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NOTES ON MODULAR PROJECTIONS

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Dedicated to the Memory of Professor Ronald E. Bruck

Abstract. Let X_ρ be a modulated convergence space, that is, a modular space equipped with a sequential convergence structure. Given an element x of X_ρ, we consider the minimisation problem of finding $x_0 \in K$ such that $\rho(x - x_o) = \inf\{\rho(x - y) : y \in K\}$, where ρ is a modular, and K is a subset of X_ρ. Such an element x_0 is called a best approximant. We prove an existence of best approximants in a large class of modulated convergence spaces. We also investigate semicontinuity of the related modular projection which in general is a nonlinear multi-valued operator. Problems of finding best approximants are important in approximation theory and probability theory. In particular, we show how our results can be applied to the approximation of functions in variable Lebesgue spaces by rational functions.

Keywords. Best approximation; Convergence spaces; Modular function spaces; Projection operator; Variable Lebesgue spaces.

1. Introduction

In the recently published paper [1], the author defines modulated LTI-spaces as modular spaces equipped with a sequential convergence structure. The cited paper considers a problem of existence and uniqueness of approximants in such spaces assuming uniform convexity of the modular. In such case, the projection operator is a single-valued non-linear operator and its continuity can be proven under some additional assumptions, [1, Theorem 3.2]. The cited paper points to several application areas, where a modular approximation, instead of a norm (or metric) approximation, occurs quite naturally, and quite often allows results, which are not available within the limitations of normed spaces. Problems of finding best approximants are important in approximation theory and probability theory, and the cited paper shows specifically how such modular approximation can be applied to the nonlinear prediction theory.

In the current paper, we remove the assumptions of uniform convexity of a modular ρ. In most cases, we do not even assume that ρ is convex at all. We consider instead adaptations of the concept of approximatively compact sets, frequently used when dealing with approximation in norm spaces; see, e.g., [2, 3, 4]. Consequently, in general we loose uniqueness in our existence result (Theorem 3.1), hence the nonlinear projection operator becomes multi-valued. To prove its continuity, we need to use multi-valued operator methods; see Theorem 3.2 and related results in Section 3. Section 4 discusses the applications of general results from Section 2 to

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the rational approximation in variable Lebesgue spaces. Variable Lebesgue spaces have been studied for many years and have several important applications, for instance, in the modelling of electrorheological fluids and in the study of image processing.

2. Modulated Convergence Spaces

Let X be a real vector space. Let us recall the definition of modular on X, [5, 6], and associated terminology.

Definition 2.1. A functional $\rho : X \to [0,\infty]$ is called a modular if

(1) $\rho(x) = 0$ if and only if $x = 0$;
(2) $\rho(-x) = \rho(x)$;
(3) $\rho(\alpha x+\beta y) \leq \rho(x)+\rho(y)$ for any $x,y \in X$, and $\alpha,\beta \geq 0$ with $\alpha+\beta = 1$,

and convex modular if instead of (3) the following holds
(3') $\rho(\alpha x+\beta y) \leq \alpha\rho(x)+\beta\rho(y)$ for any $x,y \in X$, and $\alpha,\beta \geq 0$ with $\alpha+\beta = 1$.
The vector space $X_\rho = \{x \in X : \rho(\lambda x) \to 0,\ as\ \lambda \to 0\}$ is called a modular space.

The notions shown in Definition 2.2 below were introduced in [7] for general modular spaces. They follow the same pattern as their equivalents in modular function spaces (see, e.g., [8, Def. 3.4]). For the foundational exposition of the theory of modular function spaces, the reader is referred to the book [9].

Definition 2.2. Let ρ be a modular defined on a vector spaces X.

(a) We say that $\{x_n\}$, a sequence of elements of X_ρ is ρ-convergent to x and write $x_n \overset{\rho}{\to} x$ if $\rho(x_n - x) \to 0$.
(b) A sequence $\{x_n\}$, where $x_n \in X_\rho$ is called ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n,m \to \infty$.
(c) X_ρ is called ρ-complete if every ρ-Cauchy is ρ-convergent to an $x \in X_\rho$.
(d) A set $B \subset X_\rho$ is called ρ-closed if for any sequence of $x_n \in B$, the convergence $x_n \overset{\rho}{\to} x$ implies that x belongs to B.
(e) A set $B \subset X_\rho$ is called ρ-bounded if its ρ-diameter $\delta_\rho(B) = \sup\{\rho(x-y) : x \in B, y \in B\}$ is finite.
(f) A set $K \subset X_\rho$ is called ρ-compact if, for any $\{x_n\}$ in K, there exists a subsequence $\{x_{n_k}\}$ and an $x \in K$ such that $\rho(x_{n_k} - x) \to 0$.
(g) Let $x \in X_\rho$ and $C \subset X_\rho$. The ρ-distance between x and C is defined as
$$d_\rho(x,C) = inf\{\rho(x-y) : y \in C\}.$$
(h) A ρ-ball $B_\rho(x,r)$ is defined by $B_\rho(x,r) = \{y \in X_\rho : \rho(x-y) \leq r\}$.

