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ERGODIC AND FIXED POINT THEOREMS FOR BREGMAN NONEXPANSIVE SEQUENCES AND MAPPINGS IN BANACH SPACES

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Dedicated to the memory of Professor Ronald E. Bruck

Abstract. In this paper, we introduce the notion of Bregman nonexpansive sequences and mappings in a Banach space, prove an ergodic theorem and a fixed point theorem, and some weak convergence theorems for such sequences and maps with rather mild assumptions on the Banach space. These results extend our previous results for ϕ-nonexpansive sequences and give a partial answer to the open problem raised in 1981 by Djafari Rouhani for nonexpansive sequences in a Banach space.

Keywords. Asymptotic behavior; Asymptotically regular sequence; Bregman nonexpansive sequence; Strong convergence; Weak convergence.

1. INTRODUCTION

Let E be a real Banach space with norm $|\cdot|$, and let C be a nonempty subset of E. A mapping $T : C \to C$ is said to be nonexpansive if $|Tx - Tx| \leq |x - y|$, $\forall x, y \in C$. When C is closed and convex, and E is a real Hilbert space, the first mean ergodic theorem for nonexpansive self-mappings on C was proved by Baillon [1] in 1975. He subsequently extended his theorem in [2] to L^p spaces in 1978; see also Reich [3]. In this connection, the early papers by Reich are [4, 5, 6]; see also [7]. Bruck [8] in 1979 gave a simple proof of Baillon's ergodic theorem in uniformly convex Banach spaces with a Fréchet differentiable norm. Hirano [9] in 1980 gave another proof of Bruck's mean ergodic theorem. Other results in this direction can be found, for example, in [10, 11].

Djafari Rouhani [12] in 1981 introduced the notion of nonexpansive sequences, and gave a simple proof to Baillon's ergodic theorem in Hilbert space, without assuming the convexity of the domain C of the nonexpansive mapping. He also extended this notion to almost nonexpansive sequences and curves, as well as to hybrid sequences, and applied the results to the study of the asymptotic behavior of some evolution systems in a Hilbert space; see, e.g., [13, 14, 15, 16, 17] and the references therein. It was stated in [12] as an open problem, whether the mean ergodic theorem for nonexpansive sequences can be extended to the setting of a Banach space. To the best of our knowledge, this problem is still open to this date. More recently,

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in order to study nonlinear problems in the Banach space setting, the following function ϕ was introduced: $\phi(x,y) = |x|^2 - 2\langle x, Jy\rangle + |y|^2$, $\forall x,y \in E$, where $J : E \to E^*$ is the normalized duality mapping of E. When E is a Hilbert space, $\phi(x,y) = |x-y|^2$. See [18, 19, 20] and the references therein for introducing the pseudo-distance $\phi(x,y)$ and its properties. A partial answer to the above problem was given in [21] for ϕ-nonexpansive sequences in a Banach space. A more general pseudo-distance is the Bregman distance $D_g(x,y) := g(x) - g(y) - \langle \nabla g(y), x-y\rangle$, where $g : E \to]-\infty, +\infty]$ is a suitable function. If $g(x) = |x|^2$, then $D_g(x,y) = \phi(x,y)$. While the open problem stated above is still unsolved, in this paper, we would like to give a partial answer to that problem, by introducing the notion of Bregman nonexpansive sequences and maps in a Banach space E, extending our previous results for ϕ-nonexpansive sequences, and prove an ergodic theorem, a fixed point theorem, and some weak convergence theorems for such sequences and maps in E, with rather mild assumptions on the Banach space E, weaker than those used in [2, 8, 9].

2. Preliminaries

Let E be a real Banach space with norm $|\cdot|$. E^* denotes the topological dual of E. The duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{v \in E^* : \langle x, v\rangle = |x|^2 = |v|^2\}.$$

A Banach space E is said to be *strictly convex* if $|\frac{x+y}{2}| < 1$ for all $x,y \in E$ with $|x| = |y| = 1$ and $x \neq y$. It is said to be *uniformly convex* if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that, for all $x,y \in E$ with $|x| = |y| = 1$ and $|x-y| \geq \varepsilon$, it holds that $|\frac{x+y}{2}| < 1-\delta$. It is known that uniformly convex Banach spaces are reflexive and strictly convex.

A Banach space E is said to be *smooth* if

$$\lim_{t\to 0} \frac{|x+ty| - |x|}{t} \tag{2.1}$$

exists for all $x,y \in S = \{z \in E : |z| = 1\}$. It is said to be *uniformly smooth* if the limit in (2.1) is attained uniformly for $x,y \in S$. It is well known that the spaces L^p $(1 < p < \infty)$ and the Sobolev spaces $W^{k,p}$ $(1 < p < \infty)$ are both uniformly convex and uniformly smooth.

