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AN INERTIAL SCHEME FOR SOLVING BI-LEVEL VARIATIONAL INEQUALITIES AND THE FIXED POINT PROBLEM WITH PSEUDOMONOTONE AND ρ-DEMIMETRIC MAPPINGS

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Abstract. This paper investigates the solutions of a bi-level variational inequality problem and the fixed point problem of the operators, which are pseudo-monotone and ρ-demimetric in the framework of Hilbert spaces. An iterative scheme is presented and it is proved to be strongly convergent to the solution of the two problem. Four numerical examples are presented to demonstrate the usefulness and applicability of our scheme. The result obtained in this paper extends, generalizes, and compliments several existing results in this direction of this research.
Keywords. ρ-demimetric mapping; Inertial scheme; Pseudomonotone mappings; Variational inequality.

1. INTRODUCTION

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A : H \to H$ and $T : H \to H$ be two single-valued mappings. The Bi-level Variational Inequality and Fixed Point Problem (BVIFPP) considered in this paper consist of

$$\text{finding } x^* \in VI(C,A) \text{ such that } T(x^*) = x^*. \tag{1.1}$$

where $VI(C,A)$ denotes the set of all solutions of the following Variational Inequality Problem (VIP), which consists of

$$\text{finding } z^* \in C \text{ such that } \langle Az^*, x - z^* \rangle \geq 0 \quad \forall\, x \in C. \tag{1.2}$$

Let H be a real Hilbert space with $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ as its norm and inner product, respectively. Recall that a mapping $A : H \to H$ is said to be monotone if, for any $x,y \in H$, $\langle Ax - Ay, x - y\rangle \geq 0$. A is said to be pseudomonotone if, for all $x,y \in H$, $\langle Ax, y - x\rangle \geq 0$ implies $\langle Ay, y - x\rangle \geq 0$. It is easy to see that every monotone operator is pseudomonotone. However, the converse is not always true (see [1] for an example). A is said to be L-Lipschitz continuous if there exists a positive constant L such that, for any $x,y \in H$, $\|Ax - Ay\| \leq L\|x - y\|$. If $L \in (0,1)$, then A is

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called a contraction. A point $x \in H$ is called a fixed point of A if and only if $Ax = x$. The set of fixed points of A is defined as $Fix(A) := \{x \in H : Ax = x\}$. The Fixed Point Problem (shortly FPP) enjoys numerous applications in several real world problems, such as image recovery and signal processing; see, e.g., [2, 3, 4, 5, 6]. Let $T : H \to H$ be a mapping. Recall that $T - I$ is said to be demiclosed at zero if, for any sequence $\{x_n\}_{n\geq 1}$ in H with $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$ as $n \to \infty$, then $Tx = x$. Let ρ be a real number in $(-\infty, 1)$. Recall that a mapping $T : H \to H$ with $Fix(T) \neq \emptyset$ is said to be ρ-demimetric (see Takahashi [7]) if, for any $x \in H$ and $x^* \in Fix(T)$, $\langle x - x^*, x - Tx\rangle \geq \frac{(1-\rho)\|x-Tx\|^2}{2}$. Takahashi [7] proved that the fixed point set $Fix(T)$ of a ρ-demimetric mapping T is closed and convex. From [8, 9, 10, 11, 12], we see that the class of ρ-demimetric mappings includes numerous important classes nonlinear mappings as its special case.

Recall the celebrated gradient-projection method $x_{n+1} = P_C(x_n - \tau Ax_n)$, $\forall n \geq 1$, where P_C denotes the nearest point projection (see below) and τ is a positive real constant. It was shown in [13] that this scheme converges weakly to an element of $VI(C,A)$ if A is L-Lipschitz continuous and γ-strongly monotone. The scheme failed to guarantee convergence if A is monotone. If A is monotone only, [14] and [15] independently proposed the extragradient method in $\mathbb{R}^n$ $x_0 \in C$, $x_{n+1} = P_C(x_n - \tau Ay_n)$, where $y_n = P_C(x_n - \tau Ax_n)$, $\forall n \geq 0$. They proved that the iterative scheme converges weakly to an element of $VI(C,A)$. However, the scheme is difficult in implementation as it requires two projections onto C at each iteration. For improving this drawback, the projection and contraction method was developed. The scheme, which was developed by [16] and [17], reads

$$\forall\ n \geq 1; \quad y_n = P_C(x_n - \tau_n Ax_n), x_{n+1} = x_n - \alpha\eta_n d(x_n, y_n)$$

where

$$d(x_n, y_n) := x_n - y_n - \tau_n(Ax_n - Ay_n) \text{ and } \eta_n := \frac{\langle x_n - y_n, d(x_n, y_n)\rangle}{\|d(x_n, y_n)\|^2}.$$

The projection and contraction method is an improvement of other existing methods as it requires only one projection onto C at each iteration. This led to numerous research on approximating the solutions of variational inequality problem; see, e.g., [18, 19, 20, 21, 22] and the references therein. Recently, the inertial technique has attracted much attention of due to its significant effect in speeding up the performance of iterative methods. For details and recent works on inertial methods, we refer to [23, 24, 25] and the references therein.

In this paper, we propose a new iterative scheme by using projection and contraction method with an inertial to solve the BVIFPP, where A is a Lipschitz continuous pseudomonotone operator, and T is ρ-demimetric. We prove that the scheme is strongly convergent in real Hilbert spaces. The rest of the paper is organized as follows: In Section 2, we recall some useful definitions and known lemmas which are neended in the sequel. In Section 3, we present our proposed scheme and highlight some of its features. Strong convergence analysis of our scheme is investigated in Section 4. Section 5 presents some numerical examples which support the applications and usefulness, and we conclude with some concluding remarks in Section 6.

2. Preliminaries

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. For any $x \in H$, the nearest projection mapping $P_C : H \to C$ is defined by $\|P_C x - x\| = \inf_{z\in C} \|x - z\|$. The following properties are equivalent $\forall x \in H$, $\langle x - P_C x, z - P_C x\rangle \leq 0$, $\forall z \in C$ and $\|P_C x - z\|^2 \leq \|x-z\|^2 - \|P_C x - x\|^2$, $\forall z \in C$. The following inequalities and equality hold for any $x, y \in H$ $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, $2\langle x,y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2$, and $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$, where $\alpha \in (0,1)$.

Additionally, one also needs the following tools.

