3.5) \\ &\leq ||v_n - x^*||^2. && (3.6) \end{aligned}$$

From Lemma 2.2, (3.4) and (3.6), we have

$$\begin{aligned} ||x_{n+1} - x^*|| &\leq ||[(1-a_n)u_n - b_n\sigma F(u_n)] - [(1-a_n)x^* - b_n\sigma F(x^*)]|| \\ &\quad + a_n||x_n - x^*|| + b_n\sigma||F(x^*)|| \\ &\leq (1 - a_n - b_n\delta)||u_n - x^*|| + a_n||x_n - x^*|| + b_n\sigma||F(x^*)|| \\ &\leq (1 - a_n - b_n\delta)[||x_n - x^*|| + b_n M] + a_n||x_n - x^*|| + b_n\sigma||F(x^*)|| \\ &\leq (1 - b_n\delta)||x_n - x^*|| + b_n\delta\frac{M + \sigma||F(x^*)||}{\delta} \\ &\leq \max\{||x_n - x^*||, \delta^{-1}(M + \sigma||F(x^*)||)\}. \end{aligned}$$

Hence,

$$||x_n - x^*|| \leq \max\left\{||x_1 - x^*||, \delta^{-1}(M + \sigma||F(x^*)||)\right\}.$$

This proves that $\{x_n\}$ is bounded, so are $\{u_n\}, \{v_n\}$, $\{y_n\}$, and $\{Fu_n\}$. □

Lemma 3.3. *Let $\{y_n\}$ and $\{w_n\}$ be sequences generated by Algorithm 3.1 under Assumption 3.1. If there exist subsequences $\{y_{n_k}\}$ and $\{w_{n_k}\}$ of $\{y_n\}$ and $\{w_n\}$, respectively such that $\{y_{n_k}\}$ converges weakly to a point, say z in H_1 and $\lim_{k\to\infty}||w_{n_k} - y_{n_k}|| = 0$, then $z \in VI(C,A)$.*

Proof. Since $w_{n_k} = P_C(y_{n_k} - \tau_{n_k} A y_{n_k})$, we have $\langle y_{n_k} - \tau_{n_k} A y_{n_k} - w_{n_k}, x - w_{n_k}\rangle \leq 0$, $\forall x \in C$, which implies that

$$\begin{aligned} \langle y_{n_k} - w_{n_k}, x - w_{n_k}\rangle &\leq \tau_{n_k}\langle Ay_{n_k}, x - w_{n_k}\rangle \\ &= \tau_{n_k}\langle Ay_{n_k}, y_{n_k} - w_{n_k}\rangle + \tau_{n_k}\langle Ay_{n_k}, x - y_{n_k}\rangle. \end{aligned}$$

Hence,

$$\frac{1}{\tau_{n_k}}\langle y_{n_k} - w_{n_k}, x - w_{n_k}\rangle + \langle Ay_{n_k}, w_{n_k} - y_{n_k}\rangle \leq \langle Ay_{n_k}, x - y_{n_k}\rangle. \tag{3.7}$$

Since $\{y_{n_k}\}$ converges weakly to a point $z \in H$, then it is bounded. Using the hypothesis $||w_{n_k} - y_{n_k}|| \to 0$ as $k \to \infty$, we have that $\{w_{n_k}\}$ and $\{Ay_{n_k}\}$ are also bounded. From Lemma 3.1, we have $\lim_{k\to\infty}\tau_{n_k} > 0$. Thus, passing limit as $k \to \infty$ in (3.7), we obtain $\liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \geq 0$, $\forall x \in C$. We also have

$$\begin{aligned} \langle Aw_{n_k}, x - w_{n_k}\rangle &= \langle Aw_{n_k} - Ay_{n_k}, x - y_{n_k}\rangle + \langle Ay_{n_k}, x - y_{n_k}\rangle \\ &\quad + \langle Aw_{n_k}, y_{n_k} - w_{n_k}\rangle. \end{aligned} \tag{3.8}$$

From $\lim_{k\to\infty}||y_{n_k} - w_{n_k}|| = 0$, and the fact that A is Lipschitz continuous, we have $\lim_{k\to\infty}||Ay_{n_k} - Aw_{n_k}|| = 0$, which together with (3.8) yields that

$$\liminf_{k\to\infty}\langle Aw_{n_k}, x - w_{n_k}\rangle \geq 0,\ \forall x \in C.$$