It is well known that when E is smooth the duality mapping J is single valued. Let E be a smooth Banach space. We define $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x,y) = |x|^2 - 2\langle x, J(y)\rangle + |y|^2.$$

This function can be seen as a "distance-like" function, better conditioned than the square of the metric distance, namely $|x-y|^2$; see, e.g., [18], [19], and [22].

It is easy to see that $0 \leq (|x| - |y|)^2 \leq \phi(x,y)$ for all $x,y \in E$. In Hilbert spaces, where the duality mapping J is the identity operator, it holds that $\phi(x,y) = |x-y|^2$.

Throughout this paper, when $\{x_k\}$ is a sequence in E, we denote the strong convergence of $\{x_k\}$ to $x \in E$ by $x_k \to x$, and the weak convergence by $x_k \rightharpoonup x$.

We recall that the duality mapping $J : E \to E^*$ is said to be *sequentially weak-to-weak* continuous* if, for any sequence $\{x_k\} \subset E$, which is weakly convergent to $x \in E$, it holds that $\{J(x_k)\} \subset E^*$ is weak* convergent to $J(x) \in E^*$.

Consider a function $g : E \to]-\infty, +\infty]$. The domain of g is defined by $D(g) := \{x \in E : g(x) < +\infty\}$, and g is called proper if $D(g) \neq \emptyset$. Let $x \in \mathrm{int}(D(g))$. For any $y \in E$, we define

$$g^{\circ}(x,y) := \lim_{t \to 0} \frac{g(x+ty) - g(x)}{t}.$$

If the limit exists for any $y \in E$ and is a bounded linear functional with respect to y, the function g is called Gâteaux differentiable at x. In this case, the gradient of g at x, which is denoted by $\nabla g(x)$, is defined by $\langle \nabla g(x), y \rangle = g^{\circ}(x,y)$, $\forall y \in E$. The function f is said to be Gâteaux differentiable if it has a Gâteaux differential at each $x \in \mathrm{int}(D(g))$.

A function $g : E \to]-\infty, +\infty]$ is called:
(i) convex iff

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y), \ \ \forall x,y \in E, \ \forall\, 0 < \lambda < 1$$

(ii) strictly convex iff

$$g(\lambda x + (1-\lambda)y) < \lambda g(x) + (1-\lambda)g(y), \ \ \forall x \neq y \in E, \ \forall\, 0 < \lambda < 1.$$

It is well-known that g is (strictly) convex and Gâteaux differentiable iff
(i) $g(x) - g(y) \geq (>) \langle \nabla g(y), x-y \rangle, \ \ \forall x \in E, \ \ \forall y \in \mathrm{int}(D(g))$.
(ii) $\langle \nabla g(x) - \nabla g(y), x-y \rangle \geq (>) 0, \ \ \forall x,y \in \mathrm{int}(D(g))(x \neq y)$.

Definition 2.1. (Fenchel conjugate) The Fenchel conjugate of g is the function $g^* : E^* \to]-\infty, +\infty]$ defined by

$$g^*(x^*) = \sup_{x \in E} \Big\{ \langle x, x^* \rangle - g(x) \Big\}.$$

Definition 2.2. (Legendre function) [23] The function g is called a *Legendre function* if it satisfies the following two conditions:

(i) $\mathrm{int}(D(g)) \neq \emptyset$, g is Gâteaux differentiable and $D(\nabla g) = \mathrm{int}(D(g))$.
(ii) $\mathrm{int}(D(g^*)) \neq \emptyset$, g^* is Gâteaux differentiable and $D(\nabla g^*) = \mathrm{int}(D(g^*))$.

Definition 2.3. (Bregman distance) [24] For a convex function $g : E \to]-\infty, +\infty]$, the *Bregman distance* with respect to g, is defined by $D_g(x,y) = g(x) - g(y) - \langle \nabla g(y), x-y \rangle, \ \ \forall x \in D(g), \ \forall y \in \mathrm{int}(D(g))$.

Obviously, by the above characterization in (i), $D_g(x,y) \geq 0$, but D_g does not satisfy all the properties of a metric. More information on Bregman functions and distances can be found in [25].

