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STRONG CONVERGENCE OF A PATH FOR CONVEX MINIMIZATION, GENERALIZED SPLIT FEASIBILITY, AND FIXED POINT PROBLEMS

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Dedicated to Professor Simeon Reich on the occasion of his 75th Birthday

Abstract. In this paper, we introduce a path for finding a common element of the set of minimizers of a convex function, the set of solutions of a generalized split feasibility problem, and the set of fixed points of a continuous pseudocontractive mapping in Hilbert spaces. Then we establish strong convergence of the path to a common element of these sets, which is a solution to a certain variational inequality. As a direct consequence, we obtain the unique minimum-norm common point of these sets.

Keywords. Convex minimization problem; Generalized split feasibility problem; Minimum-norm point; Pseudocontractive mapping.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and induced norm $\|\cdot\|$. Let C be a convex, closed, and nonempty subset of H, and let $T : C \to C$ be a self-mapping on set C. $Fix(T)$ is denoted by the set of fixed points of mapping T.

The minimization problem (shortly, MP) is one of most important problems in nonlinear analysis and optimization theory. The MP is defined as follows: find $x \in H$ such that

$$F(x) = \min_{y\in H} F(y), \tag{1.1}$$

where $F : H \to (-\infty,\infty]$ is a proper, convex, and lower semi-continuous. The set of MP(1.1), that is, the set of all minimizers of F is denoted by $\arg\min_{y\in H} F(y)$. A successful and powerful tool for solving MP (1.1) is well-known proximal point algorithm (shortly, the PPA) which was initiated by Martinet [1] and later studied by Rockafellar [2] in 1976.

Let D and Q be convex, closed, and nonempty subsets of two Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded and linear operator. Then the split feasibility problem (SFP) is to find a point $z \in H_1$ such that $z \in D \cap A^{-1}Q$. In 1994, the SFP was first investigated by Censor and Elfving [3], in finite-dimensional Hilbert spaces, for some inverse problems. Since 1994, the problem has been under the spotlight due to its various applications in medical image reconstruction, intensity-modulated radiation therapy(IMRT), control theory, biomedical engineering, communications and geophysics; see, e.g., [3, 4, 5, 6] and the references therein.

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In 2015, Takahashi et al. [7] considered the following generalized split feasibility problem (GSFP):

$$\text{find a point } x^* \in H_1 \text{ such that } 0 \in B(x^*), \tag{1.2}$$

and

$$y^* = Ax^* \in H_2 \text{ solves } y^* = S(y^*), \tag{1.3}$$

where $B : H_1 \to 2^{H_1}$ is a multi-valued maximal monotone mapping; $S : H_2 \to H_2$ is a nonexpansive mapping; $A : H_1 \to H_2$ is a bounded linear operator. GSFP (1.2)-(1.3) includes, as special cases, several split problems, such as the split zero problem (SZP), the split variational inclusion problem (SVIP), the SFP, and split common fixed point problem (SCFPP); see, e.g., [5, 8, 9, 10, 11, 12, 13, 14, 15] and the references therein.

A fixed point problem (FPP) is to find a fixed point z of a nonlinear mapping T with property:

$$z \in C, \ Tz = z. \tag{1.4}$$

Fixed point theory is one of the most powerful and important analysis tools of modern mathematics and may be considered a core subject of pure ad applied nonlinear analysis.

In particular, in 2015, using a generalized hybrid mapping U, Takahashi et al. [7] considered the GSFP (1.2)-(1.3) combined with FPP (1.4) for a nonexpansive mapping S and introduced an iterative algorithm for finding a common element of the solution set of GSFP (1.2)-(1.3) and the fixed point set $Fix(S)$ of S in an explicit way. In 2020, replacing a generalized hybrid mapping U and a nonexpansive mapping S in [7] by a continuous pseudocontractive mapping R and a continuous pseudocontractive mapping T, respectively, Jung [16] proposed an iterative algorithm based on Yamada's hybrid steepest descent method [17] finding a common element of the solution set of GSFP (1.2)-(1.3) and the fixed point set $Fix(T)$ of T for a continuous pseudocontractive mapping T.

In this paper, in order to study the MP (1.1) combined with the GSFP (1.2)-(1.3) and the FPP (1.4) in Hilbert spaces, we introduce a new path based on the hybrid steepest descent method for finding a common element of the minimizer set $\arg\min_{y\in H_1} F(y)$ of the MP(1.1) for F, the solution set $B^{-1}0 \cap A^{-1}(Fix(R))$ of the GSFP (1.2)-(1.3) and the fixed point set $Fix(T)$ of T, where $F : H_1 \to (-\infty, \infty]$ is a proper convex and lower semi-continuous function; $B : H_1 \to 2^{H_1}$ is a maximal monotone mapping; $A : H_1 \to H_2$ is a bounded linear operator; $R : H_2 \to H_2$ is a continuous pseudocontractive mapping; and $T : H_1 \to H_1$ is a continuous pseudocontractive mapping. Then we establish strong convergence of the path to a common element of $\Omega := \arg\min_{y\in H_1} F(y) \cap B^{-1}0 \cap A^{-1}(Fix(R)) \cap Fix(T)$, which is a solution to a certain variational inequality. As a direct consequence, we find the unique solution of the minimization-norm problem: $\|x^*\| = \min\{\|x\| : x \in \Omega\}$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, and let C be a nonempty, convex and closed subset of H.

