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SCALARIZATION AND SEMICONTINUITY OF APPROXIMATE SOLUTIONS TO SET OPTIMIZATION PROBLEMS

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Abstract. This paper deals with the approximate solutions to set optimization problems in the sense of strict upper set less order relation. First, the concept of scalar approximate solutions is introduced, and its relationship to the approximate weak minimal solutions is proposed. Second, by using the oriented distance function, a scalarization theorem is established for approximate weak minimal solutions. Finally, the upper and lower semicontinuity of approximate weak minimal solution mappings are proved for the parametric set optimization problems.

Keywords. Approximate solution; Set optimization; Scalarization; Semicontinuity; Set order relation.

1. INTRODUCTION

Set optimization is an extension of vector optimization and scalar optimization, and it is widely used in optimal control, viability theory, and applied mathematics. There are two criteria of solutions to set optimization problems: vector criterion and set criterion. The vector criterion consists of looking for efficient points of the image set of objective functions. In order to cope with set optimization problems in a natural and interesting way, set criterion was introduced and investigated by many scholars; see, e.g., [1, 2, 3, 4, 6, 5, 7, 8]. Up to now, the existence, well-posedness, optimality, duality, and stability of set optimization problems have been examined under various set order relations; see, e.g., [8, 9, 10, 11, 12, 13, 14, 15, 16]. Particularly, the upper set less order relation is the most of important set criteria, and the minimal solutions defined by it for set optimization problems were presented in [1, 16, 17, 18]. In many practical problems, the majority of the solutions obtained from numerical algorithms are approximate solutions. Therefore, it is of great theoretical value and practical significance to study approximate solutions for set optimization problems. For the lower set less order relation, the scalarization and convergence of approximate minimal solutions were discussed in [19] by Tammer's nonlinear function, and the stability and extended well-posedness were derived by the Painlevé-Kuratowski convergence in [11]. In this paper, we use the different methods to deal with the scalarization and continuity of approximate weak minimal solution with respect to upper set less order relations.

Linear and nonlinear scalarization methods were commonly used in literatures. For example, the scalarization theorems of minimal solutions were characterized in [16, 20, 21] via linear

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method, and the nonlinear scalarization theorems were showed in [7, 17, 19] by Tammer's function. It is worth to mention that the oriented distance function is another significant nonlinear function, and it was proven that it is a powerful scalarization tool for set optimization problems; see, e.g., [22, 23]. In current paper, we propose a concept of scalar approximate, which is given by linear functional, and prove that its connection with approximate solutions for set optimization problems. Further, an equivalent scalarization theorem for approximate weak minimal solutions is established by the oriented distance function.

It is well-known that the continuity of solution mappings is one of the core contents to parametric set optimization problems. For instance, the continuity and semicontinuity of minimal solutions to the parametric set optimization problems were gained in [10, 12, 15] under the assumptions of different cone quasi-convexity. In [13], the continuity of optimal solutions to the parametric optimization problems were presented with the help of the generalized monotonicity. To the best of our knowledge, there are few papers on the continuity of approximate solution mappings in set optimization. We will establish the upper and lower semicontinuity of approximate solution mapping to the parametric set optimization problems without any convexity and monotonicity assumption.

The article is arranged as follows. In Section 2, we provide some notations, definitions, and lemmas, which are used in later sections. Section 3 gives the scalarization characterizations of approximate weak minimal solutions to set optimization problems. Section 4 focuses on the continuity theorems of approximate weak minimal solution to the parametric set optimization problems. Section 5, which is the last section, ends this paper.

2. Preliminaries

Let X, Y, and Z be real Banach spaces, and let the family of all nonempty subsets in Y be denoted by $P(Y)$. Let $\mathbb{R}^n$ be the n dimensional Euclid space, and

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_i \geq 0, \ i = 1, \ldots, n\}, \ \ \mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : \ x_i > 0, \ i = 1, \ldots, n\}.$$

Let Y^* be the dual space of Y, and let $C \subset Y$ be a pointed closed and convex cone with $intC \neq \emptyset$ ($intC$ denotes the interior of C). The dual cone of C is defined as (see [24])

$$C^* = \{\xi \in Y^* : \ \xi(y) \geq 0, \quad \forall y \in C\}.$$

Let $e \in intC$ be a fixed point, and $B^*_e = \{\xi \in C^* : \ \xi(e) = 1\}$.

Lemma 2.1. *(see [24]) If $\xi \in C^* \backslash \{0\}$, $c \in intC$, then $\xi(c) > 0$.*

Let $A, \ B \in P(Y)$. The strict upper set less order relation on $P(Y)$ with respect to C is induced as follows (see [3])

$$A \prec^u_C B \Longleftrightarrow A \subset B - intC.$$

Let M be a nonempty subset of Y. It is said that M is C-compact if any cover of M of the form $\{U_\alpha + C : U_\alpha \ \textit{are open}\}$ admits a finite subcover. A point $m \in M$ is called a weak maximal point of M with respect to C if

$$(M - m) \cap intC = \emptyset.$$

Let $WMax(M)$ denote the set of all weak maximal points.

Lemma 2.2. *(see [21]) If M is nonempty and $(-C)$-compact, then $WMax(M) \neq \emptyset$.*

Next, we introduce the definition of the oriented distance function, which plays a major role in our paper.

Definition 2.1. (see [25]) Let $M \subset Y$ be a nonempty subset. The oriented distance function $\triangle_M : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined as $\triangle_M(y) = d_M(y) - d_{Y\setminus M}(y)$, $\forall y \in Y$, where $d_M(y) = \inf_{m\in M} \|y-m\|$ is the distance function from $y \in Y$ to the set M.