Now, let $\{\varepsilon_k\}$ be a decreasing sequence in $(0,1)$ such that $\varepsilon_k \to 0$ as $k \to \infty$. For each $k \geq 1$, we denote by N_k the smallest nonnegative integer such that

$$\langle Aw_{n_j}, x - w_{n_j}\rangle + \varepsilon_k \geq 0,\ \forall j \geq N_k. \tag{3.9}$$

It is clear that $\{N_k\}$ is increasing from the fact that $\{\varepsilon_k\}$ is decreasing. Also, for each $k \geq 1$, since $\{w_{N_k}\} \subset C$, we have $A(w_{N_k}) \neq 0$. Letting $z_{N_k} = \frac{A(w_{N_k})}{||A(w_{N_k})||^2}$, we obtain $\langle A(w_{N_k}), z_{N_k}\rangle = 1$ for each $k \geq 1$. Thus, for each $k \geq 1$, $\langle Aw_{N_k}, x + \varepsilon_k z_{N_k} - w_{N_k}\rangle \geq 0$. Since A is pseudomonotone on H_1, we have $\langle A(x + \varepsilon_k z_{N_k}), x + \varepsilon_k z_{N_k} - w_{N_k}\rangle \geq 0$. Hence,

$$\langle Ax, x - w_{N_k}\rangle \geq \langle Ax - A(x + \varepsilon_k z_{N_k}), x + \varepsilon_k z_{N_k} - w_{N_k}\rangle - \varepsilon_k\langle Ax, z_{N_k}\rangle. \tag{3.10}$$

Now, since $\{y_{n_k}\}$ converges weakly to z and $||w_{n_k} - y_{n_k}|| \to 0$ as $k \to \infty$, then we have that $\{w_{n_k}\}$ converges weakly to z as $k \to \infty$. Since $\{w_{n_k}\} \subset C$, and C is closed, then $z \in C$. We assume that $Az \neq 0$ (otherwise z is a solution). From (A3), we obtain that $0 < ||Az|| \leq \liminf_{k\to\infty}||Aw_{N_k}||$. Also, using the fact that $\{w_{N_k}\} \subset \{w_{n_k}\}$ and $\varepsilon_k \to 0$ as $k \to \infty$, we have

$$0 \leq \limsup_{k\to\infty} ||\varepsilon_k z_{N_k}|| = \limsup_{k\to\infty} \Big(\frac{\varepsilon_k}{||Aw_{n_k}||} \Big) \leq \frac{\limsup\limits_{k\to\infty} \varepsilon_k}{\liminf\limits_{k\to\infty} ||Aw_{n_k}||} = 0.$$

Hence, $\lim\limits_{k\to\infty} \varepsilon_k z_{N_k} = 0$. Thus, the right-hand side of (3.10) tends to zero as $k \to \infty$. That is, $\liminf\limits_{k\to\infty} \langle Ax, x - w_{N_k} \rangle \geq 0$. Therefore, for all $x \in C$,

$$\langle Ax, x - z \rangle = \lim_{k\to\infty} \langle Ax, x - w_{N_k} \rangle = \liminf_{k\to\infty} \langle Ax, x - w_{N_k} \rangle \geq 0.$$

From Lemma 2.3, we have that $z \in VI(C,A)$. This completes the proof. □

Lemma 3.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 under Assumption 3.1. Then for all $n \geq 1$, we have*

$$||x_{n+1} - x^*||^2 \leq (1 - b_n\delta)||x_n - x^*||^2 + b_n\delta \left(\frac{\theta_n}{\delta b_n} ||x_n - x_{n-1}|| M_1 + 2\frac{\sigma}{\delta} \langle F(x^*), x^* - x_{n+1} \rangle \right) \tag{3.11}$$

where x^* is the unique solution of the BSVIP (1.12).

