Appl. Set-Valued Anal. Optim. 5 (2023), No. 1, pp. 95-119
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.5.2023.1.07

# MODIFIED FORWARD-BACKWARD SPLITTING METHOD FOR SPLIT EQUILIBRIUM, VARIATIONAL INCLUSION, AND FIXED POINT PROBLEMS

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**Abstract.** In the recent time, the problem of finding common solutions of fixed point problems (FPPs) of nonlinear mappings and optimization problems (OPs) has received great research attention due to its potential applications to mathematical models whose constraints can be expressed as the FPPs and OPs. In this paper, we study the problem of finding a common solution of a split equilibrium problem (SEP), a variational inclusion problem (VIP) and the FPP with a finite family of multivalued demicontractive mappings. We propose a new inertial iterative method, which employs the forward-backward splitting technique together with the viscosity method for approximating the solution of the problem in Hilbert spaces. The proposed method uses variable step sizes, which do not depend on the norm of the bounded linear operator. We prove strong convergence results under some mild conditions. Finally, we present some numerical experiments to demonstrate the efficiency and applicability of the proposed method. Our result improves and extends several existing results in the current literature in this direction.
**Keywords.** Adaptive step size; Forward-backward splitting method; Inclusion problem; Inertial technique; Split equilibrium problem; Multivalued demicontractive mappings.

**2020 Mathematics Subject Classification.** 90C25, 47H05.

## 1. INTRODUCTION

Let $H$ be a real Hilbert space with inner project $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, and let $C$ be a nonempty, convex, and cosed subset of $H$. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The Equilibrium problem $(EP)$ is to find $x^* \in C$ such that $F(x^*, p) \geq 0$ for all $p \in C$. The $(EP)$, introduced in 1994 by Blum and Oetti [1], has a far-reaching impact and applications in several areas and fields of research, such as computers, engineering, economics, physics, and so on; see [2] and the references therein. The $EP$ is also known to be a generalisation of many important problems in nonlinear analysis, such as variational inequality problems, nonlinear complementarity problems, saddle point problems, fixed point problems, the Nash equilibra problem, and several other problems, see, e.g., [3, 4, 5, 6, 7], and the references therein.

In this work, we consider the problem of finding a point $x^* \in C_1$ satisfying:

$$F_1(x^*, x) \geq 0, \quad \forall x \in C_1, \tag{1.1}$$

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Received October 28, 2022; Accepted January 3, 2023.

such that $p^* = Gx^* \in M$ satisfies

$$F_2(p^*, p) \geq 0, \quad \forall p \in C_2, \tag{1.2}$$

where $C_1 \subseteq H_1$ and $C_2 \subseteq H_2$ are nonempty, convex, and closed subsets of Hilbert spaces $H_1$ and $H_2$, respectively, $F_1 : C_1 \times C_1 \to \mathbb{R}$ and $F_2 : C_2 \times C_2 \to \mathbb{R}$ are bifunctions, and $G : H_1 \to H_2$ is a bounded linear operator.

The problem (1.1)-(1.2) is known as the split equilibrium problem $(SEP)$. We denote the solution sets of $EP$ (1.1) and $EP$ (1.2) by $EP(F_1)$ and $EP(F_2)$, respectively and we denote the set of solution of the $SEP$ by

$$SEP(F_1, F_2) := \{p \in EP(F_1) : Gp \in EP(F_2)\}.$$

Numerous researchers presented interesting results in finding the solutions of the $SEP$ (1.1)-(1.2) and also found applications in various areas; see, e.g., [8, 9] and the references therein. Let $H$ be a real Hilbert space, $A : H \to H$ be a single-valued operator, and $B : H \to 2^H$ be a multivalued operator. The variational inclusion problem $(VIP)$, considered by Rockafellar [10], is to find a point $x^* \in H$ such that

$$0 \in (A + B)x^*. \tag{1.3}$$

The solution set of the $VIP$ (1.3) is referred to as the set of zero points of $A + B$. The problem of finding the zero points of the sum of two monotone operators is an active area of research interest and has various applications in the field of nonlinear analysis, such as convex optimization problems and variational inequality problems. It is a well known fact that the convex minimization problem can be transformed into finding zero points of a maximal monotone operator defined on Hilbert spaces (see [11] and the references). In an attempt to find the solution to the $VIP$ (1.3), several methods have been proposed and studied by researchers; see, e.g., [12, 13, 14, 15, 16] and the references therein. The most common and efficient of these methods is the famous forward-backward method introduced by [17] and defined as follows:

$$x_{n+1} = (I + \lambda_n B)^{-1}(I - \lambda_n A)(x_n),$$

where $\lambda_n$ is a positive parameter, $(I - \lambda_n A)$ is the forward operator, and $(I + \lambda_n B)^{-1}$ is called the resolvent operator, also known as the backward operator. Numerous real world problems in image processing, machine learning, linear inverse problems, and optimization problems can easily be modelled into $VIP$ (1.3), which makes it a hot area of research interest to researchers.

Let $S : C \to C$ be a non linear mapping. A point $p$ in $C$ is called a *fixed point* of $S$ if $Sp = p$. We denote the set of all fixed points of $S$ by $Fix(S)$, that is, $Fix(S) = \{p \in C : Sp = p\}$. If $S$ is a multivalued mapping, i.e., $S : C \to 2^C$, then $p \in C$ is called a fixed point of $S$ if $p \in Sp$.

Over the years, researchers have done numerous interesting works on the common solution of the fixed point problem (FPP) of nonlinear (single-valued or multi-valued) mappings and various optimization problems; see, e.g., [18, 19, 20, 21] and the references therein. It is a well known fact that several real world problems in image recovery, network resource allocation, signal processing, bandwidth theory, data compression, computerised tomography, and so on can be directly modelled into mathematical problems in the form of fixed point problems of nonlinear mappings and various optimization problems, which has made it a centre of attraction to researchers in the recent time; see, e.g., [22, 23]. This forms the bedrock of our research interest in investigating the common solution of these problems.

In 2018, Abass et al. [24] proposed an iterative algorithm for finding the common solution of the $SEP$ and the $FPP$ for an infinite family of quasi-nonexpansive multi-valued mappings $\{T_i\}_{i=1}^{+}\infty$ in real Hilbert spaces as follows:

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \xi_n G^*(T_{r_n}^{F_2} - I)Gx_n); \\ y_n = \lambda_0 u_n + \sum_{i=1}^{+}\infty \lambda_i z_n^i; \\ x_{n+1} = \gamma_n \tau f(x_n) + (I - \gamma_n D)y_n, \quad n \geq 1; \end{cases} \tag{1.4}$$

where $z_n^i \in T_i u_n$, $r_n \subset (0, +\infty)$, and the step size $\xi_n$ is chosen such that, for some $\varepsilon > 0$,

$$\xi_n \in \left( \varepsilon, \ \frac{\|(T_{r_n}^{F_2} - I)Gx_n\|^2}{\|G^*(T_{r_n}^{F_2} - I)Gx_n\|^2} - \varepsilon \right);$$

for all $T_{r_n}^{F_2} Gx_n \neq Gx_n$, and $\xi_n = \xi$, otherwise ($\varepsilon$ being any nonnegative real number). Under the conditions that $\gamma_n$ and $r_n$ satisfy the following conditions:

(i) $\lim_{n\to+\infty} \gamma_n = 0$ and $\sum_{i=1}^{+}\infty \gamma_n = +\infty$;
(ii) $\gamma_n \in (0,1), \ 0 < \tau < \frac{\tau}{\mu}$, and $0 < \gamma_n < 2\mu$;
(iii) $\liminf_{n\to+\infty} r_n > 0$;
(iv) $\lambda_0, \ \lambda_i \in (0,1)$, such that $\sum_{i=0}^{+}\infty \lambda_i = 1$;

the authors proved that the sequence generated by (1.4) converges strongly to the solution of the problem.

To accelerate the rate of convergence of iterative methods, researchers often employ the inertial extrapolation technique, which was first introduced by Polyak [25]. The inertial method is a two-step iteration where the next iterate is defined by making use of the previous two iterates. Recently, a growing interest has been paid to the study of inertial type algorithms and several authors have developed fast iterative methods by employing the inertial technique; see, e.g., [26, 27] and the references therein.

Recently, Cholamjiak et al. [28] proposed a modified inertial forward-backward splitting method for solving the common solution of the $SEP$ and the $VIP$ in Hilbert spaces as follows:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \alpha_n y_n + (1-\alpha_n)T_{r_n}^{F_1}(I - \gamma G^*(I - T_{r_n}^{F_2})G)y_n, \\ x_{n+1} = \beta_n z_n + (1-\beta_n)J_{\lambda_n}^{Q}(I - \lambda_n V)z_n, \qquad n \geq 1, \end{cases} \tag{1.5}$$

where $J_{\lambda_n}^{Q} = (I + \lambda_n Q)^{-1}, \ \{\lambda_n\} \subset (0, 2\sigma), \ \{\theta_n\} \subset [0, \theta], \ \theta \in [0,1), \{r_n\} \subset (0, +\infty)$ with $\gamma \in (0, \frac{1}{L})$ such that $L$ is the spectral radius of $G^*G$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. They proved a weak convergence result for the proposed algorithm.

A major setback with Algorithm 1.5 is the fact that the step size depends on the norm of the bounded linear operator, however, in most cases the norm of the bounded linear operator is difficult to calculate or even almost impossible to estimate in some cases. This makes the implementation of the algorithm more computationally expensive and time consuming. Moreover, the authors were only able to obtain weak convergence result for the proposed algorithm. In solving optimization problems, strong convergence results are more desirable and applicable than weak convergence results.

In this work, we propose a new iterative method, which employs the inertial technique and the step size, which is independent on the norm of the bounded linear operator, for approximating the common solution of the $SEP$, the $VIP$, and the $FPP$. Our proposed algorithm has the following features:

(i) our algorithm solves simultaneously three different problems, that is, the split equilibrium problem, the variational inclusion problem, and the fixed point problem;
(ii) our algorithm solves the fixed point problem of a family of multivalued demicontractive mappings, which is a more general problem than the results in [9, 29];
(iii) our algorithm generates a sequence, which converges strongly to the solution of the problem;
(iv) our algorithm combines the forward-backward splitting method with the inertial technique to speed up convergence rate.

The remaining part of this paper is organised as follows. In section 2, we present relevant definitions and lemmas, used in the course of this study. In section 3, we present our algorithm. Section 4 contains the convergence analysis of the proposed iterative scheme. In section 5, we present some numerical examples to test the computational efficiency of our algorithm over other methods in literature while in Section 6 we give some concluding remarks.

## 2. Preliminaries

In this section, we recall some useful definitions and lemmas needed in establishing our main results. Let $C$ be a nonempty, convex, and closed subset of a real Hilbert space $H$, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|.\|$. One knows that each Hilbert space $H$ satisfies the Opial condition [30], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to+\infty}\|x_n-n\| < \liminf_{n\to+\infty}\|x_n-y\|$ holds for every $y\in H$ with $y\neq x$. Recall that a bounded linear operator $D: C\to H$ is strongly positive if there exist a constant $\bar{\gamma}>0$ such that $\langle Dx,x\rangle \geq \bar{\gamma}\|x\|^2$ for all $x\in C$. From [31], one knows that $||I-\rho D|| \leq 1-\rho\bar{\gamma}$, where $0<\rho\leq ||D||^{-1}$. For all $x,y\in H$, one knows that

(1) $||x+y||^2 \leq ||x||^2+2\langle y,x+y\rangle$;
(2) $||x+y||^2 = ||x||^2+2\langle x,y\rangle+||y||^2$;
(3) $||\delta x+(1-\delta)y||^2 = \delta||x||^2+(1-\delta)||y||^2-\delta(1-\delta)||x-y||^2$, where $\delta\in(0,1)$.