It was pointed out in [23] that if M is a convex cone in Y with $intM \neq \emptyset$, then

$$\triangle_M(y) = \sup_{\xi \in M^*: \|\xi\|=1} \langle -\xi, y\rangle, \quad \forall y \in Y.$$

Lemma 2.3. *(see [26]) Let $M \subset Y$ be a nonempty subset with $M \neq Y$. Then*

$$\triangle_M(y) < 0 \Longleftrightarrow y \in intM, \quad \forall y \in Y.$$

Let $K \subset X$ be a nonempty convex set, and let $F : K \to 2^Y$ be a set-valued mapping. We consider the following set optimization problem

$$\text{(SOP)} \qquad \min F(x), \quad \text{s.t. } x \in K.$$

Now, we present the notion of approximate solutions in the sense of strict upper set less order relations to problem (SOP).

Definition 2.2. (see [11]) Let $\varepsilon \geq 0$ and $e \in intC$. $\bar{x} \in K$ is called a εe-u-weak minimal solution to problem (SOP) iff

$$x \in K,\ F(x) + \varepsilon e \prec^u_C F(\bar{x}) \textit{ implies } F(\bar{x}) + \varepsilon e \prec^u_C F(x).$$

Let S^u_ε denote the set of all εe-u-weak minimal solutions to problem (SOP). From now on, we always assume that $\varepsilon \geq 0$, $e \in intC$, and $S^u_\varepsilon \neq \emptyset$.

Here, we give an example, which is to illustrate the existence of εe-u-weak minimal solutions.

Example 2.1. In problem (SOP), let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $intC = \mathbb{R}^2_{++}$, $K = [1,2]$, and mapping $F : K \to 2^{\mathbb{R}^2}$ be given by

$$F(x) = \begin{cases} \{(1, x+\frac{1}{2})\}, & x \in (1,2], \\ [0,1]\times[0,1], & x = 1. \end{cases}$$

Let $\varepsilon = \frac{1}{3}$, $e = (1,1)$, and $\bar{x} = 1$. For all $x \in K$, $F(\bar{x}) + \varepsilon e \prec^u_C F(x)$. Thus $\bar{x} = 1$ is a εe-u-weak minimal solution of this problem.

The following is an equivalent description of εe-u-weak minimal solutions to problem (SOP).

Lemma 2.4. *Let $\bar{x} \in K$. If $F(\bar{x}) + \varepsilon e$ is $(-C)$-compact, then the following assertions are equivalent:*

(i) $\bar{x}$ is a εe-u-weak minimal solution to problem (SOP);

(ii) There does not exist $x \in K$ such that $F(x) + \varepsilon e \prec^u_C F(\bar{x})$.

Proof. Suppose that (i) is true, and (ii) is not valid. Then there exists $\hat{x} \in K$ such that

$$F(\hat{x}) + \varepsilon e \subset F(\bar{x}) - intC. \tag{2.1}$$

Since $\bar{x}$ is εe-u-weak minimal solution of problem (SOP), we obtain $F(\bar{x}) + \varepsilon e \prec^u_C F(\hat{x})$. Thus

$$F(\bar{x}) + \varepsilon e \subset F(\hat{x}) - intC. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned} F(\bar{x})+\varepsilon e &\subset F(\hat{x})-intC \\ &\subset F(\hat{x})+\varepsilon e-intC \\ &\subset F(\bar{x})-intC-intC \\ &\subset F(\bar{x})-intC. \end{aligned}$$

Hence

$$F(\bar{x})+\varepsilon e \subset F(\bar{x})-intC. \tag{2.3}$$

Since $F(\bar{x})+\varepsilon e$ is $(-C)$-compact, it follows from Lemma 2.2 that $WMax(F(\bar{x})+\varepsilon e)\neq\emptyset$. Letting $y_0\in WMax(F(\bar{x})+\varepsilon e)$, we have

$$(F(\bar{x})+\varepsilon e-y_0)\cap intC=\emptyset. \tag{2.4}$$

It yields from (2.3) that there exist $\bar{y}\in F(\bar{x})$ and $c\in intC$ such that $y_0=\bar{y}-c$. Hence $\bar{y}-y_0=c\in intC$. In view of $\varepsilon\geq 0$ and $e\in intC$, one has $\bar{y}-y_0+\varepsilon e\in intC+intC=intC$. This is a contradiction to (2.4).

Next, we assume that (ii) is true, but $\bar{x}$ is not a εe-u-weak minimal solution to problem (SOP). Then there exists $\hat{x}\in K$ such that $F(\hat{x})+\varepsilon e\prec_C^u F(\bar{x})$. This leads to a contradiction. □

Here, we give an example to illustrate the result of Lemma 2.4.

Example 2.2. In problem (SOP), let $X=Y=\mathbb{R}$, $C=\mathbb{R}_+$, $intC=\mathbb{R}_{++}$, and $K=[0,1]$. The mapping $F:K\to 2^{\mathbb{R}}$ is defined by

$$F(x)=\begin{cases}[-2,-1], & x=0,\\ [0,x], & x\in(0,1].\end{cases}$$

Let $\varepsilon=\frac{1}{2}$, $e=1$, and $\bar{x}=0$. It can been obtained that $F(0)+\varepsilon e$ is compact, and, for any $x\in K$, $F(x)+\varepsilon e\not\subset F(0)-intC$. Therefore, there is no $x\in K$ such that $F(x)+\varepsilon e\prec_C^u F(0)$. In addition, for all $x\in K$, $F(\bar{x})+\varepsilon e\prec_C^u F(x)$, which implies that $\bar{x}=0$ is a εe-u-weak minimal solution of this problem.

3. Scalarization

We first introduce the concept of scalar approximate solutions related to problem (SOP). Then, two characterizations of approximate weak minimal solutions are derived.