Lemma 2.1. [26] *Let $A : C \to H$ be a continuous pseudomonotone mapping. Then, $x^* \in VI(C,A)$ if and only if $\langle Ax, x - x^*\rangle \geq 0$, $\forall x \in C$.*

Lemma 2.2. [27] *Let $\{a_n\}$ be a real number sequence with a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$ $a_{m_k} \leq a_{m_k+1}$ and $a_k < a_{m_k+1}$. In fact, m_k is the largest number n in the set $\{1,2,\cdots,k\}$ such that the condition that $a_n < a_{n+1}$ holds.*

Lemma 2.3. [28] *Let $\{a_n\}$ be a nonnegative real number sequence with $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n$, $n \geq n_0$, where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ are sequences satisfying the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.*

3. The Proposed Scheme and Its Features

The following assumptions are needed in the sequel.

Assumptions

- (S_1) The feasible set C is a nonempty, closed, and convex subset of real Hilbert space H.
- (S_2) The operator $A : H \to H$ is L_1-Lipschitz continuous, pseudomonotone on H, and sequentially weakly continuous on C.
- (S_3) $T : H \to H$ is a uniformly continuous ρ-demimetric operator such that $T - I$ is demiclosed on zero, and $0 < h \leq t_n \leq (1-\rho)$ for all $n \geq 1$.
- (S_4) $f : H \to H$ is an L-contraction with $L \in (0, \frac{1}{2})$.
- (S_5) $VI(C,A) \neq \emptyset$ and $\Omega \neq \emptyset$, where $\Omega := \{x \in VI(C,A) \ : \ T(x) = x\}$.
- (S_6) $\{\alpha_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$ are positive sequences such that $\alpha_n \in (0,1)$, $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n\geq 1} \alpha_n = \infty$, and $\lim\limits_{n\to\infty} \frac{\gamma_n}{\alpha_n} = 0$.

In this paper, we focus on the following scheme.

Algorithm 3.1. Initialization: Let $\tau_0 > 0$, $\mu \in (0,1)$, and $\beta \in (0,2)$.
Iterative Steps: Given the current iterate x_n, calculate x_{n+1} as follows:
Step 1: Given x_{n-1}, x_n with $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\{\frac{n-1}{n-1+\varepsilon}, \frac{\gamma_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1} \\ \frac{n-1}{n-1+\varepsilon}, & \text{otherwise} \end{cases}$$

Step 2: Compute $y_n = P_C(w_n - \tau_n A w_n)$, where $w_n = x_n + \theta_n(x_n - x_{n-1})$.
Step 3: Compute $z_n = w_n - \beta\eta_n d_n$, where $d_n := w_n - y_n - \tau_n(Aw_n - Ay_n)$ and

$$\eta_n := \begin{cases} \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}, & \text{if } d_n \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Step 4: Compute

$$x_{n+1}=\alpha_n f(v_n)+(1-\alpha_n)v_n, \text{ where } v_n=z_n-t_n(z_n-Tz_n) \tag{3.1}$$

Update

$$\tau_{n+1}=\begin{cases}\min\{\mu\frac{\|w_n-y_n\|}{\|Aw_n-Ay_n\|},\tau_n\}, & \text{if } Aw_n\neq Ay_n,\\ \tau_n, & \text{otherwise.}\end{cases} \tag{3.2}$$

Step 5: Set $n:=n+1$ and go to **Step 1**.

Remark 3.1. (a) Assumption (S_2) requires that operator A is pseudomonotone and Lipschitz continuous, which is much weaker assumption than the monotonicity and Lipschitz continuity assumption used in [29, 30], as well as the strongly monotonicity assumptions used in [31]. Thus our scheme is applicable to a general class of pseudomonotone and Lipschitz continuous operators.
(b) In addition, our scheme only requires one projection onto the feasible set C, which can be easily computed. Hence, it is less computationally expensive than other schemes in the literature (see, e.g., [1, 32]) for solving pseudomonotone variational inequality problems.
(c) The stepsize $\{\tau_n\}$ given by (3.2) is generated at each iteration by some simple computations. Thus $\{\tau_n\}$ is easily implemented and does not depend on the Lipschitz constant of the pseudomonotone operator A.
(d) The first step of our scheme is also easily implemented since the value of $\|x_n-x_{n-1}\|$ is priori known before choosing θ_n.

Remark 3.2. The sequence $\{\tau_n\}$ generated by (3.2) is a non-increasing sequence and $\lim_{n\to\infty}\tau_n=\tau\geq\min\{\tau_0,\frac{\mu}{L}\}$. Moreover, $\|Aw_n-Ay_n\|\leq\frac{\mu}{\tau_{n+1}}\|w_n-y_n\|$, $\forall n\geq 0$.

4. Main Results

We begin with the following lemmas which are critical in obtaining our strong convergence result.

Lemma 4.1. *Let $\{x_n\}_{n\geq 0}$ be a sequence in H iteratively generated by Algorithm 3.1. If conditions (S_1)-(S_6) hold, then $\{x_n\}$ is bounded.*

Proof. Let $x^*\in \mathrm{VI}(C,A)$. We obtain from the definition of w_n that $\|w_n-x^*\|\leq\|x_n-x^*\|+\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|$. In view of **Step 1**, we observe that $\theta_n\|x_n-x_{n-1}\|\leq\gamma_n$, $\forall n\geq 1$, which implies that $\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\leq\frac{\gamma_n}{\alpha_n}\to 0$, as $n\to\infty$. Thus there exists $K_1>0$ such that $\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\leq K_1$, $\forall n\geq 1$. Hence, $\|w_n-x^*\|\leq\|x_n-x^*\|+\alpha_nK_1$, $\forall n\geq 1$. Since $y_n=P_C(w_n-\tau_nAw_n)$, we obtain that

$$\langle w_n-\tau_nAw_n-y_n, y_n-x^*\rangle\geq 0. \tag{4.1}$$

Also, since $x^*\in VI(C,A)$ and $y_n\in C$, we have that $\langle Ax^*,y_n-x^*\rangle\geq 0$. By pseudomonotonicity of A, we obtain that $\langle Ay_n,y_n-x^*\rangle\geq 0$. In view of $\tau_n>0$, we have that $\langle\tau_nA(y_n),y_n-x^*\rangle\geq 0$, which together with (4.1) we obtain that $\langle y_n-x^*,w_n-y_n-\tau_n(Aw_n-Ay_n)\rangle\geq 0$. Equivalently, $\langle y_n-x^*,d_n\rangle\geq 0$. It follows from **Step 3** that

$$\langle w_n-x^*,d_n\rangle=\langle w_n-y_n,d_n\rangle+\langle y_n-x^*,d_n\rangle\geq\langle w_n-y_n,d_n\rangle.$$

