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A SEQUENTIAL FORMULA FOR THE ε-SUBDIFFERENTIAL OF MULTI-COMPOSED FUNCTIONS VIA A PERTURBATION APPROACH WITH AN APPLICATION TO LOCATION PROBLEMS

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Abstract. Without any qualification condition, we provide a sequential formula for the ε-subdifferential of multi-composed convex functions via the perturbation theory. As an application of this formula, necessary and sufficient sequential ε-optimality conditions are obtained for the location problems with monotonic gauges.

Keywords. Location problems; Multi-composed convex functions; Perturbation approach; Sequential ε-subdifferential calculus; ε-Optimality conditions.

1. INTRODUCTION

Multi-composed optimization problems deal with optimization models whose objective functions are written as the compositions of more than two functions (see [1, 2, 3]). In fact, the study of multi-composed optimization problems has been a subject matter of great interest because this new class of mathematical optimization models can be applied to many practical problems that arise in different fields of modern research, such as deep learning [4], facility location theory [5, 6], fractional programming problems, and entropy optimization [2], etc.

Recently, Grad, Wanka, and Wilfer [2] established a formula for the ε-subdifferential of a multi-composed convex function by using the Lagrange duality approach under a qualification condition of closedness type. They also employed this formula for delivering necessary and sufficient ε-optimality conditions that characterize the ε-optimal solutions to convex multi-composed optimization problems.

Due to the fact that the qualification condition is not always satisfied, numerous authors focused on developing approximate (or exact) optimality conditions in terms of sequences (or nets) in approximate (or exact) subdifferentials at some nearby points without requiring any qualification conditions; see, e.g., [7, 8, 9, 10, 11, 12, 13, 14, 15].

Motivated by [2], the aims of this paper are twofold. The first is to establish sequential ε-subdifferential calculus for the sums of m convex and lower semicontinuous functions ($m \geq 2$) via the tools of perturbation theory without imposing any qualification condition. The second is to present a sequential formula for the ε-subdifferential of the following multi-composed convex

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function $f+\varphi\circ\psi+g\circ h_1\circ h_2\circ\cdots\circ h_p$ $(p\geq 2)$. To attain this purpose, the ε-subdifferential of $f+\varphi\circ\psi+g\circ h_1\circ h_2\circ\cdots\circ h_p$ is transformed to that of the sums of $p+4$ convex and lower semicontinuous functions. As an application, this formula is applied to the constrained location problems with monotonic gauges.

Our paper is organised as follows. In Section 2, we recall some notions and give some preliminary results. In Section 3, we derive a sequential ε-subdifferential formula for the sums of m convex and lower semicontinuous functions $(m\geq 2)$. In Section 4, we establish a sequential formula for the ε-subdifferential of $f+\varphi\circ\psi+g\circ h_1\circ h_2\circ\cdots\circ h_p$ $(p\geq 2)$. Finally, in Section 5, we derive sequential ε-optimality conditions for a constrained location problem with monotonic gauge and without any qualification condition.

2. Preliminaries

We first recall some basic definitions and present some preliminary results, which are needed throughout the paper. Let X and Y be two Hausdorff locally convex spaces paired in duality by $\langle\cdot,\cdot\rangle$, where their topological duals X^* and Y^* are endowed respectively with the weak-star topology $w(X^*,X)$ and $w(Y^*,Y)$. For a subset $A\subseteq X^*\times\mathbb{R}$, we denote by $\overline{A}^{w(X^*,X)\times\tau_{\mathbb{R}}}$ the closure of A in $(X^*\times\mathbb{R},w(X^*,X)\times\tau_{\mathbb{R}})$, where $\tau_{\mathbb{R}}$ denotes the Euclidean topology on $\mathbb{R}$. Consider a nonempty convex cone $K\subseteq Y$ with $0\in K$. We define by $K^*:=\{y^*\in Y^*:\langle y^*,y\rangle\geq 0,\quad\forall y\in K\}$ the dual cone of K. Further, on Y, we consider that the partial order " $\leqq_K$ ", induced by K, is defined by, $y_1,y_2\in Y$, $y_1\leqq_K y_2\iff y_2-y_1\in K$. With respect to " $\leqq_K$ ", the augmented set $Y\cup\{+\infty_Y\}$ is considered. Here $+\infty_Y$ is an abstract element verifying the following operations and conventions

$$y\leqq_K+\infty_Y,\quad y+(+\infty_Y):=(+\infty_Y)+y:=+\infty_Y,\quad\forall y\in Y\cup\{+\infty_Y\},$$
$$y^*(+\infty_Y):=+\infty,\quad\alpha.(+\infty_Y):=+\infty_Y,\quad\forall y^*\in K^*,\quad\forall\alpha\geq 0.$$

Let $f:X\to\overline{\mathbb{R}}:=\mathbb{R}\cup\{\pm\infty\}$ be a function. Recall that f is said to be proper if its effective domain $\text{dom}f:=\{x\in X:f(x)\in\mathbb{R}\}\neq\emptyset$ and $f(x)>-\infty$ for all $x\in X$, and it is said to be convex if $f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$ for all $x,y\in X$ and all $\lambda\in[0,1]$. Moreover, f is said to be lower semicontinuous if its epigraph $\text{epi}f:=\{(x,r)\in X\times\mathbb{R}:\ f(x)\leq r\}$ is a closed subset of $X\times\mathbb{R}$. We denote by clf the lower semicontinuous hull of f, namely the function of which epigraph is the closure of $\text{epi}f$ in $X\times\mathbb{R}$. Furthermore, the function $f^*:X^*\to\overline{\mathbb{R}}$ defined by $f^*(x^*)=\sup_{x\in X}\{\langle x^*,x\rangle-f(x)\},\quad\forall x^*\in X^*$ is called the conjugate function of f. We have the so-called Young-Fenchel inequality $f^*(x^*)+f(x)\geq\langle x^*,x\rangle$, $\forall(x,x^*)\in X\times X^*$. The conjugate function of $f^*:X^*\to\overline{\mathbb{R}}$ is $f^{**}:=(f^*)^*$, and it is called the biconjugate function of f. Let us recall that if f is convex with nonempty effective domain, and clf is proper, then (see [16, Theorem 2.3.4]) $f^{**}=clf$.

Let $\bar{x}\in\text{dom}f$. Then the ε-subdifferential of f at $\bar{x}$, where $\varepsilon\geq 0$, is defined by

$$\partial_\varepsilon f(\bar{x}):=\{x^*\in X^*:f(x)\geq f(\bar{x})+\langle x^*,x-\bar{x}\rangle-\varepsilon,\quad\forall x\in X\}.$$

It is easy to prove that

$$\partial_\varepsilon f(\bar{x})=\{x^*\in X^*:f^*(x^*)+f(\bar{x})\leq\langle x^*,\bar{x}\rangle+\varepsilon\}$$

and

$$x^*\in\partial_\varepsilon f(\bar{x})\iff f(\bar{x})-\langle x^*,\bar{x}\rangle\leq\inf_{x\in X}\{f(x)-\langle x^*,x\rangle\}+\varepsilon.$$

Let $C \subseteq X$. We denote by $\operatorname{int} C$ the topological interior of C and by

$$C^\circ := \{x^* \in X^* : \langle x^*, x\rangle \leq 1, \quad \forall\, x \in C\}$$

the polar set of C. The indicator function $\delta_C : X \to \overline{\mathbb{R}}$ is defined by

$$\delta_C(x) := \begin{cases} 0, \text{ if } x \in C, \\ +\infty, \text{ otherwise}, \end{cases}$$

and the ε-normal set $N_C^\varepsilon(\bar{x})$ of C at $\bar{x} \in C$ is defined as the ε-subdifferential of δ_C at $\bar{x}$, i.e.,

$$N_C^\varepsilon(\bar{x}) := \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq \varepsilon, \quad \forall x \in C\}.$$

The function $j_C : X \to \overline{\mathbb{R}}$ defined by $j_C(x) := \inf\{\lambda > 0 : x \in \lambda C\}$, $\forall x \in X$, is called the gauge function (also known as Minkowski functional) of C. If C is a subset of $\mathbb{R}^q$, then the function j_C is called the monotonic gauge on $\mathbb{R}^q$ provided $j_C(x_1, \cdots, x_q) \leq j_C(y_1, \cdots, y_q)$ for all $(x_1, \cdots, x_q)$ and $(y_1, \cdots, y_q)$ in $\mathbb{R}^q$ satisfying $|x_i| \leq |y_i|$, $i = 1, \cdots, q$ (see, e.g., [17]).

Let Z be another Hausdorff locally convex space partially ordered by the convex cone $Q \subseteq Z$ with $0 \in Q$ and Z^* its topological dual space endowed with the weak-star topology $w(Z^*, Z)$. Let $g : Y \to Z \cup \{+\infty_Z\}$ be a vector mapping. g is said to be proper if its effective domain

$$\operatorname{dom} g := \{y \in Y : g(y) \in Z\}$$

is nonempty, and it is said to be Q-epi closed if its epigraph

$$\operatorname{epi} g := \{(y, z) \in Y \times Z : g(y) \leqq_Q z\}$$

is a closed subset of $Y \times Z$. The mapping g is said to be Q-convex if

$$g(\lambda y_1 + (1-\lambda) y_2) \leqq_Q \lambda g(y_1) + (1-\lambda) g(y_2),$$

for all $y_1, y_2 \in Y$ and all $\lambda \in [0,1]$. Furthermore, the mapping g is said to be (K, Q)-nondecreasing on $\operatorname{dom} g$ if, for all $y_1, y_2 \in \operatorname{dom} g$, $y_1 \leqq_K y_2 \implies g(y_1) \leqq_Q g(y_2)$. If $Z = \mathbb{R}$ and $Q = \mathbb{R}_+$, then this function is said to be K-nondecreasing on $\operatorname{dom} g$.

For a given mapping $h : X \longrightarrow Y \cup \{+\infty_Y\}$, the composed mapping $g \circ h : X \longrightarrow Z \cup \{+\infty_Z\}$ is defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)), \text{ if } x \in \operatorname{dom} h, \\ +\infty_Z, \text{ otherwise}. \end{cases}$$

It is easy to prove that if $g : Y \to Z \cup \{+\infty_Z\}$ is (K, Q)-nondecreasing on $\operatorname{dom} g$ and Q-convex, and $h : X \longrightarrow Y \cup \{+\infty_Y\}$ is K-convex with $h(\operatorname{dom} h) \subseteq \operatorname{dom} g$, then $g \circ h$ is Q-convex.

The following results is needed in the following section.

Theorem 2.1 ([18])**.** *Let X be a Banach space, and let $f : X \to \overline{\mathbb{R}}$ be a proper, convex, and lower semicontinuous function. If $\bar{x} \in \operatorname{dom} f$, then*

$$\operatorname{epi} f^* = \bigcup_{\varepsilon \geq 0} \Big\{ \Big(x^*, \langle x^*, \bar{x}\rangle + \varepsilon - f(\bar{x}) \Big) : \; x^* \in \partial_\varepsilon f(\bar{x}) \Big\}.$$

Theorem 2.2 ([9])**.** *Let $(X, \|.\|_X)$ be a Banach space, and let $f : X \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function. Consider $\varepsilon, \mu \geq 0$ and assume that $x_0 \in \operatorname{dom} f$. If $x_0^* \in \partial_\mu f(x_0)$, then there exist $x_1 \in \operatorname{dom} f$ and $x_1^* \in \partial_\varepsilon f(x_1)$ such that*

$$\|x_1 - x_0\|_X \leq \sqrt{|\mu - \varepsilon|}, \; \|x_1^* - x_0^*\|_{X^*} \leq \sqrt{|\mu - \varepsilon|}$$

and

$$|f(x_1)-f(x_0)-\langle x_1^*, x_1-x_0\rangle| \leq 2|\mu-\varepsilon|.$$

3. Sequential ε-Subdifferential Calculus for the Sums of m Functions ($m \geq 2$) via a Perturbation Approach

Let $(X, \|.\|_X)$ be a reflexive Banach space and $(Y, \|.\|_Y)$ be a Banach space, both paired in duality by $\langle .,. \rangle$ and $(X^*, \|.\|_{X^*})$, $(Y^*, \|.\|_{Y^*})$ their topological dual spaces. In what follows, we write $x_n \xrightarrow[n\to+\infty]{\|.\|_X} 0$ and $x_n^* \xrightarrow[n\to+\infty]{\|.\|_{X^*}} 0$ (resp. $x_n^* \xrightarrow[n\to+\infty]{w(X^*,X)} 0$) for the case when the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to 0 in $(X, \|.\|_X)$ and $\{x_n^*\}_{n\in\mathbb{N}}$ converges to 0 in $(X^*, \|.\|_{X^*})$ (resp. $\{x_n^*\}_{n\in\mathbb{N}}$ converges to 0 in $(X^*, w(X^*,X))$).

The aim of this section is to give a sequential formula for the ε-subdifferential of the sums of m ($m \geq 2$) proper, convex, and lower semicontinuous functions $f_1, \ldots, f_m : X \to \overline{\mathbb{R}}$ via a perturbation approach. To do this, we first provide the following theorem, which enables us to derive the sequential ε-optimality conditions for a general convex optimization problem without imposing any qualification condition but using the so-called perturbation function, which is denoted by T.

