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FACIAL REDUCTION FOR THE SHOR SDP RELAXATION OF QCQPS

HAO HU[1,*], XINXIN LI[2,*]

[1] *Department of Mathematical Sciences, Clemson University, Clemson, SC, USA*
[2] *School of Mathematics, Jilin University, Changchun, China*

Abstract. We propose a special facial reduction algorithm (FRA) for the Shor semidefinite programming (SDP) relaxation of the quadratically constrained quadratic program (QCQP). Under the mild assumption, our special FRA only requires solving a linear programming problem instead of a semidefinite program. In particular, when applied to the binary quadratic program, the proposed special FRA needs fewer assumptions. From a computational perspective, this result improves the scalability and stability of the SDP approach for QCQP problems. In addition, we also discover a new class of semidefinite programs whose singularity degree can be computed easily. This new class complements the limited examples of singularity degrees in the literature. As a by-product, our special FRA can be used to upper bound the dimension of any set defined by quadratic inequalities.
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1. INTRODUCTION

SDP is a powerful optimization model with many applications in various fields, such as economics, engineering, and machine learning. The primary issue in SDP is that the time complexity of solving an SDP instance scales poorly with the size of the problem. In this work, we attempt to address this issue by examining the effectiveness of an existing regularization technique for certain special SDP instances.

Facial reduction algorithm (FRA) proposed by Borwein and Wolkowicz is an important tool in modern convex optimization, see [1, 2]. FRA is a preprocessing step for optimization problems, and it is essential in making algorithms stable and efficient for finding the optimal solution. The key idea behind FRA is to remove redundancy in the problem. For example, in linear programming, FRA is equivalent to identifying primal or dual slack variables that are identically zero on the feasible set. An interesting connection between the degeneracy and strict feasibility in linear programming can be found in [3]. For general conic optimization problems, it is important to impose additional constraint qualifications. Without constraint qualification, it could lead to the loss of stability with respect to perturbations in the data. Therefore, it is critical

*Corresponding author.
E-mail addresses: hhu2@clemson.edu (H. Hu), xinxinli@jlu.edu.cn (X. Li).
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to preprocess the problem to find its optimal solution in a numerically stable way. Through removing redundancy, FRA provides a robust framework for recovering a constraint qualification called *Slater's condition*. An excellent survey of the facial reduction algorithm can be found in [4].

In this paper, we focus on FRA applied to *semidefinite programming* (SDP) problems. SDP is a convex optimization problem with a rich history and a significant impact on numerical applications. If a given SDP problem instance does not satisfy Slater's condition, FRA can reformulate the problem iteratively until Slater's condition is restored. In fact, each reformulation step in this reduction process is more challenging than solving the original SDP problem. Therefore, one has to use special strategies in the implementation of FRA in practice. For example, an important strategy is based on the investigation of the problem structures analytically. In [5], the authors study SDP relaxation for a combinatorial optimization problem called *quadratic assignment problem*; And they implement FRA analytically to restore Slater's condition and thus require no extra computations. It is also the first study to employ this analytical strategy for combinatorial optimization problems. This idea of using analytical formula is later applied to the SDP relaxations of many other combinatorial problems and becomes a significant technique in this area, see [6, 7, 8, 9, 10, 11]. There are also special FRA strategies for general SDP problems, see [12, 13, 14, 15]. In [12], the positive semidefinite cone in the SDP is replaced by a computationally more tractable cone. This technique is called *partial facial reduction*, and it can produce approximations with different user-specified accuracy. It is effective for a wide range of problems. In [13], a special FRA called *Sieve-SDP* is proposed; it attempts to find specific, easily identifiable structures in the constraints of the SDP. A striking feature of Sieve-SDP is that it does not rely on any external solvers, and only Cholesky factorization is used as a subroutine. This simple recipe makes it very efficient for suitable SDP problem instances.