A mapping A of C into H is called *monotone* if

$$\langle x - y, Ax - Ay \rangle \geq 0, \ \ \forall x, \ y \in C.$$

A mapping A of C into H is called *α-inverse-strongly monotone* (or, *α-ism*) if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2, \quad \forall x,\ y\in C.$$

Clearly, the class of monotone mappings includes the class of α-inverse-strongly monotone mappings .

Let B be a set-valued mapping of H into 2^H. The effective domain of mapping B is denoted by $dom(B)=\{x\in H : Bx\neq\emptyset\}$. Recall that mapping B is said to be a *monotone operator* on H if $\langle x-y, u-v\rangle\geq 0$ for all $x,\ y\in dom(B)$, $u\in Bx$, and $v\in By$. B is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $\lambda>0$, one may define a single-valued operator $J_\lambda^B=(I+\lambda B)^{-1}: H\to dom(B)$, which is called the *resolvent* of B. Denote the set of zero points of B by $B^{-1}0=\{x\in H : 0\in Bx\}$. It is well-known ([18, 19]) that $B^{-1}0=Fix(J_\lambda^B)$, for all $\lambda>0$, is convex and closed, and the resolvent J_λ^B satisfies

$$\langle x-y, J_\lambda^B x-J_\lambda^B y\rangle\geq\|J_\lambda^B x-J_\lambda^B y\|^2, \quad \forall x,\ y\in H, \tag{2.1}$$

that is, it is firmly nonexpansive, and that the resolvent identity

$$J_\lambda^B x=J_\mu^B\left(\frac{\mu}{\lambda}x+\left(1-\frac{\mu}{\lambda}\right)J_\lambda^B x\right) \tag{2.2}$$

holds for all $\lambda,\ \mu>0$ and $x\in H$.

In a real Hilbert space H, the following equality hold:

$$\|x-y\|^2=\|x\|^2+\|y\|^2-2\langle x,y\rangle, \quad \forall x,\ y\in H. \tag{2.3}$$

It is also known that every nonexpansive mapping $T: H\to H$ satisfies, for all $(x,y)\in H\times H$, the inequality

$$\langle (x-Tx)-(y-Ty), Ty-Tx\rangle\leq\frac{1}{2}\|(Tx-x)-(Ty-y)\|^2,$$

and hence, for all $(x,y)\in H\times Fix(T)$,

$$\langle x-Tx, y-Tx\rangle\leq\frac{1}{2}\|Tx-x\|^2. \tag{2.4}$$

A mapping $T: H\to H$ is said to be *averaged* if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T=(1-\alpha)I+\alpha S, \tag{2.5}$$

where α is a number in $(0,1)$ and $S: H\to H$ is nonexpansive. More precisely, when (2.5) holds, we say that T is α-averaged ([20]).

We note that averaged mappings are nonexpansive, Further firmly nonexpansive mappings (in particular, projections and resolvents of maximal monotone operators) are averaged.

The following lemmas were given in [7].

Lemma 2.1. [7] *Let H_1 and H_2 be real Hilbert spaces. Let $A: H_1\to H_2$ be a linear and bounded operator such that $A\neq 0$, and let A^* be the adjoint of A. Let L is the spectral radius of the operator A^*A. Let $T: H_2\to H_2$ be a nonexpansive mapping. Then*

(i) *$\eta A^*(I-T)A$ is $\frac{1}{2\eta L}$-ism.*

(ii) *For* $\eta \in (0, \frac{1}{L})$,

(iia) $I + \eta A^*(T - I)A$ *is* ηL*-averaged ;*

(iib) $J_\lambda^B(I + \eta A^*(T - I)A)$ *is* $\frac{1+\eta L}{2}$*-averaged.*

Lemma 2.2. [7] *Let* H_1 *and* H_2 *be real Hilbert spaces. Let* $B : H_1 \to 2^{H_1}$ *be a maximal monotone operator, and let* $J_\lambda^B = (I + \lambda B)^{-1}$ *be the resolvent of* B *for* $\lambda > 0$*. Let* $A : H_1 \to H_2$ *be a linear and bounded operator such that* $A \neq 0$*, and let* A^* *be the adjoint of* A*. Let* $T : H_2 \to H_2$ *be a nonexpansive mapping. Suppose that* $B^{-1}0 \cap A^{-1}(Fix(T)) \neq \emptyset$*. Let* $\lambda, \eta > 0$ *and* $z \in H_1$*. Then the following are equivalent*:

(i) $z = J_\lambda^B(I + \eta A^*(T - I)A)z$;

(ii) $0 \in -A^*(T - I)Az + Bz$;

(iii) $z \in B^{-1}0 \cap A^{-1}(Fix(T))$.