Theorem 3.1. *Let $T : X \times Y \to \overline{\mathbb{R}}$ be a proper, convex, and lower semicontinuous function such that $\inf\limits_{x\in X} T(x,0) < +\infty$. Let $\varepsilon \geq 0$. Then the following statements are equivalent*
(i) $\bar{x} \in \mathrm{dom}T(.,0)$ *is an ε-optimal solution of the problem* $\inf_{x\in X} T(x,0)$;
(ii) *there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\{(x_n^*, y_n^*)\}_{n\in\mathbb{N}} \subseteq X^* \times Y^*$ such that*

$$(x_n^*, y_n^*) \in \partial_{\varepsilon_n} T(\bar{x},0),\ x_n^* \xrightarrow[n\to+\infty]{\|.\|_{X^*}} 0 \text{ and } \varepsilon_n \xrightarrow[n\to+\infty]{} \varepsilon;$$

(iii) *there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\{(x_n^*, y_n^*)\}_{n\in\mathbb{N}} \subseteq X^* \times Y^*$ such that*

$$(x_n^*, y_n^*) \in \partial_{\varepsilon_n} T(\bar{x},0),\ \ x_n^* \xrightarrow[n\to+\infty]{w(X^*,X)} 0 \text{ and } \varepsilon_n \xrightarrow[n\to+\infty]{} \varepsilon.$$

Proof. (i) $\Rightarrow$ (ii) Suppose that $\bar{x} \in \mathrm{dom}T(.,0)$ is an ε-optimal solution of $T(.,0)$ on X and consider the infimal value function $v : X^* \to \overline{\mathbb{R}}$ of T^*, which is defined by

$$v(x^*) := \inf_{y^*\in Y^*} T^*(x^*, y^*), \quad x^* \in X^*.$$

It is easy to see that $\mathrm{dom}v \neq \emptyset$, and v is convex with $v^*(x) = T^{**}(x,0) = T(x,0)$ for all $x \in X$. Thus v^* is proper and then clv is also proper with $v^{**} = clv$. As a consequence, $\bar{x}$ is an ε-optimal solution of v^* on X with $clv(0) + T(\bar{x},0) = v^{**}(0) + v^*(\bar{x}) \leq \varepsilon$, which yields

$$\Big(0, \varepsilon - T(\bar{x},0)\Big) \in \mathrm{epi}\, clv = \overline{\mathrm{epi}v}^{w(X^*,X)\times\tau_{\mathbb{R}}}.$$

Since v is convex and X is a reflexive Banach space, it follows that

$$\Big(0, \varepsilon - T(\bar{x},0)\Big) \in \overline{\mathrm{epi}v}^{\|.\|_{X^*}\times\tau_{\mathbb{R}}}.$$

Hence, there exists sequence $\{(x_n^*, r_n)\}_{n\in\mathbb{N}} \subseteq X^* \times \mathbb{R}$ such that, for each $n \in \mathbb{N}$,

$$v(x_n^*) \leq r_n,\ x_n^* \xrightarrow[n\to+\infty]{\|.\|_{X^*}} 0 \ \text{ and } \ r_n \xrightarrow[n\to+\infty]{} \varepsilon - T(\bar{x},0).$$

The inequality $v(x_n^*) \leq r_n$ implies that $\inf\limits_{y^* \in Y^*} T^*(x_n^*, y^*) < r_n + \frac{1}{n+1}$, $n \in \mathbb{N}$. Therefore, it follows that there exists a sequence $\{y_n^*\}_{n\in\mathbb{N}} \subseteq Y^*$ such that

$$T^*(x_n^*, y_n^*) \leq r_n + \frac{1}{n+1}, \quad \forall n \in \mathbb{N},$$

that is,

$$\left(x_n^*, y_n^*, r_n + \frac{1}{n+1}\right) \in \text{epi} T^*, \quad \forall n \in \mathbb{N}.$$

Thus by applying Theorem 2.1, one can deduce that there a exists sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ such that, for each $n \in \mathbb{N}$,

$$r_n + \frac{1}{n+1} = \langle x_n^*, \bar{x} \rangle + \varepsilon_n - T(\bar{x}, 0),\ (x_n^*, y_n^*) \in \partial_{\varepsilon_n} T(\bar{x}, 0),$$

and

$$r_n \xrightarrow[n\to+\infty]{} \varepsilon - T(\bar{x}, 0) \text{ and } x_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} 0.$$

Since for each $n \in \mathbb{N}$, $r_n + \frac{1}{n+1} = \langle x_n^*, \bar{x} \rangle + \varepsilon_n - T(\bar{x}, 0)$ and $\{r_n\}_{n\in\mathbb{N}}$ converges to $\varepsilon - T(\bar{x}, 0)$, we have $\varepsilon_n \xrightarrow[n\to+\infty]{} \varepsilon$. Hence, (i) implies (ii).

(ii) $\Rightarrow$ (iii) It is obvious.

(iii) $\Rightarrow$ (i) Suppose that there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\{(x_n^*, y_n^*)\}_{n\in\mathbb{N}} \subseteq X^* \times Y^*$ such that

$$(x_n^*, y_n^*) \in \partial_{\varepsilon_n} T(\bar{x}, 0),\ \ x_n^* \xrightarrow[n\to+\infty]{w(X^*, X)} 0 \text{ and } \varepsilon_n \xrightarrow[n\to+\infty]{} \varepsilon.$$

Since, for each $n \in \mathbb{N}$, $(x_n^*, y_n^*) \in \partial_{\varepsilon_n} T(\bar{x}, 0)$, then

$$T(x, y) \geq T(\bar{x}, 0) + \langle x_n^*, x - \bar{x} \rangle + \langle y_n^*, y \rangle - \varepsilon_n,\ \forall (x, y) \in X \times Y.$$

Hence, we have, for each $n \in \mathbb{N}$, $T(x, 0) \geq T(\bar{x}, 0) + \langle x_n^*, x - \bar{x} \rangle - \varepsilon_n$, $\forall x \in X$. Thus by letting $n \to +\infty$, we obtain $T(x, 0) \geq T(\bar{x}, 0) - \varepsilon$, $\forall x \in X$, that is, $\bar{x}$ is an ε-optimal solution of $T(., 0)$ on X. □

Remark 3.1. It is worth mentioning that the proof of the above theorem is based on the same reasonings as those in Lemma 3.1 and Theorem 3.2 presented in [8]. Here it occurs a number $\varepsilon \geq 0$ (i.e. ε-optimal solutions are considered). In [8], this number is equal to zero because exact optimal solutions are considered.

Now, let $f_1, \cdots, f_m : X \to \overline{\mathbb{R}}$ be m ($m \geq 2$) proper, convex, and lower semicontinuous functions, $\bar{x} \in \cap_{i=1}^m \text{dom} f_i$, and $x^* \in X^*$. In order to use Theorem 3.1 to obtain a sequential formula of $\partial_\varepsilon(f_1 + \cdots + f_m)(\bar{x})$, we set $Y := \underbrace{X \times \cdots \times X}_{m-times}$ and define $T : X \times \underbrace{X \times \cdots \times X}_{m-times} \to \overline{\mathbb{R}}$ as follows

$$T(x, x_1, \cdots, x_m) := \begin{cases} (f_1 - x^*)(x + x_1) + f_2(x + x_2) + \cdots + f_m(x + x_m), \\ \qquad\qquad\qquad \text{if } x + x_i \in \text{dom} f_i \ (i = 1, \cdots, m), \\ +\infty, \text{ otherwise.} \end{cases}$$

The following lemma is needed in the proof of the next theorem.

Lemma 3.1. *For any $\varepsilon \geq 0$,*

$$(y^*, y_1^*, \cdots, y_m^*) \in \partial_\varepsilon T(\bar{x}, 0, \cdots, 0) \implies \begin{cases} \exists\, \varepsilon_1, \cdots, \varepsilon_m \geq 0,\ \varepsilon_1 + \cdots + \varepsilon_m = \varepsilon, \\ y_1^* \in \partial_{\varepsilon_1}(f_1 - x^*)(\bar{x}), \\ y_i^* \in \partial_{\varepsilon_i} f_i(\bar{x})\ (i = 2, \cdots, m), \\ y^* = y_1^* + \cdots + y_m^*. \end{cases}$$

Proof. First, it is easy to check that

$$T^*(y^*, y_1^*, \cdots, y_m^*) = \begin{cases} (f_1 - x^*)^*(y_1^*) + f_2^*(y_2^*) + \cdots + f_m^*(y_m^*),\ \text{if } y^* = \sum_{i=1}^m y_i^*, \\ +\infty,\ \text{otherwise}, \end{cases}$$

for all $(y^*, y_1^*, \cdots, y_m^*) \in X^* \times \underbrace{X^* \times \cdots \times X^*}_{m-times}$. If $(y^*, y_1^*, \cdots, y_m^*) \in \partial_\varepsilon T(\bar{x}, 0, \cdots, 0)$, then it follows that $T^*(y^*, y_1^*, \cdots, y_m^*) + T(\bar{x}, 0, \cdots, 0) \leq \langle y^*, \bar{x} \rangle + \varepsilon$. This implies that

$$\begin{cases} [(f_1 - x^*)^*(y_1^*) + (f_1 - x^*)(\bar{x}) - \langle y_1^*, \bar{x} \rangle] + \sum_{i=2}^m [f_i^*(y_i^*) + f_i(\bar{x}) - \langle y_i^*, \bar{x} \rangle] \leq \varepsilon, \\ y^* = y_1^* + \cdots + y_m^*. \end{cases}$$

Now, we define $\varepsilon_1' := (f_1 - x^*)^*(y_1^*) + (f_1 - x^*)(\bar{x}) - \langle y_1^*, \bar{x} \rangle$ and $\varepsilon_i := f_i^*(y_i^*) + f_i(\bar{x}) - \langle y_i^*, \bar{x} \rangle$, $i = 2, \cdots, m$. By the Young-Fenchel inequality, it is clear that $\varepsilon_1' \geq 0$ and $\varepsilon_i \geq 0$, $i = 2, \cdots, m$, with $\varepsilon_1' + \sum_{i=2}^m \varepsilon_i \leq \varepsilon$. By setting $\varepsilon_1 := \varepsilon - \sum_{i=2}^m \varepsilon_i \geq \varepsilon_1'$, we deduce that

$$\begin{cases} y_1^* \in \partial_{\varepsilon_1}(f_1 - x^*)(\bar{x}), \\ y_i^* \in \partial_{\varepsilon_i} f_i(\bar{x})\ (i = 2, \cdots, m), \\ y^* = y_1^* + \cdots + y_m^*, \\ \varepsilon_1 + \cdots + \varepsilon_m = \varepsilon. \end{cases}$$

□

Now, we state the main result of this section.

Theorem 3.2. *Let $f_1, \cdots, f_m : X \to \overline{\mathbb{R}}$ be m proper, convex and lower semicontinuous functions, $\bar{x} \in \cap_{i=1}^m \mathrm{dom} f_i$ and $\varepsilon \geq 0$. Then the following statements are equivalent*
(i) $x^* \in \partial_\varepsilon(f_1 + \cdots + f_m)(\bar{x})$*;*
(ii) *there exist $\varepsilon_1, \cdots, \varepsilon_m \geq 0$ with $\varepsilon_1 + \cdots + \varepsilon_m = \varepsilon$, and sequences $\{y_{i,n}\}_{n \in \mathbb{N}} \subseteq \mathrm{dom} f_i$ and $\{y_{i,n}^*\}_{n \in \mathbb{N}} \subseteq X^*$, $i = 1, \cdots, m$, such that*

$$\begin{cases} y_{i,n}^* \in \partial_{\varepsilon_i} f_i(y_{i,n})\ (i = 1, \cdots, m), \\ y_{1,n}^* + \cdots + y_{m,n}^* \xrightarrow[n \to +\infty]{\|.\|_{X^*}} x^*,\ y_{i,n} \xrightarrow[n \to +\infty]{\|.\|_X} \bar{x}\ (i = 1, \cdots, m), \\ f_i(y_{i,n}) - f_i(\bar{x}) - \langle y_{i,n}^*, y_{i,n} - \bar{x} \rangle \xrightarrow[n \to +\infty]{} 0\ (i = 1, \cdots, m); \end{cases}$$

(iii) *there exist* $\varepsilon_1, \cdots, \varepsilon_m \geq 0$ *with* $\varepsilon_1 + \cdots + \varepsilon_m = \varepsilon$ *and sequences* $\{y_{i,n}\}_{n\in\mathbb{N}} \subseteq \operatorname{dom} f_i$ *and* $\{y^*_{i,n}\}_{n\in\mathbb{N}} \subseteq X^*$, $i = 1, \cdots, m$, *such that*

$$\begin{cases} y^*_{i,n} \in \partial_{\varepsilon_i} f_i(y_{i,n}) \ (i = 1, \cdots, m), \\ y^*_{1,n} + \cdots + y^*_{m,n} \xrightarrow[n\to+\infty]{w(X^*,X)} x^*, \ y_{i,n} \xrightarrow[n\to+\infty]{\|\cdot\|_X} \bar{x} \ (i = 1, \cdots, m), \\ f_i(y_{i,n}) - f_i(\bar{x}) - \langle y^*_{i,n}, y_{i,n} - \bar{x}\rangle \xrightarrow[n\to+\infty]{} 0 \ (i = 1, \cdots, m). \end{cases}$$

Proof. (i) $\Rightarrow$ (ii) Suppose that $x^* \in \partial_\varepsilon(f_1 + \cdots + f_m)(\bar{x})$. It is clear that $x^* \in \partial_\varepsilon(f_1 + \cdots + f_m)(\bar{x})$ is equivalent to the fact that $\bar{x}$ is an ε-optimal solution of the problem $\inf_{x\in X} T(x, 0, \cdots, 0)$. It is not difficult to see that T is proper, convex, and lower semicontinuous such that $\inf\limits_{x\in X} T(x, 0, \cdots, 0) < +\infty$. Hence, by Theorem 3.1, there exist sequences $\{\varepsilon_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and $\{(x^*_n, x^*_{1,n}, \cdots, x^*_{m,n})\}_{n\in\mathbb{N}} \subseteq X^* \times \underbrace{X^* \times \cdots \times X^*}_{m-times}$ such that

$$(x^*_n, x^*_{1,n}, \cdots, x^*_{m,n}) \in \partial_{\varepsilon_n} T(\bar{x}, 0, \cdots, 0), \ x^*_n \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} 0 \text{ and } \varepsilon_n \xrightarrow[n\to+\infty]{} \varepsilon.$$