Definition 2.3. [8, Definition 3.6] We say that a modular ρ satisfies the Δ_2-type condition if there exists a constant $0 < M < \infty$ such that, for every $x \in X_\rho$, $\rho(2x) \leq M\rho(x)$.

Definition 2.4. [8, Definition 5.4] A modular ρ is called uniformly continuous if, for every $\varepsilon > 0$ and every $0 < L < \infty$, there exists $\delta > 0$ such that $|\rho(x+y)-\rho(x)| < \varepsilon$, whenever $x \in X_\rho$, $y \in X_\rho$, $\rho(y) < \delta$, and $\rho(x) \leq L$.

We define the growth function $w : [0,\infty) \to [0,\infty]$ by
$$w(t) = \sup\left\{\frac{\rho(tx)}{\rho(x)},\quad 0 < \rho(x) < \infty\right\}.$$

The following properties of the growth function are direct consequence of the definition.

Lemma 2.1. *Let X_ρ be a modular vector space. Then*

(1) w is a nondecreasing function;
(2) $w(0) = 0$ and $w(1) = 1$;
(3) $w(ts) \leq w(t)w(s), \forall t, s \in [0, \infty)$;
(4) $\rho(tx) \leq w(t)\rho(x), \forall t \geq 0, \forall x \in X_\rho$.

In addition, if ρ is convex, then w is a convex function and

(5) $w(t) \leq t$ for $t \leq 1$,

which implies that w is a convex continuous function at all $t \geq 0$ such that $w(t) < \infty$.

We can characterise modulars satisfying Δ_2-type condition in terms of the growth function w. The next three results follow immediately from the relevant definitions.

Lemma 2.2. *Let ρ be a modular. Then the following statements are equivalent:*

(1) $w(t) < \infty$ for all $t \in [0, \infty)$,
(2) ρ satisfies the Δ_2-type condition.

Proposition 2.1. *If ρ satisfies the Δ_2-type condition, then $\rho(x) < \infty$ for every $x \in X_\rho$.*

Proposition 2.2. *If ρ satisfies the Δ_2-type condition, then a set $B \subset X_\rho$ is ρ-bounded if and only if $\sup_{x \in B} \rho(x) < \infty$, and hence all ρ-balls are ρ-bounded.*

Let us recall from [1] basic concepts related to the sequential convergence and modulated convergence spaces, remembering that the framework of convergence spaces was originally introduced in [10]; see also [11], and recent papers [7, 12].

Definition 2.5. Let X be any nonempty set. A relation ζ between sequences $\{x_n\}_{n=1}^{\infty}$ of elements of X and elements x of X, denoted by $x_n \xrightarrow{\zeta} x$, is called a sequential convergence on X if

(1) if $x_n = x$ for all $n \in \mathbb{N}$ then $x_n \xrightarrow{\zeta} x$;
(2) if $x_n \xrightarrow{\zeta} x$ and $\{x_{n_k}\}$ is a proper subsequence of $\{x_n\}$, then $x_{n_k} \xrightarrow{\zeta} x$.

The pair (X, ζ) (or shortly X) is called a convergence space.

Given a sequential convergence ζ on X, we can introduce notions of closed and sequentially compact sets.

Definition 2.6. Let (X, ζ) be a convergence space. A set $K \subset X$ is called closed if whenever $x_n \in K$ all $n \in \mathbb{N}$ and $x_n \xrightarrow{\zeta} x$, then $x \in K$. Similarly, K is called sequentially compact if from every sequence $\{x_n\}$ of elements of K we can choose a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \xrightarrow{\zeta} x$ for an $x \in K$.

Definition 2.7. A sequential convergence ζ is called an L-convergence on X if

(3) if $x_n \xrightarrow{\zeta} x$ and $x_n \xrightarrow{\zeta} y$, then $x = y$.

The pair (X, ζ) (or shortly X) is called an L-space.

Definition 2.8. An L-convergence ζ on X is called L^*-convergence if, in addition, it satisfies the following condition

(*) if every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ contains a subsequence $\{x_{n_{k_p}}\}$ such that $x_{n_{k_p}} \xrightarrow{\zeta} x$, then $x_n \xrightarrow{\zeta} x$.

