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UPPER ERROR BOUNDS OF DG-FUNCTIONS FOR HISTORY-DEPENDENT VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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Abstract. The aim of this paper is to study the difference gap function (for brevity, DG-function) and upper error bounds for an abstract class of variational-hemivariational inequalities with history-dependent operators (for brevity, HDVHIs). First, we propose a new concept of gap functions to the HDVHIs and consider the regularized gap function (for brevity, RG-function) for the HDVHIs using the optimality condition for the concerning minimization problem. Then, the DG-function for the HDVHIs depending on these RG-functions is constructed. Finally, we establish upper error bounds for the HDVHIs controlled by the RG-function and the DG-function under suitable conditions.

Keywords. Gap function; History-dependent operator; Upper error bound; Variational-hemivariational inequality.
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1. INTRODUCTION

The theory of hemivariational inequalities developed by Panagiotopoulos is a well known generalization of variational inequalities to address a variety of mechanical difficulties involving nonconvex and nonsmooth energy potentials. It has been extended to variational–hemivariational inequalities in the case of both convex and nonconvex potentials by using the idea of the Clarke's generalized gradient for locally Lipschitz functions; see, e.g., [29, 30]. Accordingly, many authors have extensively studied on the theory of variational–hemivariational inequalities in various directions. Especially, this theory has been applied to different fields of engineering, mechanics, nonsmooth analysis and optimization. In particular, existence results for various kinds of variational-hemivariational inequalities can be found in [17, 25, 23], the gap functions and error bounds are investigated in [10, 14, 16, 38], the well-posedness and the stability in the sense of convergence are studied in [11, 24, 41, 45, 46] and references therein.

Besides, a significant class of operators with definitions in vector-valued function spaces are history-dependent operators. These operators arise in theories of differential equations, partial differential equations, and functional analysis. In fact, history-dependent operators are particularly helpful for the analysis of models incorporating quasi-static frictional and frictionless contact conditions with elastic or viscoelastic materials in contact mechanics. The integral

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operators and Volterra-type operators are two straightforward examples of history-dependent operators. For more details, please refer to [26, 28, 33, 34, 35, 36, 47] and references therein.

On the other hand, Auslender [1] first introduced the notion of gap functions as a useful tool for resolving related optimization problems involving variational inequalities. However, Auslender's gap function is non-differentiable in general. To overcome this disadvantage, Fukushima [9] originally proposed a regularized gap function (for short, RG-function) for variational inequalities (also see, Yamashita-Fukushima [44]). It leads a variational inequality problem to an equivalent differentiable optimization problem with convex constraints. Later, Peng [31] provided a difference gap function (for short, DG-function), also known as D-gap function, which is a difference of two RG-functions, and showed that a DG-function yields a variational inequality problem to an unconstrained optimization problem. Note that the RG-function is also a gap function. Peng-Fukushima [32] established a global upper error bound result for variational inequalities by using DG-functions under the assumption of strongly monotone functions. An upper error bound explores the upper estimate of the distance between an arbitrary feasible point and the certain problem's solution set. Hence, it is very crucial in analyzing the convergence of iterative approaches to resolving variational inequalities. Moreover, DG-functions and error upper bounds have been studied for various kinds of variational inequalities and equilibrium problems, see e.g., [2, 4, 13, 15, 18, 19, 21, 22, 37, 40]. Very recently, Cen-Nguyen-Zeng [3] considered upper error bounds for the generalized variational-hemivariational inequalities with history-dependent operators by using RG-functions. Chen-Tam [5] proposed upper error bounds for a class of history-dependent variational inequalities controlled by the DG-functions. Especially, Tam-Chen [39] derived upper error bounds for abstract elliptic variational-hemivariational inequalities based on generalized DG-functions with applications to contact mechanics. Following these tracks and directions, this work aims to study DG-functions and upper error bounds for variational-hemivariational inequalities, which are new to the literature.

As mentioned above, our aim in this paper is to continue investigations on DG-functions and upper error bounds for a general class of variational-hemivariational inequalities with history-dependent operators (for brevity, HDVHIs). We outline the main idea behind as below. First, the RG-function of the Fukushima type [9] for HDVHIs is provided in terms of the optimality condition for the concerning minimization problem. Then, we construct the DG-function for HDVHIs based on two different RG-functions and give some its characterization. Finally, we represent upper error bounds for HDVHIs by employing the RG-function and the DG-function depending on the data of HDVHIs.

The rest of this paper is organized as follows. In Section 2, we review basic notations, definitions and their properties that will be used throughout the paper. We also introduce a general class of variational-hemivariational inequalities with history-dependent operators (Problem 2.1) and present its existence under some imposed hypotheses of Problem 2.1. In Section 3, we investigate the DG-function for Problem 2.1 in terms of RG-functions. Section 4 is devoted to establishing upper error bounds for Problem 2.1 by using the RG-function and the DG-function considered in Section 3 under some suitable assumptions.

2. Preliminaries and Formulations

Given a reflexive Banach space V and its topological dual V^*, the norm on V and the duality pairing of V and V^* are denoted by $\|\cdot\|_V$ and $\langle\cdot,\cdot\rangle_V$, respectively. Let two normed spaces V and Z, $L(V,Z)$ denote the space of all linear continuous operators from V to Z. We first recall some fundamental concepts and properties that will be used in the sequel. For more details, please refer to [6, 7, 8, 27].

Definition 2.1. A function $\psi\colon V\to\overline{\mathbb{R}}:=\mathbb{R}\cup\{+\infty\}$ is said to be

(a) proper if $\psi\not\equiv+\infty$;
(b) convex if $\psi(tz+(1-t)v)\le t\psi(z)+(1-t)\psi(v)$ for all z, $v\in V$ and $t\in[0,1]$;
(c) upper semicontinuous at $z_0\in V$ if, for any $\{z_n\}\subset V$ such that $z_n\to z_0$, it holds $\limsup\psi(z_n)\le\psi(z_0)$;
(d) lower semicontinuous at $z_0\in V$ if, for any $\{z_n\}\subset V$ such that $z_n\to z_0$, it holds $\psi(z_0)\le\liminf\psi(z_n)$;
(e) upper semicontinuous (resp., lower semicontinuous) on V if ψ is upper semicontinuous (resp., lower semicontinuous) at every $z_0\in V$.

Definition 2.2. An operator $a\colon V\to V^*$ is said to be

(a) bounded if a maps bounded sets of V into bounded sets of V^*;
(b) Lipschitz continuous if there exists a constant $L_a>0$ such that

$$\|a(v)-a(z)\|_{V^*}\le L_a\|v-z\|_V \text{ for all } z,v\in V.$$

Definition 2.3. Let $h\colon V\to\overline{\mathbb{R}}$ be a proper, convex, and lower semicontinuous function. The convex subdifferential $\partial^c h\colon V\rightrightarrows V^*$ of h is defined by

$$\partial^c h(z)=\left\{w^*\in V^*\mid\langle w^*,v-z\rangle_E\le h(v)-h(z)\ \text{ for all }\ v\in V\right\}\ \text{ for all }\ z\in V.$$

An element $w^*\in\partial^c h(z)$ is called a subgradient of h at $z\in V$.

Definition 2.4. A function $\psi\colon V\to\mathbb{R}$ is said to be locally Lipschitz if, for every $z\in V$, there exist a neighbourhood $\mathcal{N}_z$ of z and a constant $l_z>0$ such that

$$|\psi(u_1)-\psi(u_2)|\le l_z\|u_1-u_2\|_V\ \text{ for all }\ u_1,u_2\in\mathcal{N}_z.$$

Given a locally Lipschitz function $\psi\colon V\to\mathbb{R}$, we denote by $\psi^0(z;w)$ the Clarke's generalized directional derivative of ψ at the point $z\in V$ in the direction $w\in V$ defined by

$$\psi^0(z;w):=\limsup_{y\to z,\ t\to 0^+}\frac{\psi(y+tw)-\psi(y)}{t}.$$

The generalized gradient of ψ at $z\in V$, denoted by $\partial\psi(z)$, is a subset of V^* given by

$$\partial\psi(z)=\left\{\zeta^*\in V^*\mid\psi^0(z;w)\ge\langle\zeta^*,w\rangle_V \text{ for all } w\in V\right\}.$$

The following lemma provides some useful properties of the Clarke's generalized gradient and directional derivative of a locally Lipschitz function, see, e.g., [6, Proposition 2.1.1].

Lemma 2.1. *Let V be a real Banach space and $\psi\colon V\to\mathbb{R}$ be a locally Lipschitz function. Then, the following assertions hold.*

(i) *For each $z\in V$, the function $V\ni w\mapsto\psi^0(z;w)\in\mathbb{R}$ is finite, positively homogeneous, subadditive and Lipschitz continuous.*

(ii) *The function* $V \times V \ni (z,w) \mapsto \psi^0(z;w) \in \mathbb{R}$ *is upper semicontinuous.*
(iii) *For every* z, $w \in V$, *it holds* $\psi^0(z;w) = \max\{\langle \zeta, w\rangle_V \mid \zeta \in \partial\psi(z)\}$.

Definition 2.5 (see [20])**.** A function $\Lambda\colon V \to \mathbb{R}$ is said to be uniformly convex if there exists a continuously increasing function $\pi\colon \mathbb{R} \to \mathbb{R}$ such that $\pi(0) = 0$ and that for all $z, v \in V$ and for each $t \in [0,1]$, we have

$$\Lambda(tz + (1-t)v) \le t\Lambda(z) + (1-t)\Lambda(v) - t(1-t)\pi(\|z-v\|_V)\|z-v\|_V.$$

If $\pi(r) = kr$ for $k > 0$, then Λ is called a strongly convex function.