From Lemma 3.1, there exist sequences $\{\varepsilon_{i,n}\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$, $i = 1, \cdots, m$, such that, for each $n \in \mathbb{N}$,

$$\begin{cases} \varepsilon_{1,n} + \cdots + \varepsilon_{m,n} = \varepsilon_n, \\ x^*_{1,n} \in \partial_{\varepsilon_{1,n}}(f_1 - x^*)(\bar{x}), \ x^*_{i,n} \in \partial_{\varepsilon_{i,n}} f_i(\bar{x}) \ (i = 2, \cdots, m), \\ x^*_n = (x^*_{1,n} + \cdots + x^*_{m,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} 0. \end{cases}$$

Since $\varepsilon_n \xrightarrow[n\to+\infty]{} \varepsilon$, it follows that there exists $M \geq 0$ such that $0 \leq \varepsilon_n \leq M$, $\forall n \in \mathbb{N}$. Hence, for all $i \in \{1, \cdots, m\}$, we have $0 \leq \varepsilon_{i,n} \leq M$, $\forall n \in \mathbb{N}$. So, we can consider that $\varepsilon_{i,n} \xrightarrow[n\to+\infty]{} \varepsilon_i$, $\varepsilon_i \geq 0$, $i = 1, \cdots, m$, with $\varepsilon_1 + \cdots + \varepsilon_m = \varepsilon$. Now by applying Theorem 2.2, we deduce that, for each $n \in \mathbb{N}$, there exist $y_{1,n} \in \operatorname{dom} f_1$, $y^*_{1,n} \in \partial_{\varepsilon_1}(f_1 - x^*)(y_{1,n})$, $y_{i,n} \in \operatorname{dom} f_i$, and $y^*_{i,n} \in \partial_{\varepsilon_i} f_i(y_{i,n})$, $i = 2, \cdots, m$, such that

$$\begin{cases} \|y_{i,n} - \bar{x}\|_X \leq \sqrt{|\varepsilon_{i,n} - \varepsilon_i|} \ (i = 1, \cdots, m), & (3.1) \\ \|y^*_{i,n} - x^*_{i,n}\|_{X^*} \leq \sqrt{|\varepsilon_{i,n} - \varepsilon_i|} \ (i = 1, \cdots, m), & (3.2) \\ |f_1(y_{1,n}) - f_1(\bar{x}) - \langle y^*_{1,n}, y_{1,n} - \bar{x}\rangle - \langle x^*, y_{1,n} - \bar{x}\rangle| \leq 2|\varepsilon_{1,n} - \varepsilon_1|, & (3.3) \\ |f_i(y_{i,n}) - f_i(\bar{x}) - \langle y^*_{i,n}, y_{i,n} - \bar{x}\rangle| \leq 2|\varepsilon_{i,n} - \varepsilon_i| \ (i = 2, \cdots, m). & (3.4) \end{cases}$$

From (3.1), one can see that $y_{i,n} \xrightarrow[n\to+\infty]{\|\cdot\|_X} \bar{x}$, $i = 1, \cdots, m$. From (3.3) and the fact that $\langle x^*, y_{1,n} - \bar{x}\rangle \xrightarrow[n\to+\infty]{} 0$, we have $f_1(y_{1,n}) - f_1(\bar{x}) - \langle y^*_{1,n}, y_{1,n} - \bar{x}\rangle \xrightarrow[n\to+\infty]{} 0$. By (3.4), we have $f_i(y_{i,n}) - f_i(\bar{x}) - \langle y^*_{i,n}, y_{i,n} - \bar{x}\rangle \xrightarrow[n\to+\infty]{} 0$ for $i \in \{2, \cdots, m\}$. From (3.2) and the fact that $x^*_{1,n} + \cdots + x^*_{m,n} \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} 0$, we have $y^*_{1,n} + \cdots + y^*_{m,n} \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} 0$.

On the other hand, it is clear that, for each $n \in \mathbb{N}$, $\partial_{\varepsilon_1}(f_1 - x^*)(y_{1,n}) = \partial_{\varepsilon_1} f_1(y_{1,n}) - x^*$. By taking $\bar{y}^*_{1,n} := y^*_{1,n} + x^*$, $n \in \mathbb{N}$, it follows that, for each $n \in \mathbb{N}$,

$$\bar{y}^*_{1,n} \in \partial_{\varepsilon_1} f_1(y_{1,n}) \ \text{ and } \ \bar{y}^*_{1,n} + y^*_{2,n} + \cdots + y^*_{m,n} \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} x^*.$$

Finally, by setting again $y_{1,n}^* := \bar{y}_{1,n}^*$, we obtain the desired statement (ii).

(ii) $\Rightarrow$ (iii) It is immediate.

(iii) $\Rightarrow$ (i) Suppose that there exist $\varepsilon_1, \cdots, \varepsilon_m \geq 0$, with $\varepsilon_1 + \cdots + \varepsilon_m = \varepsilon$, and sequences $\{y_{i,n}\}_{n\in\mathbb{N}} \subseteq \mathrm{dom} f_i$, $\{y_{i,n}^*\}_{n\in\mathbb{N}} \subseteq X^*$, $i \in \{1, \cdots, m\}$, such that

$$\begin{cases} y_{i,n}^* \in \partial_{\varepsilon_i} f_i(y_{i,n}) \ (i = 1, \cdots, m), \\ y_{1,n}^* + \cdots + y_{m,n}^* \xrightarrow[n\to+\infty]{w(X^*,X)} x^*, \ y_{i,n} \xrightarrow[n\to+\infty]{\|.\|_X} \bar{x} \ (i = 1, \cdots, m), \\ f_i(y_{i,n}) - f_i(\bar{x}) - \langle y_{i,n}^*, y_{i,n} - \bar{x}\rangle \xrightarrow[n\to+\infty]{} 0 \ (i = 1, \cdots, m). \end{cases} \tag{3.5}$$

From (3.5), it follows that, for each $n \in \mathbb{N}$ and $i \in \{1, \cdots, m\}$,

$$\begin{aligned} f_i(x) &\geq f_i(y_{i,n}) + \langle y_{i,n}^*, x - y_{i,n}\rangle - \varepsilon_i \\ &= f_i(\bar{x}) + [f_i(y_{i,n}) - f_i(\bar{x}) - \langle y_{i,n}^*, y_{i,n} - \bar{x}\rangle] + \langle y_{i,n}^*, x - \bar{x}\rangle - \varepsilon_i, \ \forall x \in X. \end{aligned}$$

Thus by summing over i and letting $n \to +\infty$, we obtain

$$\sum_{i=1}^m f_i(x) \geq \sum_{i=1}^m f_i(\bar{x}) + \langle x^*, x - \bar{x}\rangle - \varepsilon, \quad \forall x \in X,$$

i.e., $x^* \in \partial_\varepsilon (f_1 + \cdots + f_m)(\bar{x})$. □

Remark 3.2. If $m = 2$ in Theorem 3.2, then we exactly obtain the formula for the case of two proper, convex, and lower semicontinuous functions established by Gutiérrez et al. [9, Theorem 3].

4. Sequential ε-Subdifferential Formula for Multi-Composed Convex Functions

In what follows, $(X, \|.\|_X)$, $(Y_i, \|.\|_{Y_i}), i = 0, \cdots, p$ $(p \geq 2)$ are reflexive Banach spaces all paired in duality by $\langle ., . \rangle$ and $(X^*, \|.\|_{X^*})$, $(Y_i^*, \|.\|_{Y_i^*}), i = 0, \cdots, p$, respectively, their topological dual spaces. Likewise, we assume that Y_i is partially ordered by the nonempty convex cone $K_i \subseteq Y_i$ with $0 \in K_i$, $i = 0, \cdots, p$. On $X \times \prod_{k=0}^p Y_k$, we use the norm

$$\|(x, y_0, y_1, \cdots, y_p)\|_{X\times\prod_{k=0}^p Y_k} = \sqrt{\|x\|_X^2 + \|y_0\|_{Y_0}^2 + \|y_1\|_{Y_1}^2 + \cdots + \|y_p\|_{Y_p}^2}.$$

Similarly, we define the norm on $X^* \times \prod_{k=0}^p Y_k^*$ to that on $X \times \prod_{k=0}^p Y_k$.

The aim of this section is to give a sequential formula for the ε-subdifferential of the function $f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ \cdots \circ h_p$, where

- $f : X \to \overline{\mathbb{R}}$ is proper, convex, and lower semicontinuous,
- $\varphi : Y_0 \to \overline{\mathbb{R}}$ is proper, convex, K_0-nondecreasing on $\mathrm{dom}\varphi$, and lower semicontinuous,
- $\psi : X \to Y_0 \cup \{+\infty_{Y_0}\}$ is proper, K_0-convex, K_0-epi closed, and $\psi(\mathrm{dom}\psi) \subseteq \mathrm{dom}\varphi$,
- $g : Y_1 \to \overline{\mathbb{R}}$ is proper, convex, K_1-nondecreasing on $\mathrm{dom}g$, and lower semicontinuous,
- $h_1 : Y_2 \to Y_1 \cup \{+\infty_{Y_1}\}$ is proper, K_1-convex, (K_2, K_1)-nondecreasing on $\mathrm{dom}h_1$, K_1-epi closed, and $h_1(\mathrm{dom}h_1) \subseteq \mathrm{dom}g$,
- $h_i : Y_{i+1} \to Y_i \cup \{+\infty_{Y_i}\}$ is proper, K_i-convex, (K_{i+1}, K_i)-nondecreasing on $\mathrm{dom}h_i$, K_i-epi closed, and $h_i(\mathrm{dom}h_i) \subseteq \mathrm{dom}h_{i-1}$, $i = 2, \cdots, p-1$,
- $h_p : X \to Y_p \cup \{+\infty_{Y_p}\}$ is proper, K_p-convex, K_p-epi closed, and $h_p(\mathrm{dom}h_p) \subseteq \mathrm{dom}h_{p-1}$,

- $\mathrm{dom} f \cap \psi^{-1}(\mathrm{dom}\varphi) \cap \mathrm{dom}\psi \cap (h_p^{-1} \circ h_{p-1}^{-1} \circ \cdots \circ h_1^{-1})(\mathrm{dom} g) \cap \mathrm{dom} h_p \neq \emptyset$,
- $\varphi(+\infty_{Y_0}) = +\infty$, $g(+\infty_{Y_1}) = +\infty$, and $h_i(+\infty_{Y_{i+1}}) = +\infty_{Y_i}, i = 1, \cdots, p-1$.

For this purpose, we introduce the following functions

$$F: \begin{array}{ccc} X \times \prod_{k=0}^{p} Y_k & \longrightarrow & \overline{\mathbb{R}} \\ (x, y_0, y_1, \cdots, y_p) & \longmapsto & f(x), \end{array} \qquad \Phi: \begin{array}{ccc} X \times \prod_{k=0}^{p} Y_k & \longrightarrow & \overline{\mathbb{R}} \\ (x, y_0, y_1, \cdots, y_p) & \longmapsto & \varphi(y_0), \end{array}$$

$$\Psi: \begin{array}{ccc} X \times \prod_{k=0}^{p} Y_k & \longrightarrow & \overline{\mathbb{R}} \\ (x, y_0, y_1, \cdots, y_p) & \longmapsto & \delta_{\mathrm{epi}\psi}(x, y_0), \end{array} \qquad G: \begin{array}{ccc} X \times \prod_{k=0}^{p} Y_k & \longrightarrow & \overline{\mathbb{R}} \\ (x, y_0, y_1, \cdots, y_p) & \longmapsto & g(y_1), \end{array}$$

for $i = 1, \cdots, p-1$,

$$H_i: \begin{array}{ccc} X \times \prod_{k=0}^{p} Y_k & \longrightarrow & \overline{\mathbb{R}} \\ (x, y_0, y_1, \cdots, y_p) & \longmapsto & \delta_{\mathrm{epi} h_i}(y_{i+1}, y_i), \end{array}$$

and

$$H_p: \begin{array}{ccc} X \times \prod_{k=0}^{p} Y_k & \longrightarrow & \overline{\mathbb{R}} \\ (x, y_0, y_1, \cdots, y_p) & \longmapsto & \delta_{\mathrm{epi} h_p}(x, y_p). \end{array}$$

Remark 4.1. It is worth noting that

- $\mathrm{dom} F = \mathrm{dom} f \times \prod_{k=0}^{p} Y_k$,
- $\mathrm{dom}\Phi = X \times \mathrm{dom}\varphi \times \prod_{k=1}^{p} Y_k$,
- $\mathrm{dom}\Psi = \{(x, y_0, y_1, \cdots, y_p) \in X \times \prod_{k=0}^{p} Y_k : (x, y_0) \in \mathrm{epi}\psi\}$,
- $\mathrm{dom} G = X \times Y_0 \times \mathrm{dom} g \times \prod_{k=2}^{p} Y_k$,
- $\mathrm{dom} H_i = \{(x, y_0, y_1, \cdots, y_p) \in X \times \prod_{k=0}^{p} Y_k : (y_{i+1}, y_i) \in \mathrm{epi} h_i\}$ $(i = 1, \cdots, p-1)$,
- $\mathrm{dom} H_p = \{(x, y_0, y_1, \cdots, y_p) \in X \times \prod_{k=0}^{p} Y_k : (x, y_p) \in \mathrm{epi} h_p\}$,
- F, Φ, Ψ, G and $H_1, \cdots, H_p$ are proper, convex, and lower semicontinuous functions.

The following results are needed in this section.