FRA is not only a computational tool but also important in theoretical analysis. In [16], Sturm defines the *singularity degree* as the fewest number of iterations required in the facial reduction algorithm to restore Slater's condition. The singularity degree can be seen as a measurement of ill-conditioning. Sturm derives a *Hölderian error bound* based on the singularity degree for the feasible region of any semidefinite programming problems. Sturm's error bound is very influential, and many exciting new results are developed subsequently. For example, Sturm's bound is extended and applied to more interesting cases in [17, 18, 19]. In [20], the bound is used to explain how the exponential size solutions arise in SDP. In [21], a strengthened Barvinok-Pataki bound on SDP rank is derived using singularity degree. The singularity degree of the linear image of a cone is investigated in [22].

As the singularity degree depends on the FRA, it is generally not possible to compute it for arbitrary SDP problem instances. There are very limited examples whose singularity degrees are known. In [16], the author provides an SDP instance with the largest possible singularity degree, and it is used as an example to demonstrate the numerical errors caused by a large singularity degree. The singularity degree of the positive semidefinite matrix completion problem is studied in [23, 24].

This paper investigates FRA for the Shor SDP relaxation of QCQP. In general, each step in FRA requires solving a semidefinite programming problem which is equally challenging to solve as the original problem. We propose a special FRA which can be applied to the Shor SDP relaxation, and under the mild assumption, it only requires solving a linear programming problem. Similar to other special FRAs, our FRA is not exact in the sense that we do not guarantee to recover Slater's condition. By relating our work with the existing special FRAs, it suggests that we achieve a good trade-off between exactness and computational efficiency. As a by-product, we also come up with a class of SDP instances whose singularity degree can be computed easily. This is a valuable addition to the limited examples in the literature. We also note that the proposed FRA can be used to upper bound the dimension of sets defined by quadratic inequalities.

2. Notations and Background on FRA

We let $\mathbb{R}^{m\times n}$ denote the set of m-by-n real matrices. For $X,Y \in \mathbb{R}^{m\times n}$, let $\langle X,Y\rangle$ denote the usual trace inner product of X and Y, $\mathrm{tr}(X^TY)$. We let $\mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$, resp.) denote the nonnegative (positive, resp.) orthant of n-coordinates. We let $\mathbb{S}^n$ denote the space of n-by-n symmetric matrices. A matrix $X \in \mathbb{S}^n$ is called *positive semidefinite* if $\langle x,Xx\rangle \geq 0$ for all $x \in \mathbb{R}^n$. The set of $n\times n$ positive semidefinite (definite, resp.) matrices is denoted by $\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$, resp.), and we use the notation $X \succeq 0$ ($X \succ 0$, resp.) to denote the membership $X \in \mathbb{S}^n_+$ ($X \in \mathbb{S}^n_{++}$, resp.).

Given a matrix X, we use $\mathrm{range}(X)$ and $\mathrm{null}(X)$ to denote the *range* and the *nullspace* of X, respectively. A face f is said to be *proper* if $f \neq \mathbb{S}^n_+$ and $f \neq \emptyset$. Given a convex set $C \subseteq \mathbb{S}^n_+$, the minimal face of $\mathbb{S}^n_+$ containing C, with respect to set inclusion, is denoted by $\mathrm{face}(C)$. A face f is said to be *exposed* if there exists $W \in \mathbb{S}^n_+ \setminus \{0\}$ such that

$$f = \left\{X \in \mathbb{S}^n_+ \mid \langle W,X\rangle = 0\right\}.$$

Every face of $\mathbb{S}^n_+$ is exposed, and the matrix W is referred to as *an exposing vector*. The faces of $\mathbb{S}^n_+$ can be characterized in terms of the range of any of its maximal rank elements. Moreover, it is well-known that each face is isomorphic to a smaller dimensional positive semidefinite cone, as is seen in the subsequent theorem. Its proof can be found in Proposition 10.1.2 from [25].

Theorem 2.1. *Let f be a face of $\mathbb{S}^n_+$ and $X \in f$ a maximal rank element with rank r and orthogonal spectral decomposition*

$$X = \begin{bmatrix} V & U \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V & U \end{bmatrix}^T \in \mathbb{S}^n_+, \quad D \in \mathbb{S}^r_{++}.$$

Then $f = V\mathbb{S}^r_+V^T$ and $\mathrm{relint}\,(f) = V\mathbb{S}^r_{++}V^T$. Moreover, $W \in \mathbb{S}^n_+$ is an exposing vector for f if and only if $W \in U\mathbb{S}^{n-r}_+U^T$.