Definition 2.4. (Total convexity) [26, 27] For a convex function $g : E \to]-\infty, +\infty]$, the modulus of total convexity of g at $x \in \mathrm{int}(D(g))$, $v_g(x, \cdot) : [0, \infty] \to [0, +\infty]$ is defined by $v_g(x,t) = \inf\{D_g(y,x) : y \in D(g), \ |y - x| = t\}$. g is said to be totally convex at $x \in \mathrm{int}(D(g))$, if $v_g(x,t) > 0$ whenever $t > 0$. g is called totally convex if it is totally convex at every point $x \in \mathrm{int}(D(g))$.

It is easily seen that if g is totally convex and Gâteaux differentiable, then

$$\langle \nabla g(x) - \nabla g(y), x-y \rangle > 0, \ \ \forall x,y \in \mathrm{int}(D(g)), \ x \neq y.$$

In this connection, see [28].

Definition 2.5. (Bregman projection) [24] Let $K \subset \operatorname{int}(D(g))$ be a nonempty, closed, and convex subset of a Banach space E. For each $x \in \operatorname{int}(D(g))$, the convex function $D_g(\cdot,x)$ has a unique minimizer over K, called the Bregman projection of x onto K, i.e.,

$$P_K^g(x) := \operatorname{Argmin}\{D_g(y,x) : \ y \in K\}.$$

Proposition 2.1. *(Bregman projection properties)* [29, Corollary 4.4, p. 23] *Let $g : E \to]-\infty,+\infty]$ be a Gâteaux differentiable and totally convex function. Let $x \in \operatorname{int}(D(g))$, and let $K \subset \operatorname{int}(D(g))$, be nonempty, closed, and convex, then the following statements are equivalent:*

(i) *$\hat{x} \in K$ is the Bregman projection of x onto K.*

(ii) *$\hat{x}$ is the unique solution of the variational inequality*

$$\langle \nabla g(x) - \nabla g(z), z - y \rangle \geq 0, \ \ \forall y \in K;$$

(iii) *$\hat{x}$ is the unique solution to the inequality*

$$D_g(y,z) + D_g(z,x) \leq D_g(y,x), \ \ \forall y \in K.$$

Definition 2.6. (Sequentially consistent) [29] The function $g : E \to]-\infty,+\infty]$ is called sequentially consistent if, for any two sequence $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded,

$$\lim_{n\to\infty} D_g(y_n,x_n) = 0 \Rightarrow \lim_{n\to\infty} |x_n - y_n| = 0.$$

Proposition 2.2. [27, Lemma 2.1.2, p. 67] *The function g is totally convex on bounded subsets if and only if it is sequentially consistent.*

Proposition 2.3. *(Boundedness property)* [30, Lemma 3.1] *Let $g : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$, and the sequence $\{D_g(x_n,x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Proposition 2.4. *(Boundedness property)* [31] *Let $g : E \to]-\infty,+\infty]$ be a Legendre function such that ∇g^* is bounded on bounded subsets of* $\operatorname{int}(D(g^*))$. *Let $x \in \operatorname{int}(D(g))$. If $D_g(x,x_n)$ is bounded, so is the sequence $\{x_n\}$.*

Definition 2.7. (g-asymptotic center) Let $g : E \to]-\infty,+\infty]$ be a proper, strictly convex, and lower semi-continuous function with domain $D(g)$. Let $\{x_n\}$ be a bounded sequence in $\operatorname{int}(D(g))$ and

$$\lim_{|y|\to\infty} \limsup_{n\to\infty} g(y,x_n) = \infty. \tag{2.2}$$

For each $y \in D(g)$, the convex function $\limsup_{n\to\infty} D_g(y,x_n)$ has a unique minimizer on $D(g)$ if E is reflexive. We call this unique minimizer, the g-asymptotic center of $\{x_n\}$.

3. Bregman Nonexpansive Sequences

Throughout the paper, we assume that $g : E \to \mathbb{R}$ is convex and Gâteaux differentiable with $D(g) = E$.

Definition 3.1. Let E be a reflexive Banach space. A sequence $\{x_k\}$ in E is called a Bregman nonexpansive sequence, whenever

$$D_g(x_{i+1},x_{j+1}) \leq D_g(x_i,x_j), \qquad \forall i,j \geq 0.$$

Let

$$F_* := \left\{ q \in E \,\middle|\, D_g(q, x_k) \text{ is nonincreasing} \right\}.$$

It is easy to see that F_* is a (possibly empty) closed and convex set. We also denote $s_n := \frac{1}{n}\sum_{k=1}^{n} x_k$.

