Appl. Set-Valued Anal. Optim. 4 (2022), No. 2, pp. 207-221
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.4.2022.2.05

GLOBAL SMOOTH LARGE SOLUTIONS TO 2D MAGNETIC BÉNARD SYSTEMS WITH MIXED PARTIAL DISSIPATION

YUGUI CAO, YINXIA WANG*

School of Mathematics and Statistics,
North China University of Water Resources and Electric Power, Zhengzhou 450011, China

Abstract. This paper focuses on the initial value problem for two-dimensional magnetic Bénard system with the partial dissipation and zero diffusivity. Based on the energy estimates and the tricky analytical skills, we prove that the problem always exist a unique global smooth solution without any smallness restriction on the initial data.
Keywords. Global smooth large solutions; Magnetic Bénard system; Mixed partial viscosity; Zero diffusivity.

1. INTRODUCTION

This paper mainly investigates global smooth large solutions to the two-dimensional (2D) magnetic Bénard system with mixed partial dissipation and zero diffusivity

$$\begin{cases} \partial_t u_1 + u\cdot\nabla u_1 - B\cdot\nabla B_1 - \mu_1\partial_y^2 u_1 + \partial_x p = 0, \\ \partial_t u_2 + u\cdot\nabla u_2 - B\cdot\nabla B_2 - \mu_2\partial_x^2 u_2 + \partial_y p = \theta, \\ \partial_t B_1 + u\cdot\nabla B_1 - B\cdot\nabla u_1 - \nu_1\partial_y^2 B_1 = 0, \\ \partial_t B_2 + u\cdot\nabla B_2 - B\cdot\nabla u_2 - \nu_2\partial_x^2 B_2 = 0, \\ \partial_t\theta + u\cdot\nabla\theta = u\cdot e_2, \\ \nabla\cdot u = \nabla\cdot B = 0 \end{cases} \tag{1.1}$$

with the initial data

$$t=0: \quad u = u_0(x,y),\ \ B = B_0(x,y),\ \ \theta = \theta_0(x,y),\ \ x,y\in\mathbb{R}, \tag{1.2}$$

where $u = (u_1(t,x,y), u_2(t,x,y))$, $B = (B_1(t,x,y), B_2(t,x,y))$, $\theta = \theta(t,x,y)$, and $p = p(t,x,y)$ denotes the velocity field, the magnetic filed, the scalar temperature and the scalar pressure, respectively, and μ_i, $\nu_i > 0 (i = 1,2)$ are the coefficients of dissipation and magnetic diffusion, respectively.

*Corresponding author.
E-mail addresses: cyg@ncwu.edu.cn (Y. Cao), yinxia117@126.com (Y. Wang).
Received March 4, 2022; Accepted July 3, 2022.

The 2D magnetic Bénard system with full viscosity and full thermal diffusivity for incompressible fluid flows in $\mathbb{R}^2$ has the following form

$$\begin{cases} \partial_t u + u\cdot\nabla u + B\cdot\nabla B - \mu\Delta u + \nabla p = \theta e_2, \\ \partial_t B + u\cdot\nabla B + B\cdot\nabla u - \nu\Delta B = 0, \\ \partial_t \theta + u\cdot\nabla\theta - \kappa\Delta\theta = u\cdot e_2, \\ \nabla\cdot u = \nabla\cdot B = 0, \end{cases} \tag{1.3}$$

where $\mu \geq 0$ is the coefficients of dissipation, $\nu \geq 0$ is magnetic diffusion, $\kappa \geq 0$ is thermal diffusivity, and $e_2 = (0,1)$. The magnetic Bénard system as a toy model comes from the convective motions in a heated and incompressible fluid, illuminates the heat convection phenomenon under the presence of the magnetic field (see [1] and [2] for details).

Let us first briefly review some global well-posedness results of (1.3). Global smooth solutions were proved by Zhou, Fan and Nakamura [3] with μ, $\nu > 0$, and $\kappa = 0$. Cheng and Du [4] established the global well-posedness of (1.3) without thermal diffusivity and with horizontal or vertical magnetic diffusion. The global regularity of the 2D magnetic Bénard system with zero thermal conductivity was established in [5]. Moreover, the global regularity of the 2D magnetic Bénard system with horizontal dissipation, horizontal magnetic diffusion and with either horizontal or vertical thermal diffusivity was proved. Ye [6] proved that the global regularity of (1.3) with vertical dissipation. Ma [7] proved that global smooth solutions to the 3D magnetic Bénard system with mixed partial dissipation, magnetic diffusion and thermal diffusivity with small initial data. We also refer to [8] for $2\frac{1}{2}$D magnetic Bénard system with mixed partial viscosity.

If we neglect the thermal effects in the fluid motion, the two-dimensional magnetic Bénard problem can be specialized to the well-known 2D MHD system

$$\begin{cases} \partial_t u + u\cdot\nabla u + B\cdot\nabla B - \mu\Delta u + \nabla p = 0, \\ \partial_t B + u\cdot\nabla B + B\cdot\nabla u - \nu\Delta B = 0, \\ \nabla\cdot u = 0 = \nabla\cdot B = 0, \end{cases}$$

Global smooth solutions to the two-dimensional MHD system with partial dissipation has attracted interest of mathematicians, and many interesting results have been established in some beautiful paper (see, e.g., [9, 10, 11, 12, 13]). The pioneer work on this topic is due to Cao and Wu [9], they obtained the global regularity for the 2D MHD equations with partial dissipation. Du and Zhou [11] established global well-posedness and conditional global well-posedness of the two-dimensional MHD equations with mixed partial dissipation and magnetic diffusion in more general cases. Recently, Zhang, Dong and Jia [13] also proved that global regularity result with the partial kinematic dissipation $(\partial_y^2 u_1, \partial_x^2 u_2)$ and the partial magnetic diffusion $(\partial_y^2 B_1, \partial_x^2 B_2)$ without using anisotropic type Sobolev inequality. Moreover, optimal large-time decay rates of both solutions and their higher order derivatives were also obtained. Global stability for the ideal MHD equations was established by Cai and Lei [14]. We also refer to [15, 16, 17, 18, 19, 20, 21] for other global well-posedness results of the MHD system.

If $B=0$, then (1.3) reduces to is nothing but the so-called Bénard system:

$$\begin{cases}\partial_t u+u\cdot\nabla u-\mu\Delta u+\nabla p=\theta e_2,\\ \partial_t\theta+u\cdot\nabla\theta-\kappa\Delta\theta=u\cdot e_2,\\ \nabla\cdot u=0.\end{cases}$$

Global existence and regularity of solutions to the Bénard problem is a classical problem in the fluid dynamics area. We could refer to [22] and [23] for related research results.

If we ignore the Rayleigh-Bénard convection term ue_2 in (1), it is reduced to the well-known Boussinesq system. Owing to the Boussinesq system has important roles in the atmospheric sciences and shares a similar vortex stretching effect as that in the 3D incompressible flow, it has been extensively studied. The global well-posedness in $\mathbb{R}^2$ with partial viscosity were established; see, e.g., [24, 25, 26, 27, 28, 29, 30, 31] and the references therein. We also have to mention the recent work of Adhikari et. al [32] and Ye [33]. They proved that global smooth solutions to the 2D Boussinesq system with partial dissipation $(\partial_y^2 u_1, \partial_x^2 u_2)$ and zero diffusivity.

In this paper, motivated by the mentioned works on the $2D$ magnetic Bénard system, the MHD system, and the Boussinesq system with partial viscosity, the main target is to prove that problem (1.1)-(1.2) possesses a unique global smooth large solution. More precisely, we state the main result of this paper as follows.