Let $\mathscr{A} : \mathbb{S}^n \to \mathbb{R}^m$ be a linear operator and $b \in \mathbb{R}^m$. The feasible region of a semidefinite programming problem can be defined as

$$F := \{X \in \mathbb{S}^n_+ \mid \mathscr{A}(X) = b\}. \tag{2.1}$$

We say that *Slater's condition* holds for the problem if there exists a matrix $\hat{X} \in F \cap \mathbb{S}^n_{++}$. When Slater's condition is failed for S, it implies that the entire feasible set S is contained in a proper face of the semidefinite cone $\mathbb{S}^n_+$. If f is a face of $\mathbb{S}^n_+$ containing the feasible set F, then we can use Theorem 2.1 to rewrite F equivalently as

$$F = \{VRV^T \in \mathbb{S}^n \mid \mathscr{A}_V(R) = b, R \in \mathbb{S}^r_+\}. \tag{2.2}$$

Here, we define $\mathscr{A}_V(\cdot) := \mathscr{A}(V \cdot V^T)$. Then, it is obvious that Slater's condition holds for optimization problems with the above constraint. Therefore, the key issue is to find (the minimal) face f. As all the proper faces of a semidefinite cone are exposed, it suffices to find the exposing vector for the minimal face f. FRA attempts to identify an exposing vector by solving the auxiliary problem

$$\{y \mid \mathscr{A}^*(y) \succeq 0,\ b^T y = 0\}. \tag{2.3}$$

Then $\mathscr{A}^*(y)$ is an exposing vector, and it defines an exposed face containing F. In general, the reformulation (2.2) after FRA may still not satisfy Slater's condition. When this is the case, one simply needs to reapply FRA for the reformulated set (2.2). At each step, the size of the positive semidefinite constraint becomes strictly smaller, and thus, this procedure stops after at most $n-1$ steps. The smallest number of steps for FRA to restore Slater's condition is called the singularity degree, which is denoted by $\mathrm{sd}(F)$. Thus, the singularity degree $\mathrm{sd}(F)$ is an integer between 0 and $n-1$. Apart from the reduction in the size of the positive semidefinite constraint, at least one linear constraint in $\mathscr{A}(X) = b$ becomes redundant and thus can be discarded, see [26]. In general, finding an exposing vector by solving the auxiliary problem (2.3) is difficult. Indeed, (2.3) is also a semidefinite program, and it may fail Slater's condition as well.

3. The Shor SDP Relaxation for QCQPs

Quadratically constrained quadratic programs(QCQPs) are an important class of nonconvex optimization problems, and they can be formulated as

$$\inf\{q_0(x) \mid q_i(x) \le 0\ \forall i = 1, \dots, m\},$$

where $q_0, \dots, q_m : \mathbb{R}^n \to \mathbb{R}$ are quadratic functions. For each i, we write

$$q_i(x) = x^T A_i x + 2b_i^T x + c_i$$

for $A_i \in \mathbb{S}^n$, $b_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$. The feasible region is denoted by

$$P = \{x \in \mathbb{R}^n \mid q_i(x) \le 0\ \forall i = 1, \dots, m\}. \tag{3.1}$$

SDP relaxation plays an important role in the study of the QCQP, [27, 28, 29]. In [30], the authors study the exactness of the SDP relaxation for the QCQP, as well as the efficient algorithms for solving them.

We consider a convex relaxation for P called the Shor SDP relaxation. Define $Q_i \in \mathbb{S}^{n+1}$ and the linear operator $\mathscr{A} : \mathbb{S}^{n+1} \to \mathbb{R}^m$ as follows:

$$Q_i := \begin{bmatrix} c_i & b_i^T \\ b_i & A_i \end{bmatrix} \text{ and } \mathscr{A}(Y) = \begin{bmatrix} \langle Q_1, Y\rangle \\ \vdots \\ \langle Q_m, Y\rangle \end{bmatrix}.$$

The Shor relaxation for (3.1) can be written as

$$S := \left\{Y \in \mathbb{S}^{n+1}_+ \mid Y_{00} = 1,\ \mathscr{A}(Y) + s = 0 \text{ for some } s \in \mathbb{R}^m_+\right\}, \tag{3.2}$$

Note that if P is full dimensional, then S always satisfies Slater's condition.