Next, we recall the existence and uniqueness of solutions for a convex optimization problem involving Λ.

Lemma 2.2 (see [43], Chapter 1, Section 3, Theorem 9)**.** *Suppose that* $K \subset V$ *is a nonempty, convex, and closed set,* $\Lambda\colon V \to \mathbb{R}$ *is a uniformly convex and lower semicontinuous function. Then, the optimization problem*

$$\min_{z\in K} \longrightarrow \Lambda(z)$$

has the unique solution $z^* \in K$.

Throughout the paper, unless otherwise specified, for each $i \in \{1,\dots,l\}$, let V be a reflexive Banach space, V_i be a Banach space, $\alpha_i > 0$, $K \subset V$, $0 < T < \infty$, $a\colon [0,T] \times V \to V^*$, $\gamma_i\colon V \to V_i$, $\mathscr{H}\colon C([0,T];V) \to C([0,T];V^*)$ be operators, $h\colon V \times V \to \mathbb{R}$, $\Psi_i\colon V_i \to \mathbb{R}$ be functions, and $f \in C([0,T];V^*)$. We now consider the following abstract variational-hemivariational inequality with history-dependent operator:

Problem 2.1. Find a function $z\colon [0,T] \to K$ such that

$$\begin{aligned}\langle a(s,z(s)) + (\mathscr{H}z)(s), v - z(s)\rangle_V + h(z(s),v)&\\ + \sum_{i=1}^{l} \alpha_i \Psi_i^0(\gamma_i z(s); \gamma_i(v - z(s))) &\ge \langle f(s), v - z(s)\rangle_V\end{aligned}$$

for all $v \in K$ and a.e. $s \in [0,T]$.

When $l = 1$, $\Psi_1 = \Psi$ and $\gamma_1 = \gamma$, Problem 2.1 is equivalent to the following class of history-dependent variational-hemivariational inequality studied by Cen-Nguyen-Zeng [3]:

Problem 2.2. Find a function $z\colon [0,T] \to K$ such that

$$\begin{aligned}\langle a(s,z(s)) + (\mathscr{H}z)(s), v - z(s)\rangle_V + h(z(s),v)&\\ + \Psi^0(\gamma z(s); \gamma(v - z(s))) &\ge \langle f(s), v - z(s)\rangle_V\end{aligned}$$

for all $v \in K$ and a.e. $s \in [0,T]$.

When $\Psi_i \equiv 0$ for all $i \in \{1,\dots,l\}$, $h(z,v) = \phi(v) - \phi(z)$ for all $z, v \in K$, Problem 2.1 reduces to the following history-dependent variational inequality considered by Sofonea-Pătrulescu [34] in the interval of time $[0,T]$:

Problem 2.3. Find a function $z\colon [0,T] \to K$ such that

$$\langle a(s,z(s)), v - z(s)\rangle_V + \langle (\mathscr{H}z)(s), v - z(s)\rangle_V + \phi(v) - \phi(z(s) \ge \langle f(s), v - z(s)\rangle_V$$

for all $v \in K$ and a.e. $s \in [0,T]$.

To proceed, we now impose the following hypotheses on the data of Problem 2.1.

$\mathfrak{H}(a)$: For the operator $a\colon [0,T]\times V \to V^*$,

(i) for all $s\in[0,T]$, the mapping $z\mapsto a(s,z)$ is continuous, and is strongly monotone, i.e., there exists $m_a>0$ such that
$$\langle a(s,z_1)-a(s,z_2),z_1-z_2\rangle_V \geq m_a\|z_1-z_2\|_V^2,\ \forall z_1,z_2\in V,\ \forall s\in[0,T];$$

(ii) there exists $l_a>0$ such that
$$\|a(s_1,z)-a(s_2,z)\|_{V^*}\leq l_a|s_1-s_2|,\ \forall z\in V,\ \forall s_1,s_2\in[0,T];$$

(iii) there exists $L_a>0$ such that
$$\|a(s,z_1)-a(s,z_2)\|_{V^*}\leq L_a\|z_1-z_2\|_V,\ \forall z_1,z_2\in V,\ \forall s\in[0,T].$$

$\mathfrak{H}(\mathscr{H})$: For the operator $\mathscr{H}\colon C([0,T];V)\to C([0,T];V^*)$, there exists a constant $L_{\mathscr{H}}$ such that
$$\|\mathscr{H}z_1)(s)-(\mathscr{H}z_2)(s)\|_{V^*}\leq L_{\mathscr{H}}\int_0^s\|z_1(t)-z_2(t)\|_V\,dt,$$
for all $z_1,z_2\in C([0,T];V)$ and for all $s\in[0,T]$.

$\mathfrak{H}(h)$: $h\colon K\times K\to\mathbb{R}$ is a bounded function such that

(i) for each $z\in K$, $v\mapsto h(z,v)$ is convex and lower semicontinuous;
(ii) for each $v\in K$, $z\mapsto h(z,v)$ is concave and upper semicontinuous;
(iii) for each $z\in K$, $h(z,z)=0$;
(iv) there exists $m_h\geq 0$ such that
$$h(z,v)+h(v,z)\leq -m_h\|z-v\|_V^2,\ \ \forall z,v\in K. \tag{2.1}$$

$\mathfrak{H}(\Psi)$: For each $i\in\{1,\ldots,l\}$, for the locally Lipschitz function $\Psi_i\colon V_i\to\mathbb{R}$,

(i) $\|\xi\|_{V_i}^*\leq c_0+c_1\|z\|_{V_i},\ \ \forall z\in V_i,\ \xi\in\partial\Psi_i(z)$ with some $c_0,c_1\geq 0$;
(ii) there exists $L_{\Psi_i}\geq 0$ such that
$$\begin{aligned}&\Psi_i^0(w_1;v_2-v_1)+\Psi_i^0(w_2;v_1-v_2)\\&\quad\leq L_{\Psi_i}\|w_1-w_2\|_{E_i}\|v_1-v_2\|_{E_i},\ \forall w_1,w_2,v_1,v_2\in E_i.\end{aligned} \tag{2.2}$$

$\mathfrak{H}(K)$: K is a nonempty, closed, and convex subset of V with $\mathbf{0}_V\in K$.

$\mathfrak{H}(K_b)$: K is a nonempty, bounded, closed, and convex subset of V with $\mathbf{0}_V\in K$.

$\mathfrak{H}(\gamma)$: For each $i\in\{1,\ldots,l\}$, for the operator $\gamma_i\in L(V,V_i)$, there exists $c_{\Psi_i}>0$,
$$\|\gamma_i v\|_{V_i}\leq c_{\Psi_i}\|v\|_V.$$

$\mathfrak{H}(f)$: $f\in C([0,T];V^*)$.

$\mathfrak{H}(0)$: $m_a+m_h-\sum_{i=1}^l\alpha_iL_{\Psi_i}c_{\Psi_i}^2>0$.

Remark 2.1. (i) The operator $\mathscr{H}\colon C([0,T];V)\to C([0,T];V^*)$ satisfying condition $\mathfrak{H}(\mathscr{H})$ is said to be the history-dependent operator.
(ii) If $w_1=v_1,w_2=v_2$, then condition (2.2) reduces to
$$\Psi_i^0(v_1;v_2-v_1)+\Psi_i^0(v_2;v_1-v_2)\leq L_{\Psi_i}\|v_1-v_2\|_{V_i}^2,\ \ \forall v_1,v_2\in V_i.$$

To end this section, we provide existence and uniqueness result for Problem 2.1. The proof is omitted because it is done by slightly modifying the arguments as in [28, 42],

Theorem 2.1. *Assume that assumptions $\mathfrak{H}(a)(i,ii)$, $\mathfrak{H}(h)$, $\mathfrak{H}(\mathscr{H})$, $\mathfrak{H}(K)$, $\mathfrak{H}(\gamma)$, $\mathfrak{H}(\Psi)$, $\mathfrak{H}(f)$, and $\mathfrak{H}(0)$ hold, then Problem 2.1 has a unique solution $z^* \in C([0,T];K)$.*

3. DG-Functions

In this section, using suitable conditions, we introduce a gap function of the Fukushima regularized type for Problem 2.1. Furthermore, the DG-function for Problem 2.1 is constructed based on different RG-functions. We begin with providing the exact definition of gap functions for Problem 2.1 as below.

Definition 3.1. A real-valued function $\mathbf{g}\colon [0,T]\times C([0,T];K)\to\mathbb{R}$ is said to be a gap function for Problem 2.1 if it satisfies the following two properties:

(a) $\mathbf{g}(s,z)\geq 0$ for all $z\in C([0,T];K)$ and $s\in[0,T]$.
(b) $z^*\in C([0,T];K)$ is such that $\mathbf{g}(s,z^*)=0$ for all $s\in[0,T]$ if and only if z^* is a solution to Problem 2.1.

Now, for each $\theta>0$ fixed, let the function $\Lambda_{\theta,f}\colon [0,T]\times C([0,T];K)\times K\to\mathbb{R}$ be defined by

$$\begin{aligned}\Lambda_{\theta,f}(s,z,v) &= \langle a(s,z(s))+(\mathscr{H}z)(s)-f(s),v-z(s)\rangle_V + h(z(s),v)\\ &\quad+\sum_{i=1}^{l}\alpha_i\Psi_i^0\left(\gamma_i z(s);\gamma_i(v-z(s))\right)+\frac{1}{2\theta}\left\|v-z(s)\right\|_V^2 \end{aligned}\tag{3.1}$$

for all $z\in C([0,T];K)$, $v\in K$, and all $s\in[0,T]$.