Let the weak and strong convergence of the sequence $\{x_n\}_{n=1}^{+}\infty$ to $x$ as $n\to+\infty$ be denoted by $x_n \rightharpoonup x$, and $x_n\to x$, respectively. We denote the *metric projection* of $H$ onto $C$ by $P_C$: $\|x-P_Cx\| \leq \|x-y\|$ for all $x\in H, y\in C$. It is known that $P_C$ is nonexpansive and has the following properties:

(i) $z=P_Cx \Longleftrightarrow \langle x-z,z-y\rangle \geq 0$ for all $x\in H$ and $y\in C$;
(ii) $\|y-P_Cx\|^2+\|x-P_Cx\|^2 \leq \|x-y\|^2$ for all $x\in H$ and $y\in C$;
(iii) $\langle x-y,P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2$ for all $x,y\in H$.

For a given sequence $\{x_n\}\subset H$, $w_\omega(x_n)$ denotes the set of weak limits of $\{x_n\}$, that is,

$$w_\omega(x_n) := \Big\{x\in H : x_{n_k} \rightharpoonup x, \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\Big\}$$

Recall that a mapping $T: H\to H$ is said to be

(1) *L- Lipschitz continuous* on $H$ if there exists a constant $L>0$ such that $\|Tx-Ty\| \leq L\|x-y\|$, $\forall x,y\in H$. If $L\in[0,1)$, then $T$ is called a *contraction*.
(2) *nonexpansive* on $H$ if $T$ is 1-Lipschitz continuous.

(3) *averaged* if it can be written as $T=(1-\alpha)I+\alpha S$, where $\alpha\in(0,1)$, $S:H\to H$ is nonexpansive, and $I$ is the identity mapping on $H$.
(4) *monotone* on $H$ if $\langle Tx-Ty,x-y\rangle\geq 0$, $\forall x,y\in H$.
(5) *k-inverse strongly monotone* ($k$-ism) on $H$ if there exists a constant $k>0$ such that $\langle Tx-Ty,x-y\rangle\geq k\|Tx-Ty\|^2$, $\forall x,y\in H$.
(6) *firmly nonexpansive* on $H$ if $\|Tx-Ty\|^2\leq\langle Tx-Ty,x-y\rangle$, $\forall x,y\in H$, or equivalently

$$\|Tx-Ty\|^2\leq\|x-y\|^2-\|(I-T)x-(I-T)y\|^2.$$

Recall that a subset $K$ of $H$ is called proximinal if, for each $x\in H$, there exists $y\in K$ such that $\|x-y\|=d(x,K)=\inf\{\|x-z\|:z\in K\}$. In the course of this work, we denote the families of all nonempty, closed, and bounded subsets, nonempty, closed, and convex subsets, nonempty and compact subsets, and nonempty, proximinal, and bounded subsets of $C$ by $CB(C)$, $CC(C)$, $KC(C)$, and $P(C)$, respectively. The *Pompeiu-Hausdorff* metric on $CB(C)$ is defined by:

$$H(A,B):=\max\left\{\sup_{x\in A}d(x,B),\ \sup_{y\in B}(y,A)\right\},\quad\forall A,B\in CB(C),$$

where $d(x,B)=\inf_{b\in B}\|x-b\|$.

Let $S:C\to 2^C$ be a multivalued mapping. $S$ is said to satisfy the *end point* condition if $Sd=\{d\}$ for all $d\in Fix(S)$. For multivalued mappings, $S_i:C\to 2^C$ ($i\in\mathbb{N}$) with $\bigcap_{i=1}^{+\infty}Fix(S_i)\neq\emptyset$, $S_i$ is said to satisfy the *common endpoint* condition if $S_i(d)=\{d\}$ for all $i\in\mathbb{N}$, $d\in\bigcap_{i=1}^{+}\infty Fix(S_i)$.

Recall that a multivalued mapping $S:C\to CB(C)$ is said to be:

(i) *nonexpansive* if $H(Sa,Sb)\leq\|a-b\|$ for all $a,b\in C$;
(ii) *quasi-nonexpansive* if $F(S)\neq\emptyset$ and $H(Sa,Sd)\leq\|a-d\|$ for all $a\in C,\ d\in Fix(S)$;
(iii) *nonspreading* if $2H(Sa,Sb)^2\leq d(b,Sa)^2+d(a,Sb)^2$ for all $a,b\in C$;
(iv) *k-hybrid* if there exists $k\in\mathbb{R}$ such that

$$(1+k)H(Sa,Sb)^2\leq(1-k)\|a-b\|^2+kd(b,Sa)^2+kd(a,Sb)^2,\quad\forall a,b\in C;$$

(iv) *$\lambda$-demicontractive* for $0\leq\lambda<1$ if $F(S)\neq\emptyset$ and

$$H(Sa,Sd)^2\leq\|a-d\|^2+\lambda d(a,Sa)^2,\quad\forall a\in C,d\in Fix(S).$$

It is easy to note from the definition above that the class of $\lambda$-demicontractive mappings is more general than all other types of mappings listed above.

Recall that the best approximation operator $P_S$ for a multivalued mapping $S:C\to P(C)$ is defined by $P_S(a):=\{b\in Sa:\|a-b\|=d(a,Sa)\}$. It is a known fact that $Fix(S)=Fix(P_S)$ and $P_S$ satisfies the endpoint condition; see [32] for an example of a best approximation operator $P_S$, which is nonexpansive but $S$ is not necessarily nonexpansive. Let $S:C\to CB(C)$ be a multivalued mapping. The multivalued mapping $I-S$ is said to be demiclosed at zero if, for any sequence $\{x_n\}\subset C$ which converges weakly to $d$ and the sequence $\{\|x_n-t_n\|\}$ converges strongly to 0, where $t_n\in Sx_n$, $d\in Fix(S)$.

**Lemma 2.1.** [33] *Let $H$ be a real Hilbert space and let $T:H\to H$ be a given operator.*

(i) *$T$ is firmly nonexpansive if and only if its complement $I-T$ is firmly nonexpansive.*
(ii) *$T$ is averaged if and only if its complement $I-T$ is $\beta$-ism for some $\beta>\frac{1}{2}$. Indeed, for $\alpha\in(0,1),T$ is $\alpha$-averaged if and only if $I-T$ is $\frac{1}{2\alpha}$-ism.*

**Lemma 2.2.** [34] *For each* $x_1, x_2, ... x_m \in H$ *and* $\alpha_1, ..., \alpha_m \in [0,1]$ *with* $\sum_{i=1}^{m} \alpha_i = 1$, *the following holds:* $\|\alpha_1 x_1 + ... + \alpha_m x_m\|^2 = \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \sum_{1 \le i < j \le m} \alpha_i \alpha_j \|x_i - x_j\|^2$.

**Lemma 2.3.** [35] *Let* $A : H \to H$ *be a k-inverse strongly mapping, then*

(1) *A is* $\frac{1}{k}$*- Lipschitz continuous and monotone mapping;*
(2) *If* $\lambda$ *is any constant in* $(0, 2k]$, *then the mapping* $(I - \lambda A)$ *is nonexpansive, where* $I$ *is the identity mapping on* $H$.

**Lemma 2.4.** [36] *Let* $\{a_n\}$, $\{c_n\} \subset \mathbb{R}_+$, $\{\sigma_n\} \subset (0,1)$, *and* $\{b_n\} \subset \mathbb{R}$ *be sequences such that* $a_{n+1} \le (1-\sigma)a_n + b_n + c_n$ *for all* $n \ge 0$. *Assume* $\sum_{n=0}^{+\infty} |c_n| < +\infty$. *Then*

(1) *If* $b_n \le \beta \sigma_n$ *for some* $\beta \ge 0$, *then* $\{a_n\}$ *is a bounded sequence.*
(2) *If* $\sum_{n=0}^{+\infty} \sigma_n = +\infty$ *and* $\limsup_{n\to+\infty} \frac{b_n}{\sigma_n} \le 0$, *then* $\lim\limits_{n\to+\infty} a_n = 0$.

**Lemma 2.5.** [37] *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers,* $\{\alpha_n\}$ *be a sequence in* $(0,1)$ *with* $\sum_{n=1}^{+\infty} \alpha_n = +\infty$, *and* $b_n$ *be a sequence of real numbers. Assume that* $a_{n+1} \le (1-\alpha)a_n + \alpha_n b_n$ *for all* $n \ge 1$, *if* $\limsup_{k\to+\infty} b_{n_k} \le 0$ *for every subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying* $\liminf_{k\to+\infty}(a_{n_{k+1}} - a_{n_k}) \ge 0$, *then* $\lim_{n\to+\infty} a_n = 0$.

**Assumption 2.1.** In solving the equilibrium problem $(EP)$, we assume that $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

$(A_1)$ $F(x,x) = 0$, for all $x \in C$;
$(A_2)$ $F(x,y) + F(y,x) \le 0$ for all $x, y \in C$;
$(A_3)$ $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \le F(x,y)$ for all $x, y, z \in C$;
$(A_4)$ $y \mapsto F(x,y)$ is convex and lower semicontinuous for each $x \in C$.

**Lemma 2.6.** [38] *Let* $C$ *be a nonempty, convex, and closed subset of a Hilbert space* $H$, *and let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying Assumption 2.1. For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r^F : H \to C$ *by* $T_r^F(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\}$. *Then,* $T_r^F$ *is well defined and the following hold:*

(1) *For each* $x \in H$, $T_r^F(x) \ne \emptyset$;
(2) $T_r^F$ *is single-valued;*
(3) $T_r^F$ *is firmly nonexpansive;*
(4) $Fix(T_r^F) = EP(F)$;
(5) $EP(F)$ *is convex and closed.*

Let $B : H \to 2^H$ be a multivalued monotone mapping. The effective domain of $B$, denoted by $Dom(B)$, is defined as $D(B) = \{x \in H : Bx \ne \emptyset\}$. The graph of $(B)$ is denoted by $G(B) = \{(p,q) \in H \times H : q \in Bp\}$. The range denoted, by $R(B)$, is defined by $R(B) = \bigcup_{x\in H} Bx$. $B$ is called *monotone* if, for all $x, y \in H$, $p \in Bx$ and $q \in By$ implies that $\langle p - q, x - y\rangle \le 0$. $B$ is called *maximal monotone* if it is monotone and if for any $(x,p) \in H \times H$, $\langle p - q, x - y\rangle \ge 0$ for every $(y,q) \in \text{Graph}(B)$ implies that $p \in Bx$. Recall that the *resolvent* mapping $J_\lambda^B : H \to H$ associated with a multivalued maximal monotone mapping $B$ is defined by $J_\lambda^B(x) = (I + \lambda B)^{-1}(x)$ for all $x \in H$, for some $\lambda > 0$, where $I$ is the identity operator on $H$. It is a known fact that if $B : H \to 2^H$ is a multivalued maximal monotone mapping and $\lambda > 0$, then $Dom(J_\lambda^B) = H$ and $J_\lambda^B$ is a single-valued and firmly nonexpansive mapping.

**Lemma 2.7.** [39] *Let $H$ be a real Hilbert space. Let $B : H \to 2^H$ be a maximal monotone operator and $A : H \to H$ be a $k$-inverse strongly monotone mapping on $H$. Define $T_\lambda = (I+\lambda B)^{-1}(I-\lambda A)$, $\lambda > 0$. Then,*

(i) $F(T_\lambda) = (A+B)^{-1}(0)$;

(ii) *for* $0 < s \leq \lambda$ *and* $x \in H$, $\|x - T_s x\| \leq 2\|x - T_\lambda x\|$.

## 3. The Algorithm

In this section, we present our proposed algorithm and highlight its main features. We also state the conditions needed to establish the strong convergence theorem for the proposed method.

**Condition B**

$(B_1)$ $H_1$ and $H_2$ are two real Hilbert spaces and $C_1 \subseteq H_1$, $C_2 \subseteq H_2$ are nonempty, closed and convex subsets of $H_1$ and $H_2$, respectively.