Proof. From Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= \Big\| \Big\{ [(1-a_n)u_n - b_n\sigma F(u_n)] - [(1-a_n)x^* - b_n\sigma F(x^*)] \Big\} \\ &\quad + a_n(x_n - x^*) - b_n\sigma F(x^*) \Big\|^2 \\ &\leq ||[(1-a_n)u_n - b_n\sigma F(u_n)] - [(1-a_n)x^* - b_n\sigma F(x^*)] + a_n(x_n - x^*)||^2 \\ &\quad + 2b_n\sigma \langle F(x^*), x^* - x_{n+1} \rangle \\ &\leq \Big\{ ||[(1-a_n)u_n - b_n\sigma F(u_n)] - [(1-a_n)x^* - b_n\sigma F(x^*)]|| + a_n||x_n - x^*|| \Big\}^2 \\ &\quad + 2b_n\sigma \langle F(x^*), x^* - x_{n+1} \rangle \\ &\leq [1 - a_n - b_n\delta]||u_n - x^*||^2 + a_n||x_n - x^*||^2 + 2b_n\sigma \langle F(x^*), x^* - x_{n+1} \rangle. \end{aligned}$$

From Step 2, we have

$$\begin{aligned} ||y_n - x^*||^2 &= ||x_n - x^*||^2 + 2\theta_n \langle x_n - x^*, x_n - x_{n-1} \rangle + \theta_n^2 ||x_n - x_{n-1}||^2 \\ &\leq ||x_n - x^*||^2 + \theta_n ||x_n - x^*|| [2||x_n - x_{n-1}|| + \theta_n ||x_n - x_{n-1}||] \\ &\leq ||x_n - x^*||^2 + \theta_n ||x_n - x_{n-1}|| M_1, \end{aligned} \tag{3.12}$$

for some constant $M_1 > 0$. So, combining (3.3), (3.5), and (3.12), we obtain

$$\begin{aligned} ||u_n - x^*||^2 &\leq ||x_n - x^*||^2 + \theta_n ||x_n - x_{n-1}|| M_1 - \frac{2-\alpha}{\alpha} ||y_n - v_n||^2 \\ &\quad + \lambda_n (\lambda_n ||B||^2 - \frac{2}{\kappa}) ||(I - S)Bv_n||^2. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), we have

$$\begin{aligned}
||x_{n+1}-x^*||^2 &\leq (1-a_n-b_n\delta)||x_n-x^*||^2+\theta_n(1-a_n-b_n\delta)||x_n-x_{n-1}||M_1\\
&\quad -\frac{2-\alpha}{\alpha}(1-a_n-b_n\delta)||y_n-v_n||^2\\
&\quad +\lambda_n(1-a_n-b_n\delta)(\lambda_n||B||^2-\frac{2}{\kappa})||(I-S)Bv_n||^2\\
&\quad +a_n||x_n-x^*||^2+2b_n\sigma\langle F(x^*),x^*-x_{n+1}\rangle\\
&\leq (1-b_n\delta)||x_n-x^*||^2+\theta_n||x_n-x_{n-1}||M_1-\frac{2-\alpha}{\alpha}(1-a_n-b_n\delta)||y_n-v_n||^2\\
&\quad +\lambda_n(1-a_n-b_n\delta)(\lambda_n||B||^2-\frac{2}{\kappa})||(I-S)Bv_n||^2+2b_n\sigma\langle F(x^*),x^*-x_{n+1}\rangle\\
&\leq (1-b_n\delta)||x_n-x^*||^2+\theta_n||x_n-x_{n-1}||M_1+2b_n\sigma\langle F(x^*),x^*-x_{n+1}\rangle,
\end{aligned}$$

which yields the desired conclusion. □

Theorem 3.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 under Assumption 3.1. Then, $\{x_n\}$ converges strongly to the unique solution of the BSVIP (1.12).*

Proof. From Lemma 2.4 and (3.11), we only need to show that

$$\limsup_{k\to\infty}\Big(\frac{\theta_{n_k}}{\delta b_{n_k}}||x_{n_k}-x_{n_k-1}||+2\frac{\sigma}{\delta}\langle F(x^*),x^*-x_{n_k+1}\rangle\Big)\leq 0,$$

for every subsequence $\{||x_{n_k}-x^*||\}$ of $\{||x_n-x^*||\}$ satisfying the following condition:

$$\liminf_{k\to\infty}(||x_{n_k+1}-x^*||-||x_{n_k}-x^*||)\geq 0. \tag{3.14}$$

Now, let $\{||x_{n_k}-x^*||\}$ be a subsequence of $\{||x_n-x^*||\}$ such that (3.14) holds. Then

$$\begin{aligned}
&\liminf_{k\to\infty}(||x_{n_k+1}-x^*||^2-||x_{n_k}-x^*||^2)\\
&= \liminf_{k\to\infty}\Big[(||x_{n_k+1}-x^*||-||x_{n_k}-x^*||)(||x_{n_k+1}-x^*||+||x_{n_k}-x^*||)\Big]\\
&\geq 0.
\end{aligned} \tag{3.15}$$