In problem (SOP), let $\bar{x}\in K$ and $f\in C^*\backslash\{0\}$. $\bar{x}$ is said to be a ε-f-minimal solution of problem (SOP) if

$$\sup_{\bar{y}\in F(\bar{x})} f(\bar{y})\leq \inf_{y\in F(x)} f(y)+\varepsilon, \quad \forall x\in K.$$

Let $\Upsilon(f)$ denote the set of all ε-f-minimal solutions of problem (SOP), i.e.,

$$\Upsilon(f)=\{\bar{x}\in K:\sup_{\bar{y}\in F(\bar{x})} f(\bar{y})\leq \inf_{y\in F(x)} f(y)+\varepsilon, \quad \forall x\in K\}.$$

Theorem 3.1. $\bigcup_{f\in B_e^*}\Upsilon(f)\subset S_\varepsilon^u$.

Proof. For any $\bar{x} \in \Upsilon(f)$ and $f \in B_e^*$, we have

$$\sup_{\bar{y}\in F(\bar{x})} f(\bar{y}) \leq \inf_{y\in F(x)} f(y) + \varepsilon, \quad \forall x \in K. \tag{3.1}$$

Suppose that $\bar{x} \notin S_\varepsilon^u$, by Lemma 2.4, then there exists $\hat{x} \in K$ such that $F(\hat{x}) + \varepsilon e \subset F(\bar{x}) - intC$. This implies that there exist $\hat{y} \in F(\hat{x})$, $\bar{y} \in F(\bar{x})$ and $c \in intC$ such that

$$\hat{y} + \varepsilon e = \bar{y} - c. \tag{3.2}$$

Noticing that $c \in intC$ and $f \in B_e^* \subset C^* \backslash \{0\}$, we obtain from Lemma 2.1 that $f(c) > 0$. From $f(e) = 1$, we conclude $f(\bar{y}) > f(\hat{y}) + \varepsilon$. Since

$$\sup_{\bar{y}\in F(\bar{x})} f(\bar{y}) \geq f(\bar{y}) > f(\hat{y}) + \varepsilon \geq \inf_{\hat{y}\in F(\hat{x})} f(\hat{y}) + \varepsilon,$$

it yields that $\sup_{\bar{y}\in F(\bar{x})} f(\bar{y}) > \inf_{\hat{y}\in F(\hat{x})} f(\hat{y}) + \varepsilon$, which contradicts to (3.1). This proves that $\bar{x} \in S_\varepsilon^u$ and $\bigcup_{f\in B_e^*} \Upsilon(f) \subset S_\varepsilon^u$. □

Remark 3.1. Theorem 3.1 provides a sufficient optimality condition for εe-u-weak minimal solutions to problem (SOP). The following example illustrates this fact.

Example 3.1. In problem (SOP), let $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $intC = \mathbb{R}_{++}$, and $K = [0,1]$. The mapping $F : \mathbb{R} \to 2^{\mathbb{R}}$ is defined by $F(x) = [x(x-1), 0]$, $\forall x \in K$. Let $\varepsilon = \frac{1}{2}$ and $e = 1$. We obtain that $C^* = \mathbb{R}_+$ and $f \in B_e^* = \{1\}$. Thus, for all $x \in [0,1]$, we obtain $\sup_{y\in F(x)} f(y) = \sup_{y\in F(x)} y = 0$ and $\inf_{y\in F(x)} f(y) + \varepsilon = \inf_{y\in F(x)} y + \varepsilon = \frac{1}{4}$. Therefore

$$\sup_{y\in F(x)} f(y) \leq \inf_{y\in F(x)} f(y) + \varepsilon, \quad \forall x \in [0,1],$$

which means that $[0,1] \subset \Upsilon(f)$. Moreover, it is obvious that $\Upsilon(f) \subset [0,1]$. Thus $\Upsilon(f) = [0,1]$. In addition, for any $x \in [0,1]$, we have $F(x) + \varepsilon e = [x(x-1) + \frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}_{++}$. By Lemma 2.4, $S_\varepsilon^u \subset [0,1]$. Moreover, since $F(x) + \varepsilon e \not\subset F(x) - intC, \quad \forall x \in [0,1]$, we obtain that $[0,1] \subset S_\varepsilon^u$. Hence, $\bigcup_{f\in B_e^*} \Upsilon(f) \subset S_\varepsilon^u$.

The following Theorem 3.2 is a scalarization theorem of approximate weak minimal solutions to problem (SOP), which is proved by using the oriented distance function. In particular, we refer to the proof method of [22, Proposition 3.3].

Theorem 3.2. *Let $\bar{x}$, $x \in K$. Assume that $\inf_{\bar{y}\in F(\bar{x})} \triangle_C(\bar{y} - y - \varepsilon e)$ is attained for any $y \in F(x)$. $\bar{x}$ is a εe-u-weak minimal solution to problem (SOP) if and only if*

$$\sup_{y\in F(x)} \inf_{\bar{y}\in F(\bar{x})} \triangle_C(\bar{y} - y - \varepsilon e) \geq 0. \tag{3.3}$$

Proof. We assume that $\bar{x}$ is not a εe-u-weak minimal solution to problem (SOP). Then there exists $\hat{x} \in K$ such that $F(\hat{x}) + \varepsilon e \subset F(\bar{x}) - intC$. This implies that, for any $\hat{y} \in F(\hat{x})$, there exists $\bar{y} \in F(\bar{x})$ such that $\hat{y} + \varepsilon e - \bar{y} \in -intC$, that is, $\bar{y} - \hat{y} - \varepsilon e \in intC$. According to Lemma 2.3, it holds that $\triangle_C(\bar{y} - \hat{y} - \varepsilon e) < 0, \quad \forall \hat{y} \in F(\hat{x}), \exists \bar{y} \in F(\bar{x})$. By the arbitrariness of $\hat{y} \in F(\hat{x})$, one has $\sup_{\hat{y}\in F(\hat{x})} \inf_{\bar{y}\in F(\bar{x})} \triangle_C(\bar{y} - \hat{y} - \varepsilon e) < 0$. This is a contradiction to (3.3). Hence, $\bar{x}$ is εe-u-weak minimal solution of problem (SOP).