Proposition 3.1. *Let E be a reflexive Banach space, and $\{x_k\}$ a Bregman nonexpansive sequence in E. If $\liminf_{n\to\infty} |s_n| < +\infty$, then every weak cluster point of s_n is in F_*; in particular $F_* \neq \emptyset$.*

Proof. Assume $s_{n_j} \rightharpoonup p$. Also, we can write

$$\begin{aligned}
&D_g(x_i, x_k) - D_g(x_i, x_{k+1}) \\
&= g(x_{k+1}) - g(x_k) - \langle \nabla g(x_k), x_i - x_k \rangle + \langle \nabla g(x_{k+1}), x_i - x_{k+1} \rangle \\
&= g(x_{k+1}) - g(x_k) - \langle \nabla g(x_k), x_i - x_k \rangle + \langle \nabla g(x_{k+1}) - \nabla g(x_k), x_i - x_{k+1} \rangle \\
&\quad + \langle \nabla g(x_k), x_i - x_{k+1} \rangle \\
&= g(x_{k+1}) - g(x_k) + \langle \nabla g(x_{k+1}) - \nabla g(x_k), x_i - x_{k+1} \rangle + \langle \nabla g(x_k), x_k - x_{k+1} \rangle.
\end{aligned}$$

From the Bregman nonexpansivity of the sequence $\{x_k\}$, we have

$$\begin{aligned}
&D_g(x_i, x_k) - D_g(x_{i-1}, x_k) \\
&\leq g(x_{k+1}) - g(x_k) + \langle \nabla g(x_{k+1}) - \nabla g(x_k), x_i - x_{k+1} \rangle + \langle \nabla g(x_k), x_k - x_{k+1} \rangle.
\end{aligned}$$

Summing up both sides of the above inequality from $i = 1$ to $i = n$ and dividing by n, we have

$$\begin{aligned}
&\frac{1}{n}\big(D_g(x_n, x_k) - D_g(x_0, x_k)\big) \\
&\leq g(x_{k+1}) - g(x_k) + \langle \nabla g(x_{k+1}) - \nabla g(x_k), s_n - x_{k+1} \rangle + \langle \nabla g(x_k), x_k - x_{k+1} \rangle.
\end{aligned}$$

Replacing n by n_j and taking the limit as $j \to \infty$, we have

$$g(x_{k+1}) - g(x_k) + \langle \nabla g(x_{k+1} - \nabla g(x_k), p - x_{k+1} \rangle + \langle \nabla g(x_k), x_k - x_{k+1} \rangle \geq 0$$

It follows that $D_g(p, x_{k+1}) \leq D_g(p, x_k)$ That is, $\{D_g(p, x_k)\}$ is nonincreasing. Thus every weak subsequential limit p of $\{s_n\}$ belongs to F_*. □

Open Problem. We do not know whether s_n converges weakly to an element of F_*?

Definition 3.2. A sequence $\{x_k\}$ is weakly asymptotically regular if $x_{k+1} - x_k \rightharpoonup 0$ as $k \to \infty$.

Theorem 3.1. *Let E be a reflexive Banach space, and $g : E \to \mathbb{R}$ be a strictly convex and Legendre function such that g and ∇g^* are bounded on bounded subsets of E. Suppose that $\{x_k\}$ is a Bregman nonexpansive and weakly asymptotically regular sequence in E. Then the following statements are equivalent:*
(i) $\liminf_{k\to\infty} |x_k| < +\infty$.
(ii) *$\{x_k\}$ is bounded.*
(iii) *each weak limit point p of $\{x_k\}$ belongs to F_*. Moreover, if $\nabla g : E \to E^*$ is sequentially weak-to-weak* continuous, then $\{x_k\}$ converges weakly to an element $p \in F_*$. Also, p is the g-asymptotic center of the sequence $\{x_k\}$, if g satisfies (2.2).*

Proof. (i)⇒ (ii): Since $\liminf_{k\to\infty}|x_k| < +\infty$, there is a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ and $p \in E$ such that $x_{k_n} \rightharpoonup p$. On the other hand, since $x_{k+1} - x_k \rightharpoonup 0$ as $k \to \infty$, it is easy to see that

$$\frac{1}{m}\sum_{j=1}^{j=m} x_{k_n+j} \rightharpoonup p, \quad \forall m \in \mathbb{N}.$$

Note that

$$\begin{aligned}
&D_g(p,x_{k+1}) - D_g(p,x_k)\\
&= g(x_k) - g(x_{k+1}) - \langle \nabla g(x_{k+1}), p - x_{k+1}\rangle + \langle \nabla g(x_k), p - x_k\rangle\\
&= g(x_k) - g(x_{k+1}) + g(x_{k_n+j}) - g(x_{k_n+j}) - \langle \nabla g(x_{k+1}), p - x_{k_n+j}\rangle\\
&\quad - \langle \nabla g(x_{k+1}), x_{k_n+j} - x_{k+1}\rangle + \langle \nabla g(x_k), p - x_{k_n+j}\rangle + \langle \nabla g(x_k), x_{k_n+j} - x_k\rangle\\
&= D_g(x_{k_n+j}, x_{k+1}) - D_g(x_{k_n+j}, x_k) + \langle \nabla g(x_k) - \nabla g(x_{k+1}), p - x_{k_n+j}\rangle.
\end{aligned}$$