Lemma 4.1 ([12]). *For any $x \in X$, one has*

$$\begin{aligned} & f(x) + (\varphi \circ \psi)(x) + (g \circ h_1 \circ h_2 \circ \cdots \circ h_p)(x) \\ & \quad = \inf_{(y_0, y_1, \cdots, y_p) \in \prod_{k=0}^{p} Y_k} \{F(x, y_0, y_1, \cdots, y_p) + \Phi(x, y_0, y_1, \cdots, y_p) + \Psi(x, y_0, y_1, \cdots, y_p) \\ & \qquad + G(x, y_0, y_1, \cdots, y_p) + \sum_{i=1}^{p} H_i(x, y_0, y_1, \cdots, y_p)\}. \end{aligned}$$

Lemma 4.2. *Consider $\varepsilon \geq 0$ and suppose that $\bar{x} \in \mathrm{dom} f \cap \psi^{-1}(\mathrm{dom}\varphi) \cap \mathrm{dom}\psi \cap (h_p^{-1} \circ h_{p-1}^{-1} \circ \cdots \circ h_1^{-1})(\mathrm{dom} g) \cap \mathrm{dom} h_p$, $\bar{y}_p := h_p(\bar{x})$, $\bar{y}_{p-1} := h_{p-1}(\bar{y}_p), \cdots,$ $\bar{y}_1 := h_1(\bar{y}_2)$ and $\bar{y}_0 := \psi(\bar{x})$. Then*

$$x^* \in \partial_\varepsilon(f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ \cdots \circ h_p)(\bar{x})$$
$$\Longleftrightarrow$$
$$(x^*, 0, 0, \cdots, 0) \in \partial_\varepsilon\Big(F + \Phi + \Psi + G + \sum_{i=1}^{p} H_i\Big)(\bar{x}, \bar{y}_0, \bar{y}_1, \cdots, \bar{y}_p).$$

Proof. ($\Rightarrow$) Let $x^* \in \partial_\varepsilon(f+\varphi\circ\psi+g\circ h_1\circ h_2\circ\cdots\circ h_p)(\bar{x})$. Then, for any $x\in X$, we have

$$\begin{aligned} f(x)&+(\varphi\circ\psi)(x)+(g\circ h_1\circ h_2\circ\cdots\circ h_p)(x)\\ &\geq f(\bar{x})+(\varphi\circ\psi)(\bar{x})+(g\circ h_1\circ h_2\circ\cdots\circ h_p)(\bar{x})+\langle x^*,x-\bar{x}\rangle-\varepsilon\\ &=\Big(F+\Phi+\Psi+G+\sum_{i=1}^{p}H_i\Big)(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)+\langle x^*,x-\bar{x}\rangle-\varepsilon. \end{aligned}$$

From Lemma 4.1, it follows that, for any $(x,y_0,y_1,\cdots,y_p)\in X\times\prod_{k=0}^{p}Y_k$,

$$\Big(F+\Phi+\Psi+G+\sum_{i=1}^{p}H_i\Big)(x,y_0,y_1,\cdots,y_p)\geq f(x)+(\varphi\circ\psi)(x)+(g\circ h_1\circ h_2\circ\cdots\circ h_p)(x).$$

Thus we obtain

$$\begin{aligned} &\Big(F+\Phi+\Psi+G+\sum_{i=1}^{p}H_i\Big)(x,y_0,y_1,\cdots,y_p)\\ &\geq\Big(F+\Phi+\Psi+G+\sum_{i=1}^{p}H_i\Big)(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)+\langle x^*,x-\bar{x}\rangle-\varepsilon, \end{aligned}$$

i.e.

$$(x^*,0,0,\cdots,0)\in\partial_\varepsilon\Big(F+\Phi+\Psi+G+\sum_{i=1}^{p}H_i\Big)(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p).$$

($\Leftarrow$) It is immediate by using Lemma 4.1. □

Lemma 4.3. (i) *Let $i\in\{1,\cdots,p-1\}$ and $(x,y_0,y_1,\cdots,y_p)\in \mathrm{dom}H_i$. Then, for any $\varepsilon\geq 0$, we have*

$$(x^*,y_0^*,y_1^*,\cdots,y_p^*)\in\partial_\varepsilon H_i(x,y_0,y_1,\cdots,y_p)\implies\begin{cases} x^*=0,\\ y_k^*=0, k\in\{0,\cdots,p\}\backslash\{i,i+1\},\\ -y_i^*\in K_i^*,\\ \langle -y_i^*,y_i-h_i(y_{i+1})\rangle\in[0,\varepsilon],\\ y_{i+1}^*\in\partial_\varepsilon(-y_i^*\circ h_i)(y_{i+1}). \end{cases}$$

(ii) *Let $(x,y_0,y_1,\cdots,y_p)\in \mathrm{dom}H_p$. Then, for any $\varepsilon\geq 0$, we have*

$$(x^*,y_0^*,y_1^*,\cdots,y_p^*)\in\partial_\varepsilon H_p(x,y_0,y_1,\cdots,y_p)\implies\begin{cases} y_k^*=0, k\in\{0,\cdots,p-1\},\\ -y_p^*\in K_p^*,\\ \langle -y_p^*,y_p-h_p(x)\rangle\in[0,\varepsilon],\\ x^*\in\partial_\varepsilon(-y_p^*\circ h_p)(x). \end{cases}$$

(iii) *Let $(x,y_0,y_1,\cdots,y_p)\in \mathrm{dom}\Psi$. Then, for any $\varepsilon\geq 0$, we have*

$$(x^*,y_0^*,y_1^*,\cdots,y_p^*)\in\partial_\varepsilon \Psi(x,y_0,y_1,\cdots,y_p)\implies\begin{cases} y_k^*=0, k\in\{1,\cdots,p\},\\ -y_0^*\in K_0^*,\\ \langle -y_0^*,y_0-\psi(x)\rangle\in[0,\varepsilon],\\ x^*\in\partial_\varepsilon(-y_0^*\circ \psi)(x). \end{cases}$$

Proof. (i) Let $i \in \{1,\cdots,p-1\}$, $(x,y_0,y_1,\cdots,y_p) \in \mathrm{dom} H_i$ and $\varepsilon \geq 0$. It is easy to check that, for any $(x^*,y_0^*,y_1^*,\cdots,y_p^*) \in X^* \times \prod_{k=0}^{p} Y_k^*$,

$$H_i^*(x^*,y_0^*,y_1^*,\cdots,y_p^*) = \delta_{\{0\}}(x^*) + \sum_{\substack{k=0\\ k\notin\{i,i+1\}}}^{p} \delta_{\{0\}}(y_k^*) + \delta_{K_i}^*(y_i^*) + (-y_i^* \circ h_i)^*(y_{i+1}^*).$$

Thus $(x^*,y_0^*,y_1^*,\cdots,y_p^*) \in \partial_\varepsilon H_i(x,y_0,y_1,\cdots,y_p)$ if and only if

$$H_i^*(x^*,y_0^*,y_1^*,\cdots,y_p^*)+H_i(x,y_0,y_1,\cdots,y_p)-\langle x^*,x\rangle - \sum_{\substack{k=0\\ k\notin\{i,i+1\}}}^{p} \langle y_k^*,y_k\rangle - \langle y_i^*,y_i\rangle - \langle y_{i+1}^*,y_{i+1}\rangle \leq \varepsilon,$$

i.e., $x^*=0,\ y_k^*=0,\ k\in\{0,\cdots,p\}\backslash\{i,i+1\}$ and

$$\begin{aligned}[\delta_{K_i}^*(y_i^*) + \delta_{K_i}(y_i - h_i(y_{i+1})) - \langle y_i^*, y_i - h_i(y_{i+1})\rangle]& \\ + [(-y_i^* \circ h_i)^*(y_{i+1}^*) + (-y_i^* \circ h_i)(y_{i+1}) - \langle y_{i+1}^*, y_{i+1}\rangle] &\leq \varepsilon. \quad (4.1)\end{aligned}$$

By the Young-Fenchel inequality and the fact that K_i is a convex cone, (4.1) implies that

$$-y_i^* \in K_i^*,\ \langle -y_i^*, y_i - h_i(y_{i+1})\rangle \in [0,\varepsilon] \text{ and } y_{i+1}^* \in \partial_\varepsilon(-y_i^* \circ h_i)(y_{i+1}).$$

Hence, the proof is complete. The proof of (ii) and (iii) is similar to (i). □

Lemma 4.4. (i) *Let $(x,y_0,y_1,\cdots,y_p) \in \mathrm{dom}F$. Then, for any $\varepsilon \geq 0$,*

$$\partial_\varepsilon F(x,y_0,y_1,\cdots,y_p) = \partial_\varepsilon f(x) \times \{0\} \times \cdots \times \{0\}.$$

(ii) *Let $(x,y_0,y_1,\cdots,y_p) \in \mathrm{dom}\Phi$. Then, for any $\varepsilon \geq 0$,*

$$\partial_\varepsilon \Phi(x,y_0,y_1,\cdots,y_p) = \{0\} \times \partial_\varepsilon \varphi(y_0) \times \{0\} \times \cdots \times \{0\}.$$

(iii) *Let $(x,y_0,y_1,\cdots,y_p) \in \mathrm{dom}G$. Then, for any $\varepsilon \geq 0$,*

$$\partial_\varepsilon G(x,y_0,y_1,\cdots,y_p) = \{0\} \times \{0\} \times \partial_\varepsilon g(y_1) \times \{0\} \times \cdots \times \{0\}.$$

Proof. The proof is straightforward. □

Now, we state the main result of this work.

Theorem 4.1. *Consider $\varepsilon \geq 0$ and suppose that $\bar{x} \in \mathrm{dom} f \cap \psi^{-1}(\mathrm{dom}\varphi) \cap \mathrm{dom}\psi \cap (h_p^{-1} \circ h_{p-1}^{-1} \circ \cdots \circ h_1^{-1})(\mathrm{dom} g) \cap \mathrm{dom} h_p$, $\bar{y}_p = h_p(\bar{x})$, $\bar{y}_{p-1} = h_{p-1}(\bar{y}_p), \cdots, \bar{y}_1 = h_1(\bar{y}_2)$ and $\bar{y}_0 = \psi(\bar{x})$. Then, the following statements are equivalent*
(i) *$x^* \in \partial_\varepsilon(f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ \cdots \circ h_p)(\bar{x})$;*
(ii) *for all $\mu_0,\cdots,\mu_p > 0$, there exist $\gamma,\gamma_0,\gamma_1 \geq 0$ and $\varepsilon_0,\cdots,\varepsilon_p \geq 0$, with $\gamma+\gamma_0+\gamma_1+(\mu_0\varepsilon_0+\cdots+\mu_p\varepsilon_p)=\varepsilon$, and sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathrm{dom} f$, $\{y_{0,n}\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}\varphi$, $\{(z_n,z_{0,n})\}_{n\in\mathbb{N}} \subseteq \mathrm{epi}\psi$, $\{u_{1,n}\}_{n\in\mathbb{N}} \subseteq \mathrm{dom} g$, $\{(v_{i+1,n}^i, v_{i,n}^i)\}_{n\in\mathbb{N}} \subseteq \mathrm{epi} h_i$, $i=1,\cdots,p-1$, $\{(v_n^p, v_{p,n}^p)\}_{n\in\mathbb{N}} \subseteq \mathrm{epi} h_p$, $\{x_n^*\}_{n\in\mathbb{N}} \subseteq X^*$, $\{y_{0,n}^*\}_{n\in\mathbb{N}} \subseteq Y_0^*$, $\{(z_n^*,\ z_{0,n}^*)\}_{n\in\mathbb{N}} \subseteq X^* \times Y_0^*$, $\{u_{1,n}^*\}_{n\in\mathbb{N}} \subseteq Y_1^*$, $\{(v_{i,n}^{i*}, v_{i+1,n}^{i*})\}_{n\in\mathbb{N}} \subseteq Y_i^* \times$*

Y^*_{i+1}, $i=1,\cdots,p-1$, $\{(v^{p*}_n, v^{p*}_{p,n})\}_{n\in\mathbb{N}} \subseteq X^*\times Y^*_p$, *satisfying*

$$(\mathscr{C}_1)\begin{cases} x^*_n \in \partial_\gamma f(x_n),\ y^*_{0,n}\in \partial_{\gamma_0}\varphi(y_{0,n}),\ u^*_{1,n}\in\partial_{\gamma_1} g(u_{1,n}),\\ -z^*_{0,n}\in K^*_0,\ \langle -z^*_{0,n},\ z_{0,n}-\psi(z_n)\rangle \in [0,\mu_0\varepsilon_0],\\ z^*_n \in \partial_{\mu_0\varepsilon_0}(-z^*_{0,n}\circ\psi)(z_n),\\ -v^{i*}_{i,n}\in K^*_i,\ \langle -v^{i*}_{i,n}, v^i_{i,n}-h_i(v^i_{i+1,n})\rangle\in[0,\mu_i\varepsilon_i]\ (i=1,\cdots,p-1),\\ v^{i*}_{i+1,n}\in\partial_{\mu_i\varepsilon_i}(-v^{i*}_{i,n}\circ h_i)(v^i_{i+1,n})\ (i=1,\cdots,p-1),\\ -v^{p*}_{p,n}\in K^*_p,\ \langle -v^{p*}_{p,n}, v^p_{p,n}-h_p(v^p_n)\rangle\in[0,\mu_p\varepsilon_p],\\ v^{p*}_n\in\partial_{\mu_p\varepsilon_p}(-v^{p*}_{p,n}\circ h_p)(v^p_n), \end{cases}$$

$$(\mathscr{C}_2)\begin{cases} x^*_n+z^*_n+v^{p*}_n \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} x^*,\ y^*_{0,n}+z^*_{0,n}\xrightarrow[n\to+\infty]{\|\cdot\|_{Y^*_0}} 0,\\ u^*_{1,n}+v^{1*}_{1,n}\xrightarrow[n\to+\infty]{\|\cdot\|_{Y^*_1}} 0,\ v^{(i-1)*}_{i,n}+v^{i*}_{i,n}\xrightarrow[n\to+\infty]{\|\cdot\|_{Y^*_i}} 0\ (i=2,\cdots,p), \end{cases}$$

