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CONSTRAINTED OPTIMIZATION PROBLEMS AND OPTIMAL TAXZATIONS

JINLU LI[1,*], GLENN STONE[2]

[1]*Department of Mathematics, Shawnee State University, Portsmouth, Ohio 45662, USA*
[2]*Department of Social Work, Ball State University, Muncie, Indiana 47306, USA*

Abstract. In this paper, by applying the Fan-KKM Theorem, we prove the existence of solutions to a constrained optimization problem. As applications, we solve some constrained optimal taxation problems. That is, we demonstrate the existence of tax rate functions that maximizes the utilities of taxpayers subjected to some government tax revenue plans.

Keywords. Constrained optimization problem; Fan-KKM Theorem; Tax rate function; Utility function.

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1. Introduction

In nonlinear analysis, fixed point theory, and optimization theory, the Fan-KKM theorem has been extensively applied to the proof of the existence of fixed points of some mappings and the solvability of some optimization problems, which include maximization and minimization problems; see, e.g., [1, 2, 3, 4, 5, 6]. We know that the underlying spaces for applying the Fan-KKM theorem are convex subsets of general Hausdorff topological vector spaces. With respect to a given underlying set, the Fan-KKM theorem helps to solve the existence of solutions to some optimization problems globally. In fact, roughly speaking, optimization problems include two aspects: one is to find a point to maximize or minimize a given function on a domain; the other one is, from a given underlying set of some functions, to seek a function, at which the considered functional takes the maximal or minimal value over the completely given domain. Hence, under the given underlying spaces, Fan-KKM theorem helps to solve some global optimization problems.

Let L_2 be the Hilbert space of square integrable functions on a finite closed interval $[0,L]$ for some $L > 0$. In this paper, we consider an optimization problem with some constraint conditions. Then, we use the Fan-KKM theorem to prove the existence of solutions to the considered constraint optimization problems (see Section 2). The motivation of this paper is, by applying the results about a constraint optimization problem studied in Section 2, to study taxation optimization problems.

For any given country or society, the optimization of the tax base (or tax rates) is an important challenge for its government. The government has to collect enough tax revenue to keep the

*Corresponding author.
E-mail addresses: jli@shawnee.edu (J. Li), gestone@bsu.edu (G. Stone).
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systems of this government smoothly perform and to keep the country's economic activities in the legal fields. Meanwhile, the government must consider maximizing the taxpayers' utilities. Precisely speaking, the goals of tax optimizations have three aspects: (1) keep the systems of this government perform well; (2) reduce the size of all taxes of the taxpayer, to minimize possible penalties, and to reduce tax risks; see, e.g., [7, 8, 9, 10]; (3) maximize the utilities of the taxpayers; see, e.g., [11, 12, 13, 14, 15].

In this paper, we suppose that, for a given country and a given time period, the incomes (including labors) of the taxpayers lay in a closed interval $[0,L]$ for some positive number L. A continuous macroeconomic model is constructed under the following assumptions:

(i) the government collects taxes from its taxpayers by its designed tax rate function;

(ii) every practical tax rate function is a function from $[0,L]$ to $[0,1]$;

(iii) every considered tax rate function is an increasing function;

(iv) we consider the case that the taxpayers' utilities in this country only depend on their incomes and the paid taxes. More precisely, the utility is defined by a function from $[0,L]\times[0,1]$ to $[0,\infty)$.

Our taxation optimization model is a constrained optimization problem. Subjected to some economic constraints of the amount of tax revenue decided by the government, the government designs a tax rate function to maximize the social welfare for the whole society, which is the total utilities of the taxpayers. In contrast to [9] with respect to discrete model, in this paper, we study the taxation optimization problem with continuous model. Since the Fan-KKM Theorem has more than one different versions, for easy reference, we briefly review the version of the Fan-KKM Theorem used in this paper. For more details regarding to KKM mappings and the Fan-KKM theorem, the readers are referred to Fan [1] and Park [6].

Let C be a nonempty subset of a vector space X. A set-valued mapping $F : C \to 2^X\backslash\{\emptyset\}$ is said to be a KKM mapping if, for any finite subset $\{y_1, y_2, \ldots, y_n\}$ of C,

$$\operatorname{co}\{y_1, y_2, \ldots, y_n\} \subseteq U_{1\le i\le n} F(y_i),$$

where $\operatorname{co}\{y_1, y_2 \ldots, y_n\}$ denotes the convex hull of $\{y_1, y_2, \ldots, y_n\}$. Indeed, in applications, n can be chosen to be 2.