Now, since $\{x_k\}$ is Bregman nonexpansive, we have

$$\begin{aligned}
&D_g(p,x_{k+1}) - D_g(p,x_k)\\
&\leq D_g(x_{k_n+j-1}, x_k) - D_g(x_{k_n+j}, x_k) + \langle \nabla g(x_k) - \nabla g(x_{k+1}), p - x_{k_n+j}\rangle.
\end{aligned} \tag{3.1}$$

Summing up both sides of (3.1) from $j = 1$ to $j = m$, and dividing by m, we obtain

$$\begin{aligned}
D_g(p,x_{k+1}) - D_g(p,x_k) \leq \frac{1}{m}\big(D_g(x_{k_n}, x_k) - D_g(x_{k_n+m}, x_k)\big)\\
+ \langle \nabla g(x_k) - \nabla g(x_{k+1}), p - \frac{1}{m}\sum_{j=1}^{m} x_{k_n+j}\rangle.
\end{aligned}$$

Taking the limsup as $n \to +\infty$ and then letting $m \to \infty$, we have $D_g(p,x_{k+1}) - D_g(p,x_k) \leq 0$. This implies that the sequence $\{D_g(p,x_k)\}$ is nonincreasing. Thus $\lim_{k\to\infty} D_g(p,x_k)$ exists. Now Proposition 2.4 shows that the sequence $\{x_k\}$ is bounded.

(ii)⇒ (iii): Note that the proof of part (i) shows that $\{D_g(p,x_k)\}$ is nonincreasing for any weak limit point p of $\{x_k\}$. Therefore, $p \in F_*$. Now, let p and q be two weak limit points of $\{x_k\}$. Then there are two subsequences $\{x_{k_n}\}$ and $\{x_{k_i}\}$ of $\{x_k\}$ such that $x_{k_n} \rightharpoonup p$ and $x_{k_i} \rightharpoonup q$. Note that

$$D_g(p,x_k) - D_g(q,x_k) = g(p) - g(q) + \langle \nabla g(x_k), q - p\rangle \tag{3.2}$$

Now, replacing k by k_n in (3.2), and then taking the limit as $n \to \infty$, since ∇g is sequentially weak-to-weak* continuous, we have

$$\lim_{n\to\infty} D_g(p,x_{k_n}) - D_g(q,x_{k_n}) = g(p) - g(q) - \langle p - q, \nabla g(p)\rangle. \tag{3.3}$$

Once again, replacing k by k_i in (3.2), and then taking the limit as $i \to \infty$, we obtain

$$\lim_{i\to\infty} D_g(p,x_{k_i}) - D_g(q,x_{k_i}) = g(p) - g(q) - \langle p - q, \nabla g(q)\rangle. \tag{3.4}$$

Now, we conclude from (3.3) and (3.4) that

$$\langle p - q, \nabla g(p) - \nabla g(q)\rangle = 0.$$

The strict convexity of g shows that $p = q$, i.e., $\{x_k\}$ has only one weak cluster point, and hence it converges weakly to a point $p \in F_*$.

Now we prove that the weak limit point p is also the g-asymptotic center of the sequence $\{x_k\}$. Note that, for any $x \in E$,

$$D_g(p,x_k) = D_g(x,x_k) + D_g(p,x) + \langle p - x, \nabla g(x) - \nabla g(x_k)\rangle.$$

Since $x_k \rightharpoonup p$, ∇g is sequentially weak-to-weak* continuous and $\lim_{k\to\infty} D_g(p,x_k)$ exists, we obtain by taking the limsup as $k \to \infty$ that

$$\lim_{k\to\infty} D_g(p,x_k) = \limsup_{k\to\infty} D_g(x,x_k) - D_g(x,p).$$

Now since $D_g(x,p) \geq 0$, it follows that p is the g-asymptotic center of $\{x_k\}$.

(iii)$\Rightarrow$ (i): It is obvious.