$$(\mathscr{C}_3)\begin{cases} x_n\xrightarrow[n\to+\infty]{\|\cdot\|_X}\overline{x},\ z_n\xrightarrow[n\to+\infty]{\|\cdot\|_X}\overline{x},\ v^p_n\xrightarrow[n\to+\infty]{\|\cdot\|_X}\overline{x},\\ y_{0,n}\xrightarrow[n\to+\infty]{\|\cdot\|_{Y_0}}\overline{y}_0,\ z_{0,n}\xrightarrow[n\to+\infty]{\|\cdot\|_{Y_0}}\overline{y}_0,\ u_{1,n}\xrightarrow[n\to+\infty]{\|\cdot\|_{Y_1}}\overline{y}_1,\\ v^p_{p,n}\xrightarrow[n\to+\infty]{\|\cdot\|_{Y_p}}\overline{y}_p,\ v^i_{i,n}\xrightarrow[n\to+\infty]{\|\cdot\|_{Y_i}}\overline{y}_i,\ v^i_{i+1,n}\xrightarrow[n\to+\infty]{\|\cdot\|_{Y_{i+1}}}\overline{y}_{i+1}\ (i=1,\cdots,p-1), \end{cases}$$

and

$$(\mathscr{C}_4)\begin{cases} f(x_n)-f(\overline{x})-\langle x^*_n, x_n-\overline{x}\rangle\xrightarrow[n\to+\infty]{} 0,\\ \varphi(y_{0,n})-\varphi(\overline{y}_0)-\langle y^*_{0,n}, y_{0,n}-\overline{y}_0\rangle\xrightarrow[n\to+\infty]{} 0,\\ \langle -z^*_n, z_n-\overline{x}\rangle+\langle -z^*_{0,n}, z_{0,n}-\overline{y}_0\rangle\xrightarrow[n\to+\infty]{} 0,\\ g(u_{1,n})-g(\overline{y}_1)-\langle u^*_{1,n}, u_{1,n}-\overline{y}_1\rangle\xrightarrow[n\to+\infty]{} 0,\\ \langle -v^{i*}_{i,n}, v^i_{i,n}-\overline{y}_i\rangle+\langle -v^{i*}_{i+1,n}, v^i_{i+1,n}-\overline{y}_{i+1}\rangle\xrightarrow[n\to+\infty]{} 0\ (i=1,\cdots,p-1),\\ \langle -v^{p*}_n, v^p_n-\overline{x}\rangle+\langle -v^{p*}_{p,n}, v^p_{p,n}-\overline{y}_p\rangle\xrightarrow[n\to+\infty]{} 0. \end{cases}$$

Proof. (i) $\Rightarrow$ (ii) Suppose that $x^*\in\partial_\varepsilon(f+\varphi\circ\psi+g\circ h_1\circ h_2\circ\cdots\circ h_p)(\overline{x})$, and let $\mu_0,\cdots,\mu_p>0$. By Lemma 4.2, it follows that

$$(x^*,0,0,\cdots,0)\in\partial_\varepsilon(F+\Phi+\Psi+G+\sum_{i=1}^p H_i)(\overline{x},\overline{y}_0,\overline{y}_1,\cdots,\overline{y}_p).$$

Therefore, by virtue of Theorem 3.2 and Remark 4.1, there exist $\gamma,\gamma_0,\gamma_1\geq 0$ and $\varepsilon_0,\cdots,\varepsilon_p\geq 0$, $\gamma+\gamma_0+\gamma_1+(\mu_0\varepsilon_0+\cdots+\mu_p\varepsilon_p)=\varepsilon$, and sequences $\{(x_n,x_{0,n},\,x_{1,n},\cdots,x_{p,n})\}_{n\in\mathbb{N}}\subseteq \operatorname{dom}F=\operatorname{dom}f\times\prod_{k=0}^p Y_k$, $\{(x^*_n,x^*_{0,n},x^*_{1,n},\cdots,x^*_{p,n})\}_{n\in\mathbb{N}}\subseteq X^*\times\prod_{k=0}^p Y^*_k$, $\{(y_n,\,y_{0,n},y_{1,n},\cdots,y_{p,n})\}_{n\in\mathbb{N}}\subseteq \operatorname{dom}\Phi=X\times\operatorname{dom}\varphi\times\prod_{k=1}^p Y_k$, $\{(y^*_n,y^*_{0,n},y^*_{1,n},\cdots,y^*_{p,n})\}_{n\in\mathbb{N}}\subseteq X^*\times\prod_{k=0}^p Y^*_k$, $\{(z_n,z_{0,n},\,z_{1,n},\cdots,z_{p,n})\}_{n\in\mathbb{N}}\subseteq\operatorname{dom}\Psi$ (i.e. $\{z_n\}_{n\in\mathbb{N}}\subseteq X$ and $\{z_{k,n}\}_{n\in\mathbb{N}}\subseteq Y_k$, $k=0,\cdots,p$, with $\{(z_n,z_{0,n})\}_{n\in\mathbb{N}}\subseteq\operatorname{epi}\psi$), $\{(z^*_n,z^*_{0,n},z^*_{1,n},\cdots,\,z^*_{p,n})\}_{n\in\mathbb{N}}\subseteq X^*\times\prod_{k=0}^p Y^*_k$, $\{(u_n,u_{0,n},\,u_{1,n},\cdots,u_{p,n})\}_{n\in\mathbb{N}}\subseteq\operatorname{dom}G=X\times Y_0\times\operatorname{dom}g\times\prod_{k=2}^p Y_k$, $\{(u^*_n,u^*_{0,n},u^*_{1,n},\cdots,u^*_{p,n})\}_{n\in\mathbb{N}}\subseteq X^*\times\prod_{k=0}^p Y^*_k$, $\{(v^i_n,v^i_{0,n},\,v^i_{1,n},\cdots,v^i_{p,n})\}_{n\in\mathbb{N}}\subseteq\operatorname{dom}H_i$, $i=1,\cdots,p-1$ (i.e. for $i=1,\cdots,p-1$, $\{v^i_n\}_{n\in\mathbb{N}}\subseteq X$ and $\{v^i_{k,n}\}_{n\in\mathbb{N}}\subseteq Y_k$,

$k=0,\cdots,p$, with $\{(v^i_{i+1,n},v^i_{i,n})\}_{n\in\mathbb{N}}\subseteq \text{epi}h_i)$, $\{(v^{i*}_n,v^{i*}_{0,n},\ v^{i*}_{1,n},\cdots,v^{i*}_{p,n})\}_{n\in\mathbb{N}}\subseteq X^*\times\prod_{k=0}^p Y^*_k$, $i=1,\cdots,p-1$, $\{(v^p_n,v^p_{0,n},v^p_{1,n},\cdots,v^p_{p,n})\}_{n\in\mathbb{N}}\subseteq \text{dom}H_p$ (i.e. $\{v^p_n\}_{n\in\mathbb{N}}\subseteq X$ and $\{v^p_{k,n}\}_{n\in\mathbb{N}}\subseteq Y_k$, $k=0,\cdots,p$, with $\{(v^p_n,v^p_{p,n})\}_{n\in\mathbb{N}}\subseteq \text{epi}h_p)$, $\{(v^{p*}_n,v^{p*}_{0,n},v^{p*}_{1,n},\cdots,v^{p*}_{p,n})\}_{n\in\mathbb{N}}\subseteq X^*\times\prod_{k=0}^p Y^*_k$ such that

$$(x^*_n,x^*_{0,n},x^*_{1,n},\cdots,x^*_{p,n})\in\partial_\gamma F(x_n,x_{0,n},x_{1,n},\cdots,x_{p,n}),\tag{4.2a}$$

$$(y^*_n,y^*_{0,n},y^*_{1,n},\cdots,y^*_{p,n})\in\partial_{\gamma_0}\Phi(y_n,y_{0,n},y_{1,n},\cdots,y_{p,n}),\tag{4.2b}$$

$$(z^*_n,z^*_{0,n},z^*_{1,n},\cdots,z^*_{p,n})\in\partial\Psi_{\mu_0\varepsilon_0}(z_n,z_{0,n},z_{1,n},\cdots,z_{p,n}),\tag{4.2c}$$

$$(u^*_n,u^*_{0,n},u^*_{1,n},\cdots,u^*_{p,n})\in\partial_{\gamma_1}G(u_n,u_{0,n},u_{1,n},\cdots,u_{p,n}),\tag{4.2d}$$

$$(v^{i*}_n,v^{i*}_{0,n},v^{i*}_{1,n},\cdots,v^{i*}_{p,n})\in\partial_{\mu_i\varepsilon_i}H_i(v^i_n,v^i_{0,n},v^i_{1,n},\cdots,v^i_{p,n})\ (i=1,\cdots,p-1),\tag{4.2e}$$

$$(v^{p*}_n,v^{p*}_{0,n},v^{p*}_{1,n},\cdots,v^{p*}_{p,n})\in\partial_{\mu_p\varepsilon_p}H_p(v^p_n,v^p_{0,n},v^p_{1,n},\cdots,v^p_{p,n}),\tag{4.2f}$$

$$\begin{aligned}\Big(x^*_n+y^*_n+z^*_n+u^*_n+\sum_{i=1}^p v^{i*}_n,x^*_{0,n}+y^*_{0,n}+z^*_{0,n}+u^*_{0,n}+\sum_{i=1}^p v^{i*}_{0,n},\\ x^*_{1,n}+y^*_{1,n}+z^*_{1,n}+u^*_{1,n}+\sum_{i=1}^p v^{i*}_{1,n},\cdots,x^*_{p,n}+y^*_{p,n}+z^*_{p,n}+u^*_{p,n}\\ +\sum_{i=1}^p v^{i*}_{p,n}\Big)\xrightarrow[n\to+\infty]{\|\cdot\|_{X^*\times\prod_{k=0}^p Y^*_k}}(x^*,0,0,\cdots,0),\end{aligned}\tag{4.3}$$

$$(x_n,x_{0,n},x_{1,n},\cdots,x_{p,n})\xrightarrow[n\to+\infty]{\|\cdot\|_{X\times\prod_{k=0}^p Y_k}}(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p),\tag{4.4a}$$

$$(y_n,y_{0,n},y_{1,n},\cdots,y_{p,n})\xrightarrow[n\to+\infty]{\|\cdot\|_{X\times\prod_{k=0}^p Y_k}}(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p),\tag{4.4b}$$

$$(z_n,z_{0,n},z_{1,n},\cdots,z_{p,n})\xrightarrow[n\to+\infty]{\|\cdot\|_{X\times\prod_{k=0}^p Y_k}}(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p),\tag{4.4c}$$

$$(u_n,u_{0,n},u_{1,n},\cdots,u_{p,n})\xrightarrow[n\to+\infty]{\|\cdot\|_{X\times\prod_{k=0}^p Y_k}}(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p),\tag{4.4d}$$

$$(v^i_n,v^i_{0,n},v^i_{1,n},\cdots,v^i_{p,n})\xrightarrow[n\to+\infty]{\|\cdot\|_{X\times\prod_{k=0}^p Y_k}}(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)\ (i=1,\cdots,p-1),\tag{4.4e}$$

$$(v^p_n,v^p_{0,n},v^p_{1,n},\cdots,v^p_{p,n})\xrightarrow[n\to+\infty]{\|\cdot\|_{X\times\prod_{k=0}^p Y_k}}(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p),\tag{4.4f}$$

and

$$F(x_n,x_{0,n},x_{1,n},\cdots,x_{p,n})-F(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)-\langle x_n^*,x_n-\bar{x}\rangle-\sum_{k=0}^{p}\langle x_{k,n}^*,x_{k,n}-\bar{y}_k\rangle\xrightarrow[n\to+\infty]{}0, \tag{4.5a}$$

$$\Phi(y_n,y_{0,n},y_{1,n},\cdots,y_{p,n})-\Phi(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)-\langle y_n^*,y_n-\bar{x}\rangle-\sum_{k=0}^{p}\langle y_{k,n}^*,y_{k,n}-\bar{y}_k\rangle\xrightarrow[n\to+\infty]{}0, \tag{4.5b}$$

$$\Psi(z_n,z_{0,n},z_{1,n},\cdots,z_{p,n})-\Psi(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)-\langle z_n^*,z_n-\bar{x}\rangle-\sum_{k=0}^{p}\langle z_{k,n}^*,z_{k,n}-\bar{y}_k\rangle\xrightarrow[n\to+\infty]{}0, \tag{4.5c}$$

$$G(u_n,u_{0,n},u_{1,n},\cdots,u_{p,n})-G(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)-\langle u_n^*,u_n-\bar{x}\rangle-\sum_{k=0}^{p}\langle u_{k,n}^*,u_{k,n}-\bar{y}_k\rangle\xrightarrow[n\to+\infty]{}0, \tag{4.5d}$$

$$\begin{aligned}H_i(v_n^i,v_{0,n}^i,v_{1,n}^i,\cdots,v_{p,n}^i)-H_i(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)-\langle v_n^{i*},v_n^i-\bar{x}\rangle&\\-\sum_{k=0}^{p}\langle v_{k,n}^{i*},v_{k,n}^i-\bar{y}_k\rangle\xrightarrow[n\to+\infty]{}0\ (i=1,\cdots,p-1),&\end{aligned} \tag{4.5e}$$

and

$$H_p(v_n^p,v_{0,n}^p,v_{1,n}^p,\cdots,v_{p,n}^p)-H_p(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)-\langle v_n^{p*},v_n^p-\bar{x}\rangle-\sum_{k=0}^{p}\langle v_{k,n}^{p*},v_{k,n}^p-\bar{y}_k\rangle\xrightarrow[n\to+\infty]{}0. \tag{4.5f}$$