Thus, from (3.14) and (3.15), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\Big(\frac{2-\alpha}{\alpha}(1-a_{n_k}-\delta b_{n_k})||y_{n_k}-v_{n_k}||^2\\
&\quad +\lambda_{n_k}(\frac{2}{\kappa}-\lambda_{n_k}||B||^2)(1-a_{n_k}-\delta b_{n_k})||(I-S)Bv_{n_k}||^2\Big)\\
&\leq \limsup_{k\to\infty}\Big((1-b_{n_k}\delta)||x_{n_k}-x^*||^2+b_{n_k}\cdot\frac{\theta_{n_k}}{b_{n_k}}||x_{n_k}-x_{n_k-1}||M_1\\
&\quad +2b_{n_k}\sigma||F(x^*)||||x_{n_k+1}-x^*||-||x_{n_k+1}-x^*||^2\Big)\\
&= -\liminf_{k\to\infty}\Big(||x_{n_k+1}-x^*||^2-||x_{n_k}-x^*||^2\Big)\leq 0,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\Big(\frac{2-\alpha}{\alpha}(1-a_{n_k}-\delta b_{n_k})||y_{n_k}-v_{n_k}||^2+\lambda_{n_k}(\frac{2}{\kappa}-\lambda_{n_k}||B||^2)(1-a_{n_k}-\delta b_{n_k})||(I-S)Bv_{n_k}||^2\Big)=0.$$

Thus,

$$\lim_{k\to\infty}||y_{n_k}-v_{n_k}||=0 \tag{3.16}$$

and

$$\lim_{k\to\infty}||(I-S)Bv_{n_k}||=0. \tag{3.17}$$

On the other hand, we have

$$\begin{aligned} ||d_{n_k}|| &\le ||y_{n_k}-w_{n_k}||+\tau_{n_k}||Ay_{n_k}-Aw_{n_k}|| \\ &\le \Big(1+\mu\frac{\tau_{n_k}}{\tau_{n_k+1}}\Big)||y_{n_k}-w_{n_k}|| \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \langle y_{n_k}-w_{n_k},d_{n_k}\rangle &= ||y_{n_k}-w_{n_k}||^2-\tau_{n_k}\langle y_{n_k}-w_{n_k},Ay_{n_k}-Aw_{n_k}\rangle \\ &\ge ||y_{n_k}-w_{n_k}||^2-\tau_{n_k}||y_{n_k}-w_{n_k}||||Ay_{n_k}-Aw_{n_k}|| \\ &\ge \Big(1-\mu\frac{\tau_{n_k}}{\tau_{n_k+1}}\Big)||y_{n_k}-w_{n_k}||^2. \end{aligned} \tag{3.19}$$

Since $v_{n_k}=y_{n_k}-\alpha\eta_{n_k}d_{n_k}$, we find from (3.18) and (3.19) that

$$||v_{n_k}-y_{n_k}||=\alpha\eta_{n_k}||d_{n_k}||=\frac{\alpha\cdot\langle y_{n_k}-w_{n_k},d_{n_k}\rangle}{||d_{n_k}||}\ge\frac{\alpha(\tau_{n_k+1}-\mu\tau_{n_k})}{\tau_{n_k+1}+\mu\tau_{n_k}}||y_{n_k}-w_{n_k}||,$$

which implies that

$$||y_{n_k}-w_{n_k}||\le\frac{\tau_{n_k+1}+\mu\tau_{n_k}}{\alpha(\tau_{n_k+1}-\mu\tau_{n_k})}||v_{n_k}-y_{n_k}||. \tag{3.20}$$

From Lemma 3.1, we have

$$\lim_{k\to\infty}\frac{\tau_{n_k+1}+\mu\tau_{n_k}}{\tau_{n_k+1}-\mu\tau_{n_k}}=\frac{1+\mu}{1-\mu},$$

which shows that $\left\{\frac{\tau_{n_k+1}+\mu\tau_{n_k}}{\tau_{n_k+1}-\mu\tau_{n_k}}\right\}$ is bounded. Hence, combining (3.16) and (3.20) yields that $\lim_{k\to\infty}||y_{n_k}-w_{n_k}||=0$. This together with (3.16) implies $\lim_{k\to\infty}||v_{n_k}-w_{n_k}||=0$. Observe that

$$||y_{n_k}-x_{n_k}||=b_{n_k}\cdot\frac{\theta_{n_k}}{b_{n_k}}||x_{n_k}-x_{n_k-1}||\to 0 \text{ as } k\to\infty. \tag{3.21}$$