Fan-KKM Theorem. *Let C be a nonempty closed convex subset of a Hausdorff topological vector space X and let $F : C \to 2^X\backslash\{\emptyset\}$ be a KKM mapping with closed values. If there exists a point $y_0 \in C$ such that $F(y_0)$ is a compact subset, then $\cap_{y\in C} F(y) \neq \emptyset$.*

2. SOME CONSTRAINTED OPTIMIZATION PROBLEMS

Let L be a positive number. Let $L_2 = (L_2[0,L], \|\cdot\|)$ denote the Hilbert space of all square integrable functions on $[0,L]$ with the ordinal norm $\|\cdot\|$. Let τ be the weak topology on L_2. Then (L_2, τ) is a Hausdorff topological vector space.

Let K denote the subset of L_2 that contains all nonnegative, increasing (no decreasing), and upper bounded by 1 functions defined on $[0,L]$. Let $f \in L_2$ be a given nonnegative function on $[0,L]$ satisfying

$$\int_0^L f(x)dx > 0. \tag{2.1}$$

Then

$$0 \le \int_0^L f(x)r(x)dx \le \int_0^L f(x)dx, \text{ for every } r \in K.$$

Take an arbitrary positive number M satisfying

$$0 < M < \int_0^L f(x)dx.$$

Denote

$$K(f,M) = \left\{ r \in K : \int_0^L f(x)r(x)dx = M \right\}, \tag{2.2}$$

where f is called a constraint function, M is called a constraint level, and $K(f,M)$ is called a constraint class with respect to this constraint function f and this constraint level M.

Lemma 2.1. *Every constraint class $K(f,M)$ is a nonempty, closed, convex, and $\|\cdot\|$-bounded subset in L_2. Therefore, $K(f,M)$ is τ-compact in L_2 (compact in the weak topology in L_2).*

Proof. From (2.1) and (2.2), we define a function h on $[0,L]$ by

$$h(y) = \int_y^L f(x)dx, \text{ for } y \in [0,L].$$

Then h is a decreasing and absolute continuous function on $[0,L]$ satisfying

$$h(0) = \int_0^L f(x)dx \text{ and } h(L) = 0,$$

which implies that, for a positive number M satisfying (2.2), there is a positive number β such that $\int_\beta^L f(x)dx = M$. Define $r_\beta = \chi_{[\beta,L]}$, which is as a characteristic function on $[\beta,L]$. One sees that $r_\beta \in K(f,M)$. Thus $K(f,M) \neq \emptyset$. Hence, one can check that $K(f,M)$ is a convex and $\|\cdot\|$-bounded subset in L_2.

Note that $K(f,M)$ is $\|\cdot\|$-closed if and only if $K(f,M)$ is sequentially $\|\cdot\|$-closed. Take an arbitrary $\|\cdot\|$-convergent sequence $\{r_m\} \subseteq K(f,M)$ with limit $r_0 \in L_2$. Then $r_n \to r_0$, a.e. on $[0,L]$ as $n \to \infty$. Let $A \subseteq [0,L]$ be the subset of all points at which $\{r_m\}$ is convergent to r_0. Then A is dense in $[0,L]$ and r_0 is increasing and upper bounded by 1 on A.

If $A = [0,L]$, then we have $r_0 \in K(f,M)$, which concludes that $K(f,M)$ is $\|\cdot\|$-closed. If $A \neq [0,L]$ for any point $x \in [0,L] \backslash (A \cup \{0\})$, we take an increasing sequence $\{x_m\} \subseteq A$ with limit x and define $r_0(x) = \lim_{m\to\infty} r_0(x_m)$. Assume that $0 \notin A$. Take a decreasing sequence $\{x_k\} \subseteq A$ with limit 0 and define $r_0(0) = \lim_{m\to\infty} r_0(x_k)$. We see that r_0 is well defined on $[0,L]$ and it is increasing with upper bound 1. Since equal, almost everywhere, elements are identical in L_2. It follows that $r_0 \in K(f,M)$. Therefore, $K(f,M)$ is $\|\cdot\|$-closed.