Conversely, if $\bar{x}$ is an εe-u-weak minimal solution to problem (SOP), by Lemma 2.4, then there is no $x \in K$ such that $F(x) + \varepsilon e \subset F(\bar{x}) - intC$. This means that

$$F(x) + \varepsilon \not\subset F(\bar{x}) - intC, \quad x \in K. \tag{3.4}$$

If (3.3) is not true, then $\sup_{y\in F(x)}\inf_{\bar{y}\in F(\bar{x})}\triangle_C(\bar{y}-y-\varepsilon e)<0$. Thus, for all $y\in F(x)$,

$$\inf_{\bar{y}\in F(\bar{x})}\triangle_C(\bar{y}-y-\varepsilon e)<0.$$

Since the condition of $\inf_{\bar{y}\in F(\bar{x})}\triangle_C(\bar{y}-y-\varepsilon e)$ is attained for all $y\in F(x)$, there exists $\hat{y}\in F(\bar{x})$ such that $\triangle_C(\hat{y}-y-\varepsilon e)<0$, It follows from Lemma 2.3 that $y-\hat{y}+\varepsilon e\in -intC$, $\forall y\in F(x), \exists \hat{y}\in F(\bar{x})$. Hence, $F(x)+\varepsilon e\subset F(\bar{x})-intC$, which contradicts with (3.4). Hence, (3.3) holds. The proof is completed. □

We give an example to illustrate the result of Theorem 3.2.

Example 3.2. In problem (SOP), let $X=Y=\mathbb{R}$, $C=\mathbb{R}_+$, $intC=\mathbb{R}_{++}$, $K=[0,1]$, and the mapping $F:\mathbb{R}\to 2^{\mathbb{R}}$ be defined by $F(x)=[x,x+\frac{1}{2}]$. Let $\varepsilon=2$, $e=\frac{1}{2}$, and $\bar{x}=0$. Then $F(0)=[0,\frac{1}{2}]$. By a simple calculation, we have that there is no $x\in K$ such that $F(x)+\varepsilon e\subset F(\bar{x})-intC$. From Lemma 2.4, one has that $\bar{x}=0$ is a εe-u-weak minimal solution to this problem. Furthermore, for all $y\in F(x)$ and all $\bar{y}\in F(\bar{x})$, by $\bar{y}-y-\varepsilon e=[-x-1,-x-1]\notin intC$, it follows from Lemma 2.3 that $\triangle_C(\bar{y}-y-\varepsilon e)\geq 0$, $\forall \bar{y}\in F(\bar{x})$, $\forall y\in F(x)$. Thus $\sup_{y\in F(x)}\inf_{\bar{y}\in F(\bar{x})}\triangle_C(\bar{y}-y-\varepsilon e)\geq 0$. This indicates that Theorem 3.2 holds.

4. Semicontinuity of Approximate Solution mappings

Let $\mathbb{N}$ be positive integer set, $\Lambda\subset Z$ be a nonempty subset, and $T:\Lambda\to 2^X$ be a set-valued mapping. Let us recall some definitions and properties of semicontinuity for set-valued mappings.

Definition 4.1. (see [3]) Let $\bar{\lambda}\in\Lambda$.

(i) T is said to be upper semicontinuous at $\bar{\lambda}$ iff, for any open set $W\subset X$ with $T(\bar{\lambda})\subset W$, there exists a neighborhood $U(\bar{\lambda})$ of $\bar{\lambda}$ such that $T(\lambda)\subset W$, $\forall\lambda\in U(\bar{\lambda})$.

(ii) T is said to be lower semicontinuous at $\bar{\lambda}$ iff, for any open set $W\subset X$ with $T(\bar{\lambda})\cap W\neq\emptyset$, there exists a neighborhood $U(\bar{\lambda})$ of $\bar{\lambda}$ such that $T(\lambda)\cap W\neq\emptyset$, $\forall\lambda\in U(\bar{\lambda})$.

(iii) T is said to be continuous at $\bar{\lambda}$ iff it is both upper semicontinuous and lower semicontinuous at $\bar{\lambda}$.

Lemma 4.1. *(see [2]) Let $\bar{\lambda}\in\Lambda$.*

(i) T is lower semicontinuous at $\bar{\lambda}$ iff, for any sequence $\{\lambda_n\}\subset\Lambda$ with $\lambda_n\to\bar{\lambda}$ and any $\bar{x}\in T(\bar{\lambda})$, there exists $x_n\in T(\bar{\lambda}_n)$ such that $x_n\to\bar{x}$.

(ii) If T has compact values at $\bar{\lambda}$ (i.e., $T(\bar{\lambda})$ is a compact set), then T is upper semicontinuous at $\bar{\lambda}$ iff, for any sequence $\{\lambda_n\}\subset\Lambda$ with $\lambda_n\to\bar{\lambda}$ and any $x_n\in T(\lambda_n)$, there exist $\bar{x}\in T(\bar{\lambda})$ and subsequence $\{x_{n_k}\}\subset\{x_n\}$ such that $x_{n_k}\to\bar{x}$.

For problem (SOP), if F and K are perturbed by parameter $\lambda\in\Lambda$, then it becomes the following parametric set optimization problem:

$$(\text{SOP})_\lambda \qquad \min F(x,\lambda), \quad \text{s.t. } x\in K(\lambda),$$

where $F:X\times Z\to 2^Y$ and $K:\Lambda\to 2^X$ are set-valued mappings. For each $\lambda\in\Lambda$, let $S^u_\varepsilon(\lambda)$ denote the approximate weak minimal solution set of problem $(\text{SOP})_\lambda$, i.e.,

$$S^u_\varepsilon(\lambda)=\{\bar{x}\in K(\lambda):\ F(x,\lambda)+\varepsilon e\prec^u_C F(\bar{x},\lambda),\ x\in K(\lambda)\ implies\ F(\bar{x},\lambda)+\varepsilon e\prec^u_C F(x,\lambda)\}.$$

It is always assumed that $S^u_\varepsilon(\lambda)\neq\emptyset$ for any $\lambda\in\Lambda$.