$(B_2)$ $G : H_1 \to H_2$ is a bounded linear operator with adjoint $G^*$, $Q : H_1 \to 2^{H_1}$ is a maximal monotone operator, and $V : H_1 \to H_1$ is a $\sigma$-ism.

$(B_3)$ $F_1 : C_1 \times C_1 \to \mathbb{R}$ and $F_2 : C_2 \times C_2 \to \mathbb{R}$ are two bifunctions satisfying conditions $(A_1) - (A_4)$ with $F_2$ being upper semicontinuous in the first argument.

$(B_4)$ $D : H_1 \to H_1$ is a strongly positive bounded linear operator with coefficient $\bar{\eta}$, and $f : H_1 \to H_1$ is a contraction with coefficient $\rho \in (0,1)$ such that $0 < \eta < \frac{\bar{\eta}}{\rho}$.

$(B_5)$ $S_i : C_1 \to CB(C_1)$ is a finite family of multivalued demicontractive mappings with constant $\kappa_i$ such that $I - S_i$ is demiclosed at zero, $S_i(p) = \{p\}$ for all $p \in \bigcap_{i=1}^m Fix(S_i)$, and $\kappa = \max\{\kappa_i\}, i = 1,2,\ldots,m$.

$(B_6)$ The solution set $\Upsilon = SEP(F_1,F_2) \cap (Q+V)^{-1}(0) \cap \bigcap_{i=1}^m Fix(S_i)$ is nonempty.

**Condition C**

$(C_1)$ $\{\psi_n\} \subset (0,1)$ such that $\lim_{n\to+\infty} \psi_n = 0$ and $\sum_{n=1}^{+}\infty \psi_n = +\infty$.

$(C_2)$ $\{\beta_{n,i}\} \subset (0,1)$, $\sum_{i=0}^m \beta_{n,i} = 1$, and $\liminf_{n\to+\infty}(\beta_{n,0} - k)\beta_{n,i} > 0$ for each $1 \leq i \leq m$.

$(C_3)$ $0 < \liminf_{n\to+\infty} \lambda_n \leq \limsup_{n\to+\infty} \lambda_n < 2\sigma$, and $0 < a \leq \phi_n \leq b < 2$.

$(C_4)$ Let $\nu > 0$ and $\{\varepsilon_n\}$ be a positive sequence such that $\varepsilon_n = o(\psi_n)$, i.e., $\lim\limits_{n\to+\infty} \frac{\varepsilon_n}{\psi_n} = 0$.

$(C_5)$ $\{r_n\} \subset (0,+\infty)$ and $\{s_n\} \subset (0,+\infty)$ such that $\liminf_{n\to+\infty} r_n > 0$ and $\liminf_{n\to+\infty} s_n > 0$.

Now, we present the algorithm as follows:

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**Algorithm 3.1.** The Inertial Forward-Backward Splitting Method

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**Step 0.** Select initial data $x_0, x_1 \in H_1$ and set $n = 1$.

**Step 1.** Given the $(n-1)th$ and $nth$ iterates, choose $\nu_n$ such that $0 \leq \nu_n \leq \hat{\nu}_n$, $\forall n \in \mathbb{N}$ with $\hat{\nu}_n$ defined by

$$\hat{\nu}_n = \begin{cases} \min\left\{\nu, \frac{\varepsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \nu, & \text{otherwise.} \end{cases}$$

**Step 2.** Compute $l_n = x_n + \nu_n(x_n - x_{n-1})$.

**Step 3.** Compute $g_n = T_{r_n}^{F_1}(l_n + \gamma_n G^*(T_{s_n}^{F_2} - I)Gl_n)$, where

$$\gamma_n = \begin{cases} \frac{\phi_n ||(T_{s_n}^{F_2} - I)Gl_n||^2}{||G^*(T_{s_n}^{F_2} - I)Gl_n||^2}, & \text{if } T_{s_n}^{F_2}Gl_n \neq Gl_n, \\ \gamma, & \text{otherwise } (\gamma \text{ being a nonnegative real number}). \end{cases}$$

**Step 4.** Compute

$$\begin{cases} t_n = \beta_{n,0} g_n + \sum_{i=1}^{m} \beta_{n,i} h_{n,i}; \quad h_{n,i} \in S_i g_n; \\ z_n = (I+\lambda_n Q)^{-1}(I-\lambda_n V)t_n = J_{\lambda_n}^{Q}(I-\lambda_n V)t_n; \\ x_{n+1} = \psi_n \eta f(l_n) + (I-\psi_n D)z_n. \end{cases}$$

Set $n := n+1$ and return to **Step 1**.

---

**Remark 3.1.** By conditions $(C_1)$ and $(C_4)$, we find from (4.26) that

$$\lim_{n\to+\infty} \nu_n \|x_n - x_{n-1}\| = 0 \quad \text{and} \quad \lim_{n\to+\infty} \frac{\nu_n}{\psi_n} \|x_n - x_{n-1}\| = 0.$$

We present some features of our proposed algorithm as following.

**Remark 3.2.** (i) The algorithm employs the inertial technique to accelerate the rate of convergence. Moreover, the first step of the algorithm is easily implemented since the value of $\|x_n - x_{n-1}\|$ is known prior to choosing $\nu_n$. These features make our algorithm is less computationally expensive.

(ii) The step size $\gamma_n$ is self-adaptive, that is, it is independent of the norm of the bounded linear operator.

(iii) Our proposed algorithm solves the problem of finding a common solution of the split equilibrium problem, the variational inclusion problem, and the common fixed point problem of a finite family of multivalued demicontractive mappings, which is a more general problem than the ones considered in [22, 29] and several other results in the literature.

Now, we demonstrate that the step size of the algorithm is well defined.

**Lemma 3.1.** *The step size $\{\gamma_n\}$ of the Algorithm 3.1 defined by (4.27) is well defined.*

*Proof.* Let $d \in \Upsilon$. Then $T_{r_n}^{F_1} d = d, T_{r_n}^{F_2} Gd = Gd$. Since $T_{r_n}^{F_2}$ is averaged, we conclude from Lemma 2.1(ii) that

$$\begin{aligned} \|G^*(I-T_{r_n}^{F_2})Gl_n\|\|l_n - d\| &\geq \langle G^*(I-T_{r_n}^{F_2})Gl_n, l_n - d\rangle \\ &= \langle (I-T_{r_n}^{F_2})Gl_n - (I-T_{r_n}^{F_2})Gd, Gl_n - Gd\rangle \\ &\geq \beta \|(I-T_{r_n}^{F_2})Gl_n\|^2, \end{aligned}$$

for some $\beta > \frac{1}{2}$. It follows that $\|G^*(I-T_{r_n}^{F_2})Gl_n\| > 0$ when $\|(I-T_{r_n}^{F_2})Gl_n\| \neq 0$. Hence, $\{\gamma_n\}$ is well defined.

□

## 4. The Convergence Analysis

First, we establish some lemmas, which are relevant in the convergence analysis of the proposed algorithm.

**Lemma 4.1.** *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then, $\{x_n\}$ is bounded.*

*Proof.* First, we need to establish that $P_\Upsilon(I-D+\eta f)$ is a contraction on $H_1$. For all $x,y\in H_1$, we have

$$\begin{aligned}\|P_\Upsilon(I-D+\eta f)(x)-P_\Upsilon(I-D+\eta f)(y)\| &\leq \|(I-D+\eta f)(x)-(I-D+\eta f)(y)\|\\ &\leq \|(I-D)x-(I-D)y\|+\eta\|fx-fy\|\\ &\leq (1-(\bar{\eta}-\eta\rho))\|x-y\|.\end{aligned}$$

This demonstrates that $P_\Upsilon(I-D+\eta f)$ is a contraction. Hence, by the Banach contraction principle, there exists an element $d\in\Upsilon$ such that $d=P_\Upsilon(I-D+\eta f)(d)$. Since $d\in\Upsilon$, then $d=T_{r_n}^{F_1}d$, $Gd=T_{s_n}^{F_2}(Gd), d=J_{\lambda_n}^{Q}(I-\lambda_nV)d$, and $S_i(d)=\{d\}$ for each $i=1,2,\ldots,m$. Since $T_{r_n}^{F_1}$ and $I-T_{r_n}^{F_1}$ are firmly nonexpansive, we obtain from Lemma 2.1(i), the definition of $\gamma_n$, and the condition on $\phi_n$ that

$$\begin{aligned}\|g_n-d\|^2 &= \|T_{r_n}^{F_1}(l_n+\gamma_nG^*(T_{s_n}^{F_2}-I)Gl_n)-d\|^2\\ &\leq \|l_n+\gamma_nG^*(T_{s_n}^{F_2}-I)Gl_n-d\|^2 &(4.1)\\ &= \|l_n-d\|^2+\gamma_n^2\|G^*(T_{s_n}^{F_2}-I)Gl_n\|^2+2\gamma_n\langle l_n-d,G^*(T_{s_n}^{F_2}-I)Gl_n\rangle\\ &= \|l_n-d\|^2+\gamma_n^2\|G^*(T_{s_n}^{F_2}-I)Gl_n\|^2+2\gamma_n\langle Gl_n-Gd,(T_{s_n}^{F_2}-I)Gl_n-(T_{s_n}^{F_2}-I)Gd\rangle\\ &\leq \|l_n-d\|^2-\gamma_n(2-\phi_n)\|(T_{s_n}^{F_2}-I)Gl_n\|^2 &(4.2)\\ &\leq \|l_n-d\|^2. &(4.3)\end{aligned}$$

From Lemma 2.2, (4.3), and the fact that $S_i$ is demicontractive for each $i=1,2,\ldots,m$, we have

$$\begin{aligned}\|t_n-d\|^2 &= \beta_{n,0}\|g_n-d\|^2+\sum_{i=1}^m\beta_{n,i}\|h_{n,i}-d\|^2-\beta_{n,0}\sum_{i=1}^m\beta_{n,i}\|h_{n,i}-g_n\|^2\\ &\leq \beta_{n,0}\|g_n-d\|^2+\sum_{i=1}^m\beta_{n,i}H(S_ig_n,S_id)-\beta_{n,0}\sum_{i=1}^m\beta_{n,i}\|h_{n,i}-g_n\|^2\\ &\leq \beta_{n,0}\|g_n-d\|^2+\sum_{i=1}^m\beta_{n,i}[\|g_n-d\|^2+\kappa_i\|g_n-h_{n,i}\|^2]-\beta_{n,0}\sum_{i=1}^m\beta_{n,i}\|h_{n,i}-g_n\|^2\\ &\leq \|g_n-d\|^2-\sum_{i=1}^m\beta_{n,i}(\beta_{n,0}-\kappa)\|g_n-h_{n,i}\|^2 &(4.4)\\ &\leq \|g_n-d\|^2. &(4.5)\end{aligned}$$

In view of the conditions imposed on the control parameters, we have

$$\begin{aligned}\|z_n-d\|^2 &= \|(I+\lambda_nQ)^{-1}(I-\lambda_nV)t_n-(I+\lambda_nQ)^{-1}(I-\lambda_nV)d\|^2\\ &\leq \|t_n-d-\lambda_n(Vt_n-Vd)\|^2\\ &= \|t_n-d\|^2-2\lambda_n\langle Vt_n-Vd,t_n-d\rangle+\lambda_n^2\|Vt_n-Vd\|^2\\ &\leq \|t_n-d\|^2-(2\sigma-\lambda_n)\lambda_n\|Vt_n-Vd\|^2 &(4.6)\\ &\leq \|t_n-d\|^2. &(4.7)\end{aligned}$$

By the definition of $l_n$, we have

$$\|l_n-d\|=\|x_n+\nu_n(x_n-x_{n-1})-d\|\leq\|x_n-d\|+\psi_n\cdot\frac{\nu_n}{\psi_n}\|x_n-x_{n-1}\|$$