Hence, $K(f,M)$ is a nonempty, convex, $\|\cdot\|$-closed, and upper bounded by $\int_0^L f(x)dx$ subset in L_2, which implies that $K(f,M)$ is τ-compact in L_2 (compact in the weak topology in L_2).

Let $g : [0,L] \times [0,1] \to [0,\infty)$ be a function. We write

$$w(g,r) = \int_0^L g(x,r(x))dx, \text{ for every } r \in K. \tag{2.3}$$

Definition 2.1. Suppose that $g : [0,L] \times [0,1] \to [0,\infty)$ is a function and $K(f,M)$ is a constraint class. The constrained optimization problem with respect to the function g subjected to the constraint class $K(f,M)$, denoted by COP(g,f,M), is to find an element $r^* \in K(f,M)$ such that $w(g,r^*)$ is finite and $w(g,r) \leq w(g,r^*)$ for every $r \in K[f,M]$.

Theorem 2.1. *Let $g : [0,L] \times [0,1] \to [0,\infty)$ be a function, and let $K(f,M)$ be a constraint class. Suppose that g satisfies the following conditions:*
(i) for any fixed $x \in [0,L]$, $g(x,t)$ is continuous, decreasing, and concave with respect to on $[0,1]$;
(ii) $g(\cdot,t) \in L_2$ for any $t \in [0,1]$.
Then $\mathrm{COP}(g,f,M)$ *has a solution.*

Proof. Let τ be the weak topology on $L_2[0,L]$. Then $(L_2[0,L],\tau)$ is a Hausdorff topological vector space. From Lemma 2.1, this constraint class $K(f,M)$ is a nonempty, closed, convex, $\|\cdot\|$-bounded, and τ-compact subset in L_2. Condition (ii) that $g(\cdot,0) \in L_2$ and condition (i) imply that, for every $r \in K(f,M)$, $w(g,r)$ is finite. Hence, Constrained optimization problem $\mathrm{COP}(g,f,M)$ is well defined.

Next, we define a set-valued mapping $F : K(f,M) \to 2^{K(f,M)}$ as follows

$$F(r) = \{q \in K(f,M) : w(g,r) \le w(g,q)\} \tag{2.4}$$

for $r \in K(f,M)$. Since $r \in F(r)$, it implies, for every $r \in K(f,M)$, $F(r) \neq \emptyset$ for every $r \in K(f,M)$. We demonstrate that, for every fixed $r \in K(f,M)$, $F(r)$ is a convex subset in $K(f,M)$. Take arbitrary $q_1, q_2 \in F(r)$, that is,

$$\int_0^L g(x,q_i(x))\,dx \ge \int_0^L g(x,r(x))dx, \text{ for } i = 1,2. \tag{2.5}$$

For any $t \in [0,1]$, we have $tq_1 + (1-t)q_2 \in K(f,M)$. By the condition (ii) in this theorem and (2.5), we obtain

$$\begin{aligned}
w(g,tq_1+(1-t)q_2) &= \int_0^L g(x,tq_1(x)+(1-t)q_2(x))\,dx \\
&\ge \int_0^L (tg(x,q_1(x))+(1-t)g(x,q_2(x))\,dx \\
&= t\int_0^L g(x,q_1(x))\,dx + (1-t)\int_0^L g(x,q_2(x))\,dx \\
&\ge \int_0^L g(x,r(x))dx,
\end{aligned}$$

which implies that $tq_1 + (1-t)q_2 \in F(r)$. Thus $F(r)$ is a convex subset in $K(f,M)$. Then we prove that, for every fixed $r \in K(f,M)$, $F(r)$ is a $\|\cdot\|$-closed subset of $K(f,M)$. To this end, we take an arbitrary sequence $\{q_n\} \subseteq F(r)$ with $\|\cdot\| - 1$ limit q_0. That is, $w(g,r) \le w(g,q_n)$ for $n = 1,2,\ldots$, and $\|q_n - q_0\| \to 0$ as $n \to \infty$. From Lemma 2.1, $q_0 \in K(f,M)$. It yields $q_n \to q_0$ a.e. as $n \to \infty$. From the condition (i) in this theorem, for any fixed $x \in [0,L]$, $g(x,t)$ is continuous, decreasing, and concave with respect to t on $[0,1]$. Thus $\{w(g,q_n)\}$ is bounded by $\int_0^L g(x,0)dx$. Hence, we have

$$\begin{aligned}
\int_0^L g(x,q_0(x))\,dx &= \int_0^L \lim_{n\to\infty} g(x,q_n(x))\,dx \\
&= \lim_{n\to\infty}\int_0^L g(x,q_n(x))\,dx \\
&\ge \int_0^L (g(x,r(x)))dx.
\end{aligned}$$