By Lemma 4.3 and Lemma 4.4, the conditions (4.2a)-(4.2f) become

$$\begin{cases}x_n^*\in\partial_\gamma f(x_n),\ y_{0,n}^*\in\partial_{\gamma_0}\varphi(y_{0,n}),\\-z_{0,n}^*\in K_0^*,\ \langle -z_{0,n}^*,\ z_{0,n}-\psi(z_n)\rangle\in[0,\mu_0\varepsilon_0],\\z_n^*\in\partial_{\mu_0\varepsilon_0}(-z_{0,n}^*\circ\psi)(z_n),\ u_{1,n}^*\in\partial_{\gamma_1}g(u_{1,n}),\\-v_{i,n}^{i*}\in K_i^*,\ \langle -v_{i,n}^{i*},v_{i,n}^i-h_i(v_{i+1,n}^i)\rangle\in[0,\mu_i\varepsilon_i]\ (i=1,\cdots,p-1),\\v_{i+1,n}^{i*}\in\partial_{\mu_i\varepsilon_i}(-v_{i,n}^{i*}\circ h_i)(v_{i+1,n}^i)\ (i=1,\cdots,p-1),\\-v_{p,n}^{p*}\in K_p^*,\ \langle -v_{p,n}^{p*},v_{p,n}^p-h_p(v_n^p)\rangle\in[0,\mu_p\varepsilon_p],\\v_n^{p*}\in\partial_{\mu_p\varepsilon_p}(-v_{p,n}^{p*}\circ h_p)(v_n^p),\end{cases}$$

with

$$\begin{cases} x_{0,n}^* = 0,\ x_{1,n}^* = 0, \dots,\ x_{p,n}^* = 0, \\ y_n^* = 0 \text{ and } y_{1,n}^* = 0,\ y_{2,n}^* = 0, \dots,\ y_{p,n}^* = 0, \\ z_{1,n}^* = 0,\ z_{2,n}^* = 0,\ \dots,\ z_{p,n}^* = 0, \\ u_n^* = 0,\ u_{0,n}^* = 0 \text{ and } u_{2,n}^* = 0,\ u_{3,n}^* = 0, \dots,\ u_{p,n}^* = 0, \\ v_n^{i*} = 0 \text{ and } v_{k,n}^{i*} = 0,\ k \in \{0, \cdots, p\} \setminus \{i, i+1\}\ (i = 1, \cdots, p-1), \\ v_{0,n}^{p*} = 0,\ v_{1,n}^{p*} = 0, \dots,\ v_{p-1,n}^{p*} = 0. \end{cases} \tag{4.6}$$

It is clear that (4.3) is equivalent to

$$\begin{cases} x_n^* + y_n^* + z_n^* + u_n^* + \sum_{i=1}^{p} v_n^{i*} \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} x^*, \\ x_{0,n}^* + y_{0,n}^* + z_{0,n}^* + u_{0,n}^* + \sum_{i=1}^{p} v_{0,n}^{i*} \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_0^*}} 0, \\ x_{1,n}^* + y_{1,n}^* + z_{1,n}^* + u_{1,n}^* + \sum_{i=1}^{p} v_{1,n}^{i*} \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_1^*}} 0, \\ \vdots \\ x_{p,n}^* + y_{p,n}^* + z_{p,n}^* + u_{p,n}^* + \sum_{i=1}^{p} v_{p,n}^{i*} \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_p^*}} 0. \end{cases}$$

By taking into account (4.6), we obtain

$$\begin{cases} x_n^* + z_n^* + v_n^{p*} \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} x^*,\ y_{0,n}^* + z_{0,n}^* \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_0^*}} 0, \\ u_{1,n}^* + v_{1,n}^{1*} \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_1^*}} 0,\ v_{i,n}^{(i-1)*} + v_{i,n}^{i*} \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_i^*}} 0\ (i = 2, \cdots, p). \end{cases}$$

From (4.4a)-(4.4f), it follows that

$$\begin{cases} x_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \bar{x},\ z_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \bar{x},\ v_n^p \xrightarrow[n\to+\infty]{\|\cdot\|_X} \bar{x}, \\ y_{0,n} \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_0}} \bar{y}_0,\ z_{0,n} \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_0}} \bar{y}_0,\ u_{1,n} \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_1}} \bar{y}_1, \\ v_{p,n}^p \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_p}} \bar{y}_p,\ v_{i,n}^i \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_i}} \bar{y}_i,\ v_{i+1,n}^i \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_{i+1}}} \bar{y}_{i+1}\ (i = 1, \cdots, p-1). \end{cases}$$

Taking into consideration (4.6) and (4.5a)-(4.5f) becomes

$$\begin{cases} f(x_n)-f(\bar{x})-\langle x_n^*,x_n-\bar{x}\rangle \xrightarrow[n\to+\infty]{} 0, \\ \varphi(y_{0,n})-\varphi(\bar{y}_0)-\langle y_{0,n}^*,y_{0,n}-\bar{y}_0\rangle \xrightarrow[n\to+\infty]{} 0, \\ \langle -z_n^*,z_n-\bar{x}\rangle+\langle -z_{0,n}^*,z_{0,n}-\bar{y}_0\rangle \xrightarrow[n\to+\infty]{} 0, \\ g(u_{1,n})-g(\bar{y}_1)-\langle u_{1,n}^*,u_{1,n}-\bar{y}_1\rangle \xrightarrow[n\to+\infty]{} 0, \\ \langle -v_{i,n}^{i*},v_{i,n}^i-\bar{y}_i\rangle+\langle -v_{i+1,n}^{i*},v_{i+1,n}^i-\bar{y}_{i+1}\rangle \xrightarrow[n\to+\infty]{} 0\ (i=1,\cdots,p-1), \\ \langle -v_n^{p*},v_n^p-\bar{x}\rangle+\langle -v_{p,n}^{p*},v_{p,n}^p-\bar{y}_p\rangle \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

(ii) $\Rightarrow$ (i) Suppose that the statement (ii) is fulfilled. Then, it follows from $(\mathscr{C}_1)$ that, for each $n\in\mathbb{N}$,

$$\begin{aligned} (x_n^*,0,0,\cdots,0) &\in \partial_\gamma F(x_n,\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p), \\ (0,y_{0,n}^*,0,\cdots,0) &\in \partial_{\gamma_0}\Phi(\bar{x},y_{0,n},\bar{y}_1,\cdots,\bar{y}_p), \\ (z_n^*,z_{0,n}^*,0,\cdots,0) &\in \partial_{2\mu_0\varepsilon_0}\Psi(z_n,z_{0,n},\bar{y}_1,\cdots,\bar{y}_p), \\ (0,0,u_{1,n}^*,0,\cdots,0) &\in \partial_{\gamma_1}G(\bar{x},\bar{y}_0,u_{1,n},\bar{y}_2,\cdots,\bar{y}_p), \end{aligned}$$

for $i=1,\cdots,p-1$,

$$(0,0,\cdots,0,v_{i,n}^{i*},v_{i+1,n}^{i*},0,\cdots,0)\in\partial_{2\mu_i\varepsilon_i}H_i(\bar{x},\bar{y}_0,\cdots,\bar{y}_{i-1},v_{i,n}^i,v_{i+1,n}^i,\bar{y}_{i+2},\cdots,\bar{y}_p)$$

and $(v_n^{p*},0,\cdots,0,v_{p,n}^{p*})\in\partial_{2\mu_p\varepsilon_p}H_p(v_n^p,\bar{y}_0,\cdots,\bar{y}_{p-1},v_{p,n}^p)$. Therefore, for all $(x,y_0,y_1,\cdots,y_p)\in X\times\prod_{k=0}^p Y_k$, we have

$$\begin{aligned} F(x,y_0,y_1,\cdots,y_p) &\geq F(x_n,\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)+\langle x_n^*,x-x_n\rangle-\gamma \\ &= f(\bar{x})+[f(x_n)-f(\bar{x})-\langle x_n^*,x_n-\bar{x}\rangle]+\langle x_n^*,x-\bar{x}\rangle-\gamma, \end{aligned} \tag{4.7a}$$

$$\begin{aligned} \Phi(x,y_0,y_1,\cdots,y_p) &\geq \Phi(\bar{x},y_{0,n},\bar{y}_1,\cdots,\bar{y}_p)+\langle y_{0,n}^*,y_0-y_{0,n}\rangle-\gamma_0 \\ &= \varphi(\bar{y}_0)+[\varphi(y_{0,n})-\varphi(\bar{y}_0)-\langle y_{0,n}^*,y_{0,n}-\bar{y}_0\rangle]+\langle y_{0,n}^*,y_0-\bar{y}_0\rangle-\gamma_0, \end{aligned} \tag{4.7b}$$

$$\begin{aligned} \Psi(x,y_0,y_1,\cdots,y_p) &\geq \Psi(z_n,z_{0,n},\bar{y}_1,\cdots,\bar{y}_p)+\langle z_n^*,x-z_n\rangle+\langle z_{0,n}^*,y_0-z_{0,n}\rangle-2\mu_0\varepsilon_0 \\ &= [\langle -z_n^*,z_n-\bar{x}\rangle+\langle -z_{0,n}^*,z_{0,n}-\bar{y}_0\rangle]+\langle z_n^*,x-\bar{x}\rangle+\langle z_{0,n}^*,y_0-\bar{y}_0\rangle-2\mu_0\varepsilon_0, \end{aligned} \tag{4.7c}$$

$$\begin{aligned} G(x,y_0,y_1,\cdots,y_p) &\geq G(\bar{x},\bar{y}_0,u_{1,n},\bar{y}_2,\cdots,\bar{y}_p)+\langle u_{1,n}^*,y_1-u_{1,n}\rangle-\gamma_1 \\ &= g(\bar{y}_1)+[g(u_{1,n})-g(\bar{y}_1)-\langle u_{1,n}^*,u_{1,n}-\bar{y}_1\rangle]+\langle u_{1,n}^*,y_1-\bar{y}_1\rangle-\gamma_1, \end{aligned} \tag{4.7d}$$

for $i=1,\cdots,p-1$,

$$\begin{aligned} H_i(x,y_0,y_1,\cdots,y_p) &\geq H_i(\bar{x},\bar{y}_0,\cdots,\bar{y}_{i-1},v_{i,n}^i,v_{i+1,n}^i,\bar{y}_{i+2},\cdots,\bar{y}_p) \\ &\quad+\langle v_{i,n}^{i*},y_i-v_{i,n}^i\rangle+\langle v_{i+1,n}^{i*},y_{i+1}-v_{i+1,n}^i\rangle-2\mu_i\varepsilon_i \\ &= [\langle -v_{i,n}^{i*},v_{i,n}^i-\bar{y}_i\rangle+\langle -v_{i+1,n}^{i*},v_{i+1,n}^i-\bar{y}_{i+1}\rangle] \\ &\quad+\langle v_{i,n}^{i*},y_i-\bar{y}_i\rangle+\langle v_{i+1,n}^{i*},y_{i+1}-\bar{y}_{i+1}\rangle-2\mu_i\varepsilon_i, \end{aligned} \tag{4.7e}$$

and

$$\begin{aligned} H_p(x,y_0,y_1,\cdots,y_p) &\geq H_p(v_n^p,\bar{y}_0,\cdots,\bar{y}_{p-1},v_{p,n}^p)+\langle v_n^{p*},x-v_n^p\rangle \\ &\quad +\langle v_{p,n}^{p*},y_p-v_{p,n}^p\rangle-2\mu_p\varepsilon_p \\ &=[\langle -v_n^{p*},v_n^p-\bar{x}\rangle+\langle -v_{p,n}^{p*},v_{p,n}^p-\bar{y}_p\rangle]+\langle v_n^{p*},x-\bar{x}\rangle \\ &\quad +\langle v_{p,n}^{p*},y_p-\bar{y}_p\rangle-2\mu_p\varepsilon_p. \end{aligned} \tag{4.7f}$$

By summing (4.7a)-(4.7f), we have, for each $n\in\mathbb{N}$ and for all $(x,y_0,y_1,\cdots,y_p)\in X\times\prod_{k=0}^p Y_k$,

$$\begin{aligned} &\Big(F+\Phi+\Psi+G+\sum_{i=1}^p H_i\Big)(x,y_0,y_1,\cdots,y_p)\geq f(\bar{x})+\varphi(\bar{y}_0)+g(\bar{y}_1) \\ &+[f(x_n)-f(\bar{x})-\langle x_n^*,x_n-\bar{x}\rangle]+[\varphi(y_{0,n})-\varphi(\bar{y}_0)-\langle y_{0,n}^*,y_{0,n}-\bar{y}_0\rangle] \\ &+[\langle -z_n^*,z_n-\bar{x}\rangle+\langle -z_{0,n}^*,z_{0,n}-\bar{y}_0\rangle]+[g(u_{1,n})-g(\bar{y}_1)-\langle u_{1,n}^*,u_{1,n}-\bar{y}_1\rangle] \\ &+\sum_{i=1}^{p-1}[\langle -v_{i,n}^{i*},v_{i,n}^i-\bar{y}_i\rangle+\langle -v_{i+1,n}^{i*},v_{i+1,n}^i-\bar{y}_{i+1}\rangle]+[\langle -v_n^{p*},v_n^p-\bar{x}\rangle \\ &+\langle -v_{p,n}^{p*},v_{p,n}^p-\bar{y}_p\rangle]+[\langle x_n^*+z_n^*+v_n^{p*},x-\bar{x}\rangle]+[\langle y_{0,n}^*+z_{0,n}^*,y_0-\bar{y}_0\rangle] \\ &+[\langle u_{1,n}^*+v_{1,n}^{1*},y_1-\bar{y}_1\rangle]+\sum_{i=2}^p[\langle v_{i,n}^{(i-1)*}+v_{i,n}^{i*},y_i-\bar{y}_i\rangle]-\varepsilon-\sum_{i=0}^p\mu_i\varepsilon_i. \end{aligned}$$