From Remark 3.1, we see that $\frac{\nu_n}{\psi_n}\|x_n - x_{n-1}\| \to 0$ as $n \to +\infty$. Thus there exists a constant $J_1 > 0$ such that $\frac{\nu_n}{\psi_n}\|x_n - x_{n-1}\| \le J_1$ for all $n \ge 1$. Hence, $\|l_n - d\| \le \|x_n - d\| + \psi_n J_1$. It follows from (4.3), (4.5), and (4.7) that

$$\begin{aligned}
&\|x_{n+1} - d\| \\
&\le \psi_n\|\eta f(l_n) - Dd\| + (1 - \psi_n\bar{\eta})\|z_n - d\| \\
&= \psi_n[\eta\|f(l_n) - f(d)\| + \|\eta f(d) - Dd\|] + (1 - \psi_n\bar{\eta})\|z_n - d\| \\
&\le \psi_n[\eta\rho\|l_n - d\| + \|\eta f(d) - Dd\|] + (1 - \psi_n\bar{\eta})\|z_n - d\| \\
&\le \psi_n\eta\rho[\|x_n - d\| + \psi_n J_1] + \psi_n\|\eta f(d) - Dd\| + (1 - \psi_n\bar{\eta})[\|x_n - d\| + \psi_n J_1] \\
&= [1 - \psi_n(\bar{\eta} - \eta\rho)]\|x_n - d\| + \psi_n(\bar{\eta} - \eta\rho)\left[\frac{\|\eta f(d) - Dd\|}{\bar{\eta} - \eta\rho} + \frac{[1 - \psi_n(\bar{\eta} - \eta\rho)]J_1}{\bar{\eta} - \eta\rho}\right] \\
&\le [1 - \psi_n(\bar{\eta} - \eta\rho)]\|x_n - d\| + \psi_n(\bar{\eta} - \eta\rho)J^*,
\end{aligned}$$

where

$$J^* = \sup_{n\in\mathbb{N}}\left\{\frac{\|\eta f(d) - Dd\|}{\bar{\eta} - \eta\rho} + \frac{[1 - \psi_n(\bar{\eta} - \eta\rho)]J_1}{\bar{\eta} - \eta\rho}\right\}.$$

Setting $a_n := \|x_n - d\|$, $b_n := \psi_n(\bar{\eta} - \eta\rho)J^*$, $c_n := 0$, and $\sigma_n := \psi_n(\bar{\eta} - \eta\rho)$, and using Lemma 2.4(1) and the assumptions on the control sequences, we have that $\{\|x_n - d\|\}$ is bounded, which implies that $\{x_n\}$ is bounded. Therefore, $\{l_n\}$, $\{g_n\}$, $\{t_n\}$, and $\{z_n\}$ are also bounded. □

**Lemma 4.2.** *The following inequality holds for all $d \in \Upsilon$ and $n \in \mathbb{N}$:*

$$\begin{aligned}
\|x_{n+1} - d\|^2 \le &\left(1 - \frac{2\psi_n(\bar{\eta} - \psi_n\eta\rho)}{(1 - \psi_n\eta\rho)}\right)\|x_n - d\|^2 \\
&+ \frac{2\psi_n(\bar{\eta} - \eta\rho)}{(1 - \psi_n\eta\rho)}\left\{\frac{\psi_n\bar{\eta}^2}{2(\bar{\eta} - \eta\rho)}J_3 + \frac{((1 - \psi_n\bar{\eta})^2 + \psi_n\eta\rho)}{2(\bar{\eta} - \eta\rho)}3J_2\frac{\nu_n}{\psi_n}\|x_n - x_{n-1}\|\right. \\
&\left.+ \frac{1}{(\bar{\eta} - \eta\rho)}\langle\eta f(d) - Dd, x_{n+1} - d\rangle\right\} \\
&- \frac{(1 - \psi_n\bar{\eta})^2}{(1 - \psi_n\eta\rho)}\left\{\gamma_n(2 - \phi_n)\|(T_{s_n}^{F_2} - I)Gl_n\|^2 + 2(\sigma_n - \lambda_n)\lambda_n\|Vt_n - Vd\|^2\right. \\
&\left.+ \sum_{i=1}^{m}\beta_{n,i}(\beta_{n,0} - \kappa)\|g_n - h_{n,i}\|^2\right\}.
\end{aligned}$$

*Proof.* Let $d \in \Upsilon$. Then, by applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|l_n - d\|^2 &= \|x_n + \nu(x_n - x_{n-1}) - d\|^2 \\
&\le \|x_n - d\|^2 + \nu_n\|x_n - x_{n-1}\|(\nu_n\|x_n - x_{n-1}\| + 2\|x_n - d\|) \\
&\le \|x_n - d\|^2 + 3J_2\psi_n\frac{\nu_n}{\psi_n}\|x_n - x_{n-1}\|, \qquad (4.8)
\end{aligned}$$

where $J_2 := \sup_{n\in\mathbb{N}}\{\|x_n-d\|, \nu_n\|x_n-x_{n-1}\|\} > 0$. Now, by using (4.2), (4.4), (4.6), and (4.8), we have

$$\begin{aligned}
\|x_{n+1}-d\|^2 &\leq (1-\psi_n\bar{\eta})^2\|z_n-d\|^2 + 2\psi_n\langle \eta f(l_n) - Dd,\ x_{n+1}-d\rangle \\
&\leq (1-\psi_n\bar{\eta})^2[\|t_n-d\|^2 - (2\sigma-\lambda_n)\lambda_n\|Vt_n-Vd\|^2] \\
&\quad + 2\psi_n\langle \eta f(l_n) - \eta f(d),\ x_{n+1}-d\rangle + 2\psi_n\langle \eta f(d) - Dd,\ x_{n+1}-d\rangle \\
&\leq (1-\psi_n\bar{\eta})^2[\|g_n-d\|^2 - \sum_{i=1}^m \beta_{n,i}(\beta_{n,0}-\kappa)\|g_n-h_{n,i}\|^2 - (2\sigma-\lambda_n)\lambda_n\|Vt_n-Vd\|^2] \\
&\quad + 2\psi_n\eta\langle f(l_n) - f(d),\ x_{n+1}-d\rangle + 2\psi_n\langle \eta f(d) - Dd,\ x_{n+1}-d\rangle \\
&\leq (1-\psi_n\bar{\eta})^2[\|l_n-d\|^2 - \gamma_n(2-\phi_n)\|(T_{s_n}^{F_2}-I)Gl_n\|^2 \\
&\quad - \sum_{i=1}^m \beta_{n,i}(\beta_{n,0}-\kappa)\|g_n-h_{n,i}\|^2 - (2\sigma-\lambda_n)\lambda_n\|Vt_n-Vd\|^2] \\
&\quad + \psi_n\eta\rho(\|l_n-d\|^2 + \|x_{n+1}-d\|^2) + 2\psi_n\langle \eta f(d) - Dd,\ x_{n+1}-d\rangle \\
&\leq [(1-\psi_n\bar{\eta})^2 + \psi_n\eta\rho]\|x_n-d\|^2 + \psi_n\eta\rho\|x_{n+1}-d\|^2 \\
&\quad + [(1-\psi_n\bar{\eta})^2 + \psi_n\eta\rho]3J_2\psi_n\frac{\nu_n}{\psi_n}\|x_n-x_{n-1}\| + 2\psi_n\langle \eta f(d) - Dd,\ x_{n+1}-d\rangle \\
&\quad - (1-\psi_n\bar{\eta})^2\{\gamma_n(2-\phi_n)\|(T_{s_n}^{F_2}-I)Gl_n\|^2 + \sum_{i=1}^m \beta_{n,i}(\beta_{n,0}-\kappa)\|g_n-h_{n,i}\|^2 \\
&\quad + (2\sigma-\lambda_n)\lambda_n\|Vt_n-Vd\|^2\},
\end{aligned} \tag{4.9}$$

which in turn implies that

$$\begin{aligned}
&\|x_{n+1}-d\|^2 \\
&\leq \frac{(1-2\psi_n\bar{\eta} + (\psi_n\bar{\eta}^2) + \psi_n\eta\rho)}{(1-\psi_n\eta\rho)}\|x_n-d\|^2 + \frac{((1-\psi_n\bar{\eta})^2 + \psi_n\eta\rho)}{(1-\psi_n\eta\rho)}3J_2\psi_n\frac{\nu_n}{\psi_n}\|x_n-x_{n-1}\| \\
&\quad + \frac{2\psi_n}{(1-\psi_n\eta\rho)}\langle \eta f(d) - Dd,\ x_{n+1}-d\rangle - \frac{(1-\psi_n\bar{\eta})^2}{(1-\psi_n\eta\rho)}\{\gamma_n(2-\phi_n)\|(T_{s_n}^{F_2}-I)Gl_n\|^2 \\
&\quad + (2\sigma-\lambda_n)\lambda_n\|Vt_n-Vd\|^2 + \sum_{i=1}^m \beta_{n,i}(\beta_{n,0}-\kappa)\|g_n-h_{n,i}\|^2\} \\
&= \big(1-\frac{2\psi_n(\bar{\eta}-\eta\rho)}{(1-\psi_n\eta\rho)}\big)\|x_n-d\|^2 + \frac{2\psi_n(\bar{\eta}-\eta\rho)}{(1-\psi_n\eta\rho)}\Big\{\frac{\psi_n\bar{\eta}^2}{2(\bar{\eta}-\eta\rho)}J_3 \\
&\quad + \frac{((1-\psi_n\bar{\eta})^2 + \psi_n\eta\rho)}{2(\bar{\eta}-\eta\rho)}3J_2\frac{\nu_n}{\psi_n}\|x_n-x_{n-1}\| + \frac{1}{\bar{\eta}-\eta\rho}\langle \eta f(d) - Dd,\ x_{n+1}-d\rangle\Big\} \\
&\quad - \frac{(1-\psi_n\bar{\eta})^2}{(1-\psi_n\eta\rho)}\{\gamma_n(2-\phi_n)\|(T_{s_n}^{F_2}-I)Gl_n\|^2 + (2\sigma-\lambda_n)\lambda_n\|Vt_n-Vd\|^2 \\
&\quad + \sum_{i=1}^m \beta_{n,i}(\beta_{n,0}-\kappa)\|g_n-h_{n,i}\|^2\},
\end{aligned}$$

where $J_3 = \sup\{\|x_n-d\|^2 : n\in\mathbb{N}\}$. This completes the proof. □

**Lemma 4.3.** *The following inequality holds for all $d \in \Upsilon$ and $n \in \mathbb{N}$*

$$\begin{aligned}\|x_{n+1}-d\|^2 \le (1-\psi_n\bar{\eta})^2\|x_n-d\|^2 &+ 3J_2(1-\psi_n\bar{\eta})^2\psi_n\frac{\nu_n}{\psi_n}\|x_n-x_{n-1}\| \\ &+ 2\psi_n\langle \eta f(d)-Dd,\ x_{n+1}-d\rangle + 2J_4(1-\psi_n\bar{\eta})^2\|Vt_n-Vd\| \\ &+ 2J_5(1-\psi_n\bar{\eta})^2\|G^*(T_{s_n}^{F_2}-I)Gl_n\| - (1-\psi_n\bar{\eta})^2\{\|g_n-l_n\|^2+\|t_n-z_n\|^2\},\end{aligned}$$

*where $J_4 := \sup_{n\in\mathbb{N}}\{\lambda_n\|t_n-z_n\|\}$ and $J_5 := \sup_{n\in\mathbb{N}}\{\gamma_n\|g_n-l_n\|\}$.*

*Proof.* Let $d \in \Upsilon$. From the fact that $(I+\lambda_n Q)^{-1}$ is firmly nonexpansive and $I-\lambda_n V$ is nonexpansive, we have

$$\begin{aligned}\|z_n-d\|^2 &= \|(I+\lambda_n Q)^{-1}(I-\lambda_n V)t_n-(I+\lambda_n Q)^{-1}(I-\lambda_n V)d\|^2 \\ &\le \langle z_n-d,\ (I-\lambda_n V)t_n-(I-\lambda_n V)d\rangle \\ &\le \frac{1}{2}\|t_n-d\|^2+\frac{1}{2}\|z_n-d\|^2-\frac{1}{2}\|t_n-z_n\|^2-\frac{1}{2}\lambda_n^2\|Vt_n-Vd\|^2+\lambda_n\|t_n-z_n\|\|Vt_n-Vd\|,\end{aligned}$$