Theorem 1.1. *Let $m\geq 3$ be integer. If $u_0, B_0\in H^m$ and $\theta_0\in H^{m-1}$ with $\nabla\cdot u_0=\nabla\cdot B_0=0$, then, for any given $T>0$, problem (1.1)-(1.2) has a unique global smooth solution*

$$u,B\in L^\infty(0,T;H^m)\bigcap L^2(0,T;H^{m+1}),\ \ \theta\in L^\infty(0,T;H^{m-1}). \tag{1.4}$$

Remark 1.1. The partial dissipation and the partial magnetic diffusion of the magnetic Bénard system in this paper act on the components of u and B (see (1.1)), which take the place of the partial dissipation and the partial magnetic diffusion that usually act on u and B (see [4, 6]).

Remark 1.2. Because the appearance of $\int_0^t\|\nabla u\|_{L^\infty}d\tau$ and $\|D\theta\|_{L^\infty}$ in closing the estimates of higher-order derivatives, the boundedness of $\int_0^t\|\nabla u\|_{L^\infty}d\tau$ and $\|D\theta\|_{L^\infty}$ become the key ingredients to establish the global smooth large solutions to the magnetic Bénard system with zero diffusivity. The boundedness of $\int_0^t\|\nabla u\|_{L^\infty}d\tau$ may be obtained by local-in-time estimate, which was used in [24] for the Boussinesq system and [4] for the magnetic Bénard system, respectively. The same techniques is still used to deal with problem (1.1)-(1.2) in this paper. Since the appearance of the term ue_2 in (1.1), it is not easy to bound $\|D\theta\|_{L^\infty}$. Then instead of bounding $\|D\theta\|_{L^\infty}$, we make use of the tricky interpolation between the bounded low derivatives and the required higher-order derivatives.

The paper is organized as follows. In Section 2, we recall on some useful Lemmas, which paly very roles in proving global smooth large solutions. Section 3 is devoted to establish global a priori H^m-bound. The proof is based on the energy estimate, local-in-time estimate, and the tricky interpolation techniques. Theorem 1.1 is completed by combining global a priori H^m-bound, local existence and uniqueness results, and standard compactness argument in Section 4, the last section.

Notations. We introduce some notations, which are used in this paper. The differential operators D^k with $k>0$ is defined by $D^k f = \mathscr{F}^{-1}[|\xi|\widehat{f}(\xi)]$. The usual Sobolev space of order m is defined by $H^m = \{u \in L^2(\mathbb{R}^2) | D^m u \in L^2\}$ with the norm $\|u\|_{H^m} = \left(\|u\|_{L^2}^2 + \|D^m u\|_{L^2}^2\right)^{\frac{1}{2}}$.

2. Preliminaries

In this section, we recall the following lemmas, which play an essential role in dealing with the nonlinear term.

Lemma 2.1. [34] *Let $p_1, q_2 \in [1,+\infty]$ and p, p_2, $q_1 \in (1,+\infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Then the following commutator estimate for $s>0$*

$$\|D^s(fg) - fD^s g\|_{L^p} \le C(\|\nabla f\|_{L^{p_1}} \|D^{s-1} g\|_{L^{q_1}} + \|D^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}) \tag{2.1}$$

holds.

Lemma 2.2. [35, 36] *Assume that $f \in H^2(\mathbb{R}^2)$. Then the following inequality*

$$\|f\|_{L^\infty} \le C_0(\|f\|_{L^2} + \|Df\|_{L^2} + 1)\left(\log(\|D^2 f\|_{L^2}^2 + \|f\|_{L^2}^2 + e)\right)^{\frac{1}{2}}$$

holds, where $C_0 > 0$ is a constant.

We also need the well-known Gagliardo-Nirenberg inequality.

Lemma 2.3. *Let j, m be any integers satisfying $0 \le j < m$, and let $1 \le q, r \le \infty$, and $p \in \mathbb{R}$, $\frac{j}{m} \le \sigma \le 1$. Then, for any $f \in W^{m,r} \bigcap L^q$, there exists $C = C(n,m,j,q,r,\sigma)$ such that the following inequality holds*

$$\|D^j f\|_{L^p} \le C\|f\|_{L^q}^{1-\sigma} \|D^m f\|_{L^r}^{\sigma} \tag{2.2}$$

with the following exception: if $1<r<\infty$ and $m-j-\frac{n}{r}$ is a nonnegative integer, then (2.2) holds only for σ satisfying $\frac{j}{m} \le \sigma < 1$.

We also need the following Jensen inequality, which plays key role in local-in-time estimate.

Lemma 2.4. *Let $\Phi(\varphi) : [\alpha,\beta] \to \mathbb{R}$ be a concave function. Suppose that $f(t) : [a,b] \to [\alpha,\beta]$, and $P(t)$ are continuous functions with $P(t) \ge 0$, and there exists $t_0 \in [a,b]$ such that $P(t_0) \ne 0$. Then the following inequality holds:*

$$\frac{\int_a^b \Phi(f(t))P(t)dt}{\int_a^b P(t)dt} \le \Phi\left(\frac{\int_a^b f(t)P(t)dt}{\int_a^b P(t)dt}\right). \tag{2.3}$$

3. A Priori H^m-Bound

To establish global existence of smooth large solutions, we need to obtain the global a priori H^m-bound. Therefore, what we should do is how to derive some a priori estimates of solutions now. Without loss of generality, we assume that $\mu_1 \ge \mu_2$ and $\nu_1 \le \nu_2$.

Taking the inner product of (1.1) with $(u_1, u_2, B_1, B_2, \theta)$ and using integration by parts and $\nabla \cdot u = \nabla \cdot B = 0$, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\|u_1(t)\|_{L^2}^2 + \|u_2(t)\|_{L^2}^2 + \|B_1(t)\|_{L^2}^2 + \|B_2(t)\|_{L^2}^2 + \|\theta\|_{L^2}^2) \\ = \ & \mu_1 \int_{\mathbb{R}^2} u_1 \partial_y^2 u_1 \, dxdy + \mu_2 \int_{\mathbb{R}^2} u_2 \partial_x^2 u_2 \, dxdy + \nu_1 \int_{\mathbb{R}^2} B_1 \partial_y^2 B_1 \, dxdy + \\ & \nu_2 \int_{\mathbb{R}^2} B_2 \partial_x^2 B_2 \, dxdy + 2\int_{\mathbb{R}^2} \theta u_2 \, dxdy. \end{aligned} \tag{3.1}$$

To deal with the first four term on the right hand side of (3.1), we claim that, for $k = 0, 1, 2, \cdots$,

$$\begin{aligned} & \mu_1 \int_{\mathbb{R}^2} D^k u_1 D^k \partial_y^2 u_1 \, dxdy + \mu_2 \int_{\mathbb{R}^2} D^k u_2 D^k \partial_x^2 u_2 \, dxdy \\ \leq \ & -\frac{\mu_2}{2}\|\nabla D^k u(t)\|_{L^2}^2 - (\mu_1 - \mu_2)\|\partial_y D^k u_1\|_{L^2}^2 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} & \nu_1 \int_{\mathbb{R}^2} D^k B_1 D^k \partial_y^2 B_1 \, dxdy + \nu_2 \int_{\mathbb{R}^2} D^k B_2 D^k \partial_x^2 B_2 \, dxdy \\ \leq \ & -\frac{\nu_1}{2}\|\nabla D^k B(t)\|_{L^2}^2 - (\nu_2 - \nu_1)\|\partial_x D^k B_2\|_{L^2}^2. \end{aligned} \tag{3.3}$$