Similarly, X is called an L^*-space.

Let ζ be a sequential convergence on X. Let us denote by $T(\zeta)$ the class of all subsets U of X such that it follows from $x \in U$ and $x_n \xrightarrow{\zeta} x$ that there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for $n \geq n_0$. We call them open sets in the sense of $T(\zeta)$. It is easy to see that these open sets form a topology as it was already observed by Birkhoff in 1936, [13]. Note also that the same topology $T(\zeta)$ can be as well determined by closed sets, where $F \subset X$ is called a closed set if $x \in F$, whenever $x_n \in F$ and $x_n \xrightarrow{\zeta} x$. Note that $T(\zeta)$ does not need to be Hausdorff even when ζ is an L^*-convergence, see [11].

Let now τ be a topology on X. We say that a sequence $\{x_n\}$ of elements of X converges to an $x \in X$ (and write $x_n \xrightarrow{\tau} x$) if it follows from $x \in U \in \tau$ that $x_n \in U$ for $n \in \mathbb{N}$ greater than some $n_0 \in \mathbb{N}$. It is easy to see that such convergence, denoted by $C(\tau)$, is a sequential convergence which, in addition, satisfies (*). If τ is Hausdorff, then the limit is unique and hence $C(\tau)$ is an L^*-convergence.

The next fundamental result was attributed by Kisyński to Urysohn [14] and can be formulated as follows.

Proposition 2.3. *Let ζ be an L-convergence on X, $x_n, x \in X$. Then $x_n \xrightarrow{C(T(\zeta))} x$ if and only if from every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ we can choose a subsequence $\{x_{n_{k_p}}\}$ such that $x_{n_{k_p}} \xrightarrow{\zeta} x$.*

An interesting and important example is when ζ is an almost everywhere convergence of equivalence classes of Lebesgue-measurable functions on $[0,1]$ (it is easy to check that this is an L-convergence). We know that the m-almost everywhere convergence implies but is not equivalent to the convergence in measure m. As a matter of fact, $f_n \to f$ in measure if and only if from every subsequence $\{f_{n_k}\}$ we can choose a subsequence $\{f_{n_{k_p}}\}$ such that $f_{n_{k_p}} \to f$ m-almost everywhere. From Proposition 2.3, we conclude immediately that the convergence almost everywhere cannot be generated from any topology.

Let us now provide precise definitions of LTI-convergence, LTI-spaces, and modulated LTI-spaces.

Definition 2.9. Let X be a real vector space, and let ζ be an L-convergence on X. We say that ζ is an LTI-convergence (translation invariant convergence) if $x_n \xrightarrow{\zeta} x$ implies that $x_n - y \xrightarrow{\zeta} x - y$ for any $y \in X$. In this case, the pair (X, ζ) is called an LTI-space.

Definition 2.10. Let ρ be a modular defined on X and let ζ be an L-convergence on X_ρ. The triplet (X_ρ, ρ, ζ) is called a modulated LTI-space if (X_ρ, ζ) is an LTI-space and the following two conditions are satisfied

(i) $x_n \xrightarrow{\zeta} x \Rightarrow \rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$;

(ii) if $x_n \xrightarrow{\rho} x$ then there exists a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\zeta} x$, where $x, x_n \in X$.

Let us note that the condition (i) of Definition 2.10 guarantees the left continuity of ρ, that is, $\rho(\lambda x) \to \rho(x)$ provided $\lambda \to 1^-$.

Proposition 2.4. *Let (X_ρ, ρ, ζ) be a modulated LTI-space. Then the following assertions are true.*

(i) Every ζ-closed set is also ρ-closed.
(ii) Every ρ-compact set is also sequentially ζ-compact.
(iii) Every ρ-ball $B_\rho(x,r)$ is ζ-closed (and hence also ρ-closed).
(iv) ρ-convergence is an L^*-convergence.

Banach spaces with ρ being a norm and ζ standing for the convergence in weak topology, and modular function spaces (including Lebesgue spaces, Orlicz spaces, variable Lebesgue spaces) are typical examples of ρ-complete modulated LTI- spaces. We refer the reader to [1] for a more comprehensive list of examples.

3. General Results

In this section, K always stand for a nonempty subset of a modular space X_ρ. We denote $X_\rho(K) = \{x \in X_\rho : d_\rho(x,K) < \infty\}$.