Now, we introduce the notions of weak approximate u-property and converse weak approximate u-property for set-valued mappings.

Definition 4.2. Let $\bar{x}$, $x \in X$, $\bar{\lambda} \in \Lambda$, $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \to \varepsilon$, and $G: X \times \Lambda \to 2^Y$ be a set-valued mapping.

(i) G is said to be admit weak ε-u-property at $(\bar{x}, \bar{\lambda})$ with respect to x iff either $G(x, \bar{\lambda}) + \varepsilon e \prec_C^u G(\bar{x}, \bar{\lambda})$ or, for any $\bar{x}_n \to \bar{x}, x_n \to x, \bar{\lambda}_n \to \bar{\lambda}$, there exists $n \in \mathbb{N}$ such that $G(x_n, \bar{\lambda}_n) + \varepsilon_n e \not\prec_C^u G(\bar{x}_n, \bar{\lambda}_n)$.

(ii) G is said to be admit the converse weak ε-u-property at $(\bar{x}, \bar{\lambda})$ with respect to x iff, either $G(x, \bar{\lambda}) + \varepsilon e \not\prec_C^u G(\bar{x}, \bar{\lambda})$ or for any $\bar{x}_n \to \bar{x}, x_n \to x, \bar{\lambda}_n \to \bar{\lambda}$, there exists $n \in \mathbb{N}$ such that $G(x_n, \bar{\lambda}_n) + \varepsilon_n e \prec_C^u G(\bar{x}_n, \bar{\lambda}_n)$.

Xu and Li [14] introduced the notion of the weak u-property as follows: G is said to be admit weak u-property at $(\bar{x}, \bar{\lambda})$ with respect to x iff either $G(x, \bar{\lambda}) \prec_C^u G(\bar{x}, \bar{\lambda})$ or, for any $\bar{x}_n \to \bar{x}, x_n \to x, \bar{\lambda}_n \to \bar{\lambda}$, there exists $n \in \mathbb{N}$ such that $G(x_n, \bar{\lambda}_n) \not\prec_C^u G(\bar{x}_n, \bar{\lambda}_n)$.

Remark 4.1. It is worth noting that if $\varepsilon = 0$, then the notion of the weak ε-u-property reduces to the weak u-property. Particularly, if G has the weak ε-u-property, then G has the weak u-property. However, the converse does not hold. This fact is illustrated by the following example.

Example 4.1. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $intC = \mathbb{R}^2_{++}$, $\Lambda = [0,1]$, and $K(\lambda) = [0,1]$. Define

$$F(x,\lambda) = \begin{cases} [-1,1] \times [0,1], & x = 0, \lambda = 0, \\ [-1,0] \times [-1,0], & x = 1, \lambda = 0. \\ [-1, 1+\lambda] \times [0,1], & \text{otherwise.} \end{cases}$$

Let $\varepsilon = 1$, $e = (1,1)$, $\bar{x} = 0$, $x = 1$, and $\bar{\lambda} = 0$. It is obtained that $F(x, \bar{\lambda}) \prec_C^u F(\bar{x}, \bar{\lambda})$. Thus, F has weak u-property $(0,0)$ with respect to $x = 1$. However, due to $F(x, \bar{\lambda}) + \varepsilon e \not\prec_C^u F(\bar{x}, \bar{\lambda})$, it follows from 4.2 that F has no weak ε-u-property $(0,0)$ with respect to $x = 1$.

Next, we provide an example to explain the notion of the converse weak ε-u-property.

Example 4.2. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $intC = \mathbb{R}^2_{++}$, $\Lambda = [0,1]$, and $K(\lambda) = [0,1]$. Define

$$F(x,\lambda) = \begin{cases} [0,2] \times [0,2], & x = 1, \lambda = 0, \\ [-2,0] \times [-1,0], & x = 0, \lambda = 0. \\ [-1-\lambda, -1] \times [-2,-1], & \text{otherwise.} \end{cases}$$

Let $\varepsilon = 1$, $e = (1,1)$, $\bar{x} = 1$, $x = 0$, and $\bar{\lambda} = 0$. Then F has the converse weak ε-u-property at point $(1,0)$ with respect to $x = 0$.

We present the upper semicontinuity of approximate weak minimal solutions to problem $(\text{SOP})_\lambda$.

Theorem 4.1. *In problem $(SOP)_\lambda$, let $\bar{\lambda} \in \Lambda$, $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \to \varepsilon$. Assume that the following conditions are satisfied*

(i) K is continuous with compact values at $\bar{\lambda}$;

(ii) F is continuous with compact values on $K(\bar{\lambda}) \times \{\bar{\lambda}\}$;

Then, $S^u_\varepsilon(\cdot)$ is upper semicontinuous at $\bar{\lambda}$.

Proof. If $S^u_\varepsilon(\cdot)$ is not upper semicontinuous at $\bar{\lambda}$, then there exist an open set W_0 satisfying $S^u_\varepsilon(\bar{\lambda}) \subset W_0$, a sequence $\{\bar{\lambda}_n\}$ with $\bar{\lambda}_n \to \bar{\lambda}$, and $\bar{x}_n \in S^u_\varepsilon(\bar{\lambda}_n)$ such that

$$\bar{x}_n \notin W_0, \quad \forall n \in N. \tag{4.1}$$

In view of $\bar{x}_n \in S^u_\varepsilon(\bar{\lambda}_n)$, one has $\bar{x}_n \in K(\bar{\lambda}_n)$. Since K is upper semicontinuous with compact values at $\bar{\lambda}$, by Lemma 4.1, then there exist $\bar{x} \in K(\bar{\lambda})$ and a subsequence $\{\bar{x}_{n_k}\} \subset \{\bar{x}_n\}$ such that $\bar{x}_{n_k} \to \bar{x}$.