Lemma 3.1 (see Tam-Chen [39])**.** *For each $i\in\{1,\dots,l\}$, suppose that $\Psi_i\colon V_i\to\mathbb{R}$ is a locally Lipschitz function and $\gamma_i\in L(V,V_i)$. Then, the function $\pi_i\colon V_i\times V_i\to\mathbb{R}$ defined by*

$$\pi_i(u_i,v_i)=\Psi_i^0\left(u_i;v_i-u_i\right)\tag{3.2}$$

satisfies the following properties:

(i) *For each $u_i\in V_i$, the function $v\mapsto\pi_i(u_i,\gamma_i v)$ is continuous and convex;*
(ii) *For each $u\in V$, $\partial_2^c(\pi_i\circ\gamma_i)(u,v)\subset\gamma_i^*\partial_2^c\Psi_i^0(\gamma_i u;\gamma_i v-\gamma_i u)$, where $\gamma_i^*\colon V_i^*\to V^*$ is the adjoint operator to γ_i, for all $i\in\{1,\dots,l\}$.*

Lemma 3.2. *Suppose that all the assumptions of Lemma 3.1, $\mathfrak{H}(h)(i)$, $\mathfrak{H}(K)$, and $\mathfrak{H}(f)$ hold. Then, for each $z\in C([0,T];K)$ and $\theta>0$ fixed, the optimization problem*

$$\min_{v\in K}\longrightarrow\Lambda_{\theta,f}(s,z,v),\ \textit{for all } s\in[0,T]\tag{3.3}$$

attains a unique solution $v_{\theta,f}(z)\in C([0,T];K)$

Proof. For each $i\in\{1,\dots,l\}$, by the condition $\mathfrak{H}(h)(i)$ and Lemma 3.1(i), we obtain that functions $v\mapsto\Psi_i^0\left(\gamma_i z(s);\gamma_i v-\gamma_i z(s)\right)$ and $v\mapsto h(z(s),v)$ are convex for all $z\in C([0,T];K)$ and all $s\in[0,T]$. Then, it can be seen easily that the function $\Lambda_{\theta,f}(s,z,\cdot)$ is a strongly convex function for all $z\in C([0,T];K)$ and all $s\in[0,T]$. Moreover, the function

$$v\mapsto\sum_{i=1}^{l}\alpha_i\Psi_i^0\left(\gamma_i z(s);\gamma_i v-\gamma_i z(s)\right)+h(z(s),v)$$

is lower semicontinuous for all $z \in C([0,T];K)$ and all $s \in [0,T]$. Hence, function $\Lambda_{\theta,f}(s,z,\cdot)$ is also lower semicontinuous for all $z \in C([0,T];K)$ and all $s \in [0,T]$. It follows from condition $\mathfrak{H}(K)$ that K is a nonempty, closed, and convex set. Thus, applying Lemma 2.2, convex minimization problem (3.3) attains a unique minimum $v_{\theta,f}(z) \in C([0,T];K)$, for any $z \in C([0,T];K)$ and $\theta > 0$ fixed. □

Next, a formulation of optimality condition for the minimization problem (3.3) is provided in the lemma below.

Lemma 3.3. *Suppose that all the conditions of Lemma 3.2 hold. Then, for each $z \in C([0,T];K)$ and $\theta > 0$ fixed, the inequality*

$$\left\langle a(s,z(s)) + (\mathscr{H}z)(s) - f(s) + \frac{1}{\theta}(v_{\theta,f}(z)(s) - z(s)), v - v_{\theta,f}(z)(s) \right\rangle_V$$
$$+ h(z(s),v) - h(z(s),v_{\theta,f}(z)(s)) + \sum_{i=1}^{l} \alpha_i \Psi_i^0(\gamma_i z(s); \gamma_i v - \gamma_i v_{\theta,f}(z)(s)) \geq 0 \tag{3.4}$$

holds for all $v \in K$ and all $s \in [0,T]$, where $v_{\theta,f}(z) \in C([0,T];K)$ is the unique solution to Problem (3.3).

Proof. For each $z \in C([0,T];K)$ and $\theta > 0$ fixed, let $v_{\theta,f}(z)$ be the unique solution to Problem (3.3). Applying the optimality condition for problem (3.3) (see [12, Theorem 1.23]), Lemma 3.1(ii) and the chain rule for generalized subgradient in [27, Proposition 3.35(ii) and Proposition 3.37(ii)] yields

$$\begin{aligned}
0 &\in \partial_3^c \Lambda_{\theta,f}(s,z,v_{\theta,f}(z)) \\
&\subset a(s,z(s)) + (\mathscr{H}z)(s) - f(s) + \partial_2^c h(z(s),v_{\theta,f}(z)(s)) \\
&\qquad + \sum_{i=1}^{l} \alpha_i \partial_2^c (\pi_i \circ \gamma_i)(z(s),v_{\theta,f}(z)(s)) + \frac{1}{\theta}(v_{\theta,f}(z)(s) - z(s)) \\
&\subset a(s,z(s)) + (\mathscr{H}z)(s) - f(s) + \partial_2^c h(z(s),v_{\theta,f}(z)(s)) \\
&\qquad + \sum_{i=1}^{l} \alpha_i \gamma_i^* \partial_2^c \Psi_i^0(\gamma_i z(s); \gamma_i(v_{\theta,f}(z)(s) - z(s))) + \frac{1}{\theta}(v_{\theta,f}(z)(s) - z(s)),
\end{aligned}$$

for a.e. $s \in [0,T]$, where π_i is defined by (3.2), $\gamma_i^* \colon V_i^* \to V^*$ is the adjoint operator to γ_i, respectively for all $i \in \{1,\ldots,l\}$. This implies that there exists $\xi(s) \in \partial_2^c h(z(s),v_{\theta,f}(z)(s))$ and

$$\zeta_i(s) \in \partial_2^c \pi_i(\gamma_i z(s), \gamma_i v_{\theta,f}(z)(s)) = \partial_2^c \Psi_i^0(\gamma_i z(s); \gamma_i(v_{\theta,f}(z)(s) - z(s)))$$

such that

$$-a(s,z(s)) - (\mathscr{H}z)(s) + f(s) - \frac{1}{\theta}(v_{\theta,f}(z)(s) - z(s)) = \xi(s) + \sum_{i=1}^{l} \alpha_i \gamma_i^* \zeta_i(s) \tag{3.5}$$

for all $s \in [0,T]$. For each $i \in \{1,\dots,l\}$, since γ_i^* is the adjoint operator to γ_i, it follows from (3.5) that, for all $v \in K$ and all $s \in [0,T]$,

$$\begin{aligned}
&\left\langle -a(s,z(s)) - (\mathscr{H}z)(s) + f(s) - \frac{1}{\theta}(v_{\theta,f}(z)(s) - z(s)), v - v_{\theta,f}(z)(s) \right\rangle_V \\
&= \langle \xi(s), v - v_{\theta,f}(z)(s) \rangle_V + \sum_{i=1}^{l} \alpha_i \langle \gamma_i^* \zeta_i(s), v - v_{\theta,f}(z)(s) \rangle_V \\
&= \langle \xi(s), v - v_{\theta,f}(z)(s) \rangle_V + \sum_{i=1}^{l} \alpha_i \langle \zeta_i(s), \gamma_i v - \gamma_i v_{\theta,f}(z)(z) \rangle_{V_i} \\
&\leq h(z(s),v) - h(z(s), v_{\theta,f}(z)(s)) \\
&\quad + \sum_{i=1}^{l} \alpha_i \left[\pi_i(\gamma_i z(s), \gamma_i v) - \pi_i(\gamma_i z(s), \gamma_i v_{\theta,f}(z)(s)) \right] \\
&= h(z(s),v) - h(z(s), v_{\theta,f}(z)(s)) \\
&\quad\quad + \sum_{i=1}^{l} \alpha_i \left[\Psi_i^0(\gamma_i z(s); \gamma_i v - \gamma_i z(s)) - \Psi_i^0(\gamma_i z(s); \gamma_i v_{\theta,f}(z)(s) - \gamma_i z(s)) \right] \\
&\quad \leq h(z(s),v) - h(z(s), v_{\theta,f}(z)(s)) + \sum_{i=1}^{l} \alpha_i \Psi_i^0(\gamma_i z(s); \gamma_i v - \gamma_i v_{\theta,f}(z)(s)),
\end{aligned}$$

that is,

$$\begin{aligned}
&\left\langle a(s,z(s)) + (\mathscr{H}z)(s) - f(s) + \frac{1}{\theta}(v_{\theta,f}(z)(s) - z(s)), v - v_{\theta,f}(z)(s) \right\rangle_V \\
&+ h(z(s),v) - h(z(s), v_{\theta,f}(z)(s)) + \sum_{i=1}^{l} \alpha_i \Psi_i^0(\gamma_i z(s); \gamma_i v - \gamma_i v_{\theta,f}(z)(s)) \geq 0.
\end{aligned}$$

Thus, for each $z \in C([0,T];K)$, inequality (3.4) holds for all $v \in K$ and all $s \in [0,T]$. □

For each $\theta > 0$ fixed, we introduce the function $\mathscr{M}_{\theta,f} \colon [0,T] \times C([0,T];K) \to \mathbb{R}$ defined by

$$\mathscr{M}_{\theta,f}(s,z) = \sup_{v \in K} \left(-\Lambda_{\theta,f}(s,z,v) \right), \tag{3.6}$$

for all $z \in C([0,T];K)$ and all $s \in [0,T]$, where the function $\Lambda_{\theta,f}$ is given by (3.1). Then, $\mathscr{M}_{\theta,f}$ can be rewritten as follows:

$$\begin{aligned}
\mathscr{M}_{\theta,f}(s,z) = \sup_{v \in K} \Big(&\langle a(s,z(s)) + (\mathscr{H}z)(s) - f(s), z(s) - v \rangle_V - h(z(s),v) \\
&- \sum_{i=1}^{l} \alpha_i \Psi_i^0 \left(\gamma_i z(s); \gamma_i (v - z(s)) \right) - \frac{1}{2\theta} \| v - z(s) \|_V^2 \Big).
\end{aligned}$$

In what follows, the function $\mathscr{M}_{\theta,f}$ is called to be a RG-function for Problem 2.1 in the form of Fukushima [9]. We now prove that $\mathscr{M}_{\theta,f}$ is a gap function for Problem 2.1.