Definition 3.1. (i) Given $x \in X_\rho(K)$, any $y \in K$ such that $\rho(y-x) = d_\rho(x,K)$ is called a best ρ-approximant of x with respect to K.
(ii) By $P_K(x)$ we will denote the set (possibly empty) of all best ρ-approximants of x with respect to K.
(iii) A set-valued operator $P_K : X_\rho(K) \to 2^K$ is called a modular projection onto K.
(iv) We say that K is ρ-proximinal at $x \in X_\rho(K)$ if $P_K(x)$ is non-empty.
(v) We say that K is ρ-proximinal if it is ρ-proximinal at every $x \in X_\rho(K)$.

The following definition of the upper semi-continuity of modular projections acting in LTI-spaces is a generalisation of the notion of the upper semi-continuity for projection onto Chebyshev spaces, that is, when P_K is single-valued, introduced in [Def. 3.3] [1].

Definition 3.2. Let (X_ρ, ρ, ζ) be a modulated LTI-space. Let $K \subset X_\rho$ be non-empty. We say that a modular projection P_K is $\rho - \zeta$ upper semicontinuous ($\rho - \zeta$ u.s.c) at $x_0 \in X_\rho(K)$ provided that for each sequence of elements $x_n \in X_\rho(K)$ with $x_n \xrightarrow{\rho} x_0$ and each $T(\zeta)$-open set V containing $P(x_0)$, $P(x_n) \subset V$ for $n \in \mathbb{N}$ bigger than some $n_0 \in \mathbb{N}$.

Definition 3.3. We say that a sequence $\{y_n\}$, where each $y_n \in K$, is minimising for $x \in X_\rho(K)$ if $\lim_{n\to\infty} \rho(x - y_n) = dist_\rho(x,K)$.

Remark 3.1. It is easy to show that if ρ satisfies the Δ_2-type condition, then every minimising sequence is ρ-bounded.

Definition 3.4. We say that $\emptyset \neq K \subset X_\rho$ is approximatively ζ-compact if, for every $x \in X_\rho(K)$ and every sequence $\{y_n\}$ minimising for x, there exists a subsequence $\{y_{n_k}\}$ and an element $y \in K$ such that $y_{n_k} \xrightarrow{\zeta} y$.

The following simple but important result will be used in the proof of the next theorem.

Lemma 3.1. *Let (X_ρ,ρ,ζ) be a modulated LTI-space and $\emptyset \neq K \subset X_\rho$. Let $x \in X_\rho(K)$ and let a sequence $\{y_n\}$ of elements of K be minimising for x with respect to K. If there exists $y \in K$ such that $y_n \xrightarrow{\zeta} y$ then $y \in P_K(x)$.*

Proof. Using the definition of LTI-spaces, the fact that $\{y_n\}$ is minimising and that $y \in K$, we have

$$\rho(y-x) \leq \liminf_{n\to\infty} \rho(y_n - x) = d_\rho(x,K) \leq \rho(y-x),$$

which means that $y \in P_K(x)$. □

Theorem 3.1. *Let (X_ρ,ρ,ζ) be a modulated LTI-space and $\emptyset \neq K \subset X_\rho$ be approximatively ζ-compact. Then*

(i) K is ρ-proximinal;
(ii) $P_K(x)$ is sequentially ζ-compact for every $x \in X_\rho(K)$.

Proof. To prove (*i*), let us fix $x \in X_\rho(K)$. We need to show that $P_K(x) \neq \emptyset$. Let $\{y_n\}$ be minimising for x . Since K is approximatively ζ-compact, there exists a subsequence $\{y_{n_k}\}$ and an element $y \in K$ such that $y_{n_k} \xrightarrow{\zeta} y$. Using Lemma 3.1, we have $y \in P_K(x)$.

Similarly, to prove (*ii*), we fix $x \in X_\rho$. It follows from (*i*) that $P_K(x) \neq \emptyset$. Let $\{y_n\}$ be a sequence of elements from $P_K(x)$. Hence, $\{y_n\}$ is trivially minimising for x in K. Because K is approximatively ζ-compact, there exists a subsequence $\{y_{n_k}\}$ and an element $y \in K$ such that $y_{n_k} \xrightarrow{\zeta} y$. Using Lemma 3.1 again, we have $y \in P_K(x)$. □

Definition 3.5. Let $\emptyset \neq K \subset X_\rho$. We say that the distance function d_ρ is upper semicontinuous in $X_\rho(K)$ if, for every $\varepsilon > 0$ and $z \in X_\rho(K)$, there exists $\delta > 0$ such that $d_\rho(w,K) \leq d_\rho(z,K)+\varepsilon$ provided that $w \in X_\rho(K)$ and $\rho(w-z) < \delta$.