In what follows, we only prove (3.2). In fact, we find that, by $\nabla \cdot u = 0$, integration by parts and Cauchy inequality

$$\begin{aligned} \|\partial_x D^k u_1\|_{L^2}^2 + \|\partial_y D^k u_2\|_{L^2}^2 &= \int_{\mathbb{R}^2} (\partial_x D^k u_1 + \partial_y D^k u_2)^2 \, dxdy - 2\int_{\mathbb{R}^2} \partial_x D^k u_1 \partial_y D^k u_2 \, dxdy \\ &\leq \|\partial_y D^k u_1\|_{L^2}^2 + \|\partial_x D^k u_2\|_{L^2}^2, \end{aligned}$$

which yields

$$\begin{aligned} \|\nabla D^k u(t)\|_{L^2}^2 &= \|\partial_x D^k u_1\|_{L^2}^2 + \|\partial_y D^k u_2\|_{L^2}^2 + \|\partial_y D^k u_1\|_{L^2}^2 + \|\partial_x D^k u_2\|_{L^2}^2 \\ &\leq 2(\|\partial_y D^k u_1\|_{L^2}^2 + \|\partial_x D^k u_2\|_{L^2}^2). \end{aligned} \tag{3.4}$$

Combining (3.4) and

$$\begin{aligned} & \mu_1 \int_{\mathbb{R}^2} D^k u_1 D^k \partial_y^2 u_1 \, dxdy + \mu_2 \int_{\mathbb{R}^2} D^k u_2 D^k \partial_x^2 u_2 \, dxdy \\ & = -\mu_2 \int_{\mathbb{R}^2} (|\partial_y D^k u_1|^2 + \partial_x D^k u_2|^2) \, dxdy - (\mu_1 - \mu_2)\int_{\mathbb{R}^2} |\partial_y D^k u_1|^2 \, dxdy \end{aligned}$$

entails that (3.2) holds. It follows from (3.1), (3.2), (3.3) with $k = 0$, and Cauchy inequality that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + \|\theta\|_{L^2}^2) + \frac{\mu_2}{2}\|\nabla u(t)\|_{L^2}^2 + (\mu_1 - \mu_2)\|\partial_y u_1(t)\|_{L^2}^2 + \\ & \frac{\nu_1}{2}\|\nabla B(t)\|_{L^2}^2 + (\nu_2 - \nu_1)\|\partial_x B_2(t)\|_{L^2}^2 \leq \|u(t)\|_{L^2}^2 + \|\theta\|_{L^2}^2. \end{aligned}$$

It follows from Gronwall inequality that

$$\begin{aligned} & \|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + \|\theta\|_{L^2}^2 + \mu_2 \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau + 2(\mu_1 - \mu_2)\int_0^t \|\partial_y u_1(\tau)\|_{L^2}^2 d\tau + \\ & \nu_1 \int_0^t \|\nabla B(\tau)\|_{L^2}^2 d\tau + 2(\nu_2 - \nu_1)\int_0^t \|\partial_x B_2(\tau)\|_{L^2}^2 d\tau \leq C. \end{aligned} \tag{3.5}$$

We apply D to the first four equation of (1.1), and then take the inner product with (Du_1, Du_2, DB_1, DB_2) to obtain

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}(\|Du_1(t)\|_{L^2}^2+\|Du_2(t)\|_{L^2}^2+\|DB_1(t)\|_{L^2}^2+\|DB_2(t)\|_{L^2}^2) \\
= \; & \mu_1\int_{\mathbb{R}^2} Du_1\partial_y^2 Du_1\,dxdy+\mu_2\int_{\mathbb{R}^2} Du_2\partial_x^2 Du_2\,dxdy+\nu_1\int_{\mathbb{R}^2} DB_1\partial_y^2 DB_1\,dxdy+ \\
& \nu_2\int_{\mathbb{R}^2} DB_2\partial_x^2 DB_2\,dxdy-\int_{\mathbb{R}^2} D(u\cdot\nabla u_1)Du_1\,dxdy-\int_{\mathbb{R}^2} D(u\cdot\nabla u_2)Du_2\,dxdy+ \\
& \int_{\mathbb{R}^2} D(B\cdot\nabla B_1)Du_1\,dxdy+\int_{\mathbb{R}^2} D(B\cdot\nabla B_2)Du_2\,dxdy-\int_{\mathbb{R}^2} D(u\cdot\nabla B_1)DB_1\,dxdy- \\
& \int_{\mathbb{R}^2} D(u\cdot\nabla B_2)DB_2\,dxdy+\int_{\mathbb{R}^2} D(B\cdot\nabla u_1)DB_1\,dxdy+\int_{\mathbb{R}^2} D(B\cdot\nabla u_2)DB_2\,dxdy \\
& +\int_{\mathbb{R}^2} D\theta Du_2\,dxdy \\
=: \; & \sum_{i=1}^{13} I_i.
\end{aligned}
\tag{3.6}
$$

(3.2) with $k=1$ gives

$$
I_1+I_2 \le -\frac{\mu_2}{2}\|\nabla Du(t)\|_{L^2}^2-(\mu_1-\mu_2)\|\partial_y Du_1\|_{L^2}^2. \tag{3.7}
$$

Resorting to (3.3) with $k=1$ yields

$$
I_3+I_4 \le -\frac{\nu_1}{2}\|\nabla DB(t)\|_{L^2}^2-(\nu_2-\nu_1)\|\partial_x DB_2\|_{L^2}^2. \tag{3.8}
$$

By using $\nabla\cdot u=0$, we have

$$
\begin{aligned}
I_5+I_6 &= -\int_{\mathbb{R}^2}[\partial_x u_1\partial_x u_1\partial_x u_1+\partial_x u_1\partial_x u_2\partial_x u_2+\partial_x u_2\partial_y u_1\partial_x u_1+\partial_x u_2\partial_y u_2\partial_x u_2 \\
&\quad +\partial_y u_1\partial_x u_1\partial_y u_1+\partial_y u_1\partial_x u_2\partial_y u_2+\partial_y u_2\partial_y u_1\partial_y u_1+\partial_y u_2\partial_y u_2\partial_y u_2]\,dxdy \\
&= -\int_{\mathbb{R}^2}[\partial_x u_1\partial_x u_1\partial_x u_1+\partial_y u_2\partial_y u_2\partial_y u_2]\,dxdy \\
&\quad -\int_{\mathbb{R}^2}[\partial_x u_1\partial_x u_2\partial_x u_2+\partial_x u_2\partial_y u_2\partial_x u_2]\,dxdy \\
&= -\int_{\mathbb{R}^2}[\partial_x u_2\partial_y u_1\partial_x u_1+\partial_y u_1\partial_x u_2\partial_y u_2]\,dxdy \\
&\quad -\int_{\mathbb{R}^2}[\partial_y u_1\partial_x u_1\partial_y u_1+\partial_y u_2\partial_y u_1\partial_y u_1]\,dxy \\
&= 0.
\end{aligned}
\tag{3.9}
$$