Since $\|d_n\| \neq 0$, we have

$$\begin{aligned}\|z_n - x^*\|^2 &= \|w_n - x^*\|^2 + \beta^2\eta_n^2\|d_n\|^2 - 2\beta\eta_n\langle w_n - x^*, d_n\rangle \\ &\le \|w_n - x^*\|^2 + \beta^2\eta_n\langle w_n - y_n, d_n\rangle - 2\beta\eta_n\langle w_n - y_n, d_n\rangle \\ &\le \|w_n - x^*\|^2 - \frac{1}{\beta}(2-\beta)\|w_n - z_n\|^2 \\ &\le \|x_n - x^*\|^2 - \frac{1}{\beta}(2-\beta)\|w_n - z_n\|^2 + \alpha_n K_1.\end{aligned}$$

From the definition of T, we obtain that

$$\begin{aligned}\|v_n - x^*\|^2 &= \|z_n - x^*\|^2 + t_n^2\|z_n - Tz_n\|^2 - 2t_n\langle z_n - Tz_n, z_n - x^*\rangle \\ &\le \|z_n - x^*\|^2 - t_n(1-\rho-t_n)\|z_n - Tz_n\|^2 \\ &\le \|x_n - x^*\|^2 - \frac{1}{\beta}(2-\beta)\|w_n - z_n\|^2 - t_n(1-\rho-t_n)\|z_n - Tz_n\|^2 + \alpha_n K_1,\end{aligned}$$

which implies that

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\le \alpha_n\|f(v_n) - f(x^*) + f(x^*) - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\ &\le 2\alpha_n[L\|v_n - x^*\|^2 + \|f(x^*) - x^*\|^2] + (1-\alpha_n)\|v_n - x^*\|^2 \\ &\le [1-\alpha_n(1-2L)]\|x_n - x^*\|^2 + \alpha_n(1-2L)\frac{2}{1-2L}\|f(x^*) - x^*\|^2 \\ &\quad - t_n[1-\alpha_n(1-2L)](1-\rho-t_n)\|z_n - Tz_n\|^2 - \frac{1}{\beta}(2-\beta)[1-\alpha_n(1-2L)]\|w_n - z_n\|^2 \\ &\quad + \alpha_n[1-\alpha_n(1-2L)]\alpha_n K_1\end{aligned}$$

Using the assumptions S_3 and S_6, we have that, for some $K_2 > 0$,

$$\|x_{n+1} - x^*\|^2 \le [1-\alpha_n(1-2L)]\big(\|x_n - x^*\|^2 + K_2\big) + \alpha_n(1-2L)\frac{2}{1-2L}\|f(x^*) - x^*\|^2.$$

If we define $M := \max\left\{\|x_0 - x^*\|^2 + K_2, \frac{2}{1-2L}\|f(x^*) - x^*\|^2\right\}$, then it is easy to see that $\|x_{n+1} - x^*\|^2 \le M$ for all $n \ge 0$. Thus $\{x_n\}_{n=1}^{\infty}$ is bounded. Consequently, $\{w_n\}$, $\{Aw_n\}$, $\{y_n\}$, $\{Ay_n\}$, $\{z_n\}$, $\{Tz_n\}$, and $\{v_n\}$ are all bounded. □

Lemma 4.2. *Suppose that conditions S_1 and S_2 hold, and $\{x_n\}_{n=1}^{\infty}$ is a sequence generated by Algorithm 3.1. Then, there exists $n_0 \in \mathbb{N}$ such that*

$$\|w_n - y_n\|^2 \le \frac{\left(1 + \mu\frac{\tau_n}{\tau_{n+1}}\right)^2}{\left[\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\beta\right]^2}\|z_n - w_n\|^2, \qquad \forall\, n \ge n_0.$$

Proof. Observe that $\|d_n\| \ge \|w_n - y_n\| - \tau_n\|Aw_n - Ay_n\| \ge \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - y_n\|$. In view of $\lim_{n\to\infty}\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right) = 1 - \mu > 0$, one sees that there exists $n_0 \in \mathbb{N}$ such that $1 - \mu\frac{\tau_n}{\tau_{n+1}} > \frac{1-\mu}{2}$,

$\forall n \geq n_0$. For each $n \geq n_0$, we have that $\|d_n\| > \frac{1-\mu}{2}\|w_n - y_n\| > 0$. In addition,

$$\begin{aligned}\|d_n\| &\leq \|w_n - y_n\| + \tau_n\|Aw_n - Ay_n\| \\ &\leq \left(1 + \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - y_n\|, \quad \forall\, n \geq n_0.\end{aligned}$$

For each $n \geq n_0$, we obtain that

$$\|d_n\|^2 \leq \left(1 + \mu\frac{\tau_n}{\tau_{n+1}}\right)^2\|w_n - y_n\|^2. \tag{4.2}$$

Observe that

$$\begin{aligned}\langle w_n - y_n, d_n\rangle &= \|w_n - y_n\|^2 - \tau_n\langle w_n - y_n, Aw_n - Ay_n\rangle \\ &\geq \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - y_n\|^2.\end{aligned} \tag{4.3}$$

For each $n \geq n_0$, we obtain from (4.2) and (4.3) that

$$\eta_n = \frac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2} \geq \frac{\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)}{\left(1 + \mu\frac{\tau_n}{\tau_{n+1}}\right)^2}. \tag{4.4}$$

It follows from (4.3) that

$$\eta_n\|d_n\|^2 = \langle w_n - y_n, d_n\rangle \geq \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - y_n\|^2. \tag{4.5}$$

From (4.5), we have that

$$\|w_n - y_n\|^2 \leq \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)^{-1}\eta_n\|d_n\|^2 = \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)^{-1}\|z_n - w_n\|^2\frac{1}{\beta^2}\frac{1}{\eta_n}. \tag{4.6}$$

Substituting (4.4) into (4.6), we obtain that

$$\|w_n - y_n\|^2 \leq \frac{\left(1 + \mu\frac{\tau_n}{\tau_{n+1}}\right)^2}{\left[\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\beta\right]^2}\|z_n - w_n\|^2, \qquad \forall\, n \geq n_0.$$

□

Lemma 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 under Assumptions S. Suppose that there exists a subsequence $\{x_{n_b}\}$ of $\{x_n\}$ which converges weakly to $\bar{x} \in H$ and $\lim_{b\to\infty}\|w_{n_b} - y_{n_b}\| = 0$. Then $\bar{x} \in VI(C,A)$.*