So, we have

$$\|z_n-d\|^2 \le \|t_n-d\|^2-\|t_n-z_n\|^2+2J_4\|Vt_n-Vd\|, \tag{4.10}$$

where $J_4 := \sup_{n\in\mathbb{N}}\{\lambda_n\|t_n-z_n\|\}$. From (4.1) and (4.3), we arrive at

$$\|l_n+\gamma_n G^*(T_{s_n}^{F_2}-I)Gl_n-d\|^2 \le \|l_n-d\|^2.$$

By the firmly nonexpansivity of $T_{r_n}^{F_1}$, we have

$$\begin{aligned}\|g_n-d\|^2 &= \|T_{r_n}^{F_1}(l_n+\gamma_n G^*(T_{s_n}^{F_2}-I)Gl_n)-T_{r_n}^{F_1}d\|^2 \\ &\le \langle g_n-d,\ l_n+\gamma_n G^*(T_{s_n}^{F_2}-I)Gl_n-d\rangle \\ &= \frac{1}{2}\{\|g_n-d\|^2+\|l_n+\gamma_n G^*(T_{s_n}^{F_2}-I)Gl_n-d\|^2 \\ &\quad -\|(g_n-d)-(l_n+\gamma_n G^*(T_{s_n}^{F_2}-I)Gl_n-d)\|^2\} \\ &\le \frac{1}{2}\{\|g_n-d\|^2+\|l_n-d\|^2-\|g_n-l_n-\gamma_n G^*(T_{s_n}^{F_2}-I)Gl_n\|^2\} \\ &= \frac{1}{2}\{\|g_n-d\|^2+\|l_n-d\|^2-(\|g_n-l_n\|^2+\gamma_n^2\|G^*(T_{s_n}^{F_2}-I)Gl_n\|^2 \\ &\quad -2\gamma_n\langle g_n-l_n,\ G^*(T_{s_n}^{F_2}-I)Gl_n\rangle)\} \\ &\le \frac{1}{2}\{\|g_n-d\|^2+\|l_n-d\|^2-\|g_n-l_n\|^2+2\gamma_n\|g_n-l_n\|\|G^*(T_{s_n}^{F_2}-I)Gl_n\|\}.\end{aligned}$$

Thus

$$\begin{aligned}\|g_n-d\|^2 &\le \|l_n-d\|^2-\|g_n-l_n\|^2+2\gamma_n\|g_n-l_n\|\|G^*(T_{s_n}^{F_2}-I)Gl_n\| \\ &\le \|l_n-d\|^2-\|g_n-l_n\|^2+2J_5\|G^*(T_{s_n}^{F_2}-I)Gl_n\|,\end{aligned} \tag{4.11}$$

where $J_5 := \sup_{n\in\mathbb{N}}\{\gamma_n\|g_n-l_n\|\}$. Using (4.5), (4.8), and (4.11) in (4.10), we arrive at

$$\begin{aligned}\|z_n-d\|^2 \le \|x_n-d\|^2 &+ 3J_2\psi_n\frac{\nu_n}{\psi_n}\|x_n-x_{n-1}\|-\|g_n-l_n\|^2 \\ &+ 2J_5\|G^*(T_{s_n}^{F_2}-I)Gl_n\|-\|t_n-z_n\|^2+2J_4\|Vt_n-Vd\|.\end{aligned} \tag{4.12}$$

Substituting (4.12) into (4.9) yields that

$$\begin{aligned}
\|x_{n+1}-d\|^2 \leq (1-\psi_n\bar{\eta})^2 \Bigg[ & \|x_n-d\|^2 + 3J_2\psi_n \frac{\nu_n}{\psi_n}\|x_n-x_{n-1}\| \\
& - \|g_n-l_n\|^2 + 2J_5\|G^*(T^{F_2}_{s_n}-I)Gl_n\| - \|t_n-z_n\|^2 + 2J_4\|Vt_n-Vd\| \Bigg] \\
& + 2\psi_n\langle \eta f(d)-Dd,\, x_{n+1}-d\rangle \\
= (1-\psi_n\bar{\eta})^2\|x_n-d\|^2 & + 3J_2(1-\psi_n\bar{\eta})^2\psi_n\frac{\nu_n}{\psi_n}\|x_n-x_{n-1}\| \\
& + 2\psi_n\langle \eta f(d)-Dd,\, x_{n+1}-d\rangle \\
& + 2J_4(1-\psi_n\bar{\eta})^2\|Vt_n-Vd\| + 2J_5(1-\psi_n\bar{\eta})^2\|G^*(T^{F_2}_{s_n}-I)Gl_n\| \\
& - (1-\psi_n\bar{\eta})^2[\|g_n-l_n\|^2 + \|t_n-z_n\|^2],
\end{aligned}$$

which is the required inequality. □

**Theorem 4.1.** *Let $H_1$ and $H_2$ be two real Hilbert spaces, and let $C_1 \subset H_1$ and $C_2 \subset H_2$ be nonempty, convex, and closed subsets. Let $G : H_1 \to H_2$ be a bounded linear operator with adjoint $G^*$. Let $F_1 : C_1 \times C_1 \to \mathbb{R}$ and $F_2 : C_2 \times C_2 \to \mathbb{R}$ be two bifunctions satisfying conditions $(A_1)$-$(A_4)$ with $F_2$ being upper semi-continuous in the first argument. Let $D : H_1 \to H_1$ be a strongly positively bounded linear operator with coefficient $\bar{\eta}$ and $f : H_1 \to H_1$ be a contraction with coefficient $\rho \in (0,1)$ such that $0 < \eta < \frac{\bar{\eta}}{\rho}$. Suppose that $Q : H_1 \to 2^{H_1}$ is a maximal monotone operator and $V : H_1 \to H_1$ is a $\sigma$-ism. Let $S_i : C_1 \to CB(C_1)$ be a finite family of multivalued demicontractive mappings with constant $\kappa_i$ such that $I - S_i$ is demiclosed at zero, $S_i(d) = \{d\}$ for all $d \in \bigcap_{i=1}^m Fix(S_i)$, and $\kappa = \max\{\kappa_i\}$ for each $i = 1,2,\ldots,m$. Suppose that Condition B and Condition C are satisfied. Then, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to a point $x^* \in \Upsilon$, where $x^* = P_\Upsilon(I - D + \eta f)(x^*)$.*

*Proof.* Let $x^* = P_\Upsilon(I - D + \eta f)(x^*)$. It follows from Lemma 4.2 that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 \leq & \left(1 - \frac{2\psi_n(\bar{\eta}-\eta\rho)}{1-\psi_n\eta\rho}\right)\|x_n-x^*\|^2 \\
& + \frac{2\psi_n(\bar{\eta}-\eta\rho)}{(1-\psi_n\eta\rho)}\left\{\frac{\psi_n\bar{\eta}^2}{2(\bar{\eta}-\eta\rho)}J_3 + \frac{(3J_2(1-\psi_n\bar{\eta})^2+\psi_n\eta\rho)}{2(\bar{\eta}-\eta\rho)}\frac{\nu_n}{\psi_n}\|x_n-x_{n-1}\| \right. \\
& \left. + \frac{1}{(\bar{\eta}-\eta\rho)}\langle \eta f(x^*)-Dx^*,\, x_{n+1}-x^*\rangle \right\}. \qquad (4.13)
\end{aligned}$$

Next, we claim $\lim_{n\to+\infty}\|x_n-x^*\| = 0$. By Lemma 2.5, it suffices to demonstrate that

$$\limsup_{k\to+\infty}\langle \eta f(x^*)-Dx^*,\, x_{n+1}-x^*\rangle \leq 0$$

for every subsequence $\{\|x_{n_k}-x^*\|\}$ of $\{\|x_n-x^*\|\}$ satisfying

$$\liminf_{k\to+\infty}(\|x_{n_k+1}-x^*\| - \|x_{n_k}-x^*\|) \geq 0.$$

So, we suppose that $\{\|x_{n_k}-x^*\|\}$ is a subsequence of $\|x_n-x^*\|$ such that

$$\liminf_{k\to\infty}(\|x_{n_k+1}-x^*\|-\|x_{n_k}-x^*\|)\geq 0. \tag{4.14}$$

Again, from lemma 4.2, we have

$$\begin{aligned}
&\frac{(1-\psi_{n_k}\bar{\eta})^2}{(1-\psi_{n_k}\eta\rho)}(2\sigma-\lambda_{n_k})\lambda_{n_k}\|Vt_{n_k}-Vd\|^2\\
&\leq\left(1-\frac{2\psi_{n_k}(\bar{\eta}-\eta\rho)}{(1-\psi_{n_k}\eta\rho)}\right)\|x_{n_k}-d\|^2-\|x_{n_k+1}-d\|^2+\frac{2\psi_{n_k}(\bar{\eta}-\eta\rho)}{(1-\psi_{n_k}\eta\rho)}\left\{\frac{\psi_{n_k}\bar{\eta}^2}{2(\bar{\eta}-\eta\rho)}J_3\right.\\
&\left.+\frac{(3J_2(1-\psi_{n_k}\bar{\eta})^2+\psi_{n_k}\eta\rho)}{2(\bar{\eta}-\eta\rho)}\frac{\nu_{n_k}}{\psi_{n_k}}\|x_{n_k}-x_{n_k-1}\|+\frac{1}{(\bar{\eta}-\eta\rho)}\langle\eta f(d)-Dd,\, x_{n_k+1}-d\rangle\right\}.
\end{aligned}$$

From the fact that $\lim_{k\to+\infty}\psi_{n_k}=0$ and (4.14), we have

$$\frac{(1-\psi_{n_k}\bar{\eta})^2}{(1-\psi_{n_k}\eta\rho)}(2\sigma-\lambda_{n_k})\lambda_{n_k}\|Vt_{n_k}-Vd\|^2\to 0,\ k\to+\infty.$$

Consequently, we obtain $\|Vt_{n_k}-Vd\|\to 0$ as $k\to+\infty$. By following a similar argument, we find from Lemma 4.2 that

$$\lim_{k\to\infty}\|g_{n_k}-h_{n_k,i}\|=0,\quad \forall i=1,2,...,m, \tag{4.15}$$

and $\lim_{k\to\infty}\gamma_{n_k}(2-\phi_{n_k})\|(T^{F_2}_{s_{n_k}}-I)Gl_{n_k}\|^2=0$. From the definition of $\gamma_n$, we have

$$\phi_{n_k}(2-\phi_{n_k})\frac{\|(T^{F_2}_{s_{n_k}}-I)Gl_{n_k}\|^4}{\|G^*(T^{F_2}_{s_{n_k}}-I)Gl^2_{n_k}\|^2}\to 0,\ k\to+\infty.$$

By the condition of $\phi_n$, we have that

$$\frac{\|(T^{F_2}_{s_{n_k}}-I)Gl_{n_k}\|^2}{\|G^*(T^{F_2}_{s_{n_k}}-I)Gl_{n_k}\|}\to 0,\ k\to+\infty.$$