Consequently, $Fix(J_\lambda^B(I + \eta A^*(T - I)A)) = (-A^*(T - I)A + B)^{-1}0 = B^{-1}0 \cap A^{-1}(Fix(T))$*. Moreover, if* $0 \in -A^*(T - I)Au + Bu$ *and* $0 \in -A^*(T - I)Av + Bv$*, then* $A^*(T - I)Au = A^*(T - I)Av$ *and* $(-A^*(T - I)A + B)^{-1}0$ *is closed and convex.*

We recall that

(i) a mapping $V : C \to H$ is said to be *l-Lipschitzian* if there exists a constant $l \geq 0$ such that
$$\|Vx - Vy\| \leq l\|x - y\| \ \text{ for all } x,\ y \in C;$$

(ii) a mapping $G : C \to H$ is said to be ρ*-strongly monotone* if there exists a constant $\rho > 0$ such that
$$\langle Gx - Gy, x - y\rangle \geq \rho\|x - y\|^2 \ \text{ for all } x,\ y \in C;$$

(iii) a mapping $T : C \to H$ is said to be *pseudocontractive* if
$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \ \text{ for all } x,\ y \in C;$$

(iv) a mapping $T : C \to H$ is said to be *k-strictly pseudocontractive* ([21]) if there exists a constant $k \in [0, 1)$ such that
$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \ \text{ for all } x,\ y \in C;$$

(v) a mapping $T : C \to H$ is said to be *nonexpansive* if
$$\|Tx - Ty\| \leq \|x - y\|, \ \text{ for all } x,\ y \in C$$
where I is the identity mapping.

Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings and the class of nonexpansive mappings as a subclass. Moreover, this inclusion is strict (see Example 5.7.1 and Example 5.7.2 in [22]).

The following lemma is due to [23].

Lemma 2.3. *Let* C *be a convex and clsoed subset of a real Hilbert space* H*. Let* $T : C \to C$ *be a continuous pseudocontractive mapping. Then, for* $r > 0$ *and* $x \in H$*, there exists* $z \in C$ *such that* $r\langle y - z, Tz\rangle - \langle y - z, (1 + r)z - x\rangle \leq 0$ *for all* $y \in C$*. For* $r > 0$ *and* $x \in H$*, define* $T_r : H \to C$ *by*
$$T_r x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C \right\}.$$

Then the following assertions hold:

(i) *T_r is single-valued*;
(ii) *T_r is firmly nonexpansive;*
(iii) *$Fix(T_r) = Fix(T)$ is a closed convex subset of C.*

The following lemma is a variant of a Minty lemma (see [24]).

Lemma 2.4. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Assume that the mapping $G : C \to H$ is monotone and weakly continuous along segments, that is, $G(x+ty) \to G(x)$ weakly as $t \to 0$. Then the variational inequality*

$$\widetilde{x} \in C, \quad \langle G\widetilde{x}, p - \widetilde{x}\rangle \geq 0 \ \text{ for all } p \in C,$$

is equivalent to the dual variational inequality

$$\widetilde{x} \in C, \quad \langle Gp, p - \widetilde{x}\rangle \geq 0 \ \text{ for all } p \in C.$$

The following lemmas can be easily proven (see [17]), and therefore, we omit their proof.

Lemma 2.5. *Let H be a real Hilbert space. Let $V : H \to H$ be an l-Lipschitzian mapping with a constant $l \geq 0$, and let $G : H \to H$ be a κ-Lipschitzian and ρ-strongly monotone mapping with constants κ, $\rho > 0$. Then, for $0 \leq \gamma l < \mu\rho$,*

$$\langle(\mu G - \gamma V)x - (\mu G - \gamma V)y, x - y\rangle \geq (\mu\rho - \gamma l)\|x - y\|^2 \ \text{ for all } x,\ y \in H.$$

That is, $\mu G - \gamma V$ is strongly monotone with constant $\mu\rho - \gamma l$.

Lemma 2.6. *Let H be a real Hilbert space H. Let $G : H \to H$ be a κ-Lipschitzian and ρ-strongly monotone mapping with constants $\kappa > 0$ and $\rho > 0$. Let $0 < \mu < \frac{2\rho}{\kappa^2}$ and $0 < t < 1$. Then $I - t\mu G : H \to H$ is a contractive mapping with a constant $1 - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\rho - \mu\kappa^2)}$.*

Lemma 2.7 ([25])**.** *Assume that T is nonexpansive self mapping of a closed convex subset of C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ and the sequence $\{(I - T)x_n\}$ converges strongly to some y, it follows that $(I - T)x = y$. Here I is the identity mapping H.*

Let $F : H \to (-\infty, \infty]$ be a proper, convex, and lower semi-continuous function. For any $\delta > 0$, define the Moreau-Yosida resolvent of F in a real Hilbert space H as follows:

$$J_\delta^F x = \arg\min_{y \in H}\left[F(y) + \frac{1}{2\delta}\|x - y\|^2\right] \tag{2.6}$$

for all $x \in H$. It was demonstrated in [26] that the set of fixed points of the resolvent associated with F coincides with the set of minimizers of F. Also the resolvent J_δ^F of F is single-valued and nonexpansive as firmly nonexpansive for all $\delta > 0$. It is also well-known ([27]) that resolvent identity (2.2) holds, that is, for any $r > 0$ and $\mu > 0$, the following holds:

$$J_r^F x = J_\mu^F\left(\frac{\mu}{r}x + \left(1 - \frac{\mu}{r}\right)J_r^F x\right). \tag{2.7}$$