Proof. From Assumption S_6, we obtain that

$$\|w_n - x_n\| = \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \frac{\gamma_n}{\alpha_n} \to 0 \text{ as } n \to \infty. \tag{4.7}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_b}\}$ of $\{x_n\}$ which converges weakly to $\bar{x} \in H$. It follows that $\langle w_{n_b} - \tau_{n_b}Aw_{n_b} - y_{n_b}, x - y_{n_b}\rangle \leq 0, \forall x \in C$. That is, $\frac{1}{\tau_{n_b}}\langle w_{n_b} - y_{n_b}, x - y_{n_b}\rangle \leq \langle Aw_{n_b}, x - y_{n_b}\rangle, \forall x \in C$. Thus

$$\frac{1}{\tau_{n_b}}\langle w_{n_b} - y_{n_b}, x - y_{n_b}\rangle + \langle Aw_{n_b}, y_{n_b} - w_{n_b}\rangle \leq \langle Aw_{n_b}, x - w_{n_b}\rangle, \quad \forall\, x \in C. \tag{4.8}$$

Fix $x \in C$. Since $\{w_{n_b}\}$ is bounded and A is Lipschitz continuous, then $\{Aw_{n_b}\}$ is bounded. Using the assumption that $\lim_{b\to\infty} \|w_{n_b} - y_{n_b}\| = 0$ and taking $\liminf$ to (4.8) yield

$$\liminf_{b\to\infty} \langle Aw_{n_b}, x - w_{n_b}\rangle \geq 0. \tag{4.9}$$

In view of the facts that $\lim\limits_{b\to\infty} \|w_{n_b} - y_{n_b}\| = 0$, and A is Lipschitz continuous, we have

$$\lim_{b\to\infty} \|Aw_{n_b} - Ay_{n_b}\| = 0. \tag{4.10}$$

Observe that

$$\langle Ay_{n_b}, x - y_{n_b}\rangle = \langle Ay_{n_b} - Aw_{n_b}, x - w_{n_b}\rangle + \langle Aw_{n_b}, x - w_{n_b}\rangle + \langle Ay_{n_b}, w_{n_b} - y_{n_b}\rangle \tag{4.11}$$

Using (4.9) and (4.10) in (4.11), we arrive at $\liminf_{b\to\infty}\langle Ay_{n_b}, x - y_{n_b}\rangle \geq 0$. Let us choose a decreasing sequence of positive numbers $\{\beta_b\}$ with $\lim\limits_{b\to\infty} \beta_b = 0$. For each $b \in \mathbb{N}$, we denote by N_b the smallest positive number such that

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \beta_b \geq 0, \quad \forall\ j \geq N_b. \tag{4.12}$$

Since $\{\beta_b\}$ is decreasing, we have that $\{N_b\}$ is increasing. Furthermore, for each $b \in \mathbb{N}$, since $\{y_{n_b}\} \subset C$, we assume that $Ay_{N_b} \neq 0$ (else, y_{N_b} is a solution in $VI(C,A)$). Set $m_{N_b} = \frac{Ay_{N_b}}{\|Ay_{N_b}\|^2}$, where $\langle Ay_{N_b}, m_{N_b}\rangle = 1$ for each $b \in \mathbb{N}$. From (4.12), we obtain that, for each $b \in \mathbb{N}$, $\langle Ay_{N_b}, x + \beta_b m_{N_b} - y_{N_b}\rangle \geq 0$. Since A is pseudomonotone, we obtain that $\langle A(x + \beta_b m_{N_b}), x + \beta_b m_{N_b} - y_{N_b}\rangle \geq 0$. Thus

$$\langle Ax, x - y_{N_b}\rangle \geq \langle Ax - A(x + \beta_b m_{N_b}), x + \beta_b m_{N_b} - y_{N_b}\rangle - \beta_b \langle Ax, m_{N_b}\rangle. \tag{4.13}$$

Since $\{x_{n_b}\}$ converges weakly to $\bar{x}$ as $b \to \infty$, then, using (4.7) and the assumption that $\lim_{b\to\infty}\|w_{n_b} - y_{n_b}\| = 0$, we easily obtain that $y_{n_b} \rightharpoonup \bar{x}$ as $b \to \infty$. By the sequentially weakly continuity of A, we have that $Ay_{n_b} \rightharpoonup A\bar{x}$ as $b \to \infty$. Assume that $A\bar{x} \neq 0$ (otherwise, $\bar{x}$ is a solution). By the sequentially weakly lower semicontinuity of norms, we have that $0 < \|A\bar{x}\| \leq \liminf_{b\to\infty}\|Ay_{n_b}\|$. Since $\{y_{N_b}\} \subset \{y_{n_b}\}$ and $\beta_b \to 0$ as $b \to \infty$, we have that

$$0 \leq \limsup_{b\to\infty} \|\beta_b m_{N_b}\| = \limsup_{b\to\infty} \left(\frac{\beta_b}{\|Ay_{n_b}\|}\right) \leq \frac{\limsup_{b\to\infty}\beta_b}{\liminf_{b\to\infty}\|Ay_{n_b}\|} \leq \frac{0}{\|A\bar{x}\|} = 0.$$

Thus $\|\beta_b m_{N_b}\| \to 0$ as $b \to \infty$. Hence, taking the limit as $b \to \infty$ in (4.13), we obtain that $\langle Ax, x - \bar{x}\rangle \geq 0$. Since $x \in H$ is arbitrary, we obtain from Lemma 2.1 that $\bar{x} \in VI(C,A)$. □

Lemma 4.4. *Let $\{x_n\}_{n\geq 0}$ be a sequence in H iteratively generated by Algorithm 3.1. Assume that $\zeta_n = (1-L)\alpha_n$, and assumptions S_1 - S_6 hold. Then*

$$\|x_{n+1} - x^*\|^2 \leq (1 - \zeta_n)\Big[\|x_n - x^*\|^2 + \alpha_n K_1\Big] + \zeta_n \frac{2}{1-L}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle.$$

Proof. From Assumption S_6, we obtain that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n[f(v_n)-f(x^*)]+(1-\alpha_n)[v_n-x^*]+\alpha_n[f(x^*)-x^*]\|^2\\ &\leq \|\alpha_n[f(v_n)-f(x^*)]+(1-\alpha_n)[v_n-x^*]\|^2+2\alpha_n\langle x_{n+1}-x^*, f(x^*)-x^*\rangle\\ &\leq [1-(1-L)\alpha_n]\|x_n-x^*\|^2+2\alpha_n\langle x_{n+1}-x^*, f(x^*)-x^*\rangle\\ &\quad+[1-(1-L)\alpha_n]\alpha_n K_1-\frac{1}{\beta}(2-\beta)[1-(1-L)\alpha_n]\|w_n-z_n\|^2\\ &\quad-t_n(1-\rho-t_n)[1-(1-L)\alpha_n]\|z_n-Tz_n\|^2.\end{aligned}$$