We claim that $\bar{x} \in S^u_\varepsilon(\bar{\lambda})$. Indeed, if $\bar{x} \notin S^u_\varepsilon(\bar{\lambda})$, then there exists $x \in K(\bar{\lambda})$ such that $F(x,\bar{\lambda}) + \varepsilon e \prec^u_C F(\bar{x},\bar{\lambda})$, that is,

$$F(x,\bar{\lambda}) + \varepsilon e \subset F(\bar{x},\bar{\lambda}) - intC. \tag{4.2}$$

Since K is lower semicontinuous at $\bar{\lambda}$, then there exists $x_n \in K(\bar{\lambda}_n)$ such that $x_n \to x$.

Next, we prove that

$$F(x_{n_k},\bar{\lambda}_{n_k}) + \varepsilon_{n_k} e \subset F(\bar{x}_{n_k},\bar{\lambda}_{n_k}) - intC. \tag{4.3}$$

If (4.3) is not true, then there exists $p_{n_k} \in F(x_{n_k},\bar{\lambda}_{n_k})$ such that

$$p_{n_k} + \varepsilon_{n_k} e \notin F(\bar{x}_{n_k},\bar{\lambda}_{n_k}) - intC. \tag{4.4}$$

Because F is upper semicontinuous with compact values at $(x,\bar{\lambda})$, we have that there exist $p \in F(x,\bar{\lambda})$ and a subsequence $\{p_{n_{k_j}}\} \subset \{p_{n_k}\}$ such that $p_{n_{k_j}} \to p$. It follows from (4.2) that there exists $\bar{z} \in F(\bar{x},\bar{\lambda})$ such that $p + \varepsilon e \in \bar{z} - intC$. Since F is lower semicontinuous at $(\bar{x},\bar{\lambda})$, then there exists a subsequence $\bar{z}_{n_{k_j}} \in F(\bar{x}_{n_{k_j}},\bar{\lambda}_{n_{k_j}})$ such that $\bar{z}_{n_{k_j}} \to \bar{z}$. Since $\varepsilon_{n_{k_j}} \to \varepsilon$, we have $p_{n_{k_j}} + \varepsilon_{n_{k_j}} e \in \bar{z}_{n_{k_j}} - intC$, for sufficiently large $j \in \mathbb{N}$. This is a contradiction to (4.4). Hence, (4.3) holds.

In addition, noticing that $\bar{x}_n \in S^u_\varepsilon(\bar{\lambda}_n)$, one has $F(\bar{x}_n,\bar{\lambda}_n) + \varepsilon_n e \subset F(x_n,\bar{\lambda}_n) - intC, \quad \forall n \in \mathbb{N}$. Since F is continuous with compact values on $K(\bar{\lambda}) \times \{\bar{\lambda}\}$, we gain $F(\bar{x},\bar{\lambda}) + \varepsilon e \subset F(x,\bar{\lambda}) - intC$, which contradicts (4.2). Therefore, $\bar{x} \in S^u_\varepsilon(\bar{\lambda})$. As $\bar{x}_n \to \bar{x}$ and $S^u_\varepsilon(\bar{\lambda}) \subset W_0$, we conclude that $\bar{x}_n \in W_0$ for sufficiently large $n \in \mathbb{N}$. This contradicts to (4.1). □

Here is an example to verify Theorem 4.1.

Example 4.3. In problem (SOP)$_\lambda$, let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $intC = \mathbb{R}^2_{++}$, $K(\lambda) = [0,1]$, and $\Lambda = [0,\frac{1}{2}]$. We define $F(x,\lambda) = [0,1] \times [0,\frac{\lambda}{2}(1+x)]$. Let $\bar{\lambda} = 0$, $\varepsilon = \frac{1}{2}$, and $e = (1,1)$. It is obvious that the assumptions of Theorem 4.1 are satisfied. Direct computation demonstrates that

$$S^u_\varepsilon(\lambda) = \begin{cases} [0,1], & \lambda = 0, \\ [0,1], & \lambda \neq 0. \end{cases}$$

Thus $S^u_\varepsilon(\cdot)$ is upper semicontinuous at $\bar{\lambda} = 0$.

Theorem 4.2. *In problem (SOP)$_\lambda$, let $\bar{\lambda} \in \Lambda$, $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \to \varepsilon$. Assume that the following conditions are satisfied*

(i) K is continuous with compact values at $\bar{\lambda}$;

(ii) F has the converse weak ε-u-property on $K(\bar{\lambda}) \times \{\bar{\lambda}\}$ with respect to each $x \in K(\bar{\lambda})$;

Then, $S^u_\varepsilon(\cdot)$ is upper semicontinuous at $\bar{\lambda}$.

Proof. If $S^u_\varepsilon(\cdot)$ is not upper semicontinuous at $\bar{\lambda}$, then there exist an open set $W_0 \subset X$ with $S^u_\varepsilon(\bar{\lambda}) \subset W_0$, a sequence $\{\bar{\lambda}_n\}$ with $\bar{\lambda}_n \to \bar{\lambda}$, and $\bar{x}_n \in S^u_\varepsilon(\bar{\lambda}_n)$ such that

$$\bar{x}_n \notin W_0, \quad \forall n \in \mathbb{N}. \tag{4.5}$$

Since $\bar{x}_n \in S^u_\varepsilon(\bar{\lambda}_n)$, then $\bar{x}_n \in K(\bar{\lambda}_n)$. Because K is upper semicontinuous with compact values at $\bar{\lambda}$, by Lemma 4.1, we have that there exist $\bar{x} \in K(\bar{\lambda})$ and a subsequence $\{\bar{x}_{n_k}\} \subset \{\bar{x}_n\}$ such that $\bar{x}_{n_k} \to \bar{x}$. Without loss of generality, let $\bar{x}_n \to \bar{x}$.