Lemma 3.1. *If P is r-dimensional, then S contains a feasible solution Y such that*

$$\mathrm{rank(Y) = r+1}.$$

In particular, P is full-dimensional implies that S satisfies Slater's condition.

Proof. If P is r-dimensional, then there exist $r+1$ affinely independent vectors $x_1,\dots,x_{r+1}$ in F such that $q_i(x) \le 0$. Define $X = \frac{1}{r+1}\sum_{i=1}^{r+1} x_i x_i^T$ and $x = \frac{1}{r+1}\sum_{i=1}^{r+1} x_i$. Then

$$Y = \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} = \frac{1}{r+1}\sum_{i=1}^{r+1} \begin{bmatrix} 1 \\ x_i \end{bmatrix} \begin{bmatrix} 1 \\ x_i \end{bmatrix}^T \succeq 0.$$

The affine independence implies that the rank of Y is exactly $r+1$. The matrix Y is clearly positive semidefinite. For $j \in \{1,\dots,m\}$, we have

$$(\mathscr{A}(Y))_j = \langle A_j, X\rangle + 2b_j^T x + c = \frac{1}{r+1}\sum_{i=1}^{r+1}\left(x_i^T A_j x_i + 2b_j^T x_i + c\right) \le 0.$$

Thus $Y \in S$ for some $s \in \mathbb{R}^m_+$. In particular, this means if P is full dimensional, then $Y \in \mathbb{S}^{n+1}_{++}$ has full rank, and thus S satisfies Slater's condition. $\square$

Remark 3.2. Slater's condition *for S in (3.2) is sometimes referred to as* partial-polyhedral Slater's (PPS) condition*, as it does not capture the polyhedron cone $\mathbb{R}^m_+$ for the variable s. To simplify the presentation, we simply use the classical term Slater's condition here. If necessary, all the results in this paper can be generalized to include the polyhedron cone $\mathbb{R}^m_+$ as well.*

4. A Special FRA for the Shor SDP Relaxations

In this section, we provide a special FRA for the Shor SDP relaxation. The key idea is to relax S further into a larger set $\tilde{S}$ whose exposing vectors can be computed via linear programming. As $S \subseteq \tilde{S}$, any exposing vector for $\tilde{S}$ is also an exposing vector for S. Moreover, we are able to find an exposing vector of maximum rank for $\tilde{S}$ at each facial reduction step, and thus this allows us to compute the singularity degree of $\tilde{S}$. Like any other special FRAs, the proposed FRA is not guaranteed to always restore Slater's condition.

Let $\mathscr{K} \subset \mathbb{S}^{n+1}$ be a convex cone such that $Y \in \mathscr{K}$ if and only if the $n \times n$ submatrix of Y formed by the last n rows and columns are positive semidefinite. It should be clear that $\mathbb{S}^{n+1} \subset \mathscr{K}$ and thus the set $\tilde{S}$ defined below is a further relaxation for S.

$$\tilde{S} := \left\{Y \in \mathscr{K} \mid Y_{00} = 1,\ \mathscr{A}(Y) + s = 0 \text{ for } s \in \mathbb{R}^m_+\right\}. \tag{4.1}$$

Proposition 4.1. *Suppose Slater's condition does not hold for $\tilde{S}$ in (4.1). Then there exists a vector y satisfying*

$$0 \ne (\mathscr{A}^*(y), y) \in \mathscr{K}^* \times \mathbb{R}^m_+. \tag{4.2}$$

In addition, $\mathscr{A}^*(y)$ in (4.2) is a non-trivial exposing vector for $\tilde{S}$.