Using assumption S_4, we obtain that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \leq{}& [1-(1-L)\alpha_n]\|x_n-x^*\|^2+[1-(1-L)\alpha_n]\alpha_n K_1\\ &+(1-L)\alpha_n\frac{2}{1-L}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle\end{aligned}$$

Setting $\zeta_n=(1-L)\alpha_n$, we obtain that

$$\|x_{n+1}-x^*\|^2 \leq (1-\zeta_n)\Big[\|x_n-x^*\|^2+\alpha_n K_1\Big]+\zeta_n\frac{2}{1-L}\langle f(x^*)-x^*, x_{n+1}-x^*\rangle.$$

□

Theorem 4.1. *Let $\{x_n\}_{n\geq 0}$ be a sequence in H iteratively generated by Algorithm 3.1. Assume that conditions (S_1)-(S_6) hold. Then $\{x_n\}$ converges strongly to $x^*=P_\Omega(f(x^*))$.*

Proof. From Lemma 4.1, we have that $\{x_n\}_{n=1}^\infty$ is bounded. Let $x^*=P_\Omega(f(x^*))$. It follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\leq \|\alpha_n f(v_n)+(1-\alpha_n)v_n-x^*-\alpha_n(f(v_n)-x^*)\|^2+2\alpha_n\langle f(v_n)-x^*, x_{n+1}-x^*\rangle\\ &\leq (1-\alpha_n)\|v_n-x^*\|^2+2\alpha_n\langle f(v_n)-x^*, x_{n+1}-x^*\rangle\\ &\leq (1-\alpha_n)\|x_n-x^*\|^2+2\alpha_n\langle f(v_n)-x^*, x_{n+1}-x^*\rangle\\ &\quad-\frac{1}{\beta}(2-\beta)(1-\alpha_n)\|w_n-z_n\|^2+\alpha_n K_1-(1-\alpha_n)t_n(1-\rho-t_n)\|z_n-Tz_n\|^2.\end{aligned} \tag{4.14}$$

To show that the sequence $\{\|x_{n+1}-x^*\|\}$ converges to zero, we consider two possible cases.

Case 1. Suppose that there exists $n_0\in\mathbb{N}$ such that the real sequence $\|x_n-x^*\|$ is decreasing for all $n\geq n_0$. It then follows that $\|x_n-x^*\|$ is convergent. Since $\{x_n\}$ is bounded, then we obtain from (4.14), assumption S_3, and the fact that $\alpha_n\to 0$ as $n\to\infty$ that

$$\lim_{n\to\infty}\|z_n-Tz_n\|=\lim_{n\to\infty}\|w_n-z_n\|=0. \tag{4.15}$$

Observe that

$$\begin{aligned}\|v_n-z_n\| &= t_n\|z_n-Tz_n\|\to 0 \text{ as } n\to\infty,\\ \|x_{n+1}-v_n\| &= \alpha_n\|f(v_n)-v_n\|\to 0 \text{ as } n\to\infty.\end{aligned} \tag{4.16}$$

Using (4.7), (4.15), and (4.16), we obtain that $\|x_{n+1}-x_n\|\leq\|x_{n+1}-v_n\|+\|v_n-z_n\|+\|z_n-w_n\|+\|w_n-x_n\|\to 0$ as $n\to\infty$. This further yields that $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$.

Next, we prove $\limsup_{n\to\infty}\Big\langle f(x^*)-x^*, x_n-x^*\Big\rangle\leq 0$. Let $\{x_{n_q}\}_{q\geq 0}$ be a subsequence of $\{x_n\}_{n\geq 0}$ such that $\limsup_{n\to\infty}\langle f(x^*)-x^*, x_n-x^*\rangle=\lim_{q\to\infty}\langle f(x^*)-x^*, x_{n_q}-x^*\rangle$. Since $\{x_{n_q}\}_{q\geq 0}$ is a bounded sequence in H, then there exists a subsequence $\{x_{n_{q_b}}\}_{b\geq 0}$ of $\{x_{n_q}\}_{q\geq 0}$ converges weakly to $\bar{x}$ in H. Hence, $x_{n_{q_b}} \rightharpoonup \bar{x}$ as $b \to \infty$. Without loss of generality, we represent $\{x_{n_{q_b}}\}_{b\geq 0}$ by $\{x_{n_b}\}_{b\geq 0}$. It follows from Lemma 4.2 that

$$\|w_{n_b} - y_{n_b}\|^2 \leq \frac{\left(1+\mu\frac{\tau_{n_b}}{\tau_{n_b+1}}\right)^2}{\left[\left(1-\mu\frac{\tau_{n_b}}{\tau_{n_b+1}}\right)\alpha\right]^2}\|z_{n_b} - w_{n_b}\|^2, \qquad \forall\, n_b \geq n_0.$$

Using (4.15), we have that $\|w_{n_b} - y_{n_b}\| \to 0$ as $n \to \infty$. Using Lemma 4.3 and the fact that $x_{n_b} \rightharpoonup \bar{x}$ as $b \to \infty$, we have that $\bar{x} \in VI(C,A)$.

Next, we prove that $\bar{x} \in \Omega$. From (4.15) and the fact that $I - T$ is demiclosed at zero, we can deduce that $\bar{x} \in F(T)$. Therefore, it follows from the argument above that $\bar{x} \in \Omega$. Thus

$$\limsup_{n\to\infty}\left\langle f(x^*) - x^*, x_n - x^*\right\rangle = \lim_{q\to\infty}\left\langle f(x^*) - x^*, x_{n_q} - x^*\right\rangle \leq 0. \tag{4.17}$$

In view of $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, we obtain from (4.17) that $\limsup_{n\to\infty} langlef(x^*) - x^*, x_{n+1} - x^*\rangle \leq 0$. In view of Lemma 4.4, we obtain that

$$\|x_{n+1} - x^*\|^2 \leq (1-\zeta_n)\left[\|x_n - x^*\|^2 + \alpha_n K_1\right] + \zeta_n\frac{2}{1-L}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle.$$

It follows from Lemma 2.3 that $\|x_n - x^*\|$ converges strongly to zero as $n \to \infty$. Hence, $\{x_n\}$ converges to $x^* \in \Omega$. This completes the proof for the first case.