Thanks to $\nabla\cdot u=\nabla\cdot B=0$, integration by parts, Hölder inequality, (2.2), and Cauchy inequality, we arrive at

$$
\begin{aligned}
I_7+I_8+I_9+I_{10}+I_{11}+I_{12} &= -\int_{\mathbb{R}^2}[DB\cdot\nabla BDu+DB\cdot\nabla uDB-Du\cdot\nabla BDB]\,dxdy \\
&\le 3\|Du\|_{L^2}\|DB\|_{L^4}^2 \\
&\le C\|Du\|_{L^2}\|DB\|_{L^2}\|D^2B\|_{L^2} \\
&\le C\|Du\|_{L^2}^2\|DB\|_{L^2}^2+\frac{\nu_1}{4}\|\nabla DB\|_{L^2}^2.
\end{aligned}
\tag{3.10}
$$

Resorting to integration by parts and Cauchy inequality, we obtain

$$I_{13} = -\int_{\mathbb{R}^2} \theta D^2 u_2\, dxdy \le C\|\theta\|_{L^2}^2 + \frac{\mu_2}{4}\|\nabla Du\|_{L^2}^2. \tag{3.11}$$

Inserting (3.7)-(3.11) into (3.6) yields

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\|Du(t)\|_{L^2}^2 + \|DB(t)\|_{L^2}^2) + \frac{\mu_2}{4}\|\nabla Du(t)\|_{L^2}^2 + \frac{\nu_1}{4}\|\nabla DB(t)\|_{L^2}^2 + \\ & (\mu_1 - \mu_2)\|\partial_y Du_1(t)\|_{L^2}^2 + (\nu_2 - \nu_1)\|\partial_x DB_2(t)\|_{L^2}^2 \\ \le\ & C(\|Du\|_{L^2}^2\|DB\|_{L^2}^2 + \|\theta\|_{L^2}^2). \end{aligned}$$

Integrating the above inequality with respect to time and using Gronwall inequality and (3.5) gives

$$\begin{aligned} & \|Du(t)\|_{L^2}^2 + \|DB(t)\|_{L^2}^2 + \frac{\mu_2}{2}\int_0^t \|\nabla Du(\tau)\|_{L^2}^2 d\tau + \frac{\nu_1}{2}\int_0^t \|\nabla DB(\tau)\|_{L^2}^2 d\tau + \\ & 2(\mu_1 - \mu_2)\int_0^t \|\partial_y Du_1(\tau)\|_{L^2}^2 d\tau + 2(\nu_2 - \nu_1)\int_0^t \|\partial_x DB_2(\tau)\|_{L^2}^2 d\tau \le C. \end{aligned} \tag{3.12}$$

We apply D^2 to the first four equation of (1.1), and then take the inner product with $(D^2u_1, D^2u_2, D^2B_1, D^2B_2)$ to obtain

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}(\|D^2u_1(t)\|_{L^2}^2 + \|D^2u_2(t)\|_{L^2}^2 + \|D^2B_1(t)\|_{L^2}^2 + \|D^2B_2(t)\|_{L^2}^2) \\ =\ & \mu_1\int_{\mathbb{R}^2} D^2u_1\partial_y^2 D^2u_1\, dxdy + \mu_2\int_{\mathbb{R}^2} D^2u_2\partial_x^2 D^2u_2\, dxdy + \nu_1\int_{\mathbb{R}^2} D^2B_1\partial_y^2 D^2B_1\, dxdy \\ & + \nu_2\int_{\mathbb{R}^2} D^2B_2\partial_x^2 D^2B_2\, dxdy - \int_{\mathbb{R}^2} D^2(u\cdot\nabla u_1)D^2u_1\, dxdy - \int_{\mathbb{R}^2} D^2(u\cdot\nabla u_2)D^2u_2\, dxdy \\ & + \int_{\mathbb{R}^2} D^2(B\cdot\nabla B_1)D^2u_1\, dxdy + \int_{\mathbb{R}^2} D^2(B\cdot\nabla B_2)D^2u_2\, dxdy - \int_{\mathbb{R}^2} D^2(u\cdot\nabla B_1)D^2B_1\, dxdy \\ & - \int_{\mathbb{R}^2} D(u\cdot\nabla B_2)D^2B_2\, dxdy + \int_{\mathbb{R}^2} D^2(B\cdot\nabla u_1)D^2B_1\, dxdy + \int_{\mathbb{R}^2} D^2(B\cdot\nabla u_2)D^2B_2\, dxdy \\ & + \int_{\mathbb{R}^2} D^2\theta D^2u_2\, dxdy \\ =:\ & \sum_{i=1}^{13} J_i. \end{aligned} \tag{3.13}$$

(3.2), (3.3) with $k = 2$ gives, respectively,

$$J_1 + J_2 \le -\frac{\mu_2}{2}\|\nabla D^2u(t)\|_{L^2}^2 - (\mu_1 - \mu_2)\|\partial_y D^2u_1\|_{L^2}^2 \tag{3.14}$$

and

$$J_3 + J_4 \le -\frac{\nu_1}{2}\|\nabla D^2B(t)\|_{L^2}^2 - (\nu_2 - \nu_1)\|\partial_x D^2B_2\|_{L^2}^2. \tag{3.15}$$

By using integration by parts, Hölder inequality, (2.2), (3.5), and Cauchy inequality, we have

$$\begin{aligned} J_5 + J_6 &\le (\|u\|_{L^4}\|D^2u\|_{L^4} + \|Du\|_{L^4}^2)\|\nabla D^2u\|_{L^2} \\ &\le (\|u\|_{L^2}^{\frac{1}{2}}\|Du\|_{L^2}^{\frac{1}{2}}\|D^2u\|_{L^2}^{\frac{1}{2}}\|\nabla D^2u\|_{L^2}^{\frac{1}{2}} + \|Du\|_{L^2}\|D^2u\|_{L^2})\|\nabla D^2u\|_{L^2} \\ &\le C\|Du\|_{L^2}^2\|D^2u\|_{L^2}^2 + \frac{\mu_2}{12}\|\nabla D^2u\|_{L^2}^2. \end{aligned} \tag{3.16}$$

The same procedure to lead to (3.16) yields

$$\begin{aligned} J_7+J_8+J_9+J_{10}+J_{11}+J_{12} &\leq C(\|DB\|_{L^2}^2+\|Du\|_{L^2}^2)(\|D^2u\|_{L^2}^2+\|D^2B\|_{L^2}^2) \\ &\quad +\frac{\mu_2}{12}\|\nabla D^2u\|_{L^2}^2+\frac{\nu_1}{4}\|\nabla D^2B\|_{L^2}^2. \end{aligned} \tag{3.17}$$

Resorting to integration by parts and Cauchy inequality, we obtain

$$J_{13} = -\int_{\mathbb{R}^2} D^2\theta D^2u_2\,dxdy \leq C\|D\theta\|_{L^2}^2+\frac{\mu_2}{12}\|\nabla D^2u\|_{L^2}^2. \tag{3.18}$$

Inserting (3.14)-(3.18) into (3.13) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|D^2u(t)\|_{L^2}^2+\|D^2B(t)\|_{L^2}^2)+\frac{\mu_2}{4}\|\nabla D^2u(t)\|_{L^2}^2+\frac{\nu_1}{4}\|\nabla D^2B(t)\|_{L^2}^2+ \\ &(\mu_1-\mu_2)\|\partial_y D^2u_1(t)\|_{L^2}^2+(\nu_2-\nu_1)\|\partial_x D^2B_2(t)\|_{L^2}^2 \\ \leq\ & C\left((\|Du\|_{L^2}^2+\|DB\|_{L^2}^2)(\|D^2u\|_{L^2}^2+\|D^2B\|_{L^2}^2)+\|D\theta\|_{L^2}^2\right). \end{aligned} \tag{3.19}$$