Since $\|G^*(T^{F_2}_{s_{n_k}}-I)Gl_{n_k}\|$ is bounded, we arrive at

$$\lim_{k\to\infty}\|(T^{F_2}_{s_{n_k}}-I)Gl_{n_k}\|=0. \tag{4.16}$$

It follows that

$$\|G^*(T^{F_2}_{s_{n_k}}-I)Gl_{n_k}\|\leq\|G^*\|\|(T^{F_2}_{s_{n_k}}-I)Gl_{n_k}\|=\|G\|\|(T^{F_2}_{s_{n_k}}-I)Gl_{n_k}\|\to 0 \tag{4.17}$$

as $k\to+\infty$. Similarly, from Lemma 4.3, we have

$$\begin{aligned}
&(1-\psi_{n_k}\bar{\eta})^2\|g_{n_k}-l_{n_k}\|^2\\
&\leq(1-\psi_{n_k}\bar{\eta})^2\|x_{n_k}-d\|^2-\|x_{n_k+1}-d\|^2+3J_2(1-\psi_{n_k}\bar{\eta})^2\psi_{n_k}\frac{\nu_{n_k}}{\psi_{n_k}}\|x_{n_k}-x_{n_k-1}\|\\
&\quad+2\psi_{n_k}\langle\eta f(d)-Dd,\, x_{n_k+1}-d\rangle+2J_4(1-\psi_{n_k}\bar{\eta})^2\|Vt_{n_k}-Vd\|\\
&\quad+2J_5(1-\psi_{n_k}\bar{\eta})^2\|G^*(T^{F_2}_{s_{n_k}}-I)Gl_{n_k}\|.
\end{aligned}$$

From (4.14), (4.17), and Remark 3.1, we have

$$\lim_{k\to\infty} \|g_{n_k} - l_{n_k}\| \to 0. \tag{4.18}$$

Following the similar argument, from Lemma 4.3, we have

$$\lim_{k\to\infty} \|t_{n_k} - z_{n_k}\| = 0. \tag{4.19}$$

It follows from Remark 3.1 that

$$\lim_{k\to\infty} \|l_{n_k} - x_{n_k}\| = \nu_{n_k}\|x_{n_k} - x_{n_k-1}\| = 0. \tag{4.20}$$

By (4.15), we have

$$\begin{aligned}\|t_{n_k} - g_{n_k}\| &= \|\beta_{n_k,0}g_{n_k} + \sum_{i=1}^{m} \beta_{n_k,i}h_{n_k,i} - g_{n,k}\| \\ &\leq \beta_{n_k,0}\|g_{n_k} - g_{n_k}\| + \sum_{i=1}^{m} \beta_{n_k,i}\|h_{n_k,i} - g_{n,k}\| \to 0,\ k \to +\infty\end{aligned}$$

Furthermore, letting $k \to +\infty$, we have

$$\|g_{n_k} - x_{n_k}\| \to 0;\ \|t_{n_k} - x_{n_k}\| \to 0;\ \|z_{n_k} - x_{n_k}\| \to 0;\ \|z_{n_k} - l_{n_k}\| \to 0;\ \|h_{n_k,i} - x_{n_k}\| \to 0. \tag{4.21}$$

Moreover, by (4.15) we have

$$\lim_{k\to+\infty} d(g_{n_k}, S_i g_{n_k}) \leq \lim_{k\to\infty} \|g_{n_k} - h_{n_k,i}\| \to 0, \quad i = 1, 2, \ldots, m.$$

From the fact that $\lim\limits_{k\to+\infty} \psi_{n_k} = 0$, we have

$$\|x_{n_k+1} - x_{n_k}\| \leq \psi_{n_k}\|\eta f(l_{n_k}) - Dx_{n_k}\| + (1 - \psi_{n_k}\bar{\eta})\|z_{n_k} - x_{n_k}\| \to 0,\ k \to +\infty. \tag{4.22}$$

To complete the proof, we need to establish that $w_\omega(x_n) \subset \Upsilon$. We first prove that $w_\omega(x_n) \subset \bigcap_{i=1}^m Fix(S_i)$. Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $\hat{x} \in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$ as $k \to +\infty$. Then, we have that $g_{n_k} \rightharpoonup \hat{x}$. Since $I - S_i$ is demiclosed at zero for each $i = 1, \ldots, m$, then it follows from (4.15) that $\hat{x} \in Fix(S_i)$ for all $i = 1, \ldots, m$. This implies that $\hat{x} \in \bigcap_{i=1}^m Fix(S_i)$. Hence, $w_\omega(x_n) \subset \bigcap_{i=1}^m Fix(S_i)$.

Next, we prove that $w_\omega(x_n) \subset (A+B)^{-1}(0)$. Let $T_{n_k} = (I + \lambda_{n_k}Q)^-1(I - \lambda_{n_k}V)$. From the definition of $z_n$ and (4.19), we have $\lim_{k\to\infty}\|T_{n_k}t_{n_k} - t_{n_k}\| = \lim_{k\to\infty}\|z_{n_k} - t_{n_k}\| = 0$. Since $\liminf_{k\to\infty}\lambda_{n_k} > 0$, there exists $\delta > 0$ such that $\lambda_{n_k} \geq \delta$ for all $k \geq 1$. By Lemma 2.7(ii), we have $\lim_{k\to\infty}\|T_\delta t_{n_k} - t_{n_k}\| \leq 2\lim_{k\to\infty}\|T_{n_k}t_{n_k} - t_{n_k}\| = 0$. By Lemma 2.7(ii), we have that $T_\delta$ is nonexpansive. In view of $t_{n_k} \rightharpoonup \hat{x}$, we obtain from the demiclosedness of $I - T_\delta$ that $\hat{x} \in F(T_\delta)$. By Lemma 2.7(i), we obtain $\hat{x} \in (A+B)^{-1}(0)$. Hence, we have that $w_\omega(x_n) \subset (A+B)^{-1}(0)$.

Finally, we prove that $w_\omega(x_n) \subset SEP(F_1, F_2)$. First, we prove that $w_\omega(x_n) \subset EP(F_1)$. Since $g_{n_k} = T_{r_{n_k}}^{F_1}(l_{n_k} + \gamma_{n_k}G^*(T_{s_{n_k}} - I)Gl_{n_k})$, we have that

$$F_1(g_{n_k}, y) + \frac{1}{r_{n_k}}\langle y - g_{n_k}, g_{n_k} - l_{n_k} - \gamma_{n_k}G^*(T_{s_{n_k}}^{F_2} - I)Gl_{n_k}\rangle \geq 0,\ y \in C_1,$$

which implies that

$$F_1(g_{n_k}, y) + \frac{1}{r_{n_k}}\langle y - g_{n_k}, g_{n_k} - l_{n_k}\rangle - \frac{1}{r_{n_k}}\langle y - g_{n_k}, \gamma_{n_k}G^*(T_{s_{n_k}}^{F_2} - I)Gl_{n_k}\rangle \geq 0,\ y \in C_1.$$

So, from the monotonicity of $F_1$, we have

$$\frac{1}{r_{n_k}}\langle y-g_{n_k},g_{n_k}-l_{n_k}\rangle-\frac{1}{r_{n_k}}\langle y-g_{n_k},\gamma_{n_k}G^*(T^{F_2}_{s_{n_k}}-I)Gl_{n_k}\rangle\geq F_1(y,g_{n_k}),\ y\in C_1.$$

Since $g_{n_k}\rightharpoonup\hat{x}$, then it follows from (4.17), and (4.18), $\liminf_{k\to+\infty}r_{n_k}>0$, and condition $(A_4)$ that

$$F_1(y,\hat{x})\leq 0,\ y\in C_1. \tag{4.23}$$

Now, for $y\in C_1$, let $y_a:=ay+(1-a)\hat{x}$, for all $a\in(0,1]$. Then we have that $y_a\in C_1$, and it follows from (4.23) that $F_1(y_a,\hat{x})\leq 0$. By applying assumptions $(A_1)$-$(A_4)$, we have

$$0=F_1(y_a,y_a)\leq aF_1(y_a,y)+(1-a)F_1(y_a,\hat{x})\leq aF_1(y_a,y).$$

It follows that $F_1(y_a,y)\geq 0, \forall y\in C_1$. Letting $a\to 0$ by Condition $(A_3)$, we have that $F_1(\hat{x},y)\geq 0$, $\forall y\in C_1$. This implies that $\hat{x}\in EP(F_1)$.

Finally, we prove that $G\hat{x}\in EP(F_2)$. Since $G$ is a bounded linear operator and $w_\omega(x_n)=w_\omega(l_n)$ by (4.20), then $Gl_{n_k}\rightharpoonup G\hat{x}$. It follows from (4.16) that

$$T^{F_2}_{s_{n_k}}Gl_{n_k}\rightharpoonup G\hat{x},\ k\to+\infty. \tag{4.24}$$

By definition of $T^{F_2}_{s_{n_k}}Gl_{n_k}$, we have

$$F_2(T^{F_2}_{s_{n_k}}Gl_{n_k},y)+\frac{1}{s_{n_k}}\langle y-T^{F_2}_{s_{n_k}}Gl_{n_k},T^{F_2}_{s_{n_k}}Gl_{n_k}-Gl_{n_k}\rangle\geq 0,\ y\in C_2.$$

Since $F_2$ is upper semi-continuous in the first argument, it follows from (4.16), (4.24), and $\limsup_{k\to\infty}s_{n_k}>0$ that $F_2(G\hat{x},y)\geq 0, \forall y\in C_2$. This proves that $G\hat{x}\in EP(F_2)$. Hence, $w_\omega(x_n)\subset\Upsilon$ as required. From (4.21), we see that $w_\omega(x_n)\{g_{n_k}\}=w_\omega(x_n)\{x_{n_k}\}$. Since $\{x_{n_k}\}$ is bounded, then there exist a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}}\rightharpoonup x^+$ and

$$\begin{aligned}\lim_{j\to\infty}\langle\eta f(x^*)-Dx^*,x_{n_{k_j}}-x^*\rangle&=\limsup_{k\to\infty}\langle\eta f(x^*)-Dx^*,x_{n_k}-x^*\rangle\\&=\limsup_{k\to\infty}\langle\eta f(x^*)-Dx^*,g_{n_k}-x^*\rangle.\end{aligned} \tag{4.25}$$

Since $x^*=P_\Upsilon(I-D+\eta f)(x^*)$, it follows from (4.25) that

$$\begin{aligned}\limsup_{k\to\infty}\langle\eta f(x^*)-Dx^*,x_{n_k}-x^*\rangle&=\lim_{j\to\infty}\langle\eta f(x^*)-Dx^*,x_{n_{k_j}}-x^*\rangle\\&=\langle\eta f(x^*)-Dx^*,x^+-x^*\rangle\leq 0,\end{aligned}$$

which together with (4.22) yields that

$$\begin{aligned}&\limsup_{k\to\infty}\langle\eta f(x^*)-Dx^*,x_{n_k+1}-x^*\rangle\\&=\limsup_{k\to\infty}\langle\eta f(x^*)-Dx^*,x_{n+1}-x_{n_k}\rangle+\limsup_{k\to+\infty}\langle\eta f(x^*)-Dx^*,x_{n_k}-x^*\rangle\\&=\langle\eta f(x^*)-Dx^*,x^+-x^*\rangle\leq 0.\end{aligned}$$

Applying Lemma 2.5 to (4.13) and using Remark 3.1 and the condition on $\psi_n$, we have that $\lim_{n\to+\infty}\|x_n-x^*\|=0$ as required. □

By the properties of best approximation operators, we have the following result.