Hence, we have $\lim_{k\to\infty}||x_{n_k}-v_{n_k}||=0$, and $\lim_{k\to\infty}||x_{n_k}-w_{n_k}||=0$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence say $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to say $z\in H_1$ as $j\to\infty$. Then, from (3.21), we have that $\{y_{n_{k_j}}\}$ converges weakly to z as $j\to\infty$. From Lemma 3.3, we obtain that $z\in VI(C,A)$.

Next, we show that $Bz\in F(S)$. Observe that $\{v_{n_{k_j}}\}$ converges weakly to z as $j\to\infty$, and B is a bounded linear operator. Then, $\{Bv_{n_{k_j}}\}$ converges weakly to $Bz\in H_2$. Combining (3.17) and demiclosedness of S, we obtain that $Bz\in F(S)$. In view of Step 4 and (3.17), we have that $||x_{n_k}-u_{n_k}||\le||x_{n_k}-v_{n_k}||+\lambda_{n_k}||B^*||||(I-S)Bv_{n_k}||\to 0$ as $k\to\infty$. Thus

$$||x_{n_k+1}-x_{n_k}||\le(1-a_{n_k})||u_{n_k}-x_{n_k}||+b_{n_k}\sigma||F(u_{n_k})||\to 0 \text{ as } k\to\infty. \tag{3.22}$$

Since $x_{n_{k_j}} \rightharpoonup z$, we have $\limsup_{k\to\infty}\langle F(x^*), x^* - x_{n_k}\rangle = \lim_{j\to\infty}\langle F(x^*), x^* - x_{n_{k_j}}\rangle = \langle F(x^*), x^* - z\rangle$. Since x^* is the unique solution of BSVIP in (1.12), we have

$$\limsup_{k\to\infty}\langle F(x^*), x^* - x_{n_k}\rangle = \langle F(x^*), x^* - z\rangle \le 0. \tag{3.23}$$

Thus, from (3.22) and (3.23), we have

$$\limsup_{k\to\infty}\langle F(x^*), x^* - x_{n_k+1}\rangle = \limsup_{k\to\infty}\langle F(x^*), x^* - x_{n_k}\rangle = \langle F(x^*), x^* - z\rangle \le 0.$$

In view of Lemma 2.4, we conclude that $\lim_{n\to\infty}||x_n - x^*|| = 0$. Therefore, $\{x_n\}$ converges strongly to x^* as $n \to \infty$. □

4. Numerical Experiments

In this section, we discuss the numerical behavior of Algorithm 3.1 in both finite and infinite dimensional Hilbert spaces. We also compare our method with the methods of Thong et al. [32] (Algorithm (1.9)) and Tan et al. [33] (Algorithm (1.10) and Algorithm (1.11)). All codes are written in Matlab 2016 (b) and performed on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30GHz and 8.00 Gb-RAM. In Tables 1-2, “Iter.” means the number of iterations while “CPU” means the CPU time in seconds. In our computations, we choose $\tau_1 = 1$, $\alpha = 1.1$, $\mu = 0.5$ and $\sigma = \frac{\beta}{L^2}$. Also, we take $b_n = \frac{1}{n+1}$, $a_n = \frac{1}{2} - b_n$, $\lambda_n = \frac{1}{\kappa||B||^2}$, and $\theta_n = \bar{\theta}_n$ with $c_n = \frac{b_n}{n^{0.01}}$ and $\theta = 0.3$.