Next, we need to estimate $\|D\theta\|_{L^2}^2$.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|D\theta(t)\|_{L^2}^2 &= -\int_{\mathbb{R}^2} D(u\cdot\nabla\theta)D\theta\,dxdy+\int_{\mathbb{R}^2} Du_2 D\theta\,dxdy \\ &\leq \|\nabla u\|_{L^\infty}\|D\theta(t)\|_{L^2}^2+\|Du\|_{L^2}^2+\|D\theta(t)\|_{L^2}^2. \end{aligned} \tag{3.20}$$

We apply Gronwall inequality to (3.20) and use (3.12) to obtain

$$\begin{aligned} \|D\theta(t)\|_{L^2}^2 &\leq C\exp\left\{C\int_0^t \|\nabla u\|_{L^\infty}d\tau\right\} \\ &\leq C\exp\left\{C\int_0^t (\|\nabla u\|_{L^2}+\|\nabla Du\|_{L^2}+1)\left(\log(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)\right)^{\frac{1}{2}}d\tau\right\} \\ &\leq C\exp\left\{C\int_0^t \left(\|\nabla u\|_{L^2}^2+\|\nabla Du\|_{L^2}^2+1\right)+\left(\log(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)\right)d\tau\right\} \\ &\leq C\exp\left\{C\int_0^t \log(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)d\tau\right\}. \end{aligned} \tag{3.21}$$

In what follows, we could close the estimate of $\|D\theta(t)\|_{L^2}^2$ by using the local-in-time analysis, which can be fund in [37]. We could refer to [4], [24], and [8] for applying the local-in-time analysis and obtaining the same estimate. For completeness, we give a detail proof.

For $0<\varepsilon\leq t$, Let

$$E = C\exp\left\{C\int_0^{t-\varepsilon} \log(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)d\tau\right\}. \tag{3.22}$$

Then we have from (3.21) and (3.22) that

$$\begin{aligned}
\|D\theta(t)\|_{L^2}^2 &= E\exp\left\{C\int_{t-\varepsilon}^{t}\log(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)d\tau\right\}\\
&= E\exp\left\{C\varepsilon\left(\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\log(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)d\tau\right)\right\}\\
&\leq E\exp\left\{C\varepsilon\log\left(\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)d\tau\right)\right\}\\
&\leq E\left(\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)d\tau\right)^{C\varepsilon}\\
&\leq CE\left(\int_{0}^{t}(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)d\tau\right)^{C\varepsilon},
\end{aligned}$$

where we have used the Jensen's inequality (see (2.3))

$$\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\log(f(\tau))d\tau\leq\log(\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}f(\tau)d\tau)$$

and $(\varepsilon)^{\varepsilon}\leq e^{\frac{1}{e}}\leq C$ for any fixed $\varepsilon>0$. Choosing $\varepsilon=\frac{1}{2C}$, one has

$$\|D\theta(t)\|_{L^2}^2\leq\begin{cases}CE\left(\int_{0}^{t}(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)d\tau\right)^{\frac{1}{2}}, & \frac{1}{2C}\leq t\leq T,\\ C\left(\int_{0}^{t}(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)d\tau\right)^{\frac{1}{2}}, & 0\leq t\leq\frac{1}{2C}.\end{cases}$$

We apply the Gronwall inequality to (3.19) and institute the above inequality into the result inequality to deduce that

$$\begin{aligned}
&\|D^2u(t)\|_{L^2}^2+\|D^2B(t)\|_{L^2}^2+\frac{\mu_2}{2}\int_0^t\|\nabla D^2u(t)\|_{L^2}^2d\tau+\frac{\nu_1}{2}\int^t\|\nabla D^2B(t)\|_{L^2}^2d\tau+\\
&2(\mu_1-\mu_2)\int_0^t\|\partial_yD^2u_1(t)\|_{L^2}^2d\tau+2(\nu_2-\nu_1)\int_0^t\|\partial_xD^2B_2(t)\|_{L^2}^2d\tau\\
\leq\ & C(\|D^2u_0\|_{L^2}^2+\|D^2B_0\|_{L^2}^2+\int_0^t\|D\theta\|_{L^2}^2d\tau)\exp\{\int_0^t(\|Du\|_{L^2}^2+\|DB\|_{L^2}^2)d\tau\}\\
\leq\ & C\exp\left\{\int_0^t(\|Du\|_{L^2}^2+\|DB\|_{L^2}^2)d\tau\right\}(\|D^2u_0\|_{L^2}^2+\|D^2B_0\|_{L^2}^2)+\\
&\exp\left\{\int_0^t(\|Du\|_{L^2}^2+\|DB\|_{L^2}^2)d\tau\right\}\left(\int_0^t CE\left(\int_0^\tau(e+\|\nabla u\|_{L^2}^2+\|\nabla D^2u\|_{L^2}^2)ds\right)^{\frac{1}{2}}d\tau\right)\\
\leq\ & CET\left(\int_0^t(\|\nabla D^2u\|_{L^2}^2+C)d\tau\right)^{\frac{1}{2}}+C\\
\leq\ & CE^2+\frac{\mu_2}{4}\int_0^t\left(\|\nabla D^2u\|_{L^2}^2+C\right)d\tau+C,
\end{aligned}$$

which yields

$$\begin{aligned}&\|D^2u(t)\|_{L^2}^2+\|D^2B(t)\|_{L^2}^2+\frac{\mu_2}{4}\int_0^t\|\nabla D^2u(t)\|_{L^2}^2d\tau+\frac{\nu_1}{2}\int^t\|\nabla D^2B(t)\|_{L^2}^2d\tau+\\&2(\mu_1-\mu_2)\int_0^t\|\partial_yD^2u_1(t)\|_{L^2}^2d\tau+2(\nu_2-\nu_1)\int_0^t\|\partial_xD^2B_2(t)\|_{L^2}^2d\tau\le CE^2+C.\end{aligned}\tag{3.23}$$

In what follows, we prove

$$\begin{aligned}&\|D^2u(t)\|_{L^2}^2+\|D^2B(t)\|_{L^2}^2+\frac{\mu_2}{4}\int_0^t\|\nabla D^2u(t)\|_{L^2}^2d\tau+\frac{\nu_1}{2}\int^t\|\nabla D^2B(t)\|_{L^2}^2d\tau+\\&2(\mu_1-\mu_2)\int_0^t\|\partial_yD^2u_1(t)\|_{L^2}^2d\tau+2(\nu_2-\nu_1)\int_0^t\|\partial_xD^2B_2(t)\|_{L^2}^2d\tau\le C\end{aligned}\tag{3.24}$$

and

$$\|D\theta(t)\|_{L^2}^2\le C\tag{3.25}$$

hold. To this end, it suffices to bound E.

Case I. When $t\le\varepsilon$, it is not difficulty to find that $E\le C$. From (3.23), we deduce that (3.24) holds for any $t\le\varepsilon$.

Case II. When $\varepsilon\le t\le 2\varepsilon$, namely, $0\le t-\varepsilon\le\varepsilon$, we have

$$\begin{aligned}E&=C\exp\left\{C\int_0^{t-\varepsilon}\log(e+\|Du\|_{L^2}^2+\|D^3u\|_{L^2}^2)d\tau\right\}\\&\le C\left(\frac{1}{t-\varepsilon}\int_0^{t-\varepsilon}(e+\|Du\|_{L^2}^2+\|D^3u\|_{L^2}^2)d\tau\right)^{C(t-\varepsilon)}\\&\le C\left(\int_0^{t-\varepsilon}(e+\|Du\|_{L^2}^2+\|D^3u\|_{L^2}^2)d\tau\right)^{C(t-\varepsilon)}\\&\le C,\end{aligned}$$

where we have used the boundedness of $\int_0^{t-\varepsilon}\|D^3u\|_{L^2}^2d\tau$ for $t-\varepsilon\le\varepsilon$ in Case I. Therefore, (3.24) and (3.25) for any $t\le 2\varepsilon$.