Now, we prove that $\bar{x} \in S^u_\varepsilon(\bar{\lambda})$. Indeed, for any $x \in K(\bar{\lambda})$, since K is lower semicontinuous at $\bar{\lambda}$, then there exists a sequence $x_n \in K(\bar{\lambda}_n)$ such that $x_n \to x$. As $\bar{x}_n \in S^u_\varepsilon(\bar{\lambda}_n)$, we have

$$F(x_n, \bar{\lambda}_n) + \varepsilon_n e \not\prec^u_C F(\bar{x}_n, \bar{\lambda}_n). \tag{4.6}$$

It follows from assumption (ii) that either

$$F(x, \bar{\lambda}) + \varepsilon e \not\prec^u_C F(\bar{x}, \bar{\lambda}), \quad \forall x \in K(\bar{\lambda}), \tag{4.7}$$

or, for any $(\bar{x}_n, x_n, \bar{\lambda}_n)$ with $\bar{x}_n \to \bar{x}$, $x_n \to x$, $\bar{\lambda}_n \to \bar{\lambda}$, there exists $k \in \mathbb{N}$ such that

$$F(x_{n_k}, \bar{\lambda}_{n_k}) + \varepsilon_{n_k} e \prec^u_C F(\bar{x}_{n_k}, \bar{\lambda}_{n_k}). \tag{4.8}$$

However, (4.8) contradicts to (4.6), which means that (4.7) holds. Hence, we have that $\bar{x} \in S^u_\varepsilon(\bar{\lambda})$. It follows from $S^u_\varepsilon(\bar{\lambda}) \subset W_0$ that $\bar{x} \in W_0$. Since $\bar{x}_n \to \bar{x}$ for the neighborhood W_0 of $\bar{x}$, then there exists $k_1 \in \mathbb{N}$ such that $\bar{x}_{n_k} \in W_0, \quad \forall k \geq k_1$. This is a contradiction to (4.5). Therefore, $S^u_\varepsilon(\cdot)$ is upper semicontinuous at $\bar{\lambda}$. □

Now, we give an example to illustrate Theorem 4.2.

Example 4.4. In problem $(\text{SOP})_\lambda$, let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $intC = \mathbb{R}^2_{++}$, $\Lambda = [0, \frac{1}{4}]$ and $K(\lambda) = [0,1]$. We define

$$F(x,\lambda) = \begin{cases} [-1,1] \times [-1,1], & \lambda = 0, \\ [-1,\lambda x] \times [-1,\lambda x], & \lambda \neq 0. \end{cases}$$

Let $\bar{\lambda} = 0$, $\varepsilon = 1$ and $e = (\frac{1}{2}, \frac{1}{2}) \in intC$. It is easy to cheak that the assumptions of Theorem 4.2 are satisfied. It follows from a direct computation that

$$S^u_\varepsilon(\lambda) = \begin{cases} [0,1], & \lambda = 0, \\ [0,1], & \lambda \neq 0. \end{cases}$$

Hence, $S^u_\varepsilon(\cdot)$ is upper semicontinuous at $\bar{\lambda} = 0$.

The following Example 4.5 indicates the assumption (ii) in Theorem 4.2 is essential.

Example 4.5. In problem $(\text{SOP})_\lambda$, let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $intC = \mathbb{R}^2_{++}$, $\Lambda = [0,1]$ and $K(\lambda) = [0,1]$. We define

$$F(x,\lambda) = \begin{cases} [x-1,x] \times [x-1,x], & \lambda = 0, \\ \frac{\lambda}{2}[x,x+1] \times \frac{\lambda}{2}[x,x+1], & \lambda \neq 0. \end{cases}$$

Let $\bar{\lambda} = 0$, $\varepsilon = \frac{1}{4}$ and $e = (1,1)$. It is easy to verify that the assumption (i) is satisfied, however, the assumption (ii) is not true. Indeed, let $\bar{x} = 1$, $x = 0$ and $\bar{\lambda} = 0$, then $F(x, \bar{\lambda}) + \varepsilon e \prec^u_C F(\bar{x}, \bar{\lambda})$. Moreover, for any sequences $\{\bar{x}_n\} \subset (\frac{1}{2}, 1)$ with $\bar{x}_n \to \bar{x}$, $\{x_n\} \subset (0, \frac{1}{2})$ with $x_n \to x$, $\{\bar{\lambda}_n\} \subset (0,1)$ with $\bar{\lambda}_n \to \bar{\lambda}$ and $\varepsilon_n \subset (0, \frac{1}{4})$ with $\varepsilon_n \to \varepsilon$, we have $F(x_n, \bar{\lambda}_n) + \varepsilon_n e \not\prec^u_C F(\bar{x}_n, \bar{\lambda}_n)$. This means that F does not have converse weak ε-u-property at point $(1,0)$ with respect to $x=0$. For any $x\in[0,1]$, one has

$$S^u_\varepsilon(\lambda)=\begin{cases}[0,\frac{1}{4}), & \lambda=0,\\ [0,\frac{1}{2}), & \lambda\neq 0.\end{cases}$$

Thus, $S^u_\varepsilon(\cdot)$ is not upper semicontinuous at $\bar\lambda=0$. Therefore, the assumption (ii) of Theorem 4.2 is necessary.

At end of this paper, we present the lower semicontinuity of $S^u_\varepsilon(\cdot)$.

Theorem 4.3. *In problem (SOP)$_\lambda$, let $\bar\lambda\in\Lambda$, $\{\varepsilon_n\}\subset\mathbb{R}_+$ with $\varepsilon_n\to\varepsilon$. Assume that the following conditions are fulfilled*
(i) K is continuous with compact values at $\bar\lambda$;
(ii) F has weak ε-u-property on $K(\bar\lambda)\times\{\bar\lambda\}$ with respect to each $x\in K(\bar\lambda)$.
Then, $S^u_\varepsilon(\cdot)$ is lower semicontinuous at $\bar\lambda$.