Theorem 3.1. *Suppose the hypotheses of Theorem 2.1 hold. Then, the function $\mathscr{M}_{\theta,f}$ defined by (3.6) for any parameter $\theta > 0$ is a gap function for Problem 2.1.*

Proof. For any fixed parameter $\theta > 0$, we verify that the conditions of Definition 3.1 are satisfied for function $\mathscr{M}_{\theta,f}$.

(a) Let $z \in C([0,T];K)$ be arbitrary. By the definition of $\mathscr{M}_{\theta,f}$, we have

$$\begin{aligned}\mathscr{M}_{\theta,f}(s,z) &= \sup_{v\in K}\big(-\Lambda_{\theta,f}(s,z,v)\big)\\ &\geq -\Lambda_{\theta,f}(s,z,z(s))\\ &= \langle a(s,z(s)) + (\mathscr{H}z)(s) - f(s), z(s)-z(s)\rangle_V - h(z(s),z(s))\\ &\quad -\sum_{i=1}^{l}\alpha_i\Psi_i^0\left(\gamma_i z(s);\gamma_i(z(s)-z(s))\right) - \frac{1}{2\theta}\|z(s)-z(s)\|_V^2\\ &= -\sum_{i=1}^{l}\alpha_i\Psi_i^0\left(\gamma_i z(s);\mathbf{0}_{V_i}\right) = 0\end{aligned}$$

for all $s\in[0,T]$. This means that $\mathscr{M}_{\theta,f}(s,z)\geq 0$ for all $s\in[0,T]$ and all $z\in C([0,T];K)$.

(b) Suppose that $z^*\in C([0,T];K)$ is a solution to Problem 2.1. From (3.6), we have

$$\begin{aligned}\mathscr{M}_{\theta,f}(s,z^*) &= \sup_{v\in K}\big(-\Lambda_{\theta,f}(s,z^*,v)\big)\\ &= -\inf_{v\in K}\Lambda_{\theta,f}(s,z^*,v)\\ &= -\Lambda_{\theta,f}(s,z^*,v_{\theta,f}(z^*)(s)),\end{aligned} \tag{3.7}$$

where $v_{\theta,f}(z^*)\in C([0,T];K)$ is the unique solution to the convex minimization problem

$$\min_{v\in K}\longrightarrow \Lambda_{\theta,f}(s,z^*,v),\ \text{for all } s\in[0,T].$$

Moreover, since $z^*\in C([0,T];K)$ is a solution to Problem 2.1, for all $v\in K$ and all $s\in[0,T]$, we obtain

$$\begin{aligned}&\langle a(s,z^*(s)) + (\mathscr{H}z^*)(s) - f(s), v_{\theta,f}(z^*)(s) - z^*(s)\rangle_V + h(z^*(s), v_{\theta,f}(z^*)(s))\\ &\quad +\sum_{i=1}^{l}\alpha_i\Psi_i^0\left(\gamma_i z^*(s);\gamma_i v_{\theta,f}(z^*)(s) - \gamma_i z^*(s)\right)\geq 0.\end{aligned} \tag{3.8}$$

Applying the result of Lemma 3.3 implies that

$$\begin{aligned}&\left\langle a(s,z^*(s)) + (\mathscr{H}z^*)(s) - f(s) + \frac{1}{\theta}(v_{\theta,f}(z^*)(s) - z^*(s)), z^*(s) - v_{\theta,f}(z^*)(s)\right\rangle_V\\ &\quad + h(z^*(s),z^*(s)) - h(z^*(s), v_{\theta,f}(z^*)(s))\\ &\quad +\sum_{i=1}^{l}\alpha_i\Psi_i^0(\gamma_i z^*(s);\gamma_i z^*(s) - \gamma_i v_{\theta,f}(z^*)(s))\geq 0.\end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), thanks to the assumption $\mathfrak{H}(\Psi)$(ii), we have

$$-\frac{1}{\theta}\|v_{\theta,f}(z^*)(s) - z^*(s)\|_V^2 \geq 0$$

for all $s\in[0,T]$. Hence, we obtain $\|v_{\theta,f}(z^*)(s) - z^*(s)\|_V^2\leq 0$, for all $s\in[0,T]$ and so $z^* = v_{\theta,f}(z^*)$. Therefore, it follows from (3.7) that $\mathscr{M}_{\theta,f}(s,z^*) = 0$ for all $s\in[0,T]$.

Conversely, let $z^* \in C([0,T];K)$ be such that $\mathscr{M}_{\theta,f}(s,z^*) = 0$ for all $s \in [0,T]$. Then, we know that $-\Lambda_{\theta,f}(s,z^*,v) \le 0$, that is, $\Lambda_{\theta,f}(s,z^*,v) \ge 0$ for all $v \in K$ and all $s \in [0,T]$. Since $\Lambda_{\theta,f}(s,z^*,z^*(s)) = 0$ for all $s \in [0,T]$, $z^*(s)$ solves the following convex minimization problem

$$\min_{v\in K} \longrightarrow \Lambda_{\theta,f}(s,z^*,v).$$

It follows from the optimality condition of this problem that

$$0 \in \partial_3^c \Lambda_{\theta,f}(s,z^*,z^*(s)).$$

Fixing the first argument of $\Lambda_{\theta,f}$, by the similar arguments for Lemma 3.3, we have

$$-a(s,z^*(s)) - (\mathscr{H}z^*)(s) + f(s) = \xi^*(s) + \sum_{i=1}^{l} \alpha_i \gamma_i^* \zeta_i^*(s)$$

where $\xi^*(s) \in \partial_2^c h(z^*(s),z^*(s))$ and $\zeta_i^*(s) \in \partial_2^c \pi_i(\gamma_i z^*(s); \gamma_i z^*(s))$ for all $i \in \{1,\dots,l\}$ and for a.e. $s \in [0,T]$. Then, for all $v \in K$ and a.e. $s \in [0,T]$,

$$\begin{aligned}
&\langle -a(s,z^*(s)) - (\mathscr{H}z^*)(s) + f(s), v - z^*(s)\rangle_V \\
&= \langle \xi^*(s), v - z^*(s)\rangle_V + \sum_{i=1}^{l} \alpha_i \langle \gamma_i^* \zeta_i^*(s), v - z^*(s)\rangle_V \\
&= \langle \xi^*(s), v - z^*(s)\rangle_V + \sum_{i=1}^{l} \alpha_i \langle \zeta_i^*(s), \gamma_i v - \gamma_i z^*(s)\rangle_{V_i} \\
&\le h(z^*(s),v) - h(z^*(s),z^*(s)) + \sum_{i=1}^{l} \alpha_i \left[\pi_i(\gamma_i z^*(s), \gamma_i v) - \pi_i(\gamma_i z^*(s), \gamma_i z^*(s))\right] \\
&= h(z^*(s),v) + \sum_{i=1}^{l} \alpha_i \left[\Psi_i^0(\gamma_i z^*(s); \gamma_i v - \gamma_i z^*(s)) - \Psi_i^0(\gamma_i z^*(s); \gamma_i z^*(s) - \gamma_i z^*(s))\right] \\
&= h(z^*(s),v) + \sum_{i=1}^{l} \alpha_i \Psi_i^0(\gamma_i z^*(s); \gamma_i v - \gamma_i z^*(s))
\end{aligned}$$

that is,

$$\begin{aligned}
&\langle a(s,z^*(s)) + (\mathscr{H}z^*)(s), v - z^*(s)\rangle_V + h(z^*(s),v) + \sum_{i=1}^{l} \alpha_i \Psi_i^0\left(\gamma_i z^*(s); \gamma_i (v - z^*(s))\right) \\
&\ge \langle f(s), v - z^*(s)\rangle_V.
\end{aligned}$$

This implies that z^* is a solution to Problem 2.1. Therefore, $\mathscr{M}_{\theta,f}$ is a gap function for Problem 2.1. □

Based on the RG-functions of the Fukushima type, we now establish DG-function for Problem 2.1.

Let $\theta > \delta > 0$ be fixed and the gap functions $\mathscr{M}_{\theta,f}$ and $\mathscr{M}_{\delta,f}$ be given in the form of (3.6). We consider the function $\mathbf{D}_{\theta,\delta}^f \colon [0,T] \times C([0,T];K) \to \mathbb{R}$ defined by

$$\mathbf{D}_{\theta,\delta}^f(s,z) = \mathscr{M}_{\theta,f}(s,z) - \mathscr{M}_{\delta,f}(s,z) \tag{3.10}$$

for all $z \in C([0,T];K)$ and all $s \in [0,T]$. Then, we achieve the following property of $\mathbf{D}_{\theta,\delta}^f$.