□

Proposition 3.2. *Let E be a reflexive Banach space, and let $g : E \to \mathbb{R}$ be totally convex, strictly convex, and Legendre. Also assume that g and ∇g^* are bounded on bounded subsets of E and E^* respectively. Let $\{x_k\}$ be any sequence in E for which $F_* \neq \emptyset$ and the weak subsequential limits of $\{x_k\}$ belong to F^*. If $y_k = P_{F_*}(x_k)$, where P_{F_*} is the generalized projection onto F_* and*

$$\lim_{|x|\to\infty} (g(x) - b|x - z|) = +\infty, \ \forall\, z \in E, \text{ and } \ \forall\, b > 0, \tag{3.5}$$

then $\{y_k\}$ converges strongly to a point $q \in F_*$. Moreover, if ∇g is sequentially weak-to-weak continuous, then the limit point q is the weak limit of x_k.*

Proof. (i) Since $y_{k+1} = P_{F_*}(x_{k+1})$, we have by Proposition 2.1 that

$$\langle z - y_{k+1}, \nabla g(x_{k+1}) - \nabla g(y_{k+1})\rangle \leq 0$$

for all $z \in F_*$. Since $y_k \in F_*$, we have by setting $z = y_k$

$$\langle y_k - y_{k+1}, \nabla g(x_{k+1}) - \nabla g(y_{k+1})\rangle \leq 0,$$

which implies that

$$D_g(y_{k+1},x_{k+1}) + D_g(y_k,y_{k+1}) \leq D_g(y_k,x_{k+1}). \tag{3.6}$$

On the other hand, we have

$$D_g(y_k,x_{k+1}) \leq D_g(y_k,x_k) \tag{3.7}$$

due to $y_k \in F_*$. Now, (3.6) and (3.7) imply that $D_g(y_{k+1},x_{k+1}) \leq D_g(y_k,x_k)$. That is, $\{D_g(y_k,x_k)\}$ is nonincreasing and therefore $\lim_{k\to\infty} D_g(y_k,x_k)$ exists. Let k and n be arbitrary. Since $y_{k+n} = P_{F_*}(x_{k+n})$, the same argument as above shows that

$$D_g(y_{k+n},x_{k+n}) + D_g(y_k,y_{k+n}) \leq D_g(y_k,x_{k+n}). \tag{3.8}$$

On the other hand, we have

$$D_g(y_k,x_{k+n}) \leq D_g(y_k,x_k), \tag{3.9}$$

Now (3.8) and (3.9) imply that

$$D_g(y_k,y_{k+n}) \leq D_g(y_k,x_k) - D_g(y_{k+n},x_{k+n}).$$

Since $\lim_{k\to\infty} D_g(y_k,x_k)$ exists, then $\lim_{k\to\infty} D_g(y_k,y_{k+n}) = 0$ uniformly for all n. Also (3.5) implies that y_k is bounded. Now Proposition 2.2 implies that $\lim_{k\to\infty} |y_k - y_{k+n}| = 0$ uniformly for all n. Thus $\{y_k\}$ is a Cauchy sequence and hence there is a $q \in F_*$ such that $\lim_{k\to\infty} y_k = q$.

(ii) Since $y_k = P_{F_*}(x_k)$, we have by Proposition 2.1 that $\langle z - y_k, \nabla g(x_k) - \nabla g(y_k)\rangle \leq 0$, $\forall z \in F_*$. Then we have

$$\langle z - y_k, \nabla g(x_k)\rangle \leq \langle z - y_k, \nabla g(y_k)\rangle,\ \forall z \in F_*. \tag{3.10}$$

Let $\{x_{k_i}\}$ be a subsequence of $\{x_k\}$ such that $x_{k_i} \rightharpoonup p \in F_*$. Replacing k by k_i in (3.10) and z by p, and taking the limit as $i \to \infty$, we have $\langle p - q, \nabla g(p)\rangle \leq \langle p - q, \nabla g(q)\rangle$ because $y_k \to q$ by part (i). This implies that $D_g(p,q) + D_g(q,p) \leq 0$. Now the strict convexity of g shows that $p = q$. This shows that x_k converges weakly to the strong limit point of y_k. □

Proposition 3.3. *Let E be a reflexive Banach space, and let $g : E \to \mathbb{R}$ be such that ∇g^* is continuous on $D(g^*)$ and $\langle \nabla g^*(x^*), x^*\rangle \geq |x^*|^2$. Suppose that $\{x_k\}$ is any sequence in E. If* $\operatorname{int}(F_*) \neq \emptyset$*, then $\{x_k\}$ converges strongly to an element of E. Moreover, if a weak limit point of $\{x_k\}$ belongs to F_*, then $\{x_k\}$ converges strongly to an element of F_*.*