Proof. Define the affine set

$$L := \left\{(Y,s) \in \mathbb{S}^{n+1} \times \mathbb{R}^m \mid Y_{00} = 1, \mathscr{A}(Y) + s = 0\right\}.$$

If Slater's condition does not hold for $\tilde{S}$, then $(\operatorname{cone} L)$ and the interior of $\left(\mathscr{K} \times \mathbb{R}^m_+\right)$ have an empty intersection. By the hyperplane separation theorem, there exists $0 \ne (W,t) \in \mathbb{S}^n \times \mathbb{R}^m$ and $\alpha \in \mathbb{R}$ such that

(1) $\langle W, Y\rangle + t^T s \ge \alpha$ for every $(Y,s) \in \operatorname{int}(\mathscr{K} \times \mathbb{R}^m_+)$.
(2) $\langle W, Y\rangle + t^T s \le \alpha$ for every $(Y,s) \in \operatorname{cone} L$.

As $0 \in \operatorname{cone} L$ and 0 is in the closure of $(\operatorname{int} \mathscr{K}) \times \mathbb{R}^m_+$, we have $\alpha = 0$. This also shows that $(W,t) \in \left(\mathscr{K} \times \mathbb{R}^m_+\right)^* = \mathscr{K}^* \times \mathbb{R}^m_+$. As $\langle W, X\rangle + t^T s \le 0$ is a valid inequality for the affine set L, we conclude that $(W,s) = (\mathscr{A}^*(y), y)$ for some vector y.

□

Note that the dual cone $\mathscr{K}^*$ is the set of matrices in $\mathbb{S}^{n+1}$ such that the first row and the first column are zeros, and the $n \times n$ submatrix formed by the last n rows and columns are positive semidefinite. This allows us to write (4.2) into a more explicit form, i.e.,

$$\begin{array}{rcl} \sum_{i=1}^m A_i y_i & \in & \mathbb{S}^n_+ \\ y^T \begin{bmatrix} B & c \end{bmatrix} & = & 0 \\ y & \ge & 0, \end{array} \tag{4.3}$$

where $B = [b_1, b_2, \dots, b_m]^T$ and $c = [c_1, c_2, \dots, c_m]^T$. Below, we obtain an interesting result for a special case; when the data matrices $A_1, \dots, A_m$ are positive semidefinite, finding a vector y satisfying (4.2) reduces to solving a linear program.

Theorem 4.2. *Let the data matrices $A_1, \dots, A_m$ in (4.3) be positive semidefinite. Then the exposing vector for $\tilde{S}$ can be obtained from any feasible solution in the set*

$$\{y \in \mathbb{R}^m_+ \mid y^T \begin{bmatrix} B & c \end{bmatrix} = 0\}. \tag{4.4}$$

Proof. Suppose that $A_1, \dots, A_m \in \mathbb{S}^n_+$. Then any $y \in \mathbb{R}^n_+$ implies that $\sum_{i=1}^m A_i y_i \in \mathbb{S}^n_+$. Thus, finding a solution to system (4.2) is equivalent to finding a nontrivial solution in the polyhedron (4.4). □

Next, we show that after FR, the facially reduced program has the same form as $\tilde{S}$ in (4.1). This allows us to restore Slater's condition for $\tilde{S}$ by applying FRA a finite number of times.

Let y be a vector satisfying (4.2). Let $V \in \mathbb{R}^{n \times r}$ be a matrix with orthonormal columns satisfying $\operatorname{range}(V) = \operatorname{null}(\mathscr{A}^*(y))$. Define $\mathscr{A}_V := \mathscr{A}(VRV^T)$ where $R \in \mathbb{S}^r$. After facial reduction, we obtain a spectrahedron $\tilde{S}^1$ defined over a smaller dimensional space

$$\tilde{S}^1 := \left\{R \in \mathbb{S}^r_+ \mid \mathscr{A}_V(R) + s = 0 \text{ for } s \in \mathbb{R}^m_+\right\}. \tag{4.5}$$