**Corollary 4.2.** Let $H_1$ and $H_2$ be two real Hilbert spaces, and let $C_1 \subset H_1$ and $C_2 \subset H_2$ be nonempty, convex, and closed subsets. Let $G: H_1 \to H_2$ be a bounded linear operator with adjoint $G^*$, and let $F_1 : C_1 \times C_1 \to \mathbb{R}$ and $F_2 : C-2 \times C_2 \to \mathbb{R}$ be two bifunctions satisfying conditions $(A_1)$-$(A_4)$ with $F_2$ being upper semi-continuous in the first argument. Let $D: H_1 \to H_1$ be a strongly positively bounded linear operator with coefficient $\bar{\eta}$, and and let $f: H_1 \to H_1$ be a contraction with coefficient $\rho \in (0,1)$ such that $0 < \eta < \frac{\bar{\eta}}{\rho}$. Suppose that $Q: H_1 \to 2^{H_1}$ is a maximal monotone operator and $V: H_1 \to H_1$ is a $\sigma$-ism. Let $S_i : C_1 \to P(C_1)$ be a finite family of multivalued mappings such that $P_{S_i}$ is $\kappa_i$- demicontractive and $I - P_{S_i}$ is demiclosed at zero with $\kappa = \max\{\kappa_i\}$ for each $i = 1, 2, \ldots, m$. Let $\{x_n\}$ be a sequence generated as follows:

---

**Step 0.** Select initial data $x_0, x_1 \in H_1$ and set $n = 1$.

**Step 1.** Given the $(n-1)th$ and $nth$ iterates, choose $\nu_n$ such that $0 \le \nu_n \le \hat{\nu}_n$, $\forall n \in \mathbb{N}$ with $\hat{\nu}_n$ defined by

$$\hat{\nu}_n = \begin{cases} \min\left\{\nu, \frac{\varepsilon_n}{||x_n - x_{n-1}||}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \nu, & \text{otherwise.} \end{cases} \tag{4.26}$$

**Step 2.** Compute $l_n = x_n + \nu_n(x_n - x_{n-1})$.

**Step 3.** Compute $g_n = T_{r_n}^{F_1}(l_n + \gamma_n G^*(T_{s_n}^{F_2} - I)Gl_n)$, where

$$\gamma_n = \begin{cases} \frac{\phi_n ||(T_{s_n}^{F_2} - I)Gl_n||^2}{||G^*(T_{s_n}^{F_2} - I)Gl_n||^2}, & \text{if } T_{s_n}^{F_2} Gl_n \neq Gl_n, \\ \gamma, & \text{otherwise } (\gamma \text{ being a nonnegative real number}). \end{cases} \tag{4.27}$$

**Step 4.** Compute

$$\begin{cases} t_n = \beta_{n,0} g_n + \sum_{i=1}^{m} \beta_{n,i} h_{n,i}; \quad h_{n,i} \in P_{S_i}(g_n); \\ z_n = (I + \lambda_n Q)^{-1}(I - \lambda_n V)t_n = J_{\lambda_n}^{Q}(I - \lambda_n V)t_n; \\ x_{n+1} = \psi_n \eta f(l_n) + (I - \psi_n D)z_n. \end{cases}$$

Set $n := n+1$ and return to **Step 1**.

---

Suppose that Condition B and Condition C are satisfied. Then, the sequence $\{x_n\}$ generated above converges strongly to a point $x^* \in \Upsilon$, where $x^* = P_\Upsilon(I - D + \eta f)(x^*)$ and $\Upsilon := SEP(F_1, F_2) \cap (Q+V)^{-1}(0) \cap \bigcap_{i=1}^{m} Fix(S_i)$ is nonempty.

*Proof.* Since $P_{S_i}$ satisfies the common endpoint condition, $F(S_i) = F(P_{S_i})$, and $I - P_{S_i}$ is demiclosed at zero for each $i = 1, 2, \ldots, m$, then the result follows immediately from Theorem 4.1. $\square$

**Remark 4.1.** Our result in Section 4 improves on existing works in the following aspects:

(i) Our result involves the problem of fixed points of a finite family of multivalued demicontractive mappings, which is a more general class of mappings than the ones studied in [9, 24, 29].

(ii) Our result improves on the results in [28, 40], which require prior knowledge of the norm of the bounded linear operator while our result employs a self-adaptive step size.

(iii) The result in this paper extends the result in [24] from the problem of common solutions of the *SEP* and the *FPP* of a family of multivalued quasi-nonexpansive mappings to the problem of common solutions of the *SEP*, the *VIP*, and the *FPP* of a family of multivalued demicontractive mappings in Hilbert spaces.

(iv) Unlike the result in [24], our algorithm utilizes the inertial technique to speed up the convergence of the proposed algorithm.
(v) We obtain the strong convergence of the proposed algorithm without following the conventional "two-case approach" employed by numerous authors in the literature.

## 5. Numerical Examples

In this section, we present some numerical experiments to demonstrate the computational efficiency of our Algorithm 3.1 in comparison with Algorithm (1.5), Appendix 6.1, Appendix 6.2, and Appendix 6.3 in the literature. All numerical computations were carried out using Matlab version R2019(b).

In the numerical computations, we choose $\eta = \frac{2}{7}, \psi_n = \frac{1}{2n+1}, \varepsilon_n = \frac{1}{(2n+1)^2}, \beta_{n,0} = \frac{n}{2n+1}, \beta_{n,i} = \frac{n+1}{5(2n+1)}, i = 1,2,\ldots,5, \lambda_n = \frac{2n}{3n+1}, \phi_n = 0.8, \nu = 0.85$, and $r_n = s_n = \frac{n+1}{3n+5}$, select $\alpha_n = \frac{2n}{3n+2}, \beta_n = \frac{3n}{5n+1}$, and $\theta_n = \frac{n}{3n+1}$ in Algorithm (1.5), Appendix 6.1, Appendix 6.2, and Appendix 6.3, and take $\phi_1 = \phi_2 = 0$ in Appendix 6.1 and Appendix 6.3.

**Example 5.1.** Let $H_1 = \mathbb{R} = H_2, C_1 = [-3,0]$ and $C_2 = (-+\infty,0]$. For all $x \in \mathbb{R}$, let the operators $G,Q,V : \mathbb{R} \to \mathbb{R}$ be defined by $G(x) = 3x$, $Qx = 4x$, and $Vx = 3x$, respectively. For $i = 1,2,\ldots,5$, define multivalued mappings $S_i : C_1 \to CB(C_1)$ by

$$S_i(x) = \begin{cases} \left[-\frac{i|x|}{i|x|+1}, 0\right], & x \in [-3,-2); \\ \{0\}, & x \in [-2,0]; \end{cases}$$

It can easily be verified that $S_i$ is a hybrid multivalued mapping with $Fix(S_i) = \{0\}$ for each $i$. Hence, $S_i$ is 0-demicontractive. Moreover, for each $x,y \in C_1$, define the bifunction $F_1 : C_1 \times C_1 \to \mathbb{R}$ by $F_1(x,y) = xy + y - x - x^2$, and for each $u,v \in C_2$, define the bifunction $F_2 : C_2 \times C_2 \to \mathbb{R}$ by $F_2(u,v) = uv + 10v - 10u - u^2$. It can easily be verified that $F_1, F_2$ both satisfy conditions $(A_1)$-$(A_4)$. After simple calculations, we obtain from Lemma 2.6 that

$$T_r^{F_1}(u) = \frac{u-r}{1+r}, \quad \forall\, u \in C_1,$$

and

$$T_s^{F_2}(v) = \frac{v-10s}{1+s}, \quad \forall\, v \in C_2.$$

We choose $f(x) = \frac{x}{3}, D(x) = \frac{x}{5}$. It can easily be checked that all the conditions of Theorem 4.1 are satisfied.

We choose four different initial values as follows with $\gamma = 0.05$ in each case:
Case I: $x_0 = 29$, $x_1 = 394$;
Case II: $x_0 = -65$, $x_1 = -500$;
Case III: $x_0 = -59.45$, $x_1 = 491$;
Case IV: $x_0 = 125$, $x_1 = -542$.

We compare the performance of Algorithm 3.1 with Algorithm (1.5), Appendix 6.1, Appendix 6.2, and Appendix 6.3. The stopping criterion used for our computation is $|x_{n+1} - x_n| < 10^{-2}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 1 and Table 1.

TABLE 1. **Numerical Results for Example 5.1**

| | | Alg. 1.5 | App 6.1 | App. 6.2 | App. 6.3 | Alg. 3.1 |
|---|---|---|---|---|---|---|
| Case I | No. of Iter. | 15 | 15 | 11 | 11 | 5 |
| | CPU time (sec) | 0.0047 | 0.0059 | 0.0249 | 0.0111 | 0.0122 |
| Case II | No. of Iter. | 15 | 15 | 11 | 11 | 5 |
| | CPU time (sec) | 0.0048 | 0.0059 | 0.0224 | 0.0106 | 0.0121 |
| Case III | No. of Iter. | 15 | 15 | 11 | 11 | 5 |
| | CPU time (sec) | 0.0055 | 0.0044 | 0.0177 | 0.0044 | 0.0055 |
| Case IV | No. of Iter. | 15 | 15 | 11 | 11 | 5 |
| | CPU time (sec) | 0.0055 | 0.0065 | 0.0325 | 0.0113 | 0.0158 |

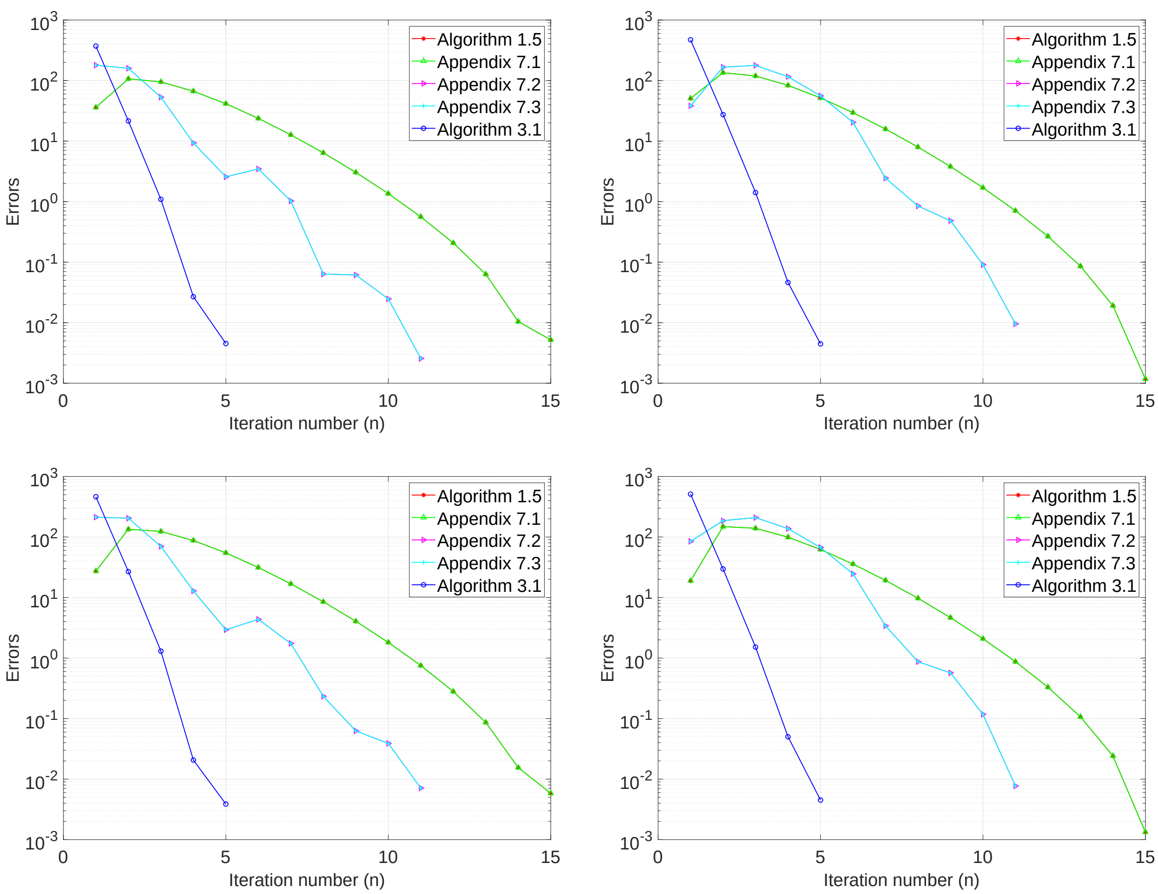


FIGURE 1. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

**Example 5.2.** Let $H_1 = H_2 = l_2(\mathbb{R})$. Define $C_1 := \{x \in l_2 : \|x\| \leq 1\}$ and $C_2 := \{y \in l_2 : \|y\| \leq 1\}$. Let $F_1 : C_1 \times C_1 \to \mathbb{R}$ be defined by $F_1(x,y) = xy - y^2$, where $x = (x_1, x_2, ..., x_i, ...)$ and $y = (y_1, y_2, ..., y_i, ...) \in C_1$,, and let $F_2 : C_2 \times C_2 \in \mathbb{R}$ be defined by $F_2(u,v) = u^2 + uv - 2v^2$, where $u = (u_1, u_2, ..., u_i, ...)$ and $v = (v_1, v_2, ..., v_i, ...) \in C_2$. One can easily verify that conditions $(A_1)$-$(A_4)$ are satisfied. After some steps of calculations with Lemma 2.6, we have

$$T_r^{F_1}(w) = \frac{w}{1-r},\ w = (v_1, w_2, ..., w_i, ...) \in C_1$$

and

$$T_s^{F_2}(z) = \frac{z}{3s+1},\ z = (z_1, z_2, ..., z_i, ...) \in C_2.$$

For all $x \in l_2$, let the operators $G, Q, V : l_2 \to l_2$ be defined by $G(x) = \frac{2}{3}x$, $Qx = 2x$, and $Vx = \frac{1}{2}x$, respectively. For $i = 1, 2, \ldots, 5$, define $S_i : H_1 \to H_1$ by $S_i x = \frac{x}{2i}$ for all $x \in l_2$. We choose $f(x) = \frac{2}{3}x$ and $D(x) = \frac{2}{7}x$. It is clear that all the conditions of Theorem 4.1 are satisfied.