It follows that $q_0 \in F(r)$, and then $F(r)$ is a convex and $\|\cdot\|$-closed subset of $K(f,M)$. Hence $F(r)$ is weak compact in subset $K(f,M)$, that is, $F(r)$ is (weak) closed in the Hausdorff topological vector space $(L_2[0,L],\tau)$.

Finally, we demonstrate that $F : K(f,M) \to 2^{K(f,M)}\backslash\{\emptyset\}$ is a KKM mapping. For an arbitrary given positive integer n, we take arbitrary n points $r_1, r_2, \ldots, r_n$ of $K(f,M)$. For any n positive numbers $t_1,\, t_2, \ldots, t_n$ satisfying $\sum_{i=1}^{n} t_i = 1$, we let $r_0 = \sum_{i=1}^{n} t_i r_i$. We need to demonstrate that $r_0 \in \bigcup_{i=1}^{n} F(r_i)$. Assume controversially that $r_o \notin F(r_i)$ for all $i = 1,2,\ldots,n$. It follows that $W(g,r_i) > W(g,r_0)$ for all $i = 1,2,\ldots,n$. That is,

$$\int_0^L g(x,r_i(x))\,dx > \int_0^L g(x,r_0(x))\,dx, \text{ for all } i = 1,2,\ldots,n. \tag{2.6}$$

Pick $j \in \{1,2,\ldots,n\}$ such that

$$\int_0^L g\left(x,r_j(x)\right)dx = \min\left\{\int_0^L g(x,r_i(x))\,dx, \text{ for } i = 1,2,\ldots n\right\}. \tag{2.7}$$

From (2.6), (2.7), and the condition (i) in this theorem, we have

$$\begin{aligned}
w\left(g,r_j\right) &= \int_0^L g\left(x,r_j(x)\right)dx > \int_0^L g(x,r_0(x))\,dx \\
&= \int_0^L g\left(x,\sum_{i=1}^{n} t_i r_i(x)\right)dx \geq \int_0^L \sum_{i=1}^{n} t_i g(x,r_i(x))\,dx \\
&= \sum_{i=1}^{n} t_i \int_0^L g(x,r_i(x))\,dx \geq \sum_{i=1}^{n} t_i \int_0^L g\left(x,r_j(x)\right)dx \\
&= \int_0^L g\left(x,r_j(x)\right)dx = w\left(g,r_j\right),
\end{aligned}$$

which is a contradiction. So there must be a positive integer $k \in \{1,2,\ldots,n\}$ such that $w(g,r_k) \leq w(g,r_0)$, that is, $r_0 = \sum_{i=1}^{n} t_i r_i \in F(r_k)$ for some $k \in \{1,2,\ldots,n\}$. It implies that $F : K(f,M) \to 2^{K(f,M)}\backslash\{\emptyset\}$ is a KKM mapping. Since $K(f,M)$ is a τ-compact subset in the Hausdorff topological vector space $(L_2[0,L],\ \tau)$, by the Fan- KKM Theorem, it yields that $\cap_{r\in K[f,M]} F(r) \neq \emptyset$. Take an arbitrary point $r^* \in \bigcap_{r\in K[f,M]} F(r)$. From (2.4), r^* satisfies $w(g,r) \leq w(g,r^*)$ for all $r \in K(f,M)$. □

Remark 2.1. Constrained optimization problem COP(g,f,M) can be extended as follows

(i) underlying space $L_2[0,L]$ can be extended to $L_p[0,L]$, for any $p > 1$;

(ii) any positive number can replace the upper bound 1 of the elements in K.

3. Applications to Optimal Taxzations

In a considered given country (society) during a given time period, let S denote the population space of the individuals (taxpayers) who pay tax to the government. In this paper, we study a continuous macroeconomic model by assuming that S is equipped with a Borel field $\mathscr{F}$ of subsets of S and a measure μ. We suppose that the size of this population S is finite, which is denoted by $|S| = \mu(S) < \infty$.