Lemma 3.4. *Under the hypotheses of Theorem 2.1, for any $\theta > \delta > 0$, we have*

$$\left\| z(s) - v_{\delta,f}(z)(s) \right\|_V^2 \le \frac{2\theta\delta}{\theta-\delta} \mathbf{D}^f_{\theta,\delta}(s,z) \le \left\| z(s) - v_{\theta,f}(z)(s) \right\|_V^2, \tag{3.11}$$

where

$$v_{\theta,f}(z)(s) = \arg\min_{v\in K} \Lambda_{\theta,f}(s,z,v) \text{ and } v_{\delta,f}(z)(s) = \arg\min_{v\in K} \Lambda_{\delta,f}(s,z,v),$$

for all $z \in C([0,T];K)$ and all $s \in [0,T]$.

Proof. In light of the definitions of the gap functions $\mathscr{M}_{\theta,f}, \mathscr{M}_{\delta,f}$ and the function $\mathbf{D}^f_{\theta,\delta}$, we know that

$$\begin{aligned}
\mathbf{D}^f_{\theta,\delta}(s,z) &= \sup_{v\in K}\{-\Lambda_{\theta,f}(s,z,v)\} - \sup_{v\in K}\{-\Lambda_{\delta,f}(s,z,v)\} \\
&\le -\Lambda_{\theta,f}(s,z,v_{\theta,f}(z)(s)) + \Lambda_{\delta,f}(s,z,v_{\theta,f}(z)(s)) \\
&= \left(\frac{1}{2\delta} - \frac{1}{2\theta}\right) \left\| z(s) - v_{\theta,f}(z)(s) \right\|_V^2.
\end{aligned}$$

Thus, the right-hand-side inequality in (3.11) holds. Similar arguments also lead to the left-hand-side inequality in (3.11). □

Theorem 3.2. *Suppose that the hypotheses of Theorem 2.1 hold. Then, the function $\mathbf{D}^f_{\theta,\delta}$ defined by (3.10) for any parameters $\theta > \delta > 0$ is a gap function for Problem 2.1.*

Proof. For any fixed parameters $\theta > \delta > 0$, we shall prove the conditions of Definition 3.1 are satisfied for $\mathbf{D}^f_{\theta,\delta}$.

(a) It clearly follows from (3.11) that $\mathbf{D}^f_{\theta,\delta}(s,z) \ge 0$, for all $z \in C([0,T];K)$ and all $s \in [0,T]$.

(b) Assume that $z^* \in C([0,T];K)$ is a solution to Problem 2.1. It follows from Theorem 3.1 that $\mathscr{M}_{\theta,f}(s,z^*) = \mathscr{M}_{\delta,f}(s,z^*) = 0$ and so $\mathbf{D}^f_{\theta,\delta}(s,z^*) = 0$ for all $s \in [0,T]$.

Conversely, suppose that $z^* \in C([0,T];K)$ with $\mathbf{D}^f_{\theta,\delta}(s,z^*) = 0$ for all $s \in [0,T]$. From (3.11), we have $z^* = v_{\delta,f}(z^*)$. Applying Lemma 3.3 with $z = z^*$ and $\theta = \delta$, there holds

$$\begin{aligned}
&\langle a(s,z^*(s)) + (\mathscr{H}z^*)(s), v - z^*(s)\rangle_V + h(z^*(s),v) \\
&\qquad + \sum_{i=1}^{l} \alpha_i \Psi_i^0(\gamma_i z^*(s); \gamma_i(v - z^*(s))) \ge \langle f(s), v - z^*(s)\rangle_V
\end{aligned}$$

for all $v \in K$ and a.e. $s \in [0,T]$, which implies that z^* is a solution to Problem 2.1. Thus, $\mathbf{D}^f_{\theta,\delta}$ is a gap function for Problem 2.1. □

The following lemma states an important property regarding the gap functions $\mathscr{M}_{\theta,f}$ and $\mathbf{D}^f_{\theta,\delta}$ defined by (3.6) and (3.10), respectively.

Lemma 3.5. *Assume that all the assumptions of Theorem 2.1 are fulfilled. If, in addition, K is bounded, then, for any parameters $\theta > \delta > 0$ fixed and for each fixed $z \in C([0,T];K)$, the functions $s \mapsto \mathscr{M}_{\theta,f}(s,z)$ and $s \mapsto \mathbf{D}^f_{\theta,\delta}(s,z)$ belong to $L^\infty_+(0,T)$.*

Proof. It follows from the proof of [3, Theorem 3.2 (ii)] that, for any parameter $\theta > 0$ fixed, function $s \mapsto \mathscr{M}_{\theta,f}(s,z)$ belongs to $L^{\infty}_{+}(0,T)$ for each fixed $z \in C([0,T];K)$. Hence, for any parameters $\theta > \delta > 0$ fixed, the function

$$z \mapsto \mathbf{D}^{f}_{\theta,\delta}(s,z) := \mathscr{M}_{\theta,f}(s,z) - \mathscr{M}_{\delta,f}(s,z)$$

also belongs to $L^{\infty}_{+}(0,T)$ for each fixed $z \in C([0,T];K)$. □

Remark 3.1. Note the constraint set K of Problem 2.1 in Lemma 3.2 is bounded, but in the general case, K is unbounded. However, Cen-Nguyen-Zeng proved that the unique solution to Problem 2.1 with a suitable bounded set coincides with the unique solution of the original problem with the constraint set K (see [3, Theorem 2.2]). Therefore, we always assume that the constraint set K is bounded in the sequel.

Remark 3.2. (i) As mentioned in the introduction, DG-functions for history-dependent variational-hemivariational inequalities have not been studied before. As a result, our Theorem 3.2 is new. (ii) On the other hand, based on a formulation of the optimality condition in Lemma 3.3, the proof method for the RG-function $\mathscr{M}_{\theta,f}$ in Theorem 3.1 is different and extends to the corresponding result on the RG-function for Problem 2.2 studied in [3].

4. Upper Error Bounds

In this section, we establish several upper error bounds for Problem 2.1 controlled by the RG-function $\mathscr{M}_{\theta,f}$ and the DG-function $\mathbf{D}^{f}_{\theta,\delta}$ which are introduced in Section 3.

Lemma 4.1. *Let $z^* \in C([0,T];K)$ be the unique solution to Problem 2.1. Assume that the hypotheses $\mathfrak{H}(a)$, $\mathfrak{H}(\mathscr{H})$, $\mathfrak{H}(\Psi)$, $\mathfrak{H}(K_b)$, $\mathfrak{H}(h)$ with $h(z,v) = \phi(v) - \phi(z)$ for all $z, v \in K$ ($\phi : K \to \mathbb{R}$ is a bounded convex and continuous function), $\mathfrak{H}(\gamma)$, $\mathfrak{H}(f)$ and $\mathfrak{H}(0)$ hold. Then, for each $z \in C([0,T];K)$,*

$$\widehat{L}_0 \|z(s) - z^*(s)\|^2_V \leq \widehat{L}_1 \|z(s) - v_{\delta,f}(z)(s)\|^2_V + T L_{\mathscr{H}} \int_0^s \|z(t) - z^*(t)\|^2_V dt, \tag{4.1}$$

for all $s \in [0,T]$, where

$$\begin{cases} \widehat{L}_0 := m_a - \frac{1}{2}\left(L_a + L_{\mathscr{H}} + \frac{1}{\delta} + 3\sum_{i=1}^{l} \alpha_i L_{\Psi_i} c^2_{\Psi_i}\right); \\ \widehat{L}_1 := \frac{1}{2}\left(L_a + L_{\mathscr{H}} + \frac{1}{\delta} + \sum_{i=1}^{l} \alpha_i L_{\Psi_i} c^2_{\Psi_i}\right), \end{cases} \tag{4.2}$$

and

$$v_{\delta,f}(z)(s) = \arg\min_{v \in K} \Lambda_{\delta,f}(s,z,v),$$

for all $z \in C([0,T];K)$ and $s \in [0,T]$.

Proof. For each $z \in C([0,T];K)$, since $z^* \in C([0,T];K)$ is a solution to Problem 2.1 and $v_{\delta,f}(z) \in C([0,T];K)$, one has

$$\langle a(s,z^*(s)) + (\mathscr{H}z^*)(s) - f(s), v_{\delta,f}(z)(s) - z^*(s)\rangle_V + h(z^*(s), v_{\delta,f}(z)(s)) \\ + \sum_{i=1}^{l} \alpha_i \Psi^0_i \left(\gamma_i z^*(s); \gamma_i v_{\delta,f}(z)(s) - \gamma_i z^*(s)\right) \geq 0, \tag{4.3}$$

for all $s \in [0,T]$. Then, we add (3.4) with $\theta = \delta, v = z^*(s)$ and achieve that

$$\begin{aligned}\Big\langle a(s,z(s)) + (\mathscr{H}z)(s) - f(s) + \frac{1}{\delta}(v_{\delta,f}(z)(s) - z(s)), z^*(s) - v_{\delta,f}(z)(s)\Big\rangle_V \\ + h(z(s), z^*(s)) - h(z(s), v_{\delta,f}(z)(s)) \\ + \sum_{i=1}^{l} \alpha_i \Psi_i^0(\gamma_i z(s); \gamma_i z^*(s) - \gamma_i v_{\delta,f}(z)(s)) \geq 0 \end{aligned} \tag{4.4}$$

for all $s \in [0,T]$. Combining (4.3) and (4.4) with $h(z,v) = \phi(v) - \phi(z)$ for all $z, v \in K$, we obtain

$$\begin{aligned} 0 \leq & \left\langle a(s,z^*(s)) - a(s,z(s)), v_{\delta,f}(z)(s) - z^*(s)\right\rangle_V \\ & + \left\langle (\mathscr{H}z^*)(s) - (\mathscr{H}z)(s), v_{\delta,f}(z)(s) - z^*(s)\right\rangle_V \\ & + \sum_{i=1}^{l} \alpha_i \left[\Psi_i^0(\gamma_i z^*(s); \gamma_i v_{\delta,f}(z)(s) - \gamma_i z^*(s)) + \Psi_i^0\left(\gamma_i z(s); \gamma_i z^*(s) - \gamma_i v_{\theta,f}(z)(s)\right)\right] \\ & + \frac{1}{\delta}\left\langle v_{\delta,f}(z)(s) - z(s), z^*(s) - v_{\delta,f}(z)(s)\right\rangle_V. \end{aligned} \tag{4.5}$$