It holds that $R \in \tilde{S}^1$ if and only if $VRV^T \in \tilde{S}$. The new feasible region $\tilde{S}^1$ has the same form as S in (4.1). (The only difference is that the constraint $Y_{00} = 1$ becomes different, but this does not affect the simplification in Theorem 4.2.) Moreover, $A_i \in \mathbb{S}^n_+$ implies that $V^T A_i V \in \mathbb{S}^r_+$. Thus if $\tilde{S}^1$ is not strictly feasible, then we can facially reduce the set (4.5) in the same way as before to obtain $\tilde{S}^2$. This yields a chain of feasible sets, say, $S^0 := S, S^1, \dots, S^k$ for some positive integer k; and the sizes of these sets are strictly decreasing. As we have a finite-dimensional problem, we restore Slater's condition after a finite number of steps. The singularity degree $\operatorname{sd}(S)$ is defined as the smallest length k of any chain of feasible sets. It is well-known that if we choose any exposing vector of maximum rank at each facial reduction step, we obtain a chain of minimum length; see [26, 31].

It turns out that we can not only compute an exposing vector efficiently by solving a linear programming problem but also obtain a maximum rank exposing vector $\mathscr{A}^*(y)$ for $\tilde{S}$. This is achieved by computing a feasible vector y for (4.2) such that the number of non-zero entries in

y is maximized, i.e,

$$\begin{array}{rl} \max & \|y\|_0 \\ \text{subject to} & y^T \begin{bmatrix} B & c \end{bmatrix} = 0 \\ & y \geq 0, \end{array} \tag{4.6}$$

where $\|y\|_0$ is the l_0-norm of y.

Lemma 4.3. *Assume $A_1, \dots, A_m$ in (4.3) are positive semidefinite. Let y' be optimal for (4.6). Then $\mathscr{A}^*(y')$ is a maximum rank exposing vector for $\tilde{S}$.*

Proof. Assume that y'' satisfies (4.2) and $\operatorname{rank}\mathscr{A}^*(y'') > \operatorname{rank}\mathscr{A}^*(y')$. This means $\operatorname{supp}(y'') \not\subseteq \operatorname{supp}(y')$. Let $\lambda \in (0,1)$ be arbitrary. Define $y^* = \lambda y' + (1-\lambda)y''$. By the convexity, we have that y^* is also a feasible solution to (4.6). Moreover $\operatorname{supp}(y') \subsetneq \operatorname{supp}(y^*)$. This is a contradiction to the optimality of y'. □

The l_0-norm maximization problem (4.6) can be converted into a linear programming problem using standard techniques in the literature, see [32, 33]. We include a self-contained proof here.

Lemma 4.4. *Let (p^*, q^*) be the optimal solution for the following linear programming problem*

$$\begin{array}{rl} \max & e^T p \\ \textit{subject to} & (p+q)^T \begin{bmatrix} B & c \end{bmatrix} = 0 \\ & 0 \leq p \leq 1 \\ & q \geq 0. \end{array} \tag{4.7}$$

It holds that $p^* + q^*$ is optimal for (4.6).

Proof. Assume that $p^* + q^*$ is not optimal for (4.6). Let y^* be any optimal solution to (4.6). Then $\|p^* + q^*\|_0 < \|y^*\|_0$, and thus the set $E = \{i \mid p_i^* = q_i^* = 0 \text{ and } y_i^* > 0\}$ is not empty. Define a new solution for (4.7) by setting

$$\tilde{p} = \min\{p^* + q^* + y^*, 1\} \text{ and } \tilde{q} = \max\{p^* + q^* + y^* - 1, 0\}.$$

Note that $(\tilde{p}, \tilde{q})$ is feasible for (4.7). Moreover, $e^T \tilde{p} > e^T p^*$ as E is not empty. This is a contradiction. □

As the l_0-norm maximization problem (4.6) can be solved using a linear program. Thus, we can compute the singularity degree of $\tilde{S}$ efficiently.

Theorem 4.5. *The singularity degree of $\tilde{S}$ in (4.1) can be computed in polynomial time.*

To the best of our knowledge, there are minimal SDP instances whose singularity degrees are known. Our example in Theorem 4.5 extends the range of information provided by the other examples. Further research is needed to comprehensively understand singularity degree, an important parameter in SDP.