Case III. Using the same arguments in Case I and Case II yields the desired estimates (3.24) and (3.25) for $t\le 3\varepsilon$. Repeating the preceding arguments, we could arrive at $E\le C$ for any $t\le T$. Therefore, (3.24) and (3.25) follows for any $t\le T$.

Thanks to Sobolev embedding theorem and (3.24), we have

$$\int_0^t\|Du(t)\|_{L^\infty}d\tau\le\int_0^t(\|Du(t)\|_{L^2}+\|D^3u(t)\|_{L^2})d\tau\le C\tag{3.26}$$

for any $t \le T$. Applying D^m and D^{m-1} to (1.1), respectively, and then taking the inner product with $(D^m u_1, D^m u_2, D^m B_1, D^m B_2, D^{m-1}\theta)$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|D^m u_1(t)\|_{L^2}^2 + \|D^m u_2(t)\|_{L^2}^2 + \|D^m B_1(t)\|_{L^2}^2 + \|D^m B_2(t)\|_{L^2}^2 + \|D^{m-1}\theta(t)\|_{L^2}^2) \\
&= \mu_1 \int_{\mathbb{R}^2} D^m u_1 \partial_y^2 D^m u_1 \, dxdy + \mu_2 \int_{\mathbb{R}^2} D^m u_2 \partial_x^2 D^m u_2 \, dxdy + \nu_1 \int_{\mathbb{R}^2} DB_1 \partial_y^2 DB_1 \, dxdy \\
&+\nu_2 \int_{\mathbb{R}^2} DB_2 \partial_x^2 DB_2 \, dxdy - \int_{\mathbb{R}^2} D^m(u\cdot\nabla u_1) D^m u_1 \, dxdy - \int_{\mathbb{R}^2} D^m(u\cdot\nabla u_2) D^m u_2 \, dxdy \\
&+ \int_{\mathbb{R}^2} D^m(B\cdot\nabla B_1) D^m u_1 \, dxdy + \int_{\mathbb{R}^2} D^m(B\cdot\nabla B_2) D^m u_2 \, dxdy - \int_{\mathbb{R}^2} D^m(u\cdot\nabla B_1) D^m B_1 \, dxdy \\
&- \int_{\mathbb{R}^2} D^m(u\cdot\nabla B_2) D^m B_2 \, dxdy + \int_{\mathbb{R}^2} D^m(B\cdot\nabla u_1) D^m B_1 \, dxdy + \int_{\mathbb{R}^2} D^m(B\cdot\nabla u_2) D^m B_2 \, dxdy \\
&+ \int_{\mathbb{R}^2} D^m\theta D^m u_2 dxdy - \int_{\mathbb{R}^2} D^{m-1}(u\cdot\nabla\theta) D^{m-1}\theta \, dxdy \\
&+ \int_{\mathbb{R}^2} D^{m-1}\theta D^{m-1} u_2 dxdy =: \sum_{i=1}^{15} K_i.
\end{aligned} \tag{3.27}$$

From (3.2) with $k = m$, we have

$$K_1 + K_2 \le -\frac{\mu_2}{2}\|\nabla D^m u(t)\|_{L^2}^2 - (\mu_1 - \mu_2)\|\partial_y D^m u_1(t)\|_{L^2}^2. \tag{3.28}$$

(3.3) with $k = m$ gives

$$K_3 + K_4 \le -\frac{\nu_1}{2}\|\nabla D^m B(t)\|_{L^2}^2 - (\nu_2 - \nu_1)\|\partial_x D^m B_2(t)\|_{L^2}^2. \tag{3.29}$$

Integration by parts, Hölder inequality, Gagaliardo-Nirenberg inequality, (2.2), and Cauchy inequality entail that

$$\begin{aligned}
K_5 + K_6 &\le \|D^m(u\otimes u)\|_{L^2}\|\nabla D^m u\|_{L^2} \\
&\le \|\sum C_m^k D^k u \otimes D^{m-k} u\|_{L^2}\|\nabla D^m u\|_{L^2} \\
&\le \sum C_m^k \|D^k u\|_{L^4}\|D^{m-k} u\|_{L^4}\|\nabla D^m u\|_{L^2} \\
&\le C\|u\|_{L^4}\|D^m u\|_{L^4}\|\nabla D^m u\|_{L^2} \\
&\le C\|u\|_{L^2}^2\|Du\|_{L^2}^2\|D^m u\|_{L^2}^2 + \frac{\mu_2}{20}\|\nabla D^m u\|_{L^2}^2
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
&K_7 + K_8 + K_{11} + K_{12} \\
\le\ & \|D^m(B\otimes B)\|_{L^2}\|D^{m+1}u\|_{L^2} + \|D^m(B\otimes u)\|_{L^2}\|\nabla D^m B\|_{L^2} \\
\le\ & \|\sum C_m^k D^k B \otimes D^{m-k} B\|_{L^2}\|\nabla D^m u\|_{L^2} + \|\sum C_m^k D^k B\otimes D^{m-k} u\|_{L^2}\|D^{m+1} B\|_{L^2} \\
\le\ & \sum C_m^k \|D^k B\|_{L^4}\|D^{m-k} B\|_{L^4}\|\nabla D^m u\|_{L^2} + \sum C_m^k \|D^k B\|_{L^4}\|D^{m-k} u\|_{L^4}\|\nabla D^m B\|_{L^2} \\
\le\ & \|B\|_{L^4}\|D^m B\|_{L^4}\|D^{m+1} u\|_{L^2} + \sum C_m^k \|B\|_{L^4}^{1-\frac{k}{m}}\|D^m B\|_{L^4}^{\frac{k}{m}}\|u\|_{L^4}^{\frac{k}{m}}\|D^m u\|_{L^4}^{1-\frac{k}{m}}\|\nabla D^m B\|_{L^2} \\
\le\ & C(\|u\|_{L^2}^2 + \|B\|_{L^2}^2)(\|Du\|_{L^2}^2 + \|DB\|_{L^2}^2)(\|D^m u\|_{L^2}^2 + \|D^m B\|_{L^2}^2) + \\
& \frac{\mu_2}{20}\|\nabla D^m u\|_{L^2}^2 + \frac{\nu_1}{8}\|\nabla D^m B\|_{L^2}^2.
\end{aligned} \tag{3.31}$$

Similarly, it holds that

$$\begin{aligned} K_9+K_{10} &= \int_{\mathbb{R}^2} D^m(u\cdot\nabla B)D^m B\,dxdy \\ &\le C(\|u\|_{L^2}^2+\|B\|_{L^2}^2)(\|Du\|_{L^2}^2+\|DB\|_{L^2}^2)(\|D^m u\|_{L^2}^2+\|D^m B\|_{L^2}^2)+ \\ &\quad \frac{\mu_2}{20}\|\nabla D^m u\|_{L^2}^2+\frac{\nu_1}{8}\|\nabla D^m B\|_{L^2}^2. \end{aligned} \tag{3.32}$$