Since a is Lipschitz continuous with constant L_a and the condition $\mathfrak{H}(a)$(ii) holds, we have

$$\begin{aligned} & \left\langle a(s,z^*(s)) - a(s,z(s)), v_{\delta,f}(z)(s) - z^*(s)\right\rangle_V \\ & = \left\langle a(s,z^*(s)) - a(s,z(s)), v_{\delta,f}(z)(s) - z(s)\right\rangle_V \\ & \quad - \langle a(s,z^*(s)) - a(s,z(s)), z^*(s) - z(s))\rangle_V \\ & \leq L_a \|z(s) - z^*(s)\|_V \|z(s) - v_{\delta,f}(z)(s)\|_V - m_a \|z(s) - z^*(s)\|_V^2. \end{aligned} \tag{4.6}$$

Moreover, we also obtain

$$\begin{aligned} & \frac{1}{\delta}\left\langle v_{\delta,f}(z)(s) - z(s), z^*(s) - v_{\delta,f}(z)(s)\right\rangle_V \\ & = \frac{1}{\delta}\left\langle v_{\delta,f}(z)(s) - z(s), z^*(s) - z(s)\right\rangle_V + \frac{1}{\delta}\left\langle v_{\delta,f}(z)(s) - z(s), z(s) - v_{\delta,f}(z)(s)\right\rangle_V \\ & \leq \frac{1}{\delta}\|z(s) - z^*(s)\|_V \|z(s) - v_{\delta,f}(z)(s)\|_V - \frac{1}{\delta}\|z(s) - v_{\delta,f}(z)(s)\|_V^2 \\ & \leq \frac{1}{\delta}\|z(s) - z^*(s)\|_V \|z(s) - v_{\delta,f}(z)(s)\|_V. \end{aligned} \tag{4.7}$$

Since $\mathscr{H}$ is the history-dependent operator, we know

$$\begin{aligned} & \left\langle (\mathscr{H}z^*)(s) - (\mathscr{H}z)(s), v_{\delta,f}(z)(s) - z^*(s)\right\rangle_V \\ & = \left\langle (\mathscr{H}z^*)(s) - (\mathscr{H}z)(s), v_{\delta,f}(z)(s) - z(s)\right\rangle_V + \langle (\mathscr{H}z^*)(s) - (\mathscr{H}z)(s), z(s)) - z^*(s)\rangle_V \\ & \leq L_{\mathscr{H}} \int_0^s \|z(t) - z^*(t)\|_V dt \|z(s) - v_{\delta,f}(z)(s)\|_V + L_{\mathscr{H}} \int_0^s \|z(t) - z^*(t)\|_V dt \|z(s) - z^*(s)\|_V. \end{aligned} \tag{4.8}$$

For each $i \in \{1,\ldots,l\}$, by the conditions $\alpha_i > 0$, $\mathfrak{H}(\Psi)$(ii) and $\mathfrak{H}(\gamma)$, we have

$$\begin{aligned}
&\alpha_i \left[\Psi_i^0(\gamma_i z^*(s); \gamma_i v_{\delta,f}(z)(s) - \gamma_i z^*(s)) + \Psi_i^0\left(\gamma_i z(s); \gamma_i z^*(s) - \gamma_i v_{\theta,f}(z)(s)\right)\right] \\
&\quad \le \alpha_i L_{\Psi_i} \|\gamma_i z^*(s) - \gamma_i z(s)\|_{V_i} \|\gamma_i v_{\theta,f}(z)(s) - \gamma_i z^*(s)\|_{V_i} \\
&\quad \le \alpha_i L_{\Psi_i} c_{\Psi_i}^2 \|z(t) - z^*(t)\|_V^2 \\
&\qquad + \alpha_i L_{\Psi_i} c_{\Psi_i}^2 \|z(t) - z^*(t)\|_V \|z(s) - v_{\delta,f}(z)(s)\|_V. \qquad (4.9)
\end{aligned}$$

From (4.5)–(4.9), employing the inequality $ab \le \dfrac{a^2+b^2}{2}$ for all $a,b \in \mathbb{R}_+$ and Hölder's inequality gives

$$\begin{aligned}
&\left(m_a - \sum_{i=1}^{l} \alpha_i L_{\Psi_i} c_{\Psi_i}^2\right) \|z(s) - z^*(s)\|_V^2 \\
&\quad \le \left(L_a + \frac{1}{\delta} + \sum_{i=1}^{l} \alpha_i L_{\Psi_i} c_{\Psi_i}^2\right) \|z(s) - z^*(s)\|_V \|z(s) - v_{\delta,f}(z)(s)\|_V \\
&\qquad + L_{\mathscr{H}} \int_0^s \|z(t) - z^*(t)\|_V dt \|z(s) - v_{\delta,f}(z)(s)\|_V \\
&\qquad + L_{\mathscr{H}} \int_0^s \|z(t) - z^*(t)\|_V dt \|z(s) - z^*(s)\|_V \\
&\quad \le \frac{1}{2}\left(L_a + \frac{1}{\delta} + \sum_{i=1}^{l} \alpha_i L_{\Psi_i} c_{\Psi_i}^2\right) (\|z(t) - z^*(t)\|_V^2 + \|z(s) - v_{\delta,f}(z)(s)\|_V^2) \\
&\qquad + \frac{L_{\mathscr{H}}}{2}\left[\left(\int_0^s \|z(t) - z^*(t)\|_V dt\right)^2 + \|z(s) - v_{\delta,f}(z)(s)\|_V^2\right] \\
&\qquad + \frac{L_{\mathscr{H}}}{2}\left[\left(\int_0^s \|z(t) - z^*(t)\|_V dt\right)^2 + \|z(s) - z^*(s)\|_V^2\right] \\
&\quad \le \frac{1}{2}\left(L_a + L_{\mathscr{H}} + \frac{1}{\delta} + \sum_{i=1}^{l} \alpha_i L_{\Psi_i} c_{\Psi_i}^2\right) (\|z(t) - z^*(t)\|_V^2 + \|z(s) - v_{\delta,f}(z)(s)\|_V^2) \\
&\qquad + T L_{\mathscr{H}} \int_0^s \|z(t) - z^*(t)\|_V^2 dt, \qquad (4.10)
\end{aligned}$$

for all $s \in [0,T]$. Set

$$\begin{aligned}
\widehat{L}_0 &:= m_a - \frac{1}{2}\left(L_a + L_{\mathscr{H}} + \frac{1}{\delta} + 3\sum_{i=1}^{l} \alpha_i L_{\Psi_i} c_{\Psi_i}^2\right); \\
\widehat{L}_1 &:= \frac{1}{2}\left(L_a + L_{\mathscr{H}} + \frac{1}{\delta} + \sum_{i=1}^{l} \alpha_i L_{\Psi_i} c_{\Psi_i}^2\right).
\end{aligned}$$

Then it follows from (4.10) that

$$\widehat{L}_0 \|z(s) - z^*(s)\|_V^2 \le \widehat{L}_1 \|z(s) - v_{\delta,f}(z)(s)\|_V^2 + T L_{\mathscr{H}} \int_0^s \|z(t) - z^*(t)\|_V^2 dt,$$

for all $s \in [0,T]$. This implies that the inequality (4.1) holds. □

From Lemma 4.1, we build up the following upper error bound for Problem 2.1 controlled by the RG-function $\mathscr{M}_{\theta,f}$.

Theorem 4.1. *Let $z^* \in C([0,T];K)$ be the unique solution to Problem 2.1, $\widehat{L}_0$ and $\widehat{L}_1$ be defined by (4.2). Suppose that the hypotheses of Lemma 4.1 hold. Assume furthermore that $\widehat{L}_0 > 0$. Then, for each $z \in C([0,T];K)$, we obtain*

$$\|z(s) - z^*(s)\|_V \le \mathscr{E}_z^{\mathscr{M}}(s) \quad \textit{for all } s \in [0,T], \tag{4.11}$$

where $\mathscr{E}_z^{\mathscr{M}} \in L_+^\infty(0,T)$ is defined by

$$\mathscr{E}_z^{\mathscr{M}} := \sqrt{\frac{2\widehat{L}_1\theta}{\widehat{L}_0}\mathscr{M}_{\theta,f}(s,z) + \frac{2\widehat{L}_1\theta T L_{\mathscr{H}}}{\widehat{L}_0^2}\int_0^s \mathscr{M}_{\theta,f}(t,z).exp\left\{\frac{TL_{\mathscr{H}}}{\widehat{L}_0}(s-t)\right\}dt}$$

for all $z \in C([0,T];K)$ and all $s \in [0,T]$.