Proof. Since $\operatorname{int}(F_*) \neq \emptyset$, there exist $r > 0$ and $p \in \operatorname{int}(F_*)$ such that $\bar{B}_r(p) \subset \operatorname{int}(F_*)$. Since E is reflexive, by [32, p. 83], $(\nabla g)^{-1} = \nabla g^* : E^* \to E$. Now, if $|\nabla g(x_k) - \nabla g(x_{k-1})| \neq 0$ by letting $q = p - r\frac{\nabla g^*(\nabla g(x_k) - \nabla g(x_{k-1}))}{|\nabla g(x_k) - \nabla g(x_{k-1})|}$, we have

$$\begin{aligned}
0 &\leq D_g(q, x_{k-1}) - D_g(q, x_k)\\
&= \langle q - x_k, \nabla g(x_k) - \nabla g(x_{k-1})\rangle + D_g(x_k, x_{k-1})\\
&= -r\langle \frac{\nabla g^*(\nabla g(x_k) - \nabla g(x_{k-1}))}{|\nabla g(x_k) - \nabla g(x_{k-1})|}, \nabla g(x_k) - \nabla g(x_{k-1})\rangle\\
&\quad + \langle p - x_k, \nabla g(x_k) - \nabla g(x_{k-1})\rangle + D_g(x_k, x_{k-1})\\
&\leq -r|\nabla g(x_k) - \nabla g(x_{k-1})| + \langle p - x_k, \nabla g(x_k) - \nabla g(x_{k-1})\rangle + D_g(x_k, x_{k-1})\\
&= -r|\nabla g(x_k) - \nabla g(x_{k-1})| + D_g(p, x_{k-1}) - D_g(p, x_k),
\end{aligned}$$

which implies that

$$r|\nabla g(x_k) - \nabla g(x_{k-1})| \leq D_g(p, x_{k-1}) - D_g(p, x_k). \tag{3.11}$$

Also, if $|\nabla g(x_k) - \nabla g(x_{k-1})| = 0$, then (3.11) is satisfied because $\{D_g(p, x_k)\}$ is nonincreasing. Summing up (3.11) from $k = 1$ to $k = n$, we obtain

$$r\sum_{k=1}^{n} |\nabla g(x_k) - \nabla g(x_{k-1})| \leq D_g(p, x_0) - D_g(p, x_n). \tag{3.12}$$

Therefore we may conclude from (3.12) that $\sum_{k=1}^{\infty} |\nabla g(x_k) - \nabla g(x_{k-1})| < +\infty$. It follows that $\nabla g(x_k) \to x^*$. Since ∇g^* is continuous, then $\{x_k\}$ converges strongly to an element in E. It is also obvious that if a weak limit point of $\{x_k\}$ belongs to F_*, then $\{x_k\}$ converges strongly to an element of F_*. □

4. Bregman Nonexpansive Mappings Defined on Convex Sets

In this section, we show that if the Bregman nonexpansive mapping is defined on a convex subset of E, then we can obtain better and stronger results than in Section 3, with weaker assumptions. Let E be a reflexive Banach space, and let C be a nonempty, closed, and convex subset of E. Throughout this section, we assume that $g : E \to \mathbb{R}$ is convex and Gâteaux differentiable. A mapping $T : C \to C$ is said to be a Bregman nonexpansive mapping if

$$D_g(Tx, Ty) \leq D_g(x, y),\ \ \forall\, x, y \in C.$$

Let $x \in C$. We denote $x_k = T^k x$, $k \geq 0$, $s_n = \frac{1}{n}\sum_{k=0}^{n-1} x_k$ (see [31, 33]), and

$$F := \left\{ q \in E \,\middle|\, D_g(q, x_k) \text{ is non-increasing} \right\}.$$

F is a closed, convex, and possibly empty subset of E. By Proposition 3.1 and Proposition 2.4, we know that if g to is Legendre, and ∇g^* is bounded on bounded subsets of $\operatorname{int}(D(g^*))$, then $\liminf_{n\to\infty} |s_n| < +\infty$ if and only if $\{x_k\}$ is bounded. Moreover, if $s_{n_i} \rightharpoonup p$, then $p \in F$. The following proposition is basically a reformulation of Proposition 3.1 for Bregman nonexpansive mappings.