Lemma 3.2. *Assume that ρ is uniformly continuous. Let $\emptyset \neq K \subset X_\rho$. Then the distance function d_ρ is upper semicontinuous in $X_\rho(K)$.*

Proof. Let $0 < \varepsilon < 1$ and fix any $z \in X_\rho(K)$. Let $0 < M = d_\rho(z,K)+1 < +\infty$ and define the set

$$C_z = \{y - z : y \in K, \rho(z-y) \leq d_\rho(z,K) + \frac{\varepsilon}{2}\} \subset B_\rho(0,M).$$

Since ρ is uniformly continuous, therefore there exists $\delta > 0$ (selected for $\frac{\varepsilon}{2}$ and M) such that, for every $y - z \in C_z$,

$$\rho(w-y) = \rho((w-z)-(y-z)) < \frac{\varepsilon}{2} + \rho(y-z),$$

whenever $\rho(w-z) < \delta$. Let us fix $w \in X_\rho(K)$ satisfying $\rho(w-z) < \delta$ and observe that for every $y \in K$ such that $y - z \in C$,

$$d_\rho(w,K) \leq \rho(w-y) < \frac{\varepsilon}{2} + \rho(z-y) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + d_\rho(z,K) = d_\rho(z,K)+\varepsilon,$$

which implies that $d_\rho(w,K) \leq d_\rho(z,K)+\varepsilon$, as claimed. □

Theorem 3.2. *Let (X_ρ,ρ,ζ) be a modulated LTI-space, where ρ is uniformly continuous. Let $\emptyset \neq K \subset X_\rho$. If K is approximatively ζ-compact, then P_K is ρ-ζ upper semicontinuous at every $x \in X_\rho(K)$.*

Proof. Observe first that by Theorem 3.1, $P_K(x) \neq \emptyset$ for every $x \in X_\rho(K)$. Assume to the contrary that there exists $x_0 \in X_\rho(K)$, a $T(\zeta)$-open set $V \supset P_K(x_0)$ and a sequence $\{x_n\}$ of elements from $X_\rho(K)$ such that $x_n \xrightarrow{\rho} x_0$ and $P_K(x_n) \setminus V \neq \emptyset$ for $n \geq 1$. Choose for every $n \geq 1$ an element $y_n \in P_K(x_n) \setminus V$. Fix arbitrarily $\varepsilon > 0$. By Lemma 3.2, the distance function d_ρ is upper semicontinuous in $X_\rho(K)$. Hence, we can choose $\delta > 0$ such that

$$d_\rho(w,K) \leq d_\rho(x_0,K) + \varepsilon \tag{3.1}$$

provided that $w \in X_\rho(K)$ and $\rho(w - x_0) < \delta$. Without loosing generality, we can assume that $\rho(x_0 - x_n) < \delta$ for every $n \in \mathbb{N}$. Using (3.1), we have

$$\rho(x_n - y_n) = d_\rho(x_n,K) = d_\rho(x_n,K) \leq d_\rho(x_0,K) + \varepsilon. \tag{3.2}$$

Fix $0 < \delta_1 < \delta$ linked to the uniform continuity of ρ applied to the constants ε and $d_\rho(x_0,K)+\varepsilon$, and take $n \in \mathbb{N}$ such that $\rho(x_0 - x_n) < \delta_1$ for every $n \geq n_0$. Using (3.2), for such n, we have

$$\begin{aligned} d_\rho(x_0,K) \leq \rho(x_0 - y_n) = \rho(x_0 - x_n + x_n - y_n) &\leq \varepsilon + \rho(x_n - y_n) \\ &= \varepsilon + d_\rho(x_n,K) \leq 2\varepsilon + d_\rho(x_0,K). \end{aligned}$$

By arbitrariness of ε, we have

$$d_\rho(x_0,K) = \lim_{n\to\infty} \rho(x_0 - y_n).$$

Hence, $\{y_n\}$ is minimising for x_0 in K. Since K is approximatively ζ-compact, there exists $y_0 \in K$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$y_{n_k} \xrightarrow{\zeta} y_0 \tag{3.3}$$

Therefore, by Lemma (3.1),

$$y_0 \in P_K(x_0) \subset V. \tag{3.4}$$

On the other hand, for every $n \in \mathbb{N}$, we have $y_n \in P_K(x_n) \subset X_\rho \setminus V$. Note that $X_\rho \setminus V$ is $T(\zeta)$-closed since V is $T(\zeta)$-open. Hence, it follows from (3.3) that $y_0 \in X_\rho \setminus V$, which contradicts (3.4), completing the proof. □