Proof. Let $z^* \in C([0,T];K)$ be the unique solution to Problem 2.1. For any $z \in C([0,T];K)$, taking $v = z(s)$ in (3.4), we obtain

$$\begin{aligned}
\Big\langle a(s,z(s)) + (\mathscr{H}z)(s) - f(s) + \frac{1}{\theta}(v_{\theta,f}(z)(s) - z(s)), z(s) - v_{\theta,f}(z)(s)\Big\rangle_V \\
h(z(s),z(s)) - h(z(s), v_{\theta,f}(z)(s)) \\
+ \sum_{i=1}^{l} \alpha_i \Psi_i^0(\gamma_i z(s); \gamma_i z(s) - \gamma_i v_{\theta,f}(z)(s)) \ge 0.
\end{aligned}$$

Applying the conditions $h(z,v) = \phi(v) - \phi(z)$ for all $z, v \in K$ and $\mathfrak{H}(\Psi)$(ii), we get

$$\begin{aligned}
&- \langle a(s,z(s)) + (\mathscr{H}z)(s) - f(s), v_{\theta,f}(z)(s) - z(s)\rangle_V - h(z(s), v_{\theta,f}(z)(s)) \\
&\quad - \sum_{i=1}^{l} \alpha_i \Psi_i^0(\gamma_i z(s); \gamma_i v_{\theta,f}(z)(s) - \gamma_i z(s)) - \frac{1}{2\theta}\|z(s) - v_{\theta,f}(z)(s)\|_V^2 \\
&\ge \frac{1}{2\theta}\|z(s) - v_{\theta,f}(z)(s)\|_V^2,
\end{aligned}$$

which implies that

$$-\Lambda_{\theta,f}(s,z,v_{\theta,f}(z)) \ge \frac{1}{2\theta}\|z(s) - v_{\theta,f}(z)(s)\|_V^2. \tag{4.12}$$

It follows from (3.6) and (4.12) that

$$\|z(s) - v_{\theta,f}(z)(s)\|_V^2 \le 2\theta \sup_{v\in K}\left(-\Lambda_{\theta,f}(s,z,v)\right) = 2\theta\mathscr{M}_{\theta,f}(s,z), \tag{4.13}$$

for all $s \in [0,T]$. Using (4.13) and taking $\delta = \theta$ in (4.1), we obtain

$$\|z(s) - z^*(s)\|_V^2 \le \frac{2\widehat{L}_1\theta\mathscr{M}_{\theta,f}(s,z)}{\widehat{L}_0} + \frac{TL_{\mathscr{H}}}{\widehat{L}_0}\int_0^s \|z(t) - z^*(t)\|_V^2 dt$$

for all $s \in [0,T]$. Invoking Gronwall's inequality for the above inequality yields

$$\begin{aligned}
&\|z(s) - z^*(s)\|_V^2 \\
&\le \frac{2\widehat{L}_1\theta}{\widehat{L}_0}\mathscr{M}_{\theta,f}(s,z) + \frac{2\widehat{L}_1\theta T L_{\mathscr{H}}}{\widehat{L}_0^2}\int_0^s \mathscr{M}_{\theta,f}(t,z).exp\left\{\frac{TL_{\mathscr{H}}}{\widehat{L}_0}(s-t)\right\}dt
\end{aligned}$$

for all $s \in [0,T]$. In addition, Lemma 3.5 indicates that $s \mapsto \mathscr{M}_{\theta,f}(s,z)$ belongs to $L_+^\infty(0,T)$.

For each function $z \in C([0,T];K)$, let the function $\mathscr{E}_z^{\mathscr{M}} : [0,T] \to \mathbb{R}_+$ be defined by

$$\mathscr{E}_z^{\mathscr{M}}(s) := \sqrt{\frac{2\widehat{L}_1\theta}{\widehat{L}_0}\mathscr{M}_{\theta,f}(s,z) + \frac{2\widehat{L}_1\theta T L_{\mathscr{H}}}{\widehat{L}_0^2}\int_0^s \mathscr{M}_{\theta,f}(t,z).exp\left\{\frac{TL_{\mathscr{H}}}{\widehat{L}_0}(s-t)\right\}dt}$$

for all $s \in [0,T]$. Whereas, from Lemma 3.5, it is easy to see that $\mathscr{E}_z^{\mathscr{M}} \in L_+^\infty(0,T)$. Then we can get

$$\|z(s) - z^*(s)\|_V \le \mathscr{E}_z^{\mathscr{M}}(s) \quad \text{for all } s \in [0,T].$$

Therefore, we conclude that inequality (4.11) is valid. □

Without using the Lipschitz continuity of a, we can also provide another upper error bound for Problem 2.1 controlled by the RG-function $\mathscr{M}_{\theta,f}$.

Theorem 4.2. *Let $z^* \in C([0,T];K)$ be the unique solution to Problem 2.1. Assume that the hypotheses $\mathfrak{H}(a)$, $\mathfrak{H}(\mathscr{H})$, $\mathfrak{H}(\Psi)$, $\mathfrak{H}(K_b)$, $\mathfrak{H}(h)$, $\mathfrak{H}(\gamma)$, $\mathfrak{H}(f)$ hold. Then, for each $z \in C([0,T];K)$ and $\theta > 0$ satisfying*

$$\mathbf{C}_0 := m_a + m_h - \sum_{i=1}^{l} \alpha_i L_{\Psi_i} c_{\Psi_i}^2 - \frac{1}{2\theta} > 0,$$

one has

$$\|z(s) - z^*(s)\|_V \le \mathscr{E}_z^{\sharp\mathscr{M}}(s) \quad \textit{for all } s \in [0,T], \tag{4.14}$$

where $\mathscr{E}_z^{\sharp\mathscr{M}} \in L_+^\infty(0,T)$ is defined by

$$\mathscr{E}_z^{\sharp\mathscr{M}}(s) := \sqrt{\frac{2}{\mathbf{C}_0}\mathscr{M}_{\theta,f}(s,z) + \frac{2TL_{\mathscr{H}}^2}{\mathbf{C}_0^3}\int_0^s \mathscr{M}_{\theta,f}(s,z).exp\left\{\frac{TL_{\mathscr{H}}^2}{\mathbf{C}_0^2}(s-t)\right\}dt}$$

for all $z \in C([0,T];K)$ and all $s \in [0,T]$.

Proof. Let $z^* \in C([0,T];K)$ be the unique solution to Problem 2.1. Fix an arbitrary $z \in C([0,T];K)$, it follows from the definition of $\mathscr{M}_{\theta,f}$ that

$$\begin{aligned}
\mathscr{M}_{\theta,f}(s,z) &= \sup_{v\in K}\{-\Lambda_{\theta,f}(s,z,v)\} \\
&\ge -\Lambda_{\theta,f}(s,z,z^*) \\
&= \langle a(s,z(s)) + (\mathscr{H}z)(s) - f(s), z(s) - z^*(s)\rangle_V - h(z(s),z^*(s)) \\
&\quad - \sum_{i=1}^{l} \alpha_i \Psi_i^0(\gamma_i z(s); \gamma_i z^*(s) - \gamma_i z(s)) - \frac{1}{2\theta}\|z(s) - z^*(s)\|_V^2.
\end{aligned} \tag{4.15}$$

Since z^* is a solution to Problem 2.1, for all $s \in [0,T]$, we obtain

$$\begin{aligned}
&\langle a(s,z^*(s)) + (\mathscr{H}z^*)(s) - f(s), z(s) - z^*(s)\rangle_V + h(z^*(s),z(s)) \\
&\qquad + \sum_{i=1}^{l} \alpha_i \Psi_i^0(\gamma_i z^*(s); \gamma_i z(s) - \gamma_i z^*(s)) \ge 0.
\end{aligned} \tag{4.16}$$

Applying hypotheses $\mathfrak{H}(a)$ and $\mathfrak{H}(\mathscr{H})$ yields

$$\begin{aligned}
&\langle a(s,z(s))+(\mathscr{H}z)(s)-f(s),z(s)-z^*(s)\rangle_V\\
&-\langle a(s,z^*(s))+(\mathscr{H}z^*)(s)-f(s),z(s)-z^*(s)\rangle_V\\
=&\langle a(s,z(s))-a(s,z^*(s)),z(s)-z^*(s)\rangle_V+\langle(\mathscr{H}z)(s)-(\mathscr{H}z^*)(s),z(s)-z^*(s)\rangle_V\\
\geq& m_a\|z(s)-z^*(s)\|_V^2-\|(\mathscr{H}z)(s)-(\mathscr{H}z^*)(s)\|_{V^*}\|z(s)-z^*(s)\|_V\\
\geq& m_a\|z(s)-z^*(s)\|_V^2-L_{\mathscr{H}}\int_0^s\|z(t)-z^*(t)\|_V dt\|z(s)-z^*(s)\|_V,
\end{aligned}\tag{4.17}$$

for all $s\in[0,T]$. It follows from the condition $h(z,v)+h(v,z)\leq -m_h\|z-v\|_V^2$ for all $z,v\in K$ that

$$-h(z(s),z^*(s))-h(z^*(s),z(s))\geq m_h\|z(t)-z^*(t)\|_V^2\tag{4.18}$$

for all $s\in[0,T]$. Moreover, using the hypothesis $\mathfrak{H}(\Psi)$ implies

$$\begin{aligned}
&-\sum_{i=1}^l\alpha_i\Psi_i^0(\gamma_iz(s);\gamma_iz^*(s)-\gamma_iz(s))-\sum_{i=1}^l\alpha_i\Psi_i^0(\gamma_iz^*(s);\gamma_iz(s)-\gamma_iz^*(s))\\
=&-\sum_{i=1}^l\alpha_i\left[\Psi_i^0(\gamma_iz(s);\gamma_iz^*(s)-\gamma_iz(s))+\Psi_i^0(\gamma_iz^*(s);\gamma_iz(s)-\gamma_iz^*(s))\right]\\
\geq&-\sum_{i=1}^l\alpha_iL_{\Psi_i}c_{\Psi_i}^2\|z(s)-z^*(s)\|_V^2.
\end{aligned}\tag{4.19}$$