In the following, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

3. Main Results

Throughout the rest of this paper, we always assume the following:

- H_1 and H_2 are real Hilbert spaces with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$;
- $F: H_1 \to (-\infty,\infty]$ is a proper convex and lower semi-continuous;
- $\arg\min_{y\in H_1} F(y)$ is the set of the MP(1.1), that is, the set of all minimizers of F;
- $A: H_1 \to H_2$ is a bounded linear operator;
- $A^*: H_2 \to H_1$ is the adjoint of A;
- L is the spectral radius of the operator A^*A
- $B: H_1 \to 2^{H_1}$ is a maximal monotone operator with $dom(B) \subset H_1$;
- $B^{-1}0$ is the set of zero points of B, that is, $B^{-1}0 = \{z \in H_1 : 0 \in Bz\}$;
- $J^B_{\lambda_t}: H_1 \to \mathrm{dom}(B)$ is the resolvent of B for $\lambda_t \in (0,\infty)$ and $\liminf_{t\to 0}\lambda_t > 0$;
- $G: H_1 \to H_1$ is a κ-Lipschitzian and ρ-strongly monotone mapping with constants $\kappa,\ \rho > 0$;
- $V: H_1 \to H_1$ is an l-Lipschitzian mapping with constant $l \in [0,\infty)$;
- Constants $\mu > 0$ and $\gamma \geq 0$ satisfy $0 < \mu < \frac{2\rho}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1-\sqrt{1-\mu(2\rho-\mu\kappa^2)}$;
- $T: H_1 \to H_1$ is a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$;
- $T_{r_t}: H_1 \to H_1$ is a mapping defined by

$$T_{r_t}x = \left\{z \in H_1 : \langle Tz, y-z\rangle - \frac{1}{r_t}\langle y-z, (1+r_t)z-x\rangle \leq 0, \quad \forall y \in H_1\right\}$$

  for $x \in H_1$ and $r_t \in (0,\infty)$, $t \in (0,1)$ and $\liminf_{t\to 0} r_t > 0$;
- $R: H_2 \to H_2$ is a continuous pseudocontractive mapping with $Fix(R) \neq \emptyset$;
- $R_{\alpha_t}: H_2 \to H_2$ is a mapping defined by

$$R_{\alpha_t}x = \left\{z \in H_2 : \langle Rz, y-z\rangle - \frac{1}{\alpha_t}\langle y-z, (1+\nu_t)z-x\rangle \leq 0, \quad \forall y \in H_2\right\}$$

  for $x \in H_2$ and $\alpha_t \in (0,\infty)$, and $\liminf_{t\to 0}\alpha_t > 0$;
- $\Omega := \arg\min_{y\in H_1} F(y) \cap B^{-1}0 \cap A^{-1}(Fix(R)) \cap Fix(T) \neq \emptyset$.

By Lemma 2.3, we note that T_{r_t} and R_{α_t} are firmly nonexpansive and hence nonexpansive, and $Fix(T_{r_t}) = Fix(T)$ and $Fix(R_{\alpha_t}) = Fix(R)$.

Now, we introduce the following path $t \to x_t$, $0 < t < 1$, defined by

$$\begin{cases} v_t &= \arg\min_{y\in H_1}[F(y) + \frac{1}{2\delta_t}\|x_t - y\|^2], \\ z_t &= J^B_{\lambda_t}(v_t + \eta_t A^*(R_{\alpha_t} - I)Av_t) \\ x_t &= t\gamma Vx_t + (I - t\mu G)(\theta_t x_t + (1-\theta_t)T_{r_t}z_t), \end{cases} \tag{3.1}$$

where $\delta_t, r_t,\ \lambda_t,\ \alpha_t \in (0,\infty)$, $\theta_t \in (0,1)$ and $\eta_t \in (0,\frac{1}{L})$ for $t \in (0,1)$. From (2.6), we note that $v_t = J^F_{\delta_t}x_t$.

For $x \in H_1$ and $t \in (0,1)$, consider the following mappings Q_t and W_t on H_1 defined by, for $x \in H_1$,

$$\begin{aligned} W_t x &= \theta_t x + (1-\theta_t)T_{r_t}J^B_{\lambda_t}(J^F_{\delta_t}x + \eta_t A^*(R_{\alpha_t} - I)AJ^F_{\delta_t}x) \\ &= \theta_t x + (1-\theta_t)T_{r_t}J^B_{\lambda_t}(I + \eta_t A^*(R_{\alpha_t} - I)A)J^F_{\delta_t}x \end{aligned}$$

and

$$Q_t x = t\gamma V x + (I - t\mu G)W_t x.$$

Since $J^B_{\lambda_t}$ and R_{α_t} are firmly nonexpansive (see (2.1)), they are averaged. For $\eta_t \in (0, \frac{1}{L})$ for $t \in (0,1)$, the mapping $I + \eta_t A^*(R_{\alpha_t} - I)A$ is averaged (see Lemma 2.2 (ii))). As a composite of averaged mappings, it follows that the mapping $J^B_{\lambda_t}(I + \eta_t A^*(R_{\alpha_t} - I)A)$ is averaged and hence nonexpansive. Noting that $J^F_{\delta_t}$ and T_{r_t} are nonexpansive, we have, for $x,\ y \in H_1$,