Proof. If $S^u_\varepsilon(\cdot)$ is not lower semicontinuous at $\bar\lambda$, then there exist $\bar x\in S^u_\varepsilon(\bar\lambda)$ and a sequence $\{\bar\lambda_n\}$ with $\bar\lambda_n\to\bar\lambda$ such that, for any $\bar x_n\in S^u_\varepsilon(\bar\lambda_n)$,

$$\bar x_n\not\to\bar x. \tag{4.9}$$

It follows from $\bar x\in S^u_\varepsilon(\bar\lambda)$ that $\bar x\in K(\bar\lambda)$. Since K is lower semicontinuous at $\bar\lambda$, then there exists $\hat x_n\in K(\bar\lambda_n)$ such that $\hat x_n\to\bar x$. This is a contradiction to (4.9). Hence, there exists a subsequence $\hat x_{n_k}$ of $\hat x_n$ such that $\hat x_{n_k}\notin S^u_\varepsilon(\bar\lambda_{n_k})$ for all $k\in\mathbb{N}$. This implies that there exists $x_{n_k}\in K(\bar\lambda_{n_k})$ such that

$$F(x_{n_k},\bar\lambda_{n_k})+\varepsilon_{n_k}e\prec^u_C F(\hat x_{n_k},\bar\lambda_{n_k}), \tag{4.10}$$

and

$$F(\hat x_{n_k},\bar\lambda_{n_k})+\varepsilon_{n_k}e\not\prec^u_C F(x_{n_k},\bar\lambda_{n_k}),\quad \forall k\in\mathbb{N}.$$

Since K is upper semicontinuous with compact values at $\bar\lambda$, then there exist $x\in K(\bar\lambda)$ and a subsequence $x_{n_{k_j}}$ of x_{n_k} such that $x_{n_{k_j}}\to x$. As $\bar x\in S^u_\varepsilon(\bar\lambda)$, then $F(x,\bar\lambda)+\varepsilon e\not\prec^u_C F(\bar x,\bar\lambda)$. By the assumption (ii) and $(\hat x_{n_{k_j}},x_{n_{k_j}},\bar\lambda_{n_{k_j}})\to(\bar x,x,\bar\lambda)$, then there exists $i\in\mathbb{N}$ such that

$$F(x_{n_{k_i}},\bar\lambda_{n_{k_i}})+\varepsilon e\not\prec^u_C F(\hat x_{n_{k_i}},\bar\lambda_{n_{k_i}}),$$

which contradicts to (4.10). □

Example 4.6. In problem (SOP)$_\lambda$, let $X=Z=\mathbb{R}$, $Y=\mathbb{R}^2$, $C=\mathbb{R}^2_+$, $intC=\mathbb{R}^2_{++}$, $\Lambda=[-1,0]$ and $K(\lambda)=[0,1]$. We define

$$F(x,\lambda)=\begin{cases}[x,x+\frac{1}{3}]\times[x,x+\frac{1}{3}], & \lambda=0,\\ \frac{\lambda}{3}[x,x+\frac{1}{2}]\times\frac{\lambda}{3}[x,x+\frac{1}{2}], & \lambda\neq 0.\end{cases}$$

Let $\bar\lambda=0$, $\varepsilon=\frac{1}{3}$ and $e=(1,1)$. It is easy to check that the assumption (i) of Theorem 4.3 is satisfied, however, assumption (ii) is not true. Indeed, taking $\bar x=0$, $x=1$ and $\bar\lambda=0$, we have $F(x,\bar\lambda)+\varepsilon e\not\prec^u_C F(\bar x,\bar\lambda)$. Moreover, for any sequences $\{\bar x_n\}\subset(0,1)$ with $\bar x_n\to\bar x$, $\{x_n\}\subset(\frac{1}{2},1)$ with $x_n\to x$, $\{\bar\lambda_n\}\subset(0,1)$ with $\bar\lambda_n\to\bar\lambda$, and $\varepsilon_n\subset(0,\frac{1}{3})$ with $\varepsilon_n\to\varepsilon$, we have $F(x_n,\bar\lambda_n)+\varepsilon_n e\prec^u_C$

$F(\bar{x}_n, \bar{\lambda}_n)$. This shows that F does not have weak ε-u-property at point $(0,0)$ with respect to $x = 1$. By a direct computation, we have

$$S_\varepsilon^u(\lambda) = \begin{cases} [0, \frac{1}{3}), & \lambda = 0, \\ \emptyset, & \lambda \neq 0. \end{cases}$$

Thus $S_\varepsilon^u(\cdot)$ is not lower semicontinuous at $\bar{\lambda} = 0$. Therefore, the assumption (ii) of Theorem 4.3 is essential.

5. Conclusions

The scalarization and semicontinuity theorems of approximate solutions to set optimization problems were established in this paper. A scalar approximate solution is an approximate weak minimal solution to problem (SOP) was proved, and a scalarization in the sense of approximate weak minimal solutions to problem (SOP) was constructed by utilizing the oriented distance function. The upper and lower semicontinuity of approximate weak minimal solution mappings for the parametric set optimization problem were obtained without any convexity and monotonicity assumptions.

In [19], the scalarization and convergence of approximate minimal solution were discussed by Tammer's nonlinear function. It is worth mentioning that we adopt the oriented distance function as a main tool to establish the scalarization and semicontinuity theorems of approximate solutions. In [10, 12, 13, 15], the semicontinuities for solution mappings were examined under the assumptions of some convexities and monotonicities. Theorem 4.2 and Theorem 4.3 obtained in this paper do not need any convexity and monotonicity. In addition, if $\varepsilon = 0$, then the approximate minimal solution reduces to minimal solutions. Thus our results generalize those of [10, 13, 14, 15].

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