Case 2. If there exists a subsequence $\{\|x_{n_j} - x^*\|\}_{j=0}^{\infty}$ of $\{\|x_n - x^*\|\}_{n\geq 0}$ such that $\|x_{n_j} - x^*\| < \|x_{n_j+1} - x^*\|$ for all $j \geq 0$, then we obtain from Lemma 2.2 that there exists a non-decreasing sequence $\{m_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$ and $\|x_{m_k} - x^*\| \leq \|x_{m_k+1} - x^*\|$ for all $k \in \mathbb{N}$. Since the sequences $\{x_{m_k}\}_{k=1}^{\infty}$ is bounded, we obtain from (4.7), (4.15), and (4.16) that, as $k \to \infty$, $\|w_{m_k} - x_{m_k}\| \to 0$, $\|w_{m_k} - z_{m_k}\| \to 0$, $\|v_{m_k} - z_{m_k}\| \to 0$, and $\|x_{m_k+1} - v_{m_k}\| \to 0$. Observe that $\|x_{m_k+1} - x_{m_k}\| \to 0$ as $k \to \infty$. Following the arguments used in Case 1, we obtain that $\limsup_{k\to\infty}\langle f(x^*) - x^*, x_{m_k} - x^*\rangle \leq 0$, which further implies that $\limsup_{k\to\infty}\langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle \leq 0$. From Lemma 4.4 and the fact that $\|x_{m_k} - x^*\|^2 \leq \|x_{m_k+1} - x^*\|^2$ for any $k \in \mathbb{N}$, we further obtain

$$\begin{aligned}\zeta_{m_k}\|x_{m_k} - x^*\|^2 \leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + \zeta_{m_k}\frac{2}{1-L}\langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle \\ + (1-\zeta_{m_k})\alpha_{m_k}K_1.\end{aligned}$$

Applying Lemma 2.2 yields

$$\begin{aligned}\zeta_{m_k}\|x_k - x^*\|^2 &\leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 \\ &\quad + \zeta_{m_k}\frac{2}{1-L}\langle f(x^*) - x^*, x_{m_k+1} - x^*\rangle + (1-\zeta_{m_k})\alpha_{m_k}K_1.\end{aligned}$$

It follows that $\limsup_{k\to\infty}\|x_k - x^*\| = 0$. Hence, $\lim_{k\to\infty}\|x_k - x^*\| = 0$. Thus $x_n \to x^*$ as $n \to \infty$. This completes the proof. $\square$

Remark 4.1. Theorem 4.1 provides a method for the approximation of solutions of bi-level variational inequalities and the fixed point problems with Lipschitz continuous and pseudomonotone mappings and ρ-demimetric mappings.

5. Numerical Experiments

In this section, we give some numerical examples to show the applicability and convergence of Algorithm 3.1. All numerical computations were carried out via Matlab version R2021(b).

In our experiments, we choose $\tau_0 = \mu = 0.1$, $\alpha_n = \frac{1}{n+1}$, and $\gamma_n = \frac{1}{(n+1)^2}$ for all $n \geq 1$ and select $\beta = \{0.1, 0.5, 1.0, 1.9\}$ in order to find the value of β that gives the best approximate solution.

In the first two examples, we consider the following quadratic programming problem

$$\begin{cases} \min \aleph = x^T \Theta x + Y^T x \\ \text{subject to } x \in \mathbb{R}^n \\ x_i \geq 0, \ i = 1, \cdots, n, \end{cases} \tag{5.1}$$

in an n-dimensional Euclidean space. If Θ is symmetric and positive-definite, then $A = \Theta x + Y$ is pseudo-monotone and Lipschitz continuous with the Lipschitz constant as $L = \|\Theta\|$.

Example 5.1. Consider the choices of Θ and Y for $H = \mathbb{R}^4$ (see [33])

$$\Theta = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \quad Y = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

Define the operators T and f by

$$Tx = \begin{pmatrix} x_1 \\ \sin x_2 \\ x_3 \cos x_3 \\ \sin x_4 \end{pmatrix}, \quad fx = \frac{1}{7}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \in H .$$

The functions, sequences, and constants satisfies conditions of Theorem 4.1 with $x^* \in \Omega$, where $x^* = (1,0,0,0)$. To approximate the solution of (5.1) via Algorithm 3.1, we take $\|x_{n+1} - x_n\| = 10^{-6}$ as the stopping criterion and choose the starting points as follows:

Case 1: Take $x_1 = (5,5,3,4)$ and $x_0 = (0,0.5,0,0.5)$.

Case 2: Take $x_1 = (4,5,4,2)$ and $x_0 = (1,0.5,1,0.5)$.

We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 1 and Table 1.

Table 1. **Numerical Results for Example 5.1**

		$\beta = 0.1$	$\beta = 0.5$	$\beta = 1.0$	$\beta = 1.9$
Case 1	No. of Iter.	61	54	35	105
	CPU time (sec)	0.0162	0.0103	0.0075	0.0133
Case 2	No. of Iter.	62	53	32	104
	CPU time (sec)	0.0148	0.0127	0.0081	0.0182

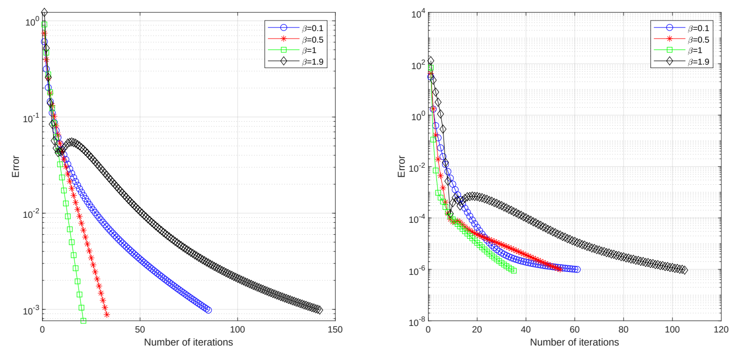

FIGURE 1. Left: Case 1 with various β; Right: Case 2 with various β.

Example 5.2. Consider the choices of Θ and Y for $H = \mathbb{R}^5$ (see [34])

$$\Theta = \begin{pmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 5 & 6 & 6 & 6 \\ 2 & 6 & 9 & 10 & 10 \\ 2 & 6 & 10 & 13 & 14 \\ 2 & 6 & 10 & 14 & 17 \end{pmatrix}, \quad Y = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

Define operators T and f by

$$Tx = \begin{pmatrix} x_1 \\ \sin x_2 \\ x_3 \\ \sin x_4 \\ x_5 \end{pmatrix}, \quad fx = \frac{1}{10}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5, \end{pmatrix}, \quad \forall x \in H \ .$$

It is obvious that the functions, sequences, and constants defined satisfies conditions of Theorem 4.1 with $x^* \in \Omega$ where $x^* = (1,0,0,0,0)$. To approximate the solution of (5.1) via Algorithm 3.1, we take $\|x_{n+1} - x_n\| = 10^{-6}$ as the stopping criterion and choose the starting points as follows:

Case 1: Take $x_1 = (0.5,5,0.5,5,0.5)$ and $x_0 = (1,0.5,1,0.5,1)$.