$$\|W_t x - W_t y\| \le \ \theta_t\|x - y\| + (1 - \theta_t)\|x - y\| = \|x - y\|.$$

and

$$\begin{aligned}\|Q_t x - Q_t y\| &= \|t\gamma V x + (I - t\mu G)W_t x - T_{r_t}(t\gamma V y + (I - t\mu G)W_t y\| \\ &\le t\|\gamma V x - \gamma V y\| + \|(I - t\mu G)W_t x - (I - t\mu G)W_t y\| \\ &\le t\gamma l\|x - y\| + (1 - t\tau)\|x - y\| \\ &= (1 - (\tau - \gamma l)t)\|x - y\|.\end{aligned}$$

Since $0 < 1 - (\tau - \gamma l)t < 1$, Q_t is a contractive mapping. By Banach contraction principle, Q_t has a unique fixed point $x_t \in H_1$, which uniquely solves the fixed point equation

$$\begin{aligned}x_t \ &= t\gamma V x_t + (I - t\mu G)(\theta_t x_t + (1 - \theta_t)T_{r_t} z_t), \\ &= t\gamma V x_t + (I - t\mu G)(\theta_t x_t + (1 - \theta_t)T_{r_t} J^B_{\lambda_t}(I + \eta_t A^*(R_{\alpha_t} - I)A)J^F_{\delta_t} x_t),\ t \in (0,1).\end{aligned}$$

We summarize the basic property of $\{x_t\}$, $\{v_t\}$, $\{u_t\}$, $\{z_t\}$ and $\{y_t\}$, where $v_t = J^F_{\delta_t} x_t$, $u_t = (I + \eta_t A^*(R_{\alpha_t} - I)A)v_t$, $z_t = J^B_{\lambda_t} u_t$ and $y_t = \theta_t x_t + (1 - \theta_t)T_{r_t} z_t$.

Proposition 3.1. *Let the path $\{x_t\}$ be defined by (3.1). Let $\{v_t\}$, $\{u_t\}$, $\{z_t\}$, and $\{y_t\}$ be defined by $v_t = J^F_{\delta_t} x_t$, $u_t = (I + \eta_t A^*(R_{\alpha_t} - I)A)v_t$, $z_t = J^B_{\lambda_t} u_t$, and $y_t = \theta_t x_t + (1 - \theta_t)T_{r_t} z_t$, respectively. Then*

(1) *$\{x_t\}$ and $\{y_t\}$ are bounded for $t \in (0,1)$;*
(2) *x_t defines a continuous path from $(0,1)$ into H_1 and so does y_t provided $\delta_t, r_t,\ \lambda_t,\ \alpha_t : (0,1) \to (0,\infty)$ are continuous, and $\eta_t : (0,1) \to (0, \frac{1}{L})$ is continuous with $0 < \delta \le \delta_t$, $0 < r \le r_t$, $0 < \lambda \le \lambda_t$, $0 < \alpha \le \alpha_t$ and $0 < \eta \le \eta_t$ for $t \in (0,1)$, and $\theta_t : (0,1) \to (0,1)$ is continuous with $0 < \lim_{t\to 0}\theta_t < 1$;*
(3) $\lim_{t\to 0}\|x_t - T_{r_t} z_t\| == 0$;
(4) $\lim_{t\to 0}\|v_t - x_t\| = \lim_{t\to 0}\|J^F_{\delta_t} x_t - x_t\| = 0$;
(5) $\lim_{t\to 0}\|u_t - J^B_{\lambda_t} u_t\| = \lim_{t\to 0}\|u_t - z_t\| = 0$;
(6) $\lim_{t\to 0}\|u_t - v_t\| = \lim_{t\to 0}\|u_t - J^F_{\delta_t} x_t\| = 0$;
(7) $\lim_{t\to 0}\|z_t - T_{r_t} z_t\| = 0$;
(8) $\lim_{t\to 0}\|x_t - z_t\| = 0$;
(9) $\lim_{t\to 0}\|x_t - u_t\| = 0$;
(10) $\lim_{t\to 0}\|x_t - T_{r_t} x_t\| = 0$;
(11) $\lim_{t\to 0}\|x_t - J^B_{\lambda_t} x_t\| = 0$.

Proof. From now, we put $K_t = I + \eta_t A^*(R_{\alpha_t} - I)A$ and $u_t = K_t v_t$ for $t \in (0,1)$. Let $p \in \Omega$.