Theorem 3.3. *If a modular ρ satisfies the Δ_2-type condition and its growth function w_ρ is right-continuous at $t = 1$ then ρ is uniformly continuous.*

Proof. Fix arbitrarily $0 < \varepsilon < 1$ and $0 < L < +\infty$, and denote $k = \max\{1, L\} < +\infty$. Using the right-continuity of w_ρ we can choose $\alpha \in (0,1)$ such that

$$k\left(w_\rho\left(\frac{1}{\alpha}\right) - 1\right) \leq \frac{\varepsilon}{3}. \tag{3.5}$$

Observe that from (3.5) and facts that $k \geq 1$ and $\varepsilon < 1$ it follows that

$$w_\rho\left(\frac{1}{\alpha}\right) - 1 \leq 1. \tag{3.6}$$

Choose $\delta > 0$ so that

$$w_\rho\left(\frac{1}{1-\alpha}\right)\delta \leq \frac{\varepsilon}{3}.$$

Let $x \in X_\rho$ and $y \in X_\rho$ be such that $\rho(y) < \delta$ and $\rho(x) \leq k$. Observe that

$$\begin{aligned}\rho(x+y) &= \rho\left(\alpha\left(\frac{x}{\alpha}\right) + (1-\alpha)\left(\frac{y}{1-\alpha}\right)\right)\\ &\leq w_\rho\left(\frac{1}{\alpha}\right)\rho(x) + w_\rho\left(\frac{1}{1-\alpha}\right)\rho(y)\\ &\leq w_\rho\left(\frac{1}{\alpha}\right)\rho(x) + w_\rho\left(\frac{1}{1-\alpha}\right)\delta\\ &\leq w_\rho\left(\frac{1}{\alpha}\right)\rho(x) + \frac{\varepsilon}{3}.\end{aligned} \tag{3.7}$$

On the other hand, using (3.5), we have

$$\begin{aligned}w_\rho\left(\frac{1}{\alpha}\right)\rho(x) &= \rho(x) + \left(w_\rho\left(\frac{1}{\alpha}\right) - 1\right)\rho(x)\\ &\leq \rho(x) + k\left(w_\rho\left(\frac{1}{\alpha}\right) - 1\right)\\ &\leq \rho(x) + \frac{\varepsilon}{3}.\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we arrive at

$$\rho(x+y) \leq \rho(x) + \frac{2\varepsilon}{3} < \rho(x) + \varepsilon. \tag{3.9}$$

Similarly,

$$\begin{aligned}\rho(x) &= \rho(x+y-y)\\ &\leq w_\rho\left(\frac{1}{\alpha}\right)\rho(x+y) + w_\rho\left(\frac{1}{1-\alpha}\right)\rho(y)\\ &\leq w_\rho\left(\frac{1}{\alpha}\right)\rho(x+y) + \frac{\varepsilon}{3}.\end{aligned} \tag{3.10}$$

Using (3.9), (3.5), and (3.6), we have

$$\begin{aligned}w_\rho\left(\frac{1}{\alpha}\right)\rho(x+y) &= \rho(x+y) + \left(w_\rho\left(\frac{1}{\alpha}\right) - 1\right)\rho(x+y)\\ &\leq \rho(x+y) + \left(w_\rho\left(\frac{1}{\alpha}\right) - 1\right)\left(\rho(x) + \frac{2\varepsilon}{3}\right)\\ &\leq \rho(x+y) + k\left(w_\rho\left(\frac{1}{\alpha}\right) - 1\right) + \left(w_\rho\left(\frac{1}{\alpha}\right) - 1\right)\frac{2\varepsilon}{3}\\ &\leq \rho(x+y) + \frac{\varepsilon}{3} + \frac{2\varepsilon}{3}\\ &= \rho(x+y) + \varepsilon.\end{aligned} \tag{3.11}$$

Substituting (3.11) into (3.10), we have $\rho(x) \leq \rho(x+y) + \varepsilon$, which together with (3.9) gives us the desired inequality $|\rho(x) - \rho(x+y)| < \varepsilon$. □

Corollary 3.1. *Theorem 3.3 assures us that there are many interesting examples of LTI-spaces, where Theorem 3.2 can be applied. The class of such LTI-spaces include for instance all such spaces where ρ is convex and satisfies the Δ_2-type condition (use Lemma 2.1 and Lemma 2.2), which gives us all classical Lebesgue spaces, Orlicz spaces with Δ_2, and in general all modular function spaces with ρ convex and satisfying Δ_2-type condition. This will lead us to important classes of variable Lebesgue spaces and spaces built upon them, a topic which will be developed* *further in the next section. However, it is important to remark that convexity is not necessary, with L^p for $0 < p < 1$ being a classical example of a non-convex modular satisfying the Δ_2-type condition and with the continuous growth function $w_\rho(t) = t^p$.*

To identify a large class of approximatively compact sets, let us introduce the following definition, which relates to notions used in norm spaces; see, e.g., [3].