Example 4.1. Let $H_1 = \mathbb{R}^N$ and $H_2 = \mathbb{R}^m$. Define $A : \mathbb{R}^N \to \mathbb{R}^N$ by $A(x) = Mx + q$, where the matrix M is formed as: $M = V\sum V'$, where $V = I - \frac{2vv'}{||v||^2}$ and $\sum = \mathrm{diag}(\sigma_{11}, \sigma_{12}, \cdots, \sigma_{1N})$ are the Householder and the diagonal matrix, and

$$\sigma_{1j} = \cos\frac{j\pi}{N+1} + 1 + \frac{\cos\frac{\pi}{N+1} + 1 - \widehat{C}(\cos\frac{N\pi}{N+1} + 1)}{\widehat{C} - 1},\ j = 1, 2, \cdots, N,$$

with $\widehat{C}$ being the present condition number of M ([40, Example 5.2]). In the numerical computation, we choose $\widehat{C} = 10^4$, $q = 0$, and uniformly take the vector $v \in \mathbb{R}^N$ in $(-1, 1)$. Thus, A is pseudomonotone and Lipschitz continuous with $K = ||M||$ (see [40]). Furthermore, we generate the bounded linear operator $B \in \mathbb{R}^{M\times N}$ with independent Gaussian components distributed in the interval $(0, 1)$, and then normalize each column of B with the unit norm. We set $C = \{x \in \mathbb{R}^N : ||x|| \le 1\}$. The projection onto C is effectively computed in Matlab. We define $F : \mathbb{R}^N \to \mathbb{R}^N$ by $Fx = Dx + p$, where D is a symmetric and positive-definite matrix of size $N \times N$ and p is a vector in $\mathbb{R}^N$. Clearly, F is $||D||$-Lipschitz continuous. Moreover, since D is symmetric and positive-definite, we have that $\langle Dx - Dy\rangle = \langle D(x-y)\rangle \ge ||D||^{-1}||x-y||^2$. Hence, F is strongly monotone. Also, define $S : \mathbb{R}^m \to \mathbb{R}^m$ by $Sx = 2x$. Since the solution of the problem is unknown, we define the sequence $\mathrm{TOL}_n := ||x_{n+1} - x_n||^2$, and use the stopping criterion $\mathrm{TOL}_n < \varepsilon$ for the iterative processes, where ε is the predetermined error. Moreover, we consider different scenarios of the dimensions. That is, $N = 100, 300, 500, 1000$, and $m = N/2$ with starting points $x_1 = (1, 1, \dots, 1)'$ and $x_0 = (0, 0, \dots, 0)'$. For this example, we take $\varepsilon = 10^{-5}$ as the stopping criterion and obtain the numerical results reported in Table 1 and Figure 1.