(1) First, we note that $p = T_{r_t}p$, $p = J^F_{\delta_t}p$, $p = J^B_{\lambda_t}p$, $R_{\alpha_t}(Ap) = Ap$, $K_tp = (I + \eta_t A^*(R_{\alpha_t} - I)A)p$, and $J^B_{\lambda_t}((I + \eta_t A^*(R_{\alpha_t} - I)A)p) = J^B_{\lambda_t}(K_tp)$. Since

$$\begin{aligned} \|z_t - p\|^2 &= \|J^B_{\lambda_t}(v_t + \eta_t A^*(R_{\alpha_t} - I)Av_t) - J^B_{\lambda_t}p\|^2 \\ &\le \|v_t + \eta_t A^*(R_{\alpha_t} - I)Av_t - p\|^2 \\ &= \|v_t - p\|^2 + \eta_t^2\|A^*(R_{\alpha_t} - I)Av_t\|^2 + 2\eta_t\langle v_t - p, A^*(R_{\alpha_t} - I)Av_t\rangle, \end{aligned} \tag{3.2}$$

we have

$$\begin{aligned} \|z_t - p\|^2 &\le \|v_t + \eta_t A^*(R_{\alpha_t} - I)Av_t - p\|^2 \\ &= \|v_t - p\|^2 + \eta_t^2\langle (R_{\alpha_t} - I)Av_t, AA^*(R_{\alpha_t} - I)Av_t\rangle \\ &\quad + 2\eta_t\langle v_t - p, A^*(R_{\alpha_t} - I)Av_t\rangle. \end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned} \eta_t^2\langle (R_{\alpha_t} - I)Av_t, AA^*(R_{\alpha_t} - I)Av_t\rangle &\le L\eta_t^2\langle (R_{\alpha_t} - I)Av_t, (R_{\alpha_t} - I)Av_t\rangle \\ &= L\eta^2\|(R_{\alpha_t} - I)Av_t\|^2. \end{aligned} \tag{3.4}$$

Moreover, from (2.4), we obtain

$$\begin{aligned} &2\eta_t\langle v_t - p, A^*(R_{\alpha_t} - I)Av_t\rangle \\ &= 2\eta_t\langle A(v_t - p), (R_{\alpha_t} - I)Av_t\rangle \\ &= 2\eta_t\langle A(v_t - p) + (R_{\alpha_t} - I)Av_t - (R_{\alpha_t} - I)Av_t, (R_{\alpha_t} - I)Av_t\rangle \\ &= 2\eta_t[\langle (R_{\alpha_t}(Av_t) - Ap, (R_{\alpha_t} - I)Av_t\rangle - \|(R_{\alpha_t} - I)Av_t\|^2] \\ &\le 2\eta\left(\frac{1}{2}\|(R_{\alpha_t} - I)Av_t\|^2 - \|(R_{\alpha_t} - I)Av_t\|^2\right) \\ &= -\eta_t\|(R_{\alpha_t} - I)Av_t\|^2. \end{aligned} \tag{3.5}$$

Therefore, from (3.2), (3.3), (3.4), and (3.5), we derive

$$\begin{aligned} \|z_t - p\|^2 &\le \|u_t - p\|^2 \\ &\le \|v_t - p\|^2 + \eta_t(L\eta_t - 1)\|(R_{\alpha_t} - I)Av_t\|^2 \\ &\le \|v_t - p\|^2 \ \ (\text{by } \eta_t \in (0, \frac{1}{L})). \end{aligned} \tag{3.6}$$

We also have

$$\|v_t - p\| = \|J^F_{\delta_t}x_t - J^F_{\delta_t}p\| \le \|x_t - p\|. \tag{3.7}$$

Observing that the mapping $K_t = I + \eta_t A^*(R_{\alpha_t} - I)A$ and $J^B_{\lambda_t}(I + \eta_t A^*(R_{\alpha_t} - I)A)$ both are nonexpansive as averaged (Lemma 2.1 (ii)) and T_{r_t} is nonexpansive, from (3.6) and (3.7), we derive

$$\begin{aligned} \|y_t - p\| &\le \theta_t\|x_t - p\| + (1 - \theta_t)\|T_{r_t}z_t - T_{r_t}p\| \\ &\le \theta_t\|x_t - p\| + (1 - \theta_t)\|z_t - p\| \\ &= \theta_t\|x_t - p\| + (1 - \theta_t)\|J^B_{\lambda_t}u_t - p\| \\ &\le \theta_t\|x_t - p\| + (1 - \theta_t)\|u_t - p\| \\ &\le \theta_t\|x_t - p\| + (1 - \theta_t)\|v_t - p\| \\ &\le \|x_t - p\|. \end{aligned} \tag{3.8}$$

Therefore, it follows from (3.1), (3.8), and Lemma 2.6 that

$$\begin{aligned}\|x_t - p\| &\le t\|\gamma V x_t - \gamma V p\| + \|(I - t\mu G)y_t - (I - t\mu G)p\| + t\|\gamma V p - \mu G p\| \\ &\le t\gamma l\|x_t - p\| + (1 - t\tau)\|y_t - p\| + t(\gamma\|Vp\| + \mu\|Gp\|) \\ &\le (1 - (\tau - \gamma l)t)\|x_t - p\| + t(\gamma\|Vp\| + \mu\|Gp\|).\end{aligned}$$

So, we obtain

$$\|x_t - p\| \le \frac{\gamma\|Vp\| + \mu\|Gp\|}{\tau - \gamma l}.$$

Hence $\{x_t\}$ is bounded and so are $\{y_t\}$, $\{v_t\}$, $\{Vx_t\}$, $\{z_t\}$, $\{T_{r_t} z_t\}$ $\{u_t\}$, $\{Gy_t\}$, and $\{J^B_{\lambda_t} u_t\}$.