Suppose that, with respect to this given period, there is a positive number L such that the taxpayers' monetary incomes (numeric quantized) lay in a closed interval $[0,L]$, where L is called the upper bound of the taxpayers' incomes in this society. Let J denote the income function on this population space S, that is, for any taxpayer $\omega \in S$, $J(\omega)$ is the income of this taxpayer ω satisfying $J(\omega) \in [0,L]$. Suppose that J is $\mathscr{F}$-measurable, which follows that, for any $0 \leq a < b \leq L$, one has $\{\omega \in S : a \leq J(\omega) \leq b\} \in \mathscr{F}$. Let p denote the density function of the income functional J on $[0,L]$ induced from the measure space $(S, \mathscr{F}, \mu)$. Then, for any $0 \leq a < b \leq L$, we have

$$\mu\{\omega \in S : a \leq J(\omega) \leq b\} = \int_a^b p(x)dx.$$

In particular, by the density function p, the size of S is calculated by

$$|S| = \mu(S) = \int_0^L p(x)dx < \infty.$$

With respect to the taxpayers' incomes, we denote the total gross domestic product (GDP) of this country in the given time period by G. By the density function p of the income functional J, G is accumulated by

$$G = \int_S J(\omega)\mu(d\omega) = \int_0^L xp(x)dx.$$

Assume that the government of this country establishes its tax policy that is actually performed for collecting income tax according to a tax rate function $r : [0,L] \to [0,1]$. We assume that r is an increasing (no decreasing) function. For a given income tax rate function r, let $R(r)$ be the total national revenue of this government from taxation with respect to r. Then

$$R(r) = \int_S J(\omega)r(J(\omega))\mu(d\omega) = \int_0^L xr(x)p(x)dx.$$

Since $0 \leq r(x) \leq 1$ for all $x \in [0,L]$, it yields that $0 \leq R(r) \leq G$ for any given tax rate function r.

In this paper, we consider the case that the taxpayers' utilities in this country only depend on their incomes and their taxes paid. The utilities are defined by a utility function $u : [0,L] \times [0,1] \to [0,\infty)$ such that the utility of an taxpayer $\omega \in S$ with income $J(\omega)$, under a given tax rate $r(J(\omega))$, is defined by the utility function $u(J(\omega), r(J(\omega)))$. Following the traditions in economic theory regarding to utility functions (see [11, 12, 13, 14]), throughout this paper, unless otherwise it is stated, we suppose that the utility function $u : [0,L] \times [0,1] \to [0,\infty)$ is continuous.

A society with individuals' (taxpayers') population S, income function J and utility function u is denoted by society (S,J,u). Under the utility function u, we denote W for the national social welfare (the total utilities of all taxpayers in this country) with respect to a given tax rate function r. Similarly, to (2.3), it is accumulated by

$$W(u,r) = \int_S u(J(\omega), r(J(\omega))\mu(d\omega) = \int_0^L u(x, r(x))p(x)dx.$$

The government of this country sets up a target for national revenue R_0 (tax collection from its taxpayers) for its fiscal plan, which satisfies $0 < R_0 < G$. Such a national revenue R_0 is called a practical national revenue. Subjected to a given practical national revenue R_0, this government seeks a most utility promising income tax rate function r^* satisfying

$$R(r^*) = \int_S f(\omega)r^*(f(\omega))\mu(d\omega) = \int_0^L xr^*(x)p(x)dx = R_0, \tag{3.1}$$

such that

$$W(u,r) \leq W(u,r^*) \text{ for all tax rate function } r \text{ satisfying (3.1).} \tag{3.2}$$

Definition 3.1. For a given society (S,J,u) at a given time period, let R_0 be a given practical national revenue. The national revenue constrained taxation optimization problem for this society (S,J,u) subject to the national revenue R_0, denoted by $\mathrm{CTOP}(S,J,u,R_0)$, is to find a tax rate function r^* such that both (3.1) and (3.2) simultaneously hold.

Theorem 3.1. *In a given society (S,J,u) at a given time period and subject to a practical national revenue R_0, suppose that the utility function u satisfies the following conditions:*
(I). $u:[0,L]\times[0,1]\to[0,\infty)$ is a continuous function;
(II). For any fixed $x\in[0,L], u(x,t)$ is decreasing and concave with respect to t on $[0,1]$.
Then the problem $\mathrm{CTOP}(S,J,u,R_0)$ has a solution.