Case 2: Take $x_1 = (2,1,0.5,5,1)$ and $x_0 = (2,1,1,0.5,2)$.

We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 2 and Table 2.

TABLE 2. **Numerical Results for Example 5.2**

		$\beta = 0.1$	$\beta = 0.5$	$\beta = 1.0$	$\beta = 1.9$
Case 1	No. of Iter.	62	29	21	172
	CPU time (sec)	0.0480	0.0183	0.0162	0.0331
Case 2	No. of Iter.	62	29	21	172
	CPU time (sec)	0.0175	0.0105	0.0084	0.0359

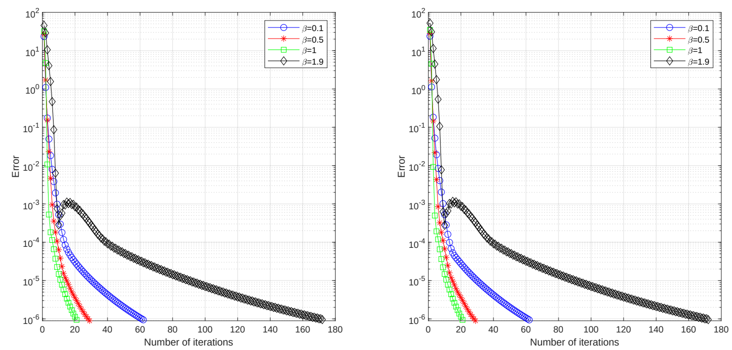

FIGURE 2. Left: Case 1 with various β; Right: Case 2 with various β.

Finally, we give two examples are in infinite dimensional spaces.

Example 5.3. Let $H = (\ell_2(\mathbb{R}), \|.\|_{l_2})$, where $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \dots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} < \infty\}$ and $\|x\|_{\ell_2} := (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$, $\forall x \in \ell_2(\mathbb{R})$. Let $C = \{x \in \ell_2(\mathbb{R}) : |x_i| \leq \frac{1}{i}, i = 1, 2, 3, \dots\}$. Thus, we have explicit formulas for P_C. Now, define the operator $A : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ by

$$Ax = \left(\|x\| + \frac{1}{\|x\| + \alpha}\right) x,$$

for some $\alpha > 0$. Then, it is easy to see that A is pseudomonotone on $\ell_2(\mathbb{R})$ (see [35]). Additionally, define the mapping $T : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ by $Tx = (0, x_1, x_2, \dots)$. Then, T is demimetric on $\ell_2(\mathbb{R})$. We take $\|x_{n+1} - x_n\| = 10^{-3}$ as the stopping criterion and choose the starting points as follows:

Case 1: Take $x_1 = (2, 1, \frac{1}{2}, \dots)$ and $x_0 = (3, 1, \frac{1}{3}, \dots)$.
Case 2: Take $x_1 = (-2, 1, -\frac{1}{2}, \dots)$ and $x_0 = (4, 1, \frac{1}{4}, \dots)$.
Case 3: Take $x_1 = (-2, 1, -\frac{1}{2}, \dots)$ and $x_0 = (-3, 1, -\frac{1}{3}, \dots)$.
Case 4: Take $x_1 = (2, 1, \frac{1}{2}, \dots)$ and $x_0 = (-4, 1, -\frac{1}{4}, \dots)$.

We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 3 and Table 3.

Example 5.4. Consider the infinite dimensional Hilbert space $H = L^2[0,1]$ with inner product (see [36]) $\langle u, v \rangle = \int_0^1 u(t)v(t)dt$, $\forall u, v \in H$ and corresponding norm $\|u\| = \left(\int_0^1 u(t)^2 dt\right)^{\frac{1}{2}}$, $\forall u \in H$. Let p, R be two real numbers such that $R > p > \left(\frac{k}{k+1}\right) R > 0$ for some $k > 1$. Take the feasible set $C = \{u \in H : \|u\| \leq p\}$ and the operator $A : H \to H$ given by $A(x) = (R - \|x\|)x$, $\forall x \in H$. Note that the operator A is not monotone. Since $\frac{R}{k+1} < \frac{p}{k} < p$, we can choose $\bar{x} \in C$ such that $\frac{R}{k+1} < \|\bar{x}\| < \frac{p}{k}$ and then set $\bar{y} = k\bar{x}$. Also, since $\|\bar{y}\| = k\|\bar{x}\| < k.(\frac{p}{k}) = p$, it is easy to see that $\bar{y} \in C$. Moreover, since $\|\bar{x}\| > \frac{R}{k+1} > 0$ and $k > 1$, we obtain $\langle A(\bar{x}) - A(\bar{y}), \bar{x} - \bar{y} \rangle = (1-k)^2 \|\bar{x}\|^2 (R - (1+k)\|\bar{x}\|) < 0$. It is easy to see that A is pseudomonotone on C. Indeed, if $\langle A(x), y - x \rangle \geq 0$ for all $x, y \in C$, i.e., $\langle (R - \|x\|)x, y - x \rangle \geq 0$, then $\langle x, y - x \rangle \geq 0$ due to $\|x\| \leq p < R$. Thus $\langle A(y), y - x \rangle \geq (R - \|y\|) y (\langle y, y - x \rangle - \langle x, y - x \rangle) (R - \|y\|) \|y - x\|^2 \geq 0$, due to $\|y\| < p < R$. Define the mapping $T : H \to H$ by $Tx = \frac{x}{2}$. Then, it is easy to show that T is demimetric on H.

TABLE 3. **Numerical Results for Example 5.3**

		$\beta = 0.1$	$\beta = 0.5$	$\beta = 1.0$	$\beta = 1.9$
Case 1	No. of Iter.	85	35	21	142
	CPU time (sec)	0.0836	0.0220	0.0271	0.0725
Case 2	No. of Iter.	84	33	21	142
	CPU time (sec)	0.0309	0.0146	0.0101	0.0392
Case 3	No. of Iter.	84	33	21	142
	CPU time (sec)	0.0168	0.0063	0.0047	0.0266
Case 4	No. of Iter.	85	33	21	142
	CPU time (sec)	0.0167	0.0048	0.0028	0.0221