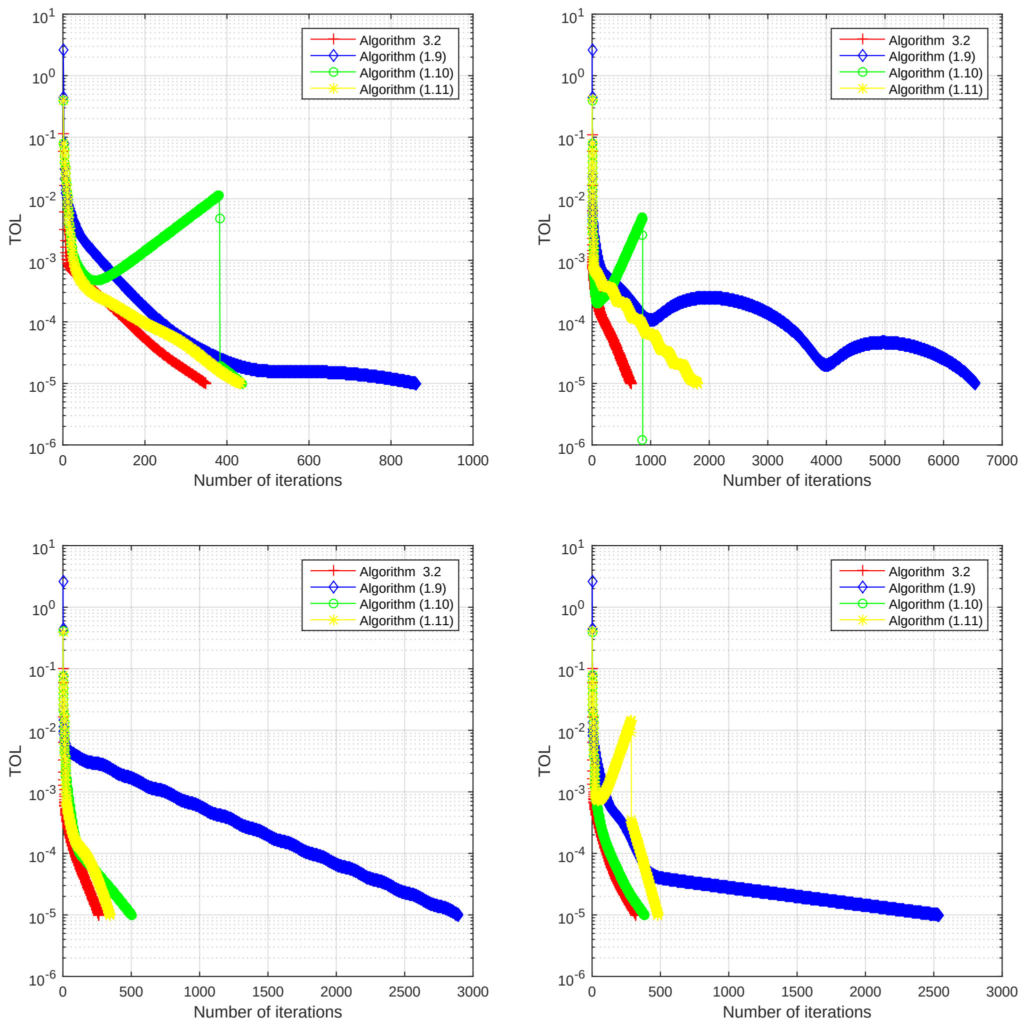

FIGURE 1. The behavior of TOL_n with $\varepsilon = 10^{-5}$ for Example 4.1: Top Left: $(N,m) = (100,50)$; Top Right: $(N,m) = (300,150)$; Bottom Left: $(N,m) = (500,250)$; Bottom Right: $(N,m) = (1000,500)$.

Table 1. Numerical results for Example 4.1 with $\varepsilon = 10^{-5}$.

(N,m)		Algorithm 3.1	Algorithm (1.9)	Algorithm (1.10)	Algorithm (1.11)
$(100,50)$	CPU Iter.	0.0371 348	1.1456 862	0.4401 437	0.4236 433
$(300,150)$	CPU Iter.	0.0385 662	1.2824 6548	0.4426 865	0.5568 1797
$(500,250)$	CPU Iter.	0.0180 265	1.1178 2893	0.4499 501	0.1124 344
$(1000,500)$	CPU Iter.	0.0259 322	1.1948 2540	0.1185 384	0.1134 483

Example 4.2. Let $H_1 = (l_2(\mathbb{R}), ||.||_{l_2}) = H_2$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \dots),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $||x||_{l_2} := \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$, $\forall x \in l_2(\mathbb{R})$. Now, define the operator $B : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by $Bx = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right)$, $\forall x \in l_2(\mathbb{R})$. Then, B is a bounded linear operator on $l_2(\mathbb{R})$ with adjoint $B^*y = \left(y_2, \frac{y_3}{2}, \frac{y_4}{3}, \dots\right)$, $\forall y \in l_2(\mathbb{R})$. Let $C = \{x \in l_2(\mathbb{R}) : |x_i| \le \frac{1}{i},\ i = 1, 2, 3, \dots\}$. Thus, we have explicit formula for P_C. Now, define the operator $A : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by

$$Ax = \left(||x|| + \frac{1}{||x|| + \alpha}\right)\alpha,$$

for some $\alpha > 0$. Then, A is pseudomonotone on $l_2(\mathbb{R})$ (see [41]). Furthermore, define the mapping $S : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by $Sx = (0, x_1, x_2, \dots)$, and $F : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by $Fx = x - x_0$. Then, F is strongly monotone and Lipschitz continuous.

We take $\varepsilon = 10^{-8}$ as the stopping criterion and choose the starting points as follows:
Case 1: Take $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ and $x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$.
Case 2: Take $x_1 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$ and $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.
Case 3: Take $x_1 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$ and $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$.
Case 4: Take $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$ and $x_0 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$.
The numerical results reported in Table 2 and Figure 2.

Table 2. Numerical results for Example 4.2 with $\varepsilon = 10^{-8}$.