We derive that by Cauchy inequality

$$\begin{aligned} K_{13} &= -\int_{\mathbb{R}^2} D^{m-1}\theta D^{m+1}u_2\,dxdy \\ &\le C\|D^{m-1}\theta\|_{L^2}^2+\frac{\mu_2}{20}\|\nabla D^m u(t)\|_{L^2}^2. \end{aligned} \tag{3.33}$$

Thanks to $\nabla\cdot u=0$, integration by parts, Hölder inequality, (2.1) with $p=2$, $p_1=\infty$, $q_1=2$, $p_2=q_2=4$, Gagaliardo-Nirenberg inequality, and Cauchy inequality, we arrive at

$$\begin{aligned} K_{14} &= -\int_{\mathbb{R}^2}[D^{m-1}(u\cdot\nabla\theta)-u\cdot\nabla D^{m-1}\theta]D^{m-1}\theta\,dxdy \\ &\le \|D^{m-1}(u\cdot\nabla\theta)-u\cdot\nabla D^{m-1}\theta\|_{L^2}\|D^{m-1}\theta\|_{L^2} \\ &\quad C(\|\nabla u\|_{L^\infty}\|D^{m-1}\theta\|_{L^2}+\|D\theta\|_{L^4}\|D^{m-1}u\|_{L^4})\|D^{m-1}\theta\|_{L^2} \\ &\le C\|\nabla u\|_{L^\infty}\|D^{m-1}\theta\|_{L^2}^2+C\|D\theta\|_{L^2}^{\frac{2m-5}{2(m-2)}}\|D^{m-1}\theta\|_{L^2}^{\frac{2m-3}{2(m-2)}}\|D^{m-1}u\|_{L^2}^{\frac{3}{4}}\|D^{m+1}u\|_{L^2}^{\frac{1}{4}} \\ &\le C\|\nabla u\|_{L^\infty}\|D^{m-1}\theta\|_{L^2}^2+\frac{\mu_2}{20}\|\nabla D^m u\|_{L^2}^2+C\|D^{m-1}\theta\|_{L^2}^2+ \\ &\quad C\|D\theta\|_{L^2}^{\frac{8(2m-5)}{3m-8}}\|D^{m-1}u\|_{L^2}^{\frac{6m-12}{3m-8}} \\ &\le C\|\nabla u\|_{L^\infty}\|D^{m-1}\theta\|_{L^2}^2+\frac{\mu_2}{20}\|\nabla D^m u\|_{L^2}^2+C\|D^{m-1}\theta\|_{L^2}^2+ \\ &\quad C\|D^m u\|_{L^2}^2+C\|D\theta\|_{L^2}^{8(2m-5)}\|D^2u\|_{L^2}^6 \\ &\le C\|\nabla u\|_{L^\infty}\|D^{m-1}\theta\|_{L^2}^2+\frac{\mu_2}{20}\|\nabla D^m u\|_{L^2}^2+C\|D^{m-1}\theta\|_{L^2}^2+C\|D^m u\|_{L^2}^2+C. \end{aligned} \tag{3.34}$$

Cauchy inequality implies that

$$\begin{aligned} K_{15} &= C\|D^{m-1}\theta\|_{L^2}^2+C\|D^{m-1}u(t)\|_{L^2}^2 \\ &\le C\|u_0\|_{L^2}^2+C\|D^{m-1}\theta\|_{L^2}^2+C\|D^m u(t)\|_{L^2}^2. \end{aligned} \tag{3.35}$$

Combining (3.28)-(3.35) and (3.27) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|D^m u(t)\|_{L^2}^2+\|D^m B(t)\|_{L^2}^2+\|D^{m-1}\theta(t)\|_{L^2}^2)+\frac{\mu_2}{4}\|\nabla D^m u(t)\|_{L^2}^2+ \\ &\frac{\nu_1}{4}\|\nabla D^m B(t)\|_{L^2}^2+(\mu_1-\mu_2)\|\partial_y D^m u_1(t)\|_{L^2}^2+(\nu_2-\mu_1)\|\partial_x D^m B_2(t)\|_{L^2}^2 \\ \le\ & C\Big(1+(\|Du\|_{L^2}^2+\|DB\|_{L^2}^2)(\|D^m u\|_{L^2}^2+\|D^m B\|_{L^2}^2)+ \\ &\|D^{m-1}\theta\|_{L^2}^2+\|D^m u\|_{L^2}^2+\|\nabla u\|_{L^\infty}\|D^{m-1}\theta\|_{L^2}^2\Big). \end{aligned}$$

which together with Gronwall equality, (3.5), and (3.26) entails that

$$\begin{aligned}&\|D^m u(t)\|_{L^2}^2+\|D^m B(t)\|_{L^2}^2+\|D^{m-1}\theta(t)\|_{L^2}^2+\frac{\mu_2}{2}\int_0^t\|\nabla D^m u(\tau)\|_{L^2}^2 d\tau+\\&\frac{\nu_1}{2}\int_0^t\|\nabla D^m B(\tau)\|_{L^2}^2 d\tau\le C.\end{aligned}\tag{3.36}$$

It follows from (3.12) and (3.36) that

$$\|u(t)\|_{H^m}^2+\|B(t)\|_{H^m}^2+\|\theta(t)\|_{H^{m-1}}^2+\int_0^t(\|\nabla u(\tau)\|_{H^m}^2+\|\nabla B(\tau)\|_{H^m}^2)d\tau\le C$$

for any $T>0$ and $0\le t\le T$.

4. Main Results

In this section, our main aim is to complete the proof of Theorem 1.1 by combining global a priori H^m-bound, local existence and uniqueness results, and standard compactness argument.

Proof. Let $N>0$ be an integer and

$$\widehat{J_N f}(\xi)=\chi_{B(0,N)}(\xi)\widehat{f}(\xi),$$

where

$$\chi_{B(0,N)}(\xi)=\begin{cases}1, & \xi\in B(0,N),\\ 0, & \xi\ni B(0,N),\end{cases}$$

and $B(0,N)=\{\xi\in\mathbb{R}^2:|\xi|\le N\}$ and $\widehat{f}(\xi)$ denotes the Fourier transform of f. Let

$$L_N^2=\{f\in L^2(\mathbb{R}^2):supp\widehat{f}\subset B(0,N)\}$$

Let $\mathbb{P}$ denote the Leray projection onto divergence-free vector fields. We investigate the following approximate equations in the space L_N^2

$$\begin{cases}\partial_t u_1^N+\mathbb{P}J_N(\mathbb{P}J_N u^N\cdot\nabla\mathbb{P}J_N u_1^N)-\mathbb{P}J_N(J_N B^N\cdot\nabla J_N B_1^N)-\mu_1\partial_y^2\mathbb{P}J_N^2 u_1^N=0,\\ \partial_t u_2^N+\mathbb{P}J_N(\mathbb{P}J_N u^N\cdot\nabla\mathbb{P}J_N u_2^N)-\mathbb{P}J_N(J_N B^N\cdot\nabla J_N B_2^N)-\mu_2\partial_x^2\mathbb{P}J_N^2 u_2^N=\mathbb{P}J_N\theta^N,\\ \partial_t B_1^N+J_N(\mathbb{P}J_N u^N\cdot\nabla J_N B_1^N)-J_N(J_N B^N\cdot\nabla\mathbb{P}J_N u_1^N)-\nu_1\partial_y^2 J_N^2 B_1^N=0,\\ \partial_t B_2^N+J_N(\mathbb{P}J_N u^N\cdot\nabla J_N B_2^N)-J_N(J_N B^N\cdot\nabla\mathbb{P}J_N u_2^N)-\nu_2\partial_x^2 J_N^2 B_2^N=0,\\ \partial_t\theta^N+\mathbb{P}J_N(\mathbb{P}J_N u^N\cdot\nabla J_N\theta^N)=\mathbb{P}J_N u^N e_2\end{cases}\tag{4.1}$$

with the initial value

$$t=0:\ \ u^N=J_N u_0(x,y),\theta^N=J_N\theta_0(x,y),B^N=J_N B_0(x,y).\tag{4.2}$$