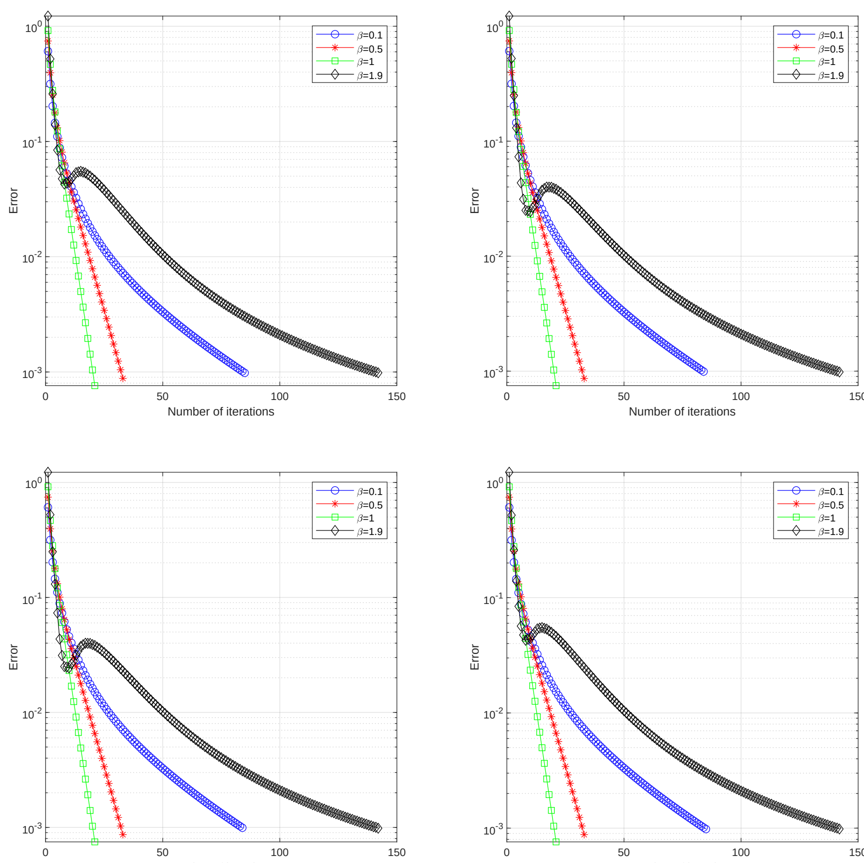

FIGURE 3. Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

For our numerical result, we choose $R = 1.5$, $p = 1$, and $k = 1.1$. The exact solution of the problem is $x^*(t) = 0$. Since the solution is known, we use the sequence $E(x_n) = \|x_n - x^*\|^2$

for each $n \geq 0$ to illustrate the numerical behaviour of our algorithm. The projection on C is computed by the explicit formula $P_C(x) = x$ if $\|x\| \leq p$ and $P_C(x) = \frac{px}{\|x\|}$ if $\|x\| > p$. We take $\|x_{n+1} - x_n\| = 10^{-3}$ as the stopping criterion and choose the starting points as follows:

Case 1: Take $x_1 = t^2$ and $x_0 = t^3 \cos t$.
Case 2: Take $x_1 = t^3$ and $x_0 = exp(-2t)$.
Case 3: Take $x_1 = t^4$ and $x_0 = \sin 2t$.
Case 4: Take $x_1 = t^5$ and $x_0 = t^2 \exp(-t)$.

We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 4 and Table 4.

TABLE 4. **Numerical Results for Example 5.4**

		$\beta = 0.1$	$\beta = 0.5$	$\beta = 1.0$	$\beta = 1.9$
Case 1	No. of Iter.	9	8	7	6
	CPU time (sec)	2.2148	0.8072	0.6564	0.6117
Case 2	No. of Iter.	9	8	7	6
	CPU time (sec)	1.0137	0.6741	0.6032	0.5348
Case 3	No. of Iter.	9	8	7	6
	CPU time (sec)	1.1558	0.7063	0.6036	0.5523
Case 4	No. of Iter.	9	8	7	6
	CPU time (sec)	1.2158	0.8600	0.6336	0.5204

Remark 5.1. Examples 5.1, 5.2, 5.3, and 5.4 demonstrate that even though our assumption is that $\beta \in (0, 2)$, our proposed method perform best in terms of number of iterations and CPU time when $\beta \approx 1$.

6. CONCLUSION

In this paper, we proposed an inertia projection contraction viscosity type scheme to find the solution of a bi-level variational inequality and a fixed point problem with the operators that are Lipschitz pseudo-monotone and ρ-demimetric. We proved that our iterative scheme is strongly convergent in the framework of Hilbert spaces. We also gave several numerical examples to illustrate the performance and applicability of our method as well as testing the effect of key parameters.

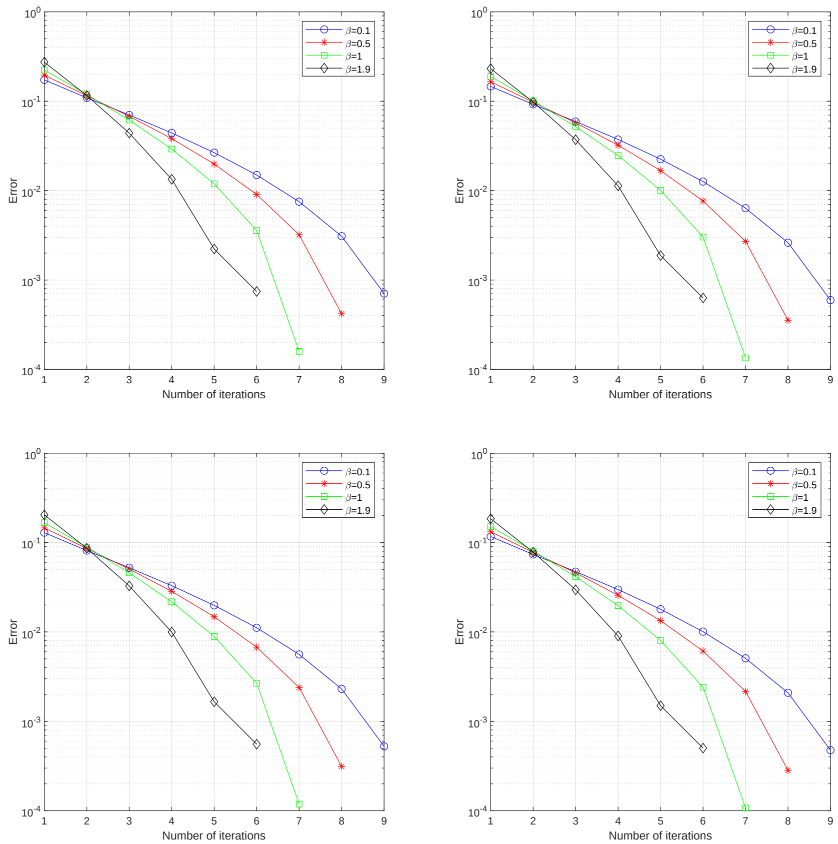

FIGURE 4. Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

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