Cases		Algorithm 3.1	Algorithm (1.9)	Algorithm (1.10)	Algorithm (1.11)
1	CPU Iter.	0.0654 330	1.0643 667	1.1005 679	1.0513 579
2	CPU Iter.	0.0340 410	1.0520 809	1.0709 817	1.0311 705
3	CPU Iter.	0.0349 292	1.0517 587	1.0998 609	1.0290 506
4	CPU Iter.	0.0377 329	1.0722 668	1.0572 673	1.0298 582

5. Conclusion

A modified projection and contraction method with inertial extrapolation steps was introduced and studied for solving the bilevel split variational inequality problem in two real Hilbert spaces when one of the cost operators is pseudomonotone, Lipschitz continuous, but not sequentially weakly continuous. We proved that the proposed algorithm converges strongly to the unique solution of the bilevel split variational inequality problem. Finally, we considered some numerical examples of our proposed method in comparison with other state-of-the-art methods for solving the bilevel split variational inequality problem. In all our comparisons, the numerical results reveal that our method performs better than these other methods.

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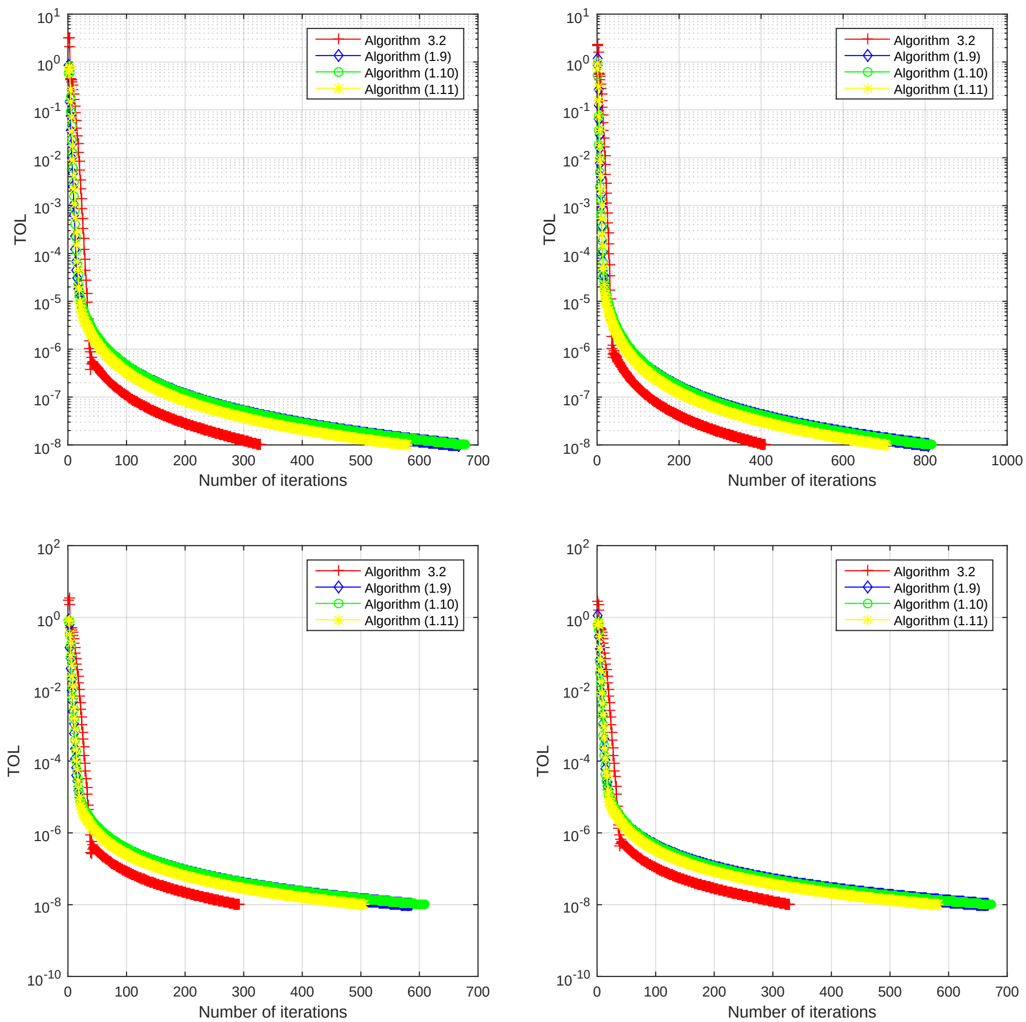

FIGURE 2. The behavior of TOL_n with $\varepsilon = 10^{-8}$ for Example 4.2: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

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