Proof. By using the income density function p, we define a function f on $[0,L]$ by $f(x)=xp(x)$ for $x\in[0,L]$. It is clear that $f\in L_2$ and it is a nonnegative function on $[0,L]$ satisfying (2.1). It follows that the collection of all income tax rate functions providing the given practical national revenue R_0 satisfying (3.1) is the (national revenue) constrained class $K(f,R_0)$, which is a nonempty T-compact subset in L_2. From the utility function u and the income density function p, we define a function $g:[0,L]\times[0,1]\to[0,\infty)$ by

$$g(x,r(x)) = p(x)u(x,r(x)) \text{ for any } x\in[0,L] \text{ and for any tax rate function } r.$$

The continuity of u on $[0,L]\times[0,1]$ and the increasing and upper bounded by 1 properties of tax rate functions imply that this function g satisfies all the conditions (i-ii) in Theorem 2.1. Then the national revenue constrained taxation optimization problem CTOP (S,J,u,R_0) becomes the constrained optimization problem $\mathrm{COP}\left(pu, I_{[0,L]}p, R_0\right)$ under constrained class $K\left(I_{[0,L]}p, R_0\right)$, where $I_{[0,L]}$ is the identity function on $[0,L]$. From Theorem 2.1, it immediately follows that the national revenue constrained taxation optimization problem $\mathrm{CTOP}(S,J,u,R_0)$ has a solution. □

Remarks on utility functions (see [11, 12, 13, 14]).

(a) In macroeconomic theory, authors commonly assume that the utility function in a continuous economic model is as smooth as one needs. Hence the condition (I) in Theorem 3.1 is very practical.

(b) The condition (II) for utility functions in Theorem 3.1 means that in the real lives, for any taxpayer $\omega\in S$ with income $J(\omega)\in[0,L]$, the taxpayer's utility $u(J(\omega),r(J(\omega)))$ decreases faster as $r(J(\omega))$ increasing closer to 1.

(c) The practical utility function u in the real world should satisfy $u(x,1)=0$ for any $x\in[0, L]$. It means that if the government collects all incomes as tax from a taxpayer $\omega\in S$ with income $J(\omega)\in[0,L]$, then this taxpayer ω has no utility to enjoy (living).

4. Conclusions

Optimal taxation theory is a major branch in macroeconomic theory and economic theory. Many papers have been published on this topic during last half century (see [7, 8, 9, 11, 10, 12, 13, 14, 15]). Roughly speaking, for a given society during a given time period, taxation optimization means to design a maximizing utility tax policy subjected to some constraints. Such policy is implemented by collecting taxes according to a tax rate function that maximizes the social welfare for the whole group of taxpayers with respect to a given utility function. In traditional economic theory, considered utility functions are defined on finite dimensional Euclidean spaces that can be assumed as smooth as one needs. So, in where, optimization problems including maximizing and minimizing problems are normally solved by applying Lagrange multiplier optimization methods, Pontryagin maximum principle, and so on.

Every optimal taxation topic is a global optimization problem. Since the Fan-KKM theorem is applied in global analysis, in this paper, it is used to study the constrained optimal taxation problem. Our mathematical social utility optimization model is very general with respect to the following practical reasons.

1. The social utility function, a two variables function, is only assumed continuous, decreasing, and concave with respect to the second variable.

2. Tax rate functions are assumed increasing, which means that the more incomes of taxpayers, the higher ratio to pay tax.

3. Utility function is decreasing to 0 as the second variable (with respect to tax) increases to 1 (it is 100%). That is, taxpayers will have no utility if the government tax 100% of their incomes.

Some questions regarding to tax rate functions for consideration:

1. How do we prove that, with a given utility function (it may satisfy the conditions in Theorem 3.1), the optimized tax rate function must be increasing instead of assuming the increasing property?

2. Suppose that it is proved that the optimized tax rate function is an increasing function and it is differentiable. How do we find its concavity?

In this paper, we use the Fan-KKM Theorem to prove an existence theorem for the optimal taxation problem. As this theorem has been widely applied to global optimization theory, equilibrium theory, fixed point theory, and variational inequality theory in nonlinear analysis, it would be useful in some fields of applied mathematics such as, economic theory, business management, finance, engineering, and so on.

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