Having in mind relations (4.16)–(4.19), it follows that

$$\begin{aligned}
&\langle a(s,z(s))+(\mathscr{H}z)(s)-f(s),z(s)-z^*(s)\rangle_V-h(z(s),z^*(s))\\
&\quad-\sum_{i=1}^l\alpha_i\Psi_i^0(\gamma_iz(s);\gamma_iz^*(s)-\gamma_iz(s))\\
\geq&\left(m_a+m_h-\sum_{i=1}^l\alpha_iL_{\Psi_i}c_{\Psi_i}^2\right)\|z(s)-z^*(s)\|_V^2\\
&-L_{\mathscr{H}}\int_0^s\|z(t)-z^*(t)\|_V dt\|z(s)-z^*(s)\|_V.
\end{aligned}\tag{4.20}$$

Combining (4.15) and (4.20), we have

$$\begin{aligned}
&\mathscr{M}_{\theta,f}(s,z)\\
\geq&\mathbf{C}_0\|z(s)-z^*(s)\|_V^2-L_{\mathscr{H}}\int_0^s\|z(t)-z^*(t)\|_V dt\|z(s)-z^*(s)\|_V,
\end{aligned}\tag{4.21}$$

for all $s\in[0,T]$, where

$$\mathbf{C}_0=m_a+m_h-\sum_{i=1}^l\alpha_iL_{\Psi_i}c_{\Psi_i}^2-\frac{1}{2\theta}.$$

Employing Young's inequality with $\varepsilon = \frac{\mathbf{C}_0}{2}$ and Hölder's inequality leads to

$$\begin{aligned} & L_{\mathscr{H}} \int_0^s \|z(t)-z^*(t)\|_V dt \|z(s)-z^*(s)\|_V \\ & \leq \varepsilon \|z(s)-z^*(s)\|_V^2 + \frac{L_{\mathscr{H}}^2}{4\varepsilon} \left(\int_0^s \|z(t)-z^*(t)\|_V dt \right)^2 \\ & \leq \frac{\mathbf{C}_0}{2} \|z(s)-z^*(s)\|_V^2 + \frac{T L_{\mathscr{H}}^2}{2\mathbf{C}_0} \int_0^s \|z(t)-z^*(t)\|_V^2 dt. \end{aligned} \tag{4.22}$$

for all $s \in [0,T]$. Inserting (4.22) into (4.21), we have

$$\|z(s)-z^*(s)\|_V^2 \leq \frac{2}{\mathbf{C}_0} \mathscr{M}_{\theta,f}(s,z) + \frac{T L_{\mathscr{H}}^2}{\mathbf{C}_0^2} \int_0^s \|z(t)-z^*(t)\|_V^2 dt$$

for all $s \in [0,T]$. Invoking Gronwall's inequality for the above inequality yields

$$\|z(s)-z^*(s)\|_V^2 \leq \frac{2}{\mathbf{C}_0} \mathscr{M}_{\theta,f}(s,z) + \frac{2T L_{\mathscr{H}}^2}{\mathbf{C}_0^3} \int_0^s \mathscr{M}_{\theta,f}(s,z).exp\left\{ \frac{T L_{\mathscr{H}}^2}{\mathbf{C}_0^2}(s-t) \right\} dt$$

for all $s \in [0,T]$. For each function $z \in C([0,T];K)$, let the function $\mathscr{E}_z^{\sharp\mathscr{M}} : [0,T] \to \mathbb{R}_+$ be defined by

$$\mathscr{E}_z^{\sharp\mathscr{M}}(s) := \sqrt{\frac{2}{\mathbf{C}_0} \mathscr{M}_{\theta,f}(s,z) + \frac{2T L_{\mathscr{H}}^2}{\mathbf{C}_0^3} \int_0^s \mathscr{M}_{\theta,f}(s,z).exp\left\{ \frac{T L_{\mathscr{H}}^2}{\mathbf{C}_0^2}(s-t) \right\} dt}$$

for all $s \in [0,T]$. We can see easily that $\mathscr{E}_z^{\sharp\mathscr{M}} \in L_+^\infty(0,T)$. Then we have

$$\|z(s)-z^*(s)\|_V \leq \mathscr{E}_z^{\sharp\mathscr{M}}(s) \quad \text{for all } s \in [0,T].$$

Therefore, we obtain inequality (4.14). □

Now, we conclude this section with the upper error bound for Problem 2.1 associated with the DG-function $\mathbf{D}_{\theta,\delta}^f$.

Theorem 4.3. *Let $z^* \in C([0,T];K)$ be the unique solution to Problem 2.1, $\widehat{L}_0$, and $\widehat{L}_1$ be defined by (4.2). Assume that the hypotheses of Theorem 4.1 hold. Then, for each $z \in C([0,T];K)$, we have the following upper error bound for Problem 2.1 controlled by the* DG*-function $\mathbf{D}_{\theta,\delta}^f$:*

$$\|z(s)-z^*(s)\|_V \leq \digamma_z^{\mathbf{D}}(s) \quad \text{for all } s \in [0,T], \tag{4.23}$$

where $\digamma_z^{\mathbf{D}} \in L_+^\infty(0,T)$ is defined by

$$\digamma_z^{\mathbf{D}}(s) := \sqrt{\frac{2\theta\delta\widehat{L}_1}{(\theta-\delta)\widehat{L}_0} \mathbf{D}_{\theta,\delta}^f(s,z) + \frac{2\theta\delta\widehat{L}_1 T L_{\mathscr{H}}}{(\theta-\delta)\widehat{L}_0^2} \int_0^s \mathbf{D}_{\theta,\delta}^f(s,z).exp\left\{ \frac{T L_{\mathscr{H}}}{\widehat{L}_0}(s-t) \right\} dt}$$

for all $z^* \in C([0,T];K)$ and all $s \in [0,T]$.

Proof. Let $z^* \in C([0,T];K)$ be the unique solution to Problem 2.1. For any $z \in C([0,T];K)$, it follows from (3.11) and (4.1) that

$$\|z(s)-z^*(s)\|_V^2 \leq \frac{2\theta\delta\widehat{L}_1}{(\theta-\delta)\widehat{L}_0} \mathbf{D}_{\theta,\delta}^f(s,z) + \frac{T L_{\mathscr{H}}}{\widehat{L}_0} \int_0^s \|z(t)-z^*(t)\|_V^2 dt \tag{4.24}$$

for all $s \in [0,T]$, where $\widehat{L}_0$ and $\widehat{L}_1$ are defined by (4.2). Using Gronwall's inequality for the inequality (4.24), we have

$$\begin{aligned}&\|z(s)-z^*(s)\|_V^2\\ &\leq \frac{2\theta\delta\widehat{L}_1}{(\theta-\delta)\widehat{L}_0}\mathbf{D}^f_{\theta,\delta}(s,z)+\frac{2\theta\delta\widehat{L}_1 T L_{\mathscr{H}}}{(\theta-\delta)\widehat{L}_0^2}\int_0^s \mathbf{D}^f_{\theta,\delta}(s,z).exp\left\{\frac{TL_{\mathscr{H}}}{\widehat{L}_0}(s-t)\right\}dt\end{aligned}$$

for all $s \in [0,T]$. For each function $z \in C([0,T];K)$, let the function $\digamma_z^{\mathbf{D}}\colon [0,T] \to \mathbb{R}_+$ be defined by

$$\digamma_z^{\mathbf{D}}(s) := \sqrt{\frac{2\theta\delta\widehat{L}_1}{(\theta-\delta)\widehat{L}_0}\mathbf{D}^f_{\theta,\delta}(s,z)+\frac{2\theta\delta\widehat{L}_1 T L_{\mathscr{H}}}{(\theta-\delta)\widehat{L}_0^2}\int_0^s \mathbf{D}^f_{\theta,\delta}(s,z).exp\left\{\frac{TL_{\mathscr{H}}}{\widehat{L}_0}(s-t)\right\}dt}$$

for all $s \in [0,T]$. Since $z \mapsto \mathbf{D}^f_{\theta,\delta}(s,z)$ belongs to $L_+^\infty(0,T)$ (Lemma 3.5), we can conclude that $\digamma_z^{\mathbf{D}} \in L_+^\infty(0,T)$. Then we obtain that

$$\|z(s)-z^*(s)\|_V \leq \digamma_z^{\mathbf{D}}(s) \quad \text{for all } s \in [0,T].$$

Therefore, inequality (4.23) is valid. □

Remark 4.1. (i) Like what we mentioned in Remark 3.2(i), the upper error bound established in Theorem 4.3 for Problem 2.1 controlled by DG-function $\mathbf{D}^f_{\theta,\delta}$ is new.
(ii) Furthermore, new upper error bounds in Theorem 4.1 and Theorem 4.2 controlled by the RG-function $\mathscr{M}_{\theta,f}$ are extensions of the corresponding ones in [3, Theorem 3.3].

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