Definition 3.6. We say that $\emptyset \neq K \subset X_\rho$ is ρ-boundedly ζ-compact if, for every ρ-bounded sequence $\{y_n\}$ of elements of K, there exists a subsequence $\{y_{n_k}\}$ and an element $y \in K$ such that $y_{n_k} \xrightarrow{\zeta} y$.

Proposition 3.1. *Since, in case of the Δ_2-type condition, every minimising sequence is ρ-bounded, it follows immediately from the definitions that every ρ-boundedly ζ-compact subset of a modulated LTI-space (X_ρ, ρ, ζ), with ρ satisfying Δ_2-type condition, is approximatively ζ-compact. This property will be used in the next section, where we show a class of ρ-boundedly ζ-compact subsets of variable Lebesgue spaces, which is important from the approximation perspective.*

4. Application to Variable Lebesgue Spaces $L^{p(\cdot)}$

The results of this section generalise to variable Lebesgue space $L^{p(\cdot)}$ results in the classical L^p spaces, [3]. For the purpose of our paper, we define the variable Lebesgue space $L^{p(\cdot)}$ as the modular space (actually a modular function space, see [8, 9]) defined by the modular

$$\rho(x) = \int_{[0,1]} |x(t)|^{p(t)} dm(t),$$

where m denotes the Lebesgue measure in the interval $[0,1]$, $x : [0,1] \to \mathbb{R}$ denotes a Lebesgue measurable function, and $p : [0,1] \to [0,+\infty)$ is a given measurable function. For more general definitions and properties of variable Lebesgue spaces; see, e.g., [15, 16, 17, 18]. Variable Lebesgue spaces have been studied for many years both from the perspective of the pure mathematics and several applications. The theory of these spaces, built upon solid foundations of the theory of Orlicz spaces, modular spaces, and modular function spaces; (see e.g. [5, 6, 9]), have found important applications, for instance, in the modelling of electrorheological fluids and in the study of image processing; see, e.g., [15, 19, 20, 21].

In this section, we assume that $1 \leq p_0 \leq p(t) \leq p_1 < +\infty$ for every $t \in [0,1]$, which - as it is easy to check - makes ρ a convex modular satisfying the Δ_2-type condition. Let us denote by $\mathscr{P}_i$ set of all polynomials of degree less than or equal to i. Define

$$\mathscr{R}_m^k = \left\{ \frac{g}{h} : g \in \mathscr{P}_k, h \in \mathscr{P}_m, h > 0 \, in \, [0,1] \right\}.$$

Lemma 4.1. *Under the assumptions of this section, $\mathscr{R}_m^k$ is ρ-boundedly a.e.-compact.*

Proof. Let $\{r_n\}$ be a ρ-bounded sequence of elements from $\mathscr{R}_m^k$, that is, $r_n = \frac{g_n}{h_n}$, where $g_n \in \mathscr{P}_k$, $h_n \in \mathscr{P}_m$, $h_n > 0$ in $[0,1]$. Using Remark 3.1 and Proposition 2.2, we conclude that $\sup_{n\geq 1} \rho(r_n) = M < \infty$ which, in view of the left continuity and convexity of ρ, implies from [6, Theorem 1.8], that $\sup_{n\geq 1} \|r_n\|_\rho \leq M < \infty$. All h_n are actually polynomials and hence elements of $L^\infty[0,1]$. Therefore, after a suitable rescaling, we can represent every $r_n = \dfrac{\bar{g}_n}{\bar{h}_n}$, where $\|\bar{h}_n\|_{L^\infty} = 1$ and $\bar{h}_n > 0$ in $[0,1]$ for every $n \in \mathbb{N}$. Let ρ' be a conjugate modular to ρ, which means that

$$\rho'(x) = \int_{[0,1]} |x(t)|^{p'(t)} dm(t),$$

where

$$\frac{1}{p(t)} + \frac{1}{p'(t)} = 1$$

for every $t \in [0,1]$, using the convention $1/\infty = 0$. Observing that

$$\rho'(\bar{h}_n) = \int_{[0,1]} |\bar{h}_n(t)|^{p'(t)} dm(t) \le \|\bar{h}_n\|_{L^\infty} = 1$$