The local existence and uniqueness results to problem (4.1)-(4.2) can be obtained by the method similar to [38, Chapter 3]. Then The energy inequality

$$\begin{aligned}&\|u^N(t)\|_{H^m}^2+\|B^N(t)\|_{H^m}^2+\|\theta^N(t)\|_{H^{m-1}}^2+\\&\int_0^t(\|\nabla u^N(\tau)\|_{H^m}^2+\|\nabla B^N(\tau)\|_{H^m}^2)d\tau\le C\end{aligned}$$

follows from a priori estimates in Section 3 for any $T > 0$ and $0 \leq t \leq T$. Then standard compactness argument allows us to obtain the global existence and uniqueness of smooth solutions (u, B, θ) to problem (1.1)-(1.2). Thus we complete the proof of Theorem 1.1. □

Acknowledgements

This work was supported in part by the Basic Research Project of Key Scientific Research Project Plan of Universities in Henan Province (Grant No. 20ZX002).

References

[1] G. Mulone, S. Rionero, Necessary and sufficient conditions for nonlinear stability in the magnetic Bénard problem, Arch. Ration. Mech. Anal. 166 (2003), 197-218.
[2] M. Nakamura, On the magnetic Bénard problem, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 38 (1991), 359-393.
[3] Y. Zhou, J. Fan, G. Nakamura, Global Cauchy problem for a 2D magnetic Bénard problem with zero thermal conductivity, Appl. Math. Lett. 26 (2013), 627-630.
[4] J. Cheng, L. Du, On two-dimensional magnetic Bénard problem with mixed partial viscosity, J. Math Fluid Mech. 17 (2015), 769-797.
[5] Z. Ye, Global regularity of the 2D magnetic Bénard system with partial dissipation, Adv. Differential Equations 23 (2018), 193-238.
[6] Z. Ye, Global regularity of the 2D anisotropic magnetic Bénard system with vertical dissipation, Nonlinear Anal. Real World Appl. 43 (2018), 407-427.
[7] L. Ma, Global existence of smooth solutions for three-dimensional magnetic Bénard system with mixed partial dissipation, magnetic diffusion and thermal diffusivity, J. Math. Anal. Appl. 461 (2018), 1639-1652.
[8] L. Ma, Global regularity results for the $2\frac{1}{2}$ D magnetic Bénard system with mixed partial viscosity, Appl. Anal. 98 (2019), 1143-1164.
[9] C. Cao, J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, Adv. Math. 226 (2011), 1803-1822.
[10] C. Cao, D. Regmi, J. Wu, The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion, J. Differential Equations 254 (2013), 2661-2681.
[11] L. Du, D. Zhou, Global well-posedness of two-dimensional magnetohydrodynamic flows with partial dissipation and magnetic diffusion, SIAM J. Math. Anal. 47 (2015), 1562-1589.
[12] X. Ren, J. Wu, Z. Xiang, Z. Zhang, Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion, J. Funct. Anal. 267 (2014), 503-541.
[13] Z. Zhang, B. Dong, Y. Jia, Remarks on the global regularity and time decay of the 2D MHD equations with partial dissipation, Math. Meth. Appl Sci. 42 (2019), 3388-3399.
[14] Y. Cai, Z. Lei, Global well-posedness of the incompressible magnetohydrodynamics, Arch. Ration Mech. Anal. 228 (2018), 969-993.
[15] C. Cao, J. Wu, B. Yuan, The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion, SIAM J. Math. Anal. 46 (2014), 588-602.
[16] L. He, L. Xu, P. Yu, On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves, Ann. PDE 4 (2018), 5.
[17] Q. Jiu, D. Niu, J. Wu, X. Xu, Y. Yu, The 2D magnetohydrodynamic equations with magnetic diffusion, Nonlinearity 28 (2015), 3935-3955.
[18] Z. Lei, On axially symmetric incompressible Magnetohydrodynamics in three dimensions, J. Differential Equations 259 (2015), 3202-3215.
[19] F. Lin, L. Xu, P. Zhang, Global small solutions to 2-D incompressible MHD system, J. Differential Equations 259 (2015), 5540-5485.
[20] B. Yuan, J. Zhao, Global regularity of 2D almost resistive MHD equations, Nonlinear Anal. Real World Appl. 41 (2018), 53-65.

[21] J. Wu, Y. Wu, Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion, Adv Math. 310 (2017), 759-888.
[22] G. Wu, L. Xue, Global well-posedness for the 2D inviscid Bénard system with fractional diffusivity and Yudovich's type data, J. Differential Equations 253 (2012), 100-125.
[23] Z. Ye, Regularity criterion of the 2D Bénard equations with critical and supercritical dissipation, Nonlinear Anal. 156 (2017), 111-143.
[24] T.Y. Hou, C. Li, Global well-posedness of the viscous Boussinesq equations, Discr. Cont. Dyn. Sys. 12 (2005), 1-12.
[25] H. Abidi, T. Hmidi, On the global well-posedness for Boussinesq system, J. Differential Equations 233 (2007), 199-220
[26] D. Adhikari, C. Cao, J. Wu, Global regularity results for the 2D Boussinesq equations with vertical dissipation, J. Differential Equations 251 (2011), 1637-1655.
[27] D. Chae, Global regularity for the 2-D Boussinesq equations with partial viscous terms, Adv. Math. 203 (2006), 497-513.
[28] C. Cao, J. Wu, Global regularity for the two-dimensional anisotropic Boussinesq equations with vertical dissipation, Arch. Ration. Mech. Anal. 208 (2013), 985-1004.
[29] T. Hmidi, S. Keraani, On the global well-posedness of the two-diemnsional Boussinesq system with a zero diffusivity, Adv. Differential Equ. 12 (2007), 461-480.
[30] J. Li, E.S. Titi, Global well-posedness of the 2D Boussinesq equations with vertical dissipation, Arch. Ration. Mech. Anal. 220 (2015), 983-1001.
[31] C. Miao, X. Zheng, On the global well-posedness for the Boussinesq system with horizontal dissipation, Comm. Math. Phys. 321 (2013), 33-67.
[32] D. Adhikari, C. Cao, H. Shang, J. Wu, X. Xu, Z. Ye, Global regularity results for the 2D Boussinesq equations with partial dissipation, J. Differential Equations 260 (2016), 1893-1917.
[33] Z. Ye, Global regularity results for the 2D Boussinesq equations and micropolar equations with partial dissipation, J. Differential Equations 268 (2020), 910-944.
[34] T. Kato, G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Commun Pure Appl Math. 41 (1988), 891-907.
[35] H. Brezis, S. Wainger, A note on limiting cases of Sobolev embedding and convolution inequalities, Comm. Partial Differential Equations 5 (1987), 773-789.
[36] H. Engler, An alternative proof of the Brezis-Wainger inequality, Comm. Partial Differential Equations 14 (1989), 541-544.
[37] C. He, New sufficient conditions for regularity of solutions to the Navier-Stokes equations, Adv. Math. Sci. Appl. 12 (2002), 535-548.
[38] A. Majda, A. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, 2002.