(2) Let $t,\ t_0 \in (0,1)$. Since $v_t = J^F_{\delta_t} x_t$ and $v_{t_0} = J^F_{\delta_{t_0}} x_{t_0}$, we derive from (2.7) that

$$\begin{aligned}\|v_t - v_{t_0}\| &= \left\| J^F_{\delta_{t_0}}\left(\frac{\delta_{t_0}}{\delta_t} x_t + \left(1 - \frac{\delta_{t_0}}{\delta_t}\right) J^F_{\delta_t} x_t\right) - J^F_{\delta_{t_0}} x_{t_0}\right\| \\ &\le \left\| \frac{\delta_{t_0}}{\delta_t}(x_t - x_{t_0}) + \left(1 - \frac{\delta_{t_0}}{\delta_t}\right)(J^F_{\delta_t} x_t - x_{t_0})\right\| \\ &\le \|x_t - x_{t_0}\| + \frac{|\delta_t - \delta_{t_0}|}{\delta}\|J^F_{\delta_t} x_t - x_t\| \\ &\le \|x_t - x_{t_0}\| + \frac{|\delta_t - \delta_{t_0}|}{\delta} M_1,\end{aligned} \tag{3.9}$$

where $M_1 > 0$ is an appropriate constant. From (2.2), we induce that

$$\begin{aligned}\|z_t - z_{t_0}\| &= \left\| J^B_{\lambda_{t_0}}\left(\frac{\lambda_{t_0}}{\lambda_t} u_t + \left(1 - \frac{\lambda_{t_0}}{\lambda_t}\right) J^B_{\lambda_t} u_t\right) - J^B_{\lambda_{t_0}} u_{t_0}\right\| \\ &\le \left\| \frac{\lambda_{t_0}}{\lambda_t}(u_t - u_{t_0}) + \left(1 - \frac{\lambda_{t_0}}{\lambda_t}\right)(J^B_{\lambda_t} u_t - u_{t_0})\right\| \\ &\le \|u_t - u_{t_0}\| + \frac{\lambda_t - \lambda_{t_0}|}{\lambda_t}\|J^B_{\lambda_t} u_t - u_t\| \\ &\le \|u_t - u_{t_0}\| + \frac{|\lambda_t - \lambda_{t_0}|}{\lambda} M_2,\end{aligned} \tag{3.10}$$

where $M_2 > 0$ is an appropriate constant. Again, since $K_t = I + \eta_t A^*(R_{\alpha_t} - I)A$ is nonexapnsive as averaged (Lemma 2.1 (ii)), we calculate that

$$\begin{aligned}\|u_t - u_{t_0}\| &= \|(I + \eta_t A^*(R_{\alpha_t} - I)A)v_t - (I + \eta_{t_0} A^*(R_{\alpha_{t_0}} - I)A)v_{t_0}\| \\ &= \|K_t v_t - K_{t_0} v_{t_0}\| \\ &\le \|K_t v_t - K_t v_{t_0}\| + \|K_t v_{t_0} - K_{t_0} v_{t_0}\| \\ &\le \|v_t - v_{t_0}\| + \|(v_{t_0} + \eta_t A^*(R_{\alpha_t} - I)Av_{t_0}) - (v_{t_0} + \eta_{t_0} A^*(R_{v_{t_0}} - I)Av_{t_0})\| \\ &\le \|v_t - v_{t_0}\| + \|\eta_t A^*(R_{\alpha_t} - I)Av_{t_0} - \eta_{t_0} A^*(R_{\alpha_t} - I)Av_{t_0}\| \\ &\quad + \|\eta_{t_0} A^*(R_{\alpha_t} - I)Av_{t_0} - \eta_{t_0} A^*(R_{\alpha_{t_0}} - I)Av_{t_0}\| \\ &\le \|v_t - v_{t_0}\| + |\eta_t - \eta_{t_0}|\|A^*(R_{\alpha_t} - I)Av_{t_0}\| \\ &\quad + \|\eta_{t_0} A^*(R_{\alpha_t}(Av_{t_0}) - R_{\alpha_{t_0}}(Av_{t_0}))\| \\ &\le \|v_t - v_{t_0}\| + |\eta_t - \eta_{t_0}| M_3 + \frac{1}{L}\|A^*\|\|R_{\alpha_t}(Av_{t_0}) - R_{\alpha_{t_0}}(Av_{t_0})\|,\end{aligned} \tag{3.11}$$

where $M_3 > 0$ is an appropriate constant. Let $R_{\alpha_t}(Av_{t_0}) = d'_t$ and $R_{\alpha_{t_0}}(Av_{t_0}) = d_{t_0}$. Then, by Lemma 2.3, we obtain

$$\langle y - d'_t, Rd'_t\rangle - \frac{1}{\alpha_t}\langle y - d'_t, (1+\alpha_t)d'_t - Av_{t_0}\rangle \le 0 \text{ for all } y \in H_2 \tag{3.12}$$

and

$$\langle y - d_{t_0}, Rd_{t_0}\rangle - \frac{1}{\alpha_{t_0}}\langle y - d_{t_0}, (1+\alpha_{t_0})d_{t_0} - Av_{t_0}\rangle \le 0 \text{ for all } y \in H_2. \tag{3.13}$$