and using Hölder's inequality for variable Lebesgue spaces (see [16, Theorem 2.33]), there exists a constant $1 \le N < \infty$ such that

$$\int_{[0,1]} |\bar{g}_n(t)| dm(t) = \int_{[0,1]} |r_n(t)| \bar{h}_n(t) dm(t) \le N \|r_n\|_\rho \|\bar{h}_n\|_{\rho'} \le NM < \infty,$$

which means that all polynomials $\bar{g}_n$ are contained in an L^1-ball in a finite dimensional space $\mathscr{P}_k$. Since all norms in finite dimensional vector spaces are equivalent, the sequence $\{\bar{g}_n\}$ is $\|\cdot\|_k$-bounded where $\|\cdot\|_k$ denotes Euclidean norm in $\mathbb{R}^k$. Noticing that $\mathscr{P}_k$ is a closed subspace of $\mathbb{R}^k$, we conclude that there exists a subsequence $\{\bar{g}_{n_p}\}$ and $g_0 \in \mathscr{P}_k$ such that

$$\|\bar{g}_{n_p} - g_o\|_k \to 0.$$

Similarly, since the sequence $\{\bar{h}_{n_p}\}$ is bounded in the sense of max norm in $\mathscr{P}_m$, there exists a subsequence $\{\bar{h}_{n_{p_s}}\}$ and $h_0 \in \mathscr{P}_m$ such that $\|\bar{h}_{n_{p_s}} - h_o\|_k \to 0$. Thus

$$r_{n_{p_s}}(t) \to \frac{g_0(t)}{h_0(t)}$$

for every $t \in [0,1]$ except $t \in Z(h_0) = \{u \in [0,1] : h_0(u) = 0\}$, which is a finite set, because h_0 is a polynomial. To prove that $\frac{g_0}{h_0} \in \mathscr{R}_m^k$, we need to show that $\dfrac{g_0}{h_0} = \dfrac{\widetilde{g}_0}{\widetilde{h}_0}$, where $\widetilde{g}_0 \in \mathscr{P}_k$, $\widetilde{h}_0 \in \mathscr{P}_m$, and $\widetilde{h}_0 > 0$ in $[0,1]$. Assume that, for some $t_0 \in [0,1]$, $h_0(t_0) = 0$ with multiplicity $\mu \ge 1$. We want to show that $g_0(t_0) = 0$ with multiplicity $\nu \ge \mu$. Assume to the contrary that $\nu < \mu$, that is,

$$\frac{g_0(t)}{h_0(t)} = \frac{g(t)}{(t-t_0)^\lambda h(t)},$$

where $\lambda \ge 1$, $g(t_0) \ne 0 \ne h(t_0)$. Since g and h are polynomials and hence continuous functions, there exists $\delta > 0$ and $0 < \varepsilon < \frac{1}{2}$ such that for every $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$

$$\left| \frac{g(t)}{h(t)} \right| \ge \delta > 0.$$

Note that $\dfrac{g_0}{h_0} \in L^{p(\cdot)}$. Indeed, using Fatou's Lemma, we have

$$\int_{[0,1]} \left| \frac{g_0(t)}{h_0(t)} \right|^{p(t)} dm(t) \le \liminf_{s\to\infty} \int_{[0,1]} |r_{n_{p_s}}(t)|^{p(t)} dm(t) \le M^{p_1} < \infty. \tag{4.1}$$

On the other hand,

$$\int_{[0,1]} \left|\frac{g_0(t)}{h_0(t)}\right|^{p(t)} dm(t) \geq \int_{[t_0-\varepsilon,t_0+\varepsilon]} \frac{\delta^{p(t)}}{|t-t_0|^{\lambda p(t)}} dm(t)$$
$$\geq \min(\delta^{p_0}, \delta^{p_1}) \int_{[t_0-\varepsilon,t_0+\varepsilon]} \frac{dm(t)}{|t-t_0|^{\lambda p_0}} = +\infty,$$

because $\lambda p_0 \geq 1$ and $|t-t_0| \leq \varepsilon < \frac{1}{2}$. Contradiction with (4.1) completes the proof. □

The proof of the next theorem follows immediately from Theorems 3.1 and 3.2 using Corollary 3.1, Proposition 3.1, and Lemma 4.1.

Theorem 4.1. *Under the assumptions used throughout this section, $\mathscr{R}_m^k$ is ρ-proximinal, $P_K(x)$ is sequentially ζ-compact for every $x \in L^{p(\cdot)}$, and P_K is $\rho - \zeta$ upper semicontinuous, where ζ denotes convergence a.e. in* $[0,1]$.

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