Remark 4.6. *Note that if B has full row rank, then Slater's condition holds for $\tilde{S}$ trivially*[1]. *Thus, we receive no dimensional reduction by considering $\tilde{S}$. Moreover, the assumption on the data matrices can be considered stringent. This represents one of the worst-case scenarios in our approach. Fortunately, for certain practical cases like binary quadratic programs, this issue can be alleviated. This is discussed in the next section.*

[1]We would like to thank Haesol Im for her helpful comments.

5. Binary Quadratic Programs

Binary quadratic programs (BQPs) are an important special case of QCQPs, and it is widely used to model combinatorial optimization problems. In this section, we show that the assumption in Theorem 4.2 on the positive semidefiniteness of the data matrices $A_1,\dots,A_m$ can be partially discarded without loss of generality for BQP.

For BQP, the feasible set P can be written as

$$P = \{x \in \{0,1\}^n \mid q_i(x) \le 0\ \forall i = 1,\dots,m\}.$$

The binary constraint $x \in \{0,1\}^n$ allows us to reformulate the quadratic constraints q_i such that the data matrices A_i are positive semidefinite. To be more precise, we simply note that

$$q_i(x) = x^T(A_i - r_i I)x + 2(b_i + r_i)^T x + c_i,$$

for any constant r_i. In particular, we may choose r_i to be the smallest eigenvalue of A_i, which makes $A_i - r_i I$ positive semidefinite. In a similar fashion, we can assume B does not have full row rank to avoid the worst-case scenario in Remark 4.6.

In this case, the relaxation $\tilde{S}$ boils down into

$$\tilde{S} := \left\{Y \in \mathcal{K} \mid \operatorname{arrow}(Y) = e_0,\ \mathcal{A}(Y) + s = 0 \text{ for some } s \in \mathbb{R}^m_+\right\},$$

where e_0 is the first unit vector and $\operatorname{arrow} : \mathbb{S}^{n+1} \to \mathbb{R}^{n+1}$ is a linear operator defined by

$$\operatorname{arrow}(Y) = \begin{bmatrix} Y_{00} \\ Y_{11} - \frac{1}{2}(Y_{01} + Y_{10}) \\ \vdots \\ Y_{nn} - \frac{1}{2}(Y_{0n} + Y_{n0}) \end{bmatrix} \in \mathbb{R}^{n+1}.$$

Here the arrow operator is used as a relaxation for the binary constraint $x_i^2 = x_i$.

Similar to Proposition 4.1, if $\tilde{S}$ does not satisfy Slater's condition, then there exists a solution $(y,z) \in \mathbb{R}^m \times \mathbb{R}^{n+1}$ to the system

$$0 \neq (\mathcal{A}^*(y) + \operatorname{arrow}^*(z), y) \in \mathcal{K}^* \times \mathbb{R}^m_+ \text{ and } z_0 = 0, \tag{5.1}$$

where the dual operator $\operatorname{arrow}^* : \mathbb{R}^n \to \mathbb{S}^{n+1}$ is given by

$$\operatorname{arrow}^*(z) = \begin{bmatrix} z_0 & -\frac{1}{2}z_1 & -\frac{1}{2}z_2 & \cdots & -\frac{1}{2}z_n \\ -\frac{1}{2}z_1 & z_1 & 0 & \cdots & 0 \\ -\frac{1}{2}z_2 & 0 & z_2 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -\frac{1}{2}z_n & 0 & \cdots & 0 & z_n \end{bmatrix}.$$

Denote by $\tilde{z} \in \mathbb{R}^n$ the last n entries in $z \in \mathbb{R}^{n+1}$. Then (5.1) can be written as a system in variable $(y,\tilde{z}) \in \mathbb{R}^m \times \mathbb{R}^n$, i.e.,

$$\begin{array}{rcl} \sum_{i=1}^m A_i y_i + \operatorname{Diag}(\tilde{z}) & \in & \mathbb{S}^n_+ \\ y^T \begin{bmatrix} B & c \end{bmatrix} - \frac{1}{2}\tilde{z} & = & 0 \\ y & \ge & 0. \end{array} \tag{5.2}$$

Since we can assume $A_1,\dots,A_m$ are positive semidefinite without loss of generality, it is always possible to set $\tilde{z} = 0$ and solve (5.2) as a linear programming problem as in Theorem 4.2 for (4.3). However, we would not be able to identify an exposing vector from (5.2) of maximum rank for $\tilde{S}$, as $\tilde{z}$ is a free variable.