Since the conditions $(\mathscr{C}_2)$, $(\mathscr{C}_3)$ and $(\mathscr{C}_4)$ are satisfied, then by taking the limit when $n\to+\infty$, we obtain

$$\begin{aligned} &\Big(F+\Phi+\Psi+G+\sum_{i=1}^p H_i\Big)(x,y_0,y_1,\cdots,y_p) \\ &\geq f(\bar{x})+\varphi(\bar{y}_0)+g(\bar{y}_1)+\langle x^*,x-\bar{x}\rangle-\varepsilon-\sum_{i=0}^p\mu_i\varepsilon_i \\ &=\Big(F+\Phi+\Psi+G+\sum_{i=1}^p H_i\Big)(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)+\langle x^*,x-\bar{x}\rangle-\varepsilon-\sum_{i=0}^p\mu_i\varepsilon_i. \end{aligned}$$

Hence, by letting $\mu_0\to 0^+,\cdots,\mu_p\to 0^+$, we have, for all $(x,y_0,y_1,\cdots,y_p)\in X\times\prod_{k=0}^p Y_k$,

$$\begin{aligned} &\Big(F+\Phi+\Psi+G+\sum_{i=1}^p H_i\Big)(x,y_0,y_1,\cdots,y_p) \\ &\qquad\geq\Big(F+\Phi+\Psi+G+\sum_{i=1}^p H_i\Big)(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)+\langle x^*,x-\bar{x}\rangle-\varepsilon, \end{aligned}$$

i.e., $(x^*,0,0,\cdots,0)\in\partial_\varepsilon\Big(F+\Phi+\Psi+G+\sum_{i=1}^p H_i\Big)(\bar{x},\bar{y}_0,\bar{y}_1,\cdots,\bar{y}_p)$. This is equivalent to $x^*\in\partial_\varepsilon(f+\varphi\circ\psi+g\circ h_1\circ h_2\circ\cdots\circ h_p)(\bar{x})$. Thus (ii) implies (i). □

Remark 4.2. (1) Let us note that Theorem 4.1 holds if we replace respectively the strong convergence $\xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}}$ and $\xrightarrow[n\to+\infty]{\|\cdot\|_{Y_i^*}}$ by the weak star convergence $\xrightarrow[n\to+\infty]{w(X^*,X)}$ and $\xrightarrow[n\to+\infty]{w(Y_i^*,Y_i)}$ with $i\in\{0,\cdots,p\}$.

(2) If $\varepsilon=0$, then Theorem 4.1 reduces to then interesting result established by Laghdir et al. [12, Theorem 4.6].

The following corollary is obtained by taking $g \equiv 0$, $h_i \equiv 0$ and $K_i = Y_i$, $i = 1, \cdots, p$, in Theorem 4.1.

Corollary 4.1. *Consider $\varepsilon \geq 0$ and suppose that $\bar{x} \in \mathrm{dom} f \cap \psi^{-1}(\mathrm{dom}\varphi) \cap \mathrm{dom}\psi$ and $\bar{y}_0 = \psi(\bar{x})$. Then, $x^* \in \partial_\varepsilon(f + \varphi \circ \psi)(\bar{x})$ if and only if for all $\mu_0 > 0$, there exist $\gamma, \gamma_0 \geq 0$ and $\varepsilon_0 \geq 0$, with $\gamma + \gamma_0 + \mu_0\varepsilon_0 = \varepsilon$, and sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathrm{dom} f$, $\{y_{0,n}\}_{n\in\mathbb{N}} \subseteq \mathrm{dom}\varphi$, $\{(z_n, z_{0,n})\}_{n\in\mathbb{N}} \subseteq \mathrm{epi}\psi$, $\{x_n^*\}_{n\in\mathbb{N}} \subseteq X^*$, $\{y_{0,n}^*\}_{n\in\mathbb{N}} \subseteq Y_0^*$, $\{(z_n^*, z_{0,n}^*)\}_{n\in\mathbb{N}} \subseteq X^* \times Y_0^*$, such that*

$$\begin{cases} x_n^* \in \partial_\gamma f(x_n),\ y_{0,n}^* \in \partial_{\gamma_0}\varphi(y_{0,n}), \\ -z_{0,n}^* \in K_0^*,\ \langle -z_{0,n}^*,\ z_{0,n} - \psi(z_n)\rangle \in [0, \mu_0\varepsilon_0], \\ z_n^* \in \partial_{\mu_0\varepsilon_0}(-z_{0,n}^* \circ \psi)(z_n), \end{cases}$$

$$\begin{cases} x_n^* + z_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} x^*, \\ y_{0,n}^* + z_{0,n}^* \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_0^*}} 0, \end{cases}$$

$$\begin{cases} x_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \bar{x},\ z_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \bar{x}, \\ y_{0,n} \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_0}} \bar{y}_0,\ z_{0,n} \xrightarrow[n\to+\infty]{\|\cdot\|_{Y_0}} \bar{y}_0, \end{cases}$$

and

$$\begin{cases} f(x_n) - f(\bar{x}) - \langle x_n^*, x_n - \bar{x}\rangle \xrightarrow[n\to+\infty]{} 0, \\ \varphi(y_{0,n}) - \varphi(\bar{y}_0) - \langle y_{0,n}^*, y_{0,n} - \bar{y}_0\rangle \xrightarrow[n\to+\infty]{} 0, \\ \langle -z_n^*, z_n - \bar{x}\rangle + \langle -z_{0,n}^*, z_{0,n} - \bar{y}_0\rangle \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

Remark 4.3. Corollary 4.1 can be used for deriving the result given by Gutiérrez et al. [9, Theorem 6].

5. Sequential ε-Optimality Conditions for Constrained Location Problems with Monotonic Gauges

Motivated by the location models examined in [17, 3], the aim of this section is to give the sequential ε-optimality conditions of the following location problem with geometric constraint

$$(\mathscr{L}\mathscr{P}) \quad \inf_{x\in C} \quad j_{C_0}^+\Big(f_1(j_{C_1}(x - e_1)), \cdots, f_q(j_{C_q}(x - e_q))\Big),$$

where

- $C_0 \subseteq \mathbb{R}^q$ and $C, C_1, \cdots, C_q \subseteq X$ are nonempty, closed, and convex with $0 \in \mathrm{int}\, C_i$, $i = 1, \cdots, q$;
- $e_1, \cdots, e_q \in X$ are distinct points;
- $j_{C_0} : \mathbb{R}^q \to \mathbb{R}$ is a monotonic gauge and $j_{C_0}^+ : \mathbb{R}^q \to \mathbb{R}$ is defined by

$$j_{C_0}^+(x_1, \cdots, x_q) := j_{C_0}(x_1^+, \cdots, x_q^+),\ x_i^+ := \max\{0, x_i\}, i = 1, \cdots, q;$$

- $f_i : \mathbb{R} \to \overline{\mathbb{R}}$ with $f_i(x) \geq 0$, if $x \geq 0$, $f_i(x) = +\infty$, otherwise, is a proper, convex, lower semicontinuous and nondecreasing function on $\mathbb{R}_+$, $i = 1, \cdots, q$.

Since the functions $j^+_{C_0}$ and $f_1, \cdots, f_q$ are defined in a general way, problem $(\mathscr{L}\mathscr{P})$ covers a large class in location problems, such as the Weber problem with gauges of closed convex sets [17], single minimax location problems with or without set-up costs [6] and so on. In order to justify this point, we consider, for instance, the Weber problem with gauges of closed convex sets studied in [17]

$$(\mathscr{W}\mathscr{P}) \quad \inf_{x\in\mathbb{R}^m} \sum_{i=1}^{q} w_{a_i} j_{C_{a_i}}(x-a_i),$$

where $a_1, \cdots, a_q \in \mathbb{R}^m$, $C_{a_1}, \cdots, C_{a_q} \subseteq \mathbb{R}^m$ are nonempty, closed, and convex, $0 \in \operatorname{int} C_{a_i}$, $i = 1, \cdots, q$, and $w_{a_1}, \cdots, w_{a_q}$ are positive weights. To see the Weber problem $(\mathscr{W}\mathscr{P})$ as a special case of $(\mathscr{L}\mathscr{P})$, it suffices to set $X := \mathbb{R}^m$, $C := \mathbb{R}^m$, $e_i := a_i$, $C_i := C_{a_i}$, $f_i : \mathbb{R} \to \overline{\mathbb{R}}$, $f_i(x) := w_{a_i}x$, if $x \geq 0$, $f_i(x) = +\infty$, otherwise, $i = 1, \cdots, q$, and $j_{C_0}(x_1, \cdots, x_q) := \sum_{i=1}^{q} |x_i|$, $(x_1, \cdots, x_q) \in \mathbb{R}^q$.

It is worth noting that the defined functions $j^+_{C_0}$ and $j_{C_1}, \cdots, j_{C_q}$ are convex and continuous with $\operatorname{dom} j^+_{C_0} = \mathbb{R}^q$ and $\operatorname{dom} j_{C_1} = \cdots = \operatorname{dom} j_{C_q} = X$ (see [17, Proposition 4.1] and [6, Theorem 1 and Remark 2]). Thus problem $(\mathscr{L}\mathscr{P})$ is a convex optimization problem.

In order to characterize sequential ε-optimality conditions of problem $(\mathscr{L}\mathscr{P})$, we set $Y_1 = Y_2 := \mathbb{R}^q$ and $K_1 = K_2 := \mathbb{R}^q_+$. To write the considered problem as a convex multi-composed optimization problem, we introduce the following functions $h_1 : \mathbb{R}^q \to \mathbb{R}^q \cup \{+\infty_{\mathbb{R}^q}\}$ and $h_2 : X \to \mathbb{R}^q$, where

$$h_1(x_1, \cdots, x_q) := \begin{cases} (f_1(x_1), \cdots, f_q(x_q)), \text{ if } x_i \geq 0, i = 1, \cdots, q, \\ +\infty_{\mathbb{R}^q}, \text{ otherwise}, \end{cases}$$

and

$$h_2(x) := (j_{C_1}(x-e_1), \cdots, j_{C_q}(x-e_q)),\ x \in X.$$

Therefore, problem $(\mathscr{L}\mathscr{P})$ can be rewriting as

$$(\mathscr{L}\mathscr{P}) \quad \inf_{x\in X} \{\delta_C(x) + (j^+_{C_0} \circ h_1 \circ h_2)(x)\}.$$

Remark 5.1. Let us note that

- $\operatorname{epi} h_1 = \mathbb{E}_1$, where

$$\mathbb{E}_1 := \{(x_1, \cdots, x_q, r_1, \cdots, r_q) \in \mathbb{R}^q_+ \times \mathbb{R}^q_+ : f_i(x_i) \leq r_i, i = 1, \cdots, q\};$$

- $\operatorname{epi} h_2 = \mathbb{E}_2$, where

$$\mathbb{E}_2 := \{(x, r_1, \cdots, r_q) \in X \times \mathbb{R}^q_+ : j_{C_i}(x-e_i) \leq r_i, i = 1, \cdots, q\}.$$

Before deriving the sequential ε-optimality conditions for problem $(\mathscr{L}\mathscr{P})$, we also need the following lemmas.

Lemma 5.1. *Let* $(x_1, \cdots, x_q) \in \mathbb{R}^q$ *and* $\varepsilon \geq 0$. *Then*

$$\partial_\varepsilon(j^+_{C_0})(x_1, \cdots, x_q) = \Big\{(x^*_1, \cdots, x^*_q) \in \mathbb{R}^q_+ \cap C^\circ_0 : j^+_{C_0}(x_1, \cdots, x_q) \leq \sum_{i=1}^{q} x^*_i x_i + \varepsilon\Big\}.$$

Proof. Let $(x^*_1, \cdots, x^*_q) \in \mathbb{R}^q$. Then $(x^*_1, \cdots, x^*_q) \in \partial_\varepsilon(j^+_{C_0})(x_1, \cdots, x_q)$ if and only if

$$(j^+_{C_0})^*(x^*_1, \cdots, x^*_q) + j^+_{C_0}(x_1, \cdots, x_q) \leq \sum_{i=1}^{q} x^*_i x_i + \varepsilon.$$

Hence, we easily obtain the desired statement since the conjugate function of $j^+_{C_0}$ is (see [17, Proposition 4.2])

$$(j^+_{C_0})^*(x_1^*,\cdots,x_q^*) = \begin{cases} 0, \text{ if } (x_1^*,\cdots,x_q^*) \in \mathbb{R}^q_+ \cap C_0^\circ, \\ +\infty, \text{ otherwise.} \end{cases}$$

□

Lemma 5.2. *Let* $(x_1,\cdots,x_q) \in \mathbb{R}^q_+$, $(y_1,\cdots,y_q) \in \mathbb{R}^q_+$, *and* $\varepsilon \geq 0$. *Then* $\partial_\varepsilon\Big((y_1,\cdots,y_q)\circ h_1\Big)(x_1,\cdots,x_q) = \mathscr{S}_1^\varepsilon(y_1,\cdots,y_q,x_1,\cdots,x_q)$, *where* $\mathscr{S}_1^\varepsilon(y_1,\cdots,y_q,x_1,\cdots,x_q) := \big\{(x_1^*,\cdots,x_q^*) \in \mathbb{R}^q : \exists\, \varepsilon_1,\cdots,\varepsilon_q \geq 0, \sum_{i=1}^q \varepsilon_i = \varepsilon$ *such that* $x_i^* \in \partial_{\varepsilon_i}(y_i \circ f_i)(x_i), i = 1,\cdots,q\big\}$.