Proposition 4.1. *Assume that E is a reflexive Banach space and C is an arbitrary nonempty subset of E. $T : C \to C$ is a Bregman nonexpansive map with a bounded orbit $x_k = T^k x$, $k \geq 0$. Then $\lim_{k\to\infty} D_g(p, T^k x)$ exists (in fact $D_g(p, T^k x)$ is nonincreasing), $\forall\, x \in C$, where p is any weak cluster point of $s_n = \frac{1}{n}\sum_{k=0}^{n-1} x_k$.*

Theorem 4.1. *(Fixed point theorem) Assume that E is a reflexive Banach space and $g : E \to \mathbb{R}$ is strictly convex and totally convex on bounded subsets. Now assume that C is nonempty, closed, and convex, and $T : C \to C$ is a Bregman nonexpansive map. If every orbit $x_k = T^k x$, $k \geq 0$ is bounded, then T has a fixed point in C. Moreover, $\lim_{k\to\infty} D_g(x_k, p)$ exists.*

Proof. We show that every weak subsequential limit p of $s_n = \frac{1}{n}\sum_{k=0}^{n-1} x_k$ is a fixed point of T. Let $s_{n_i} \rightharpoonup p$ as $i \to \infty$. Since C is closed and convex, then $p \in C$ so that Tp is well-defined. By a similar argument as in the proof of Proposition 3.1, we have $D_g(p, Ty) \leq D_g(p, y)$, $\forall y \in C$. Since $p \in C$, we can choose $y = p$. It follows that $D_g(p, Tp) \leq D_g(p, p) = 0$, and hence p is a fixed point of T by Proposition 2.2. Now by the Bregman nonexpansivity of T, we have $D_g(x_{k+1}, p) \leq D_g(x_k, p)$, which implies that $\lim_{k\to\infty} D_g(x_k, p)$ exists. □

Theorem 4.2. *(Weak ergodic theorem) Assume that E is a reflexive Banach space, and $g : E \to \mathbb{R}$ is a strictly convex and Legendre function such that ∇g^* is bounded on bounded subsets of E^*. Assume that C is a nonempty, closed, and convex subset of E and $T : C \to C$ is Bregman nonexpansive. Then the following are equivalent:*

(i) $\liminf_{n\to\infty} |s_n| < +\infty$;

(ii) $\{x_k\}$ *is bounded;*

(iii) $\{s_n\}$ *is weakly convergent to a fixed point of T.*

Proof. (i) $\Rightarrow$ (ii) was already proved in Proposition 3.1, even without assuming C to be closed and convex. In fact, since $D_g(p, x_k)$ is bounded, by Proposition 2.4, we obtain the boundedness of $\{x_k\}$.

(ii) $\Rightarrow$ (iii): Assume that $s_{n_i} \rightharpoonup p$ and $s_{m_l} \rightharpoonup q$. Then by Theorem 4.1 and the relation

$$D_g(x_k, p) - D_g(x_k, q) = g(q) - g(p) + \langle \nabla g(q) - \nabla g(p), x_k - p \rangle - \langle \nabla g(q), q - p \rangle,$$

it follows that $\lim_{k\to\infty} \langle \nabla g(q) - \nabla g(p), x_k \rangle$ exists. Hence, $\lim_{n\to\infty} \langle \nabla g(q) - \nabla g(p), s_n \rangle$ exists. This implies that $\langle \nabla g(p) - \nabla g(q), p - q \rangle = 0$. By the strict convexity of g, we have $p = q$. Hence s_n converges weakly to an element $p \in \operatorname{Fix}(T)$.

(iii) $\Rightarrow$ (i) is trivial. □

Theorem 4.3. *Assume that E is a reflexive Banach space, and $g : E \to \mathbb{R}$ is strictly convex and Legendre function such that ∇g^* is bounded on bounded subsets of E^* and $\nabla g : E \to E^*$ is* *sequentially weak to weak* * *continuous. Assume that* C *is a nonempty, closed, and convex subset of* E *and* $T : C \to C$ *is a Bregman nonexpansive mapping. Let* $x \in C$ *and* $x_k = T^k x,\ k \geq 0$*, where* $\{x_k\}$ *is weakly asymptotically regular (i.e.,* $x_{k+1} - x_k \rightharpoonup 0$*). Then the following are equivalent:*

(i) $\liminf_{k\to\infty} |x_k| < +\infty$;
(ii) $\{x_k\}$ *is bounded;*
(iii) $\{x_k\}$ *is weakly convergent to a fixed point of* T.

Proof. We only need to show (ii) $\Rightarrow$ (iii). The proof is by combining the proofs of Theorem 3.1 and Theorem 4.2. □

Remark 4.1. If $g(x) := |x|^2$, then $D_g(x,y) = \phi(x,y)$ and the Bregman nonexpansive mappings are reduced to ϕ-nonexpansive mappings defined by $\phi(Tx,Ty) \leq \phi(x,y)$. It is not difficult to see that the results of Sections 3 and 4 hold for ϕ-nonexpansive mappings in a reflexive, smooth, and strictly convex Banach space, implying therefore our previous results in [21].

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