We choose four different initials as follows with $\gamma = 0.08$ in each case:
Case I: $x_0 = (2, -1, -\frac{1}{2}, \cdots)$, $x_1 = (\frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \cdots)$,
Case II: $x_0 = (2, \frac{4}{5}, \frac{8}{25}, \cdots)$, $x_1 = (1, \frac{1}{10}, \frac{1}{100}, \cdots)$,
Case III: $x_0 = (2, 1, \frac{1}{2}, \cdots)$, $x_1 = (\frac{1}{5}, -\frac{1}{10}, \frac{1}{20}, \cdots)$,
Case IV: $x_0 = (\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \cdots)$, $x_1 = (2, \frac{1}{5}, \frac{1}{50}, \cdots)$.

We compare the performance of Algorithm 3.1 with Appendix 6.2, and Appendix 6.3. The stopping criterion used for our computation is $\|x_{n+1} - x_n\| < 10^{-2}$. We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figure 2 and Table 2.

TABLE 2. **Numerical results for Example 5.2**

| | | Alg. 1.5 | App 6.1 | App. 6.2 | App. 6.3 | Alg. 3.1 |
|---|---|---|---|---|---|---|
| Case I | No. of Iter. | 8 | 8 | 13 | 13 | 4 |
| | CPU time (sec) | 0.0079 | 0.0067 | 0.0124 | 0.0140 | 0.0144 |
| Case II | No. of Iter. | 7 | 7 | 13 | 13 | 5 |
| | CPU time (sec) | 0.0070 | 0.0111 | 0.0101 | 0.0231 | 0.0298 |
| Case III | No. of Iter. | 8 | 8 | 13 | 13 | 4 |
| | CPU time (sec) | 0.0069 | 0.0112 | 0.0100 | 0.0138 | 0.0196 |
| Case IV | No. of Iter. | 8 | 8 | 14 | 14 | 5 |
| | CPU time (sec) | 0.0063 | 0.0056 | 0.0067 | 0.0067 | 0.0145 |

**Remark 5.1.** We point out that the algorithms in Appendix 6.1 and Appendix 6.3 can solve the common solution of split mixed equilibrium problems and variational inclusion problems. But for fair comparison in the numerical examples, we take $\phi_1 = \phi_2 = 0$ in the two algorithms so that the problem reduces to the problem of finding a common solution of split equilibrium problems and variational inclusion problems.

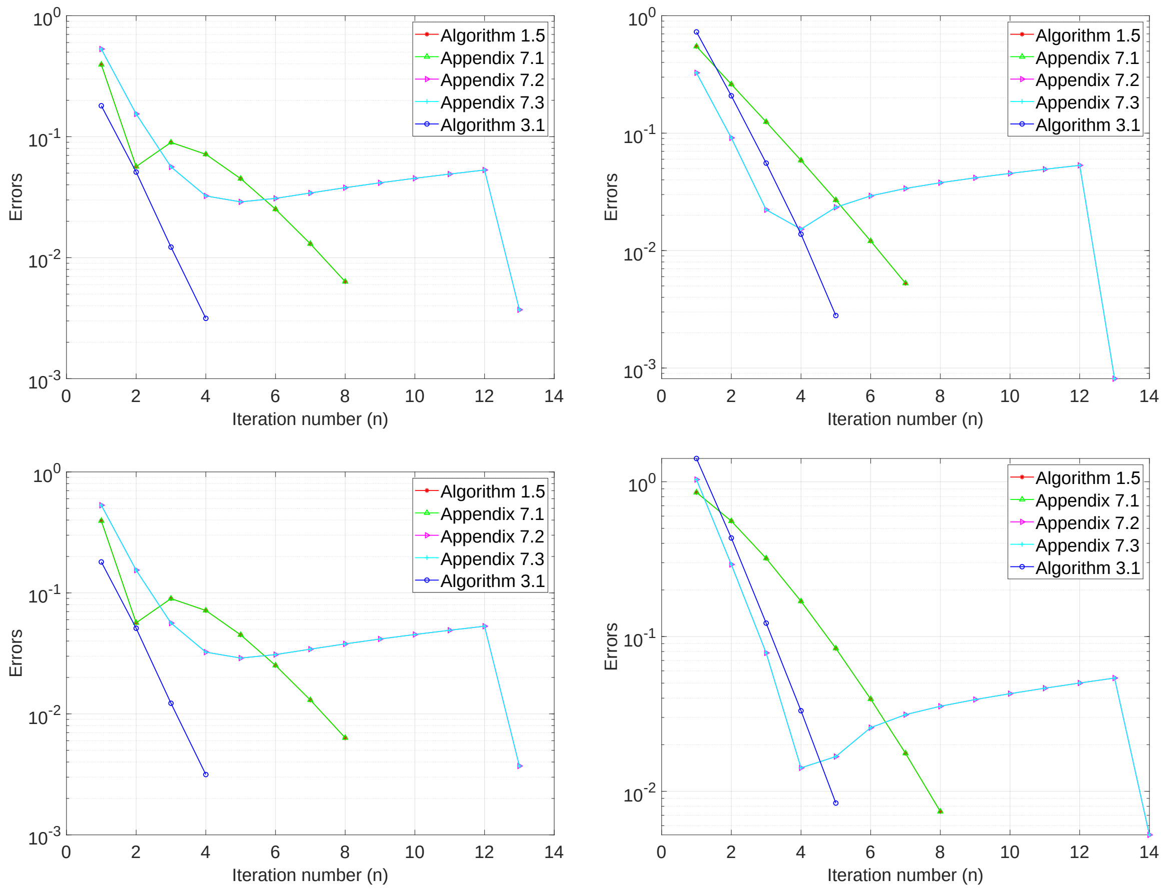


FIGURE 2. Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV.

**Remark 5.2.** By using different initials in each of Examples 5.1 and 5.2, we obtain the numerical results displayed in Tables 1 and 2, and Figures 1 and 2. We compared our proposed Algorithm 3.1 with the methods of Cholamjiak *et al.* and Arfat *et al.* (Algorithm 1.5, Appendix 6.1, Appendix 6.2, and Appendix 6.3) in Examples 5.1 and 5.2.

Furthermore, we note the following from our numerical examples:

- In Examples 5.1 and 5.2, we can see from the tables and graphs that the number of iterations for our proposed method is almost the same (well-behaved) for all initials. Also, there is no significant difference in the CPU time as we vary the initials.
- From the tables and figures, we can clearly see that, in terms of number of iterations, our Algorithm 3.1 outperforms the methods of Cholamjiak *et al.* and Arfat *et al.* (Algorithm 1.5, Appendix 6.1, Appendix 6.2, and Appendix 6.3) in Examples 5.1 and 5.2.

## 6. CONCLUSION

We studied the problem of finding a common solution of the split equilibrium problem, the variational inclusion problem, and the common fixed point problem of multivalued demicontractive mappings. We introduced a new iterative scheme, which combines inertial technique and adaptive step sizes with viscosity and forward-backward splitting methods for approximating the common solution of the aforementioned problem in Hilbert spaces. We proved strong convergence of the proposed algorithm without a prior knowledge of the operator norm. We also presented some numerical experiments to illustrate the efficiency of our method in comparison with some other methods in literature.

**Acknowledgments**
The authors sincerely thank the reviewer for his careful reading, constructive comments, and useful suggestions. The research of the second author was wholly supported by the University of KwaZulu-Natal, Durban, South Africa Postdoctoral Fellowship. He is grateful for the funding and financial support. The third author was supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant Number 119903). Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the NRF.

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**Appendix 6.1.** [40, Theorem 3.1]

Let $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ be sequences generated by $x_0, x_1 \in H_1$ and

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = \alpha_n y_n + (1-\alpha_n) T_{r_n}^{(F_1,\phi_1)}(I - \gamma G^*(I - T_{r_n}^{(F_2,\phi_2)})G)y_n, \\ x_{n+1} = \beta_n u_n + (1-\beta_n) J_n u_n, \qquad n \geq 1, \end{cases} \tag{6.1}$$

where $J_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)$, with $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\theta_n\} \subset [0, \theta]$, for some $\theta \in [0,1)$, $\{r_n\} \subset (0, +\infty)$ with $\gamma \in (0, \frac{1}{L})$, such that $L$ is the spectral radius of $G^*G$, and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$.

**Appendix 6.2.** [28, Theorem 3.2]

Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences generated by $x_0, x_1 \in H_1$ (satisfying the assumptions made on the control parameters) and

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \alpha_n y_n + (1-\alpha_n) T_{r_n}^{F_1}(I - \gamma G^*(I - T_{r_n}^{F_2})G)y_n, \\ w_n = \beta_n z_n + (1+\beta_n) J_{\lambda_n}^{Q}(I - \lambda_n V) z_n, \\ C_{n+1} = \{z \in C_n : ||w_n - z||^2 \leq ||x_n - z||^2 + 2\theta_n^2 ||x_n - x_{n-1}||^2 - 2\theta_n \langle x_n - z, x_{n-1} - x_n \rangle\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \qquad n \geq 1, \end{cases} \tag{6.2}$$

where $J_{\lambda_n}^{Q} = (I + \lambda_n Q)^{-1}$, $\{\lambda_n\} \subset (0, 2\sigma)$, $\{\theta_n\} \subset [0, \theta]$, $\theta \in [0,1)$, $\{r_n\} \subset (0, +\infty)$ with $\gamma \in (0, \frac{1}{L})$, such that $L$ is the spectral radius of $G^*G$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$.

**Appendix 6.3.** [40, Algorithm 4.1]

Let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ be sequences generated by $x_0, x_1 \in H_1$ and:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ u_n = \alpha_n y_n + (1-\alpha_n) T_{r_n}^{(F_1,\phi_1)}(I - \gamma G^*(I - T_{r_n}^{(F_2,\phi_2)})G)y_n, \\ z_n = \beta_n u_n + (1-\beta_n) J_n u_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2\} \leq \|x_n - z\|^2 + 2\theta_n^2 \|x_n - x_{n-1}\|^2 - 2\theta_n \langle x_n - z, x_{n-1} - x_n \rangle\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \qquad n \geq 1, \end{cases} \tag{6.3}$$

where $J_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)$, with $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\theta_n\} \subset [0, \theta]$, for some $\theta \in [0,1)$, $\{r_n\} \subset (0, +\infty)$ with $\gamma \in (0, \frac{1}{L})$, such that $L$ is the spectral radius of $G^*G$, and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$.