Proof. By a simple calculation, the conjugate function of $(y_1,\cdots,y_q)\circ h_1$ is

$$\Big((y_1,\cdots,y_q)\circ h_1\Big)^*(x_1^*,\cdots,x_q^*) = \sum_{i=1}^q (y_i f_i)^*(x_i^*), \quad \forall (x_1^*,\cdots,x_q^*) \in \mathbb{R}^q.$$

Therefore,

$$(x_1^*,\cdots,x_q^*) \in \partial_\varepsilon\Big((y_1,\cdots,y_q)\circ h_1\Big)(x_1,\cdots,x_q)$$

$\Longleftrightarrow$

$$\sum_{i=1}^q [(y_i f_i)^*(x_i^*) + (y_i f_i)(x_i) - x_i^* x_i] \leq \varepsilon$$

$\Longleftrightarrow$

$$\exists\, \varepsilon_1,\cdots,\varepsilon_q \geq 0,\ \varepsilon_1+\cdots+\varepsilon_q = \varepsilon,\ (y_i f_i)^*(x_i^*) + (y_i f_i)(x_i) - x_i^* x_i \leq \varepsilon_i, i = 1,\cdots,q$$

$\Longleftrightarrow$

$$\exists\, \varepsilon_1,\cdots,\varepsilon_q \geq 0,\ \varepsilon_1+\cdots+\varepsilon_q = \varepsilon,\ x_i^* \in \partial_{\varepsilon_i}(y_i \circ f_i)(x_i), i = 1,\cdots,q.$$

□

Lemma 5.3. *Let* $x \in X$, $(y_1,\cdots,y_q) \in \mathbb{R}^q_+$, *and* $\varepsilon \geq 0$. *Then* $\partial_\varepsilon\Big((y_1,\cdots,y_q)\circ h_2\Big)(x) = \mathscr{S}_2^\varepsilon(y_1,\cdots,y_q,x)$, *where* $\mathscr{S}_2^\varepsilon(y_1,\cdots,y_q,x) := \bigcup_{\substack{\varepsilon_1,\cdots,\varepsilon_q \geq 0, \\ \varepsilon_1+\cdots+\varepsilon_q = \varepsilon}} \Big\{\partial_{\varepsilon_1}(y_1 j_{C_1})(x-e_1)+\cdots+\partial_{\varepsilon_q}(y_q j_{C_q})(x-e_q)\Big\}$.

Proof. Let $x \in X$, $(y_1,\cdots,y_q) \in \mathbb{R}^q_+$ and $\varepsilon \geq 0$. First, observe that the gauges $j_{C_1},\cdots,j_{C_q} : X \to \mathbb{R}$ are proper, convex, and continuous. Hence, by applying [19, Proposition 1.3], we deduce that

$$\begin{aligned} \partial_\varepsilon\Big((y_1,\cdots,y_q)\circ h_2\Big)(x) &= \partial_\varepsilon\Big(y_1 j_{C_1}(.-e_1)+\cdots+y_q j_{C_q}(.-e_q)\Big)(x) \\ &= \bigcup_{\substack{\varepsilon_1,\cdots,\varepsilon_q \geq 0, \\ \varepsilon_1+\cdots+\varepsilon_q = \varepsilon}} \Big\{\partial_{\varepsilon_1}\Big(y_1 j_{C_1}(.-e_1)\Big)(x)+\cdots+\partial_{\varepsilon_q}\Big(y_q j_{C_q}(.-e_q)\Big)(x)\Big\} \\ &= \bigcup_{\substack{\varepsilon_1,\cdots,\varepsilon_q \geq 0, \\ \varepsilon_1+\cdots+\varepsilon_q = \varepsilon}} \Big\{\partial_{\varepsilon_1}(y_1 j_{C_1})(x-e_1)+\cdots+\partial_{\varepsilon_q}(y_q j_{C_q})(x-e_q)\Big\}. \end{aligned}$$

□

We state now the sequential ε-optimality conditions for problem $(\mathscr{L}\mathscr{P})$ by applying Theorem 4.1.

Theorem 5.1. *Let $\bar{x} \in C$ and $\varepsilon \geq 0$. Then, $\bar{x}$ is an ε-optimal solution of problem $(\mathscr{LP})$ if and only if, for all $\mu_1, \mu_2 > 0$, there exist $\gamma_1, \gamma_2, \varepsilon_1, \varepsilon_2 \geq 0$, with $\gamma_1 + \gamma_2 + \mu_1\varepsilon_1 + \mu_2\varepsilon_2 = \varepsilon$, and sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq C$, $\{(y_{1,n}, \cdots, y_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^q$, $\{(z_{1,n}, \cdots, z_{q,n}, \alpha_{1,n}, \cdots, \alpha_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{E}_1$, $\{(z_n, \beta_{1,n}, \cdots, \beta_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{E}_2$, $\{x_n^*\}_{n\in\mathbb{N}} \subseteq X^*$, $\{(y_{1,n}^*, \cdots, y_{q,n}^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+^q \cap C_0^\circ$, $\{(z_{1,n}^*, \cdots, z_{q,n}^*, \alpha_{1,n}^*, \cdots, \alpha_{q,n}^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^q \times \mathbb{R}_+^q$ and $\{(z_n^*, \beta_{1,n}^*, \cdots, \beta_{q,n}^*)\}_{n\in\mathbb{N}} \subseteq X^* \times \mathbb{R}_+^q$ such that*

$$\begin{cases} x_n^* \in N_C^{\gamma_1}(x_n),\ j_{C_0}^+(y_{1,n}, \cdots, y_{q,n}) \leq \displaystyle\sum_{i=1}^q y_{i,n}^* y_{i,n} + \gamma_2, \\ \displaystyle\sum_{i=1}^q \alpha_{i,n}^* \Big(\alpha_{i,n} - f_i(z_{i,n})\Big) \in [0, \mu_1\varepsilon_1], \\ (z_{1,n}^*, \cdots, z_{q,n}^*) \in \mathscr{S}_1^{\mu_1\varepsilon_1}(\alpha_{1,n}^*, \cdots, \alpha_{q,n}^*, z_{1,n}, \cdots, z_{q,n}), \\ \displaystyle\sum_{i=1}^q \beta_{i,n}^* \Big(\beta_{i,n} - j_{C_i}(z_n - e_i)\Big) \in [0, \mu_2\varepsilon_2],\ z_n^* \in \mathscr{S}_2^{\mu_2\varepsilon_2}(\beta_{1,n}^*, \cdots, \beta_{q,n}^*, z_n), \end{cases}$$

$$\begin{cases} x_n^* + z_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} 0,\ (y_{1,n}^* - \alpha_{1,n}^*, \cdots, y_{q,n}^* - \alpha_{q,n}^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (0, \cdots, 0), \\ (z_{1,n}^* - \beta_{1,n}^*, \cdots, z_{q,n}^* - \beta_{q,n}^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (0, \cdots, 0), \end{cases}$$

$$\begin{cases} x_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \bar{x},\ z_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \bar{x}, \\ (y_{1,n}, \cdots, y_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (f_1(j_{C_1}(\bar{x} - e_1)), \cdots, f_q(j_{C_q}(\bar{x} - e_q))), \\ (z_{1,n}, \cdots, z_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (j_{C_1}(\bar{x} - e_1), \cdots, j_{C_q}(\bar{x} - e_q)), \\ (\alpha_{1,n}, \cdots, \alpha_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (f_1(j_{C_1}(\bar{x} - e_1)), \cdots, f_q(j_{C_q}(\bar{x} - e_q))), \\ (\beta_{1,n}, \cdots, \beta_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (j_{C_1}(\bar{x} - e_1), \cdots, j_{C_q}(\bar{x} - e_q)), \end{cases}$$

and

$$\begin{cases} \langle x_n^*, x_n - \bar{x} \rangle \xrightarrow[n\to+\infty]{} 0, \\ j_{C_0}^+(y_{1,n}, \cdots, y_{q,n}) - j_{C_0}^+(f_1(j_{C_1}(\bar{x} - e_1)), \cdots, f_q(j_{C_q}(\bar{x} - e_q))) \\ \qquad\qquad - \displaystyle\sum_{i=1}^q y_{i,n}^* \Big(y_{i,n} - f_i(j_{C_i}(\bar{x} - e_i))\Big) \xrightarrow[n\to+\infty]{} 0, \\ \displaystyle\sum_{i=1}^q \alpha_{i,n}^* \Big(\alpha_{i,n} - f_i(j_{C_i}(\bar{x} - e_i))\Big) - \sum_{i=1}^q z_{i,n}^* \Big(z_{i,n} - j_{C_i}(\bar{x} - e_i)\Big) \xrightarrow[n\to+\infty]{} 0, \\ \displaystyle\sum_{i=1}^q \beta_{i,n}^* \Big(\beta_{i,n} - j_{C_i}(\bar{x} - e_i)\Big) - \langle z_n^*, z_n - \bar{x} \rangle \xrightarrow[n\to+\infty]{} 0. \end{cases}$$

Proof. It is clear that $\bar{x}$ is an ε-optimal solution to problem $(\mathscr{LP})$ is equivalent to

$$0 \in \partial_\varepsilon(\delta_C + j_{C_0}^+ \circ h_1 \circ h_2)(\bar{x}).$$

Since C is nonempty, closed, and convex, it follows that δ_C is proper, convex, and lower semicontinuous function. As the finite functions $j_{C_1}(.-e_1),\cdots,j_{C_q}(.-e_q)$ are convex and continuous, one can easily see that h_2 is proper, $\mathbb{R}^q_+$-convex and $\mathbb{R}^q_+$-epi closed with $\text{dom}h_2 = X$ and $h_2(\text{dom}(h_2)) \subseteq \mathbb{R}^q_+$. Since $f_i : \mathbb{R} \to \overline{\mathbb{R}}$ is proper, convex, lower semicontinuous, and nondecreasing on $\text{dom}f_i = \mathbb{R}_+$, $i = 1,\cdots,q$, it follows that h_1 is proper, $\mathbb{R}^q_+$-convex, $(\mathbb{R}^q_+,\mathbb{R}^q_+)$-nondecreasing on $\text{dom}h_1 = \mathbb{R}^q_+$, and $\mathbb{R}^q_+$-epi closed with $h_1(\text{dom}h_1) \subseteq \mathbb{R}^q_+$. As regards $j^+_{C_0}$, it is proper, convex, $\mathbb{R}^q_+$-nondecreasing on $\text{dom}\, j^+_{C_0} = \mathbb{R}^q$, and lower semicontinuous (see [17, Proposition 4.1]). Hence, the functions $f := \delta_C$, $\varphi \equiv 0$, $\psi \equiv 0$, $g := j^+_{C_0}$, h_1, and h_2 verify all the conditions considered in Theorem 4.1. Therefore, by applying Theorem 4.1 and Lemma 5.1 - 5.3, we can directly obtain the desired statement. □

REFERENCES

[1] G. Wanka, O. Wilfer, A Lagrange duality approach for multi-composed optimization problems, TOP 25 (2017), 288-313.

[2] S.M. Grad, G. Wanka, O. Wilfer, Duality and ε-optimality conditions for multi-composed optimization problems with applications to fractional and entropy optimization, Pure Appl. Funct. Anal. 2 (2017), 43-63.

[3] O. Wilfer, Multi-composed Programming with Applications to Facility Location, Springer, 2020.

[4] M. Kraus, S. Feuerriegel, A. Oztekin, Deep learning in business analytics and operations research: Models, applications and managerial implications, Eur. J. Oper. Res. 281 (2020), 628-641.

[5] S.M. Grad, O. Wilfer, A proximal method for solving nonlinear minmax location problems with perturbed minimal time functions via conjugate duality, J. Global Optim. 74 (2019), 121-160.

[6] G. Wanka, O. Wilfer, Duality results for nonlinear single minimax location problems via multi-composed optimization, Math. Methods Oper. Res. 86 (2017), 401–439.

[7] R.I. Boţ, E.R. Csetnek, G. Wanka, Sequential optimality conditions for composed convex optimization problems, J. Math. Anal. Appl. 342 (2008), 1015-1025.

[8] R.I. Boţ, E.R. Csetnek, G. Wanka, Sequential optimality conditions in convex programming via perturbation approach, J. Convex Anal. 15 (2008), 149-164.

[9] C. Gutiérrez, L. Huerga, V. Novo, L. Thibault, Sequential ε-subdifferential calculus for scalar and vector mappings, Set-Valued Var. Anal. 25 (2017), 383-403.

[10] M. Laghdir, A. Rikouane, Y.A. Fajri, E. Tazi, Sequential Pareto subdifferential sum rule and sequential efficiency, Appl. Math. E-Notes 6 (2016), 133-143.

[11] M. Laghdir, A. Rikouane, Y.A. Fajri, E. Tazi, Sequential Pareto subdifferential composition rule and sequential efficiency, J. Nonlinear Convex Anal. 18 (2017), 2177-2187.

[12] M. Laghdir, I. Dali, M.B. Moustaid, A generalized sequential formula for subdifferential of multi-composed functions defined on Banach spaces and applications, Pure Appl. Funct. Anal. 5 (2020), 999-1023.

[13] J.P. Penot, Subdifferential calculus without qualification assumptions, J. Convex Anal. 3 (1996), 207-219.

[14] L. Thibault, A generalized sequential formula for subdifferentials of sums of convex functions defined on Banach spaces, In: R. Durier, C. Michelot, (ed.) Recent Developments in Optimization, Lecture Notes in Economics and Mathematical Systems, vol. 429, pp. 340-345, Springer, Berlin, Heidelberg, 1995.

[15] L. Thibault, Sequential convex subdifferential calculus and sequential Lagrange multipliers, SIAM J. Control Optim. 35 (1997), 1434-1444.

[16] C. Zălinescu, Convex Analysis in General Vector Spaces, World Scientific, Singapore, 2002.

[17] G. Wanka, R.I. Boţ, E. Vargyas, Duality for location problems with unbounded unit balls, Eur. J. Oper. Res. 179 (2007), 1252-1265.

[18] V. Jeyakumar, Asymptotic dual conditions characterizing optimality for infinite convex programs, J. Optim. Theory Appl. 93 (1997), 153-165.

[19] J.B. Hiriart-Urruty, Lipschitz r-continuity of the approximate subdifferential of a convex function, Math. Scand. 47 (1980), 123-134.