6. Applications and Related Works

6.1. **Upper bounds.** In view of Lemma 3.1, if the Shor SDP relaxation S in (3.2) does not satisfy Slater's condition, then P in (3.1) is not full dimensional. The dimension of P can be upper bounded using the exposing vector W for $\tilde{S}$. To be more precise, if $W \in \mathbb{S}_+^{n+1}$ is an exposing vector for S, then $\dim(P) \leq n - \text{rank}(W)$. Since it is difficult to find an exposing vector for S, we can replace the Shor relaxation S by $\tilde{S}$ in (4.1). Indeed, as $S \subseteq \tilde{S}$, it holds that $\dim(P) \leq n - \text{rank}(W)$ for any exposing vector W of $\tilde{S}$ as well. More importantly, we can compute a maximum rank exposing vector W for $\tilde{S}$ Lemma 4.3 efficiently. This forms a procedure to upper bound the dimension of any set defined by quadratic inequalities through semidefinite programming.

6.2. **Relation with partial facial reduction.** If we compare the sets S in (3.2) and $\tilde{S}$ in (4.1), the key difference is that the positive semidefinite constraint $Y \in \mathbb{S}_+^{n+1}$ is replaced by a weaker constraint $Y \in \mathscr{K}$, see the definition of $\mathscr{K}$ before (4.1). This replacement can be viewed as a special case of an existing method in [12]. Permenter and Parrilo introduce a special FRA called *partial facial reduction algorithm*. The main idea is to replace the positive semidefinite cone in (2.1) with a user-specified computational tractable cone. The philosophy behind their method is an argument by Andersen and Andersen, which can be stated as "the most effective preprocessing approach should be straightforward and efficient". Therefore, they replace the positive semidefinite constraints $Y \in \mathbb{S}_+^{n+1}$ by polyhedron cones or positive semidefinite constraints on all 2×2 principal submatrices of Y. This yields a computationally tractable FRA.

Under their framework, it is also possible to replace the constraints $Y \in \mathbb{S}_+^{n+1}$ by positive semidefinite constraints on all $k \times k$ principal submatrices of Y for any $k \leq n+1$. This is related to the Factor-width-k matrices; see [34]. When $k = n+1$, we solve the original FR auxiliary problem. Thus, when $k = n$, this stands for the best approximation of FRA under this framework in some sense. However, it is also computationally more expensive for large k, which does not fit the philosophy of Andersen and Andersen anymore. Indeed, it is mentioned in [12] that using this representation is not always practical when k is too large.

In this work, we consider a special relaxation $\tilde{S}$ in (4.1) whose computational complexity is close to the case of using factor-width-k matrices with $k = n$. Thus, our special FRA applied to $\tilde{S}$ is expected to have a good chance to capture the exposing vectors from the original intractable auxiliary problem (2.3). Our key contribution shows that FRA applied to $\tilde{S}$ is significantly easier than for S.

7. Conclusion

In this work, we address the issue of poor scalability in SDPs by examining the effectiveness of the facial reduction algorithm. We apply FRA to the Shor SDP relaxation of QCQPs. Under the mild assumption, we show that the auxiliary problem in FRA collapses into a linear programming problem that can be solved very efficiently. Thus, we obtain a partial facial reduction at a very low cost, which is otherwise intractable. We also note that the proposed FRA requires fewer assumptions when applied to the binary quadratic programs. We use the framework of an existing special FRA to argue that our algorithm makes a good trade-off between the exactness of facial reduction and computational complexity. In addition, we construct new semidefinite programming problems whose singularity degrees are known, and this is a good addition to the list of limited examples in the literature. We also note that our special FRA can be used to find an upper bound for the dimension of any set defined by quadratic inequalities.

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