Putting $y = d_{t_0}$ in (3.12) and $y = d'_t$ in (3.13), we obtain

$$\langle d_{t_0} - d'_t, Rd'_t\rangle - \frac{1}{\alpha_t}\langle d_{t_0} - d'_t, (1+\alpha_t)d'_t - Av_{t_0}\rangle \le 0 \tag{3.14}$$

and

$$\langle d'_t - d_{t_0}, Rd_{t_0}\rangle - \frac{1}{\alpha_{t_0}}\langle d'_t - d_{t_0}, (1+\alpha_{t_0})d_{t_0} - Av_{t_0}\rangle \le 0. \tag{3.15}$$

Adding up (3.14) and (3.15), we obtain

$$\langle d_{t_0} - d'_t, Rd'_t - Rd_{t_0}\rangle - \langle d_{t_0} - d'_t, \frac{(1+\alpha_t)d'_t - Av_{t_0}}{\alpha_t} - \frac{(1+\alpha_{t_0})d_{t_0} - Av_{t_0}}{\alpha_{t_0}}\rangle \le 0. \tag{3.16}$$

Since R is pseudocontractive, by (3.16), we deduce

$$\left\langle d_{t_0} - d'_t, \frac{d'_t - Av_{t_0}}{\alpha_t} - \frac{d_{t_0} - Av_{t_0}}{\alpha_{t_0}} \right\rangle \ge 0,$$

and hence

$$\langle d_{t_0} - d'_t, d'_t - d_{t_0} + d_{t_0} - Av_{t_0} - \frac{\alpha_t}{\alpha_{t_0}}(d_{t_0} - Av_{t_0})\rangle \ge 0. \tag{3.17}$$

From (3.17), we derive

$$\begin{aligned}\|d'_t - d_{t_0}\|^2 &\le \left\langle d_{t_0} - d'_t, \left(1 - \frac{\alpha_t}{\alpha_{t_0}}\right)(d_{t_0} - Av_{t_0}) \right\rangle \\ &\le \|d_{t_0} - d'_1\| \left|\frac{\alpha_t - \alpha_{t_0}}{\alpha_{t_0}}\right| \|d_{t_0} - Av_{t_0}\|,\end{aligned}$$

and hence

$$\|R_{\alpha_t}(Av_{t_0}) - R_{\alpha_{t_0}}(Av_{t_0})\| = \|d'_t - d_{t_0}\| \le |\alpha_t - \alpha_{t_0}|\frac{M_4}{\alpha}, \tag{3.18}$$

where $M_4 > 0$ is an appropriate constant,

Now, substituting (3.9) and (3.18) into (3.11), we have

$$\begin{aligned}\|u_t - u_{t_0}\| &\le \|v_t - v_{t_0}\| + |\eta_t - \eta_{t_0}|M_3 + \frac{1}{L}\|A^*\||\alpha_t - \alpha_{t_0}|\frac{M_4}{\alpha} \\ &\le \|x_t - x_{t_0}\| + |\delta_t - \delta_{t_0}|\frac{M_1}{\delta} + |\eta_t - \eta_{t_0}|M_3 + \frac{1}{L}\|A^*\||\alpha_t - \alpha_{t_0}|\frac{M_4}{\alpha}.\end{aligned} \tag{3.19}$$

On another hand, let $w_t = T_{r_t}z_t$ and $w_{t_0} = T_{r_{t_0}}z_{t_0}$. Then, from Lemma 2.3, we see that

$$\langle y - w_{t_0}, Tw_{t_0}\rangle - \frac{1}{r_{t_0}}\langle y - w_{t_0}, (1+r_{t_0})w_{t_0} - z_{t_0}\rangle \le 0 \text{ for all } y \in H_1, \tag{3.20}$$

and

$$\langle y - w_t, Tw_t\rangle - \frac{1}{r_t}\langle y - w_t, (1+r_t)w_t - z_t\rangle \le 0 \text{ for all } y \in H_1. \tag{3.21}$$

Putting $y = w_t$ in (3.20) and $y = w_{t_0}$ in (3.21), we induce

$$\langle w_t - w_{t_0}, Tw_{t_0}\rangle - \frac{1}{r_{t_0}}\langle w_t - w_{t_0}, (1+r_{t_0})w_{t_0} - z_{t_0}\rangle \leq 0, \tag{3.22}$$

and

$$\langle w_{t_0} - w_t, Tw_t\rangle - \frac{1}{r_t}\langle w_{t_0} - w_t, (1+r_t)w_t - z_t\rangle \leq 0. \tag{3.23}$$

Adding up (3.22) and (3.24), we obtain

$$\langle w_t - w_{t_0}, Tw_t - Tw_{t_0}\rangle - \left\langle w_t - w_{t_0}, \frac{(1+r_{t_0})w_{t_0} - z_{t_0}}{r_{t_0}} - \frac{(1+r_t)w_t - z_t}{r_t}\right\rangle \leq 0,$$

which implies that

$$\langle w_t - w_{t_0}, (w_t - Tw_t) - (w_{t_0} - Tw_{t_0})\rangle - \left\langle w_t - w_{t_0}, \frac{w_{t_0} - z_{t_0}}{r_{t_0}} - \frac{w_t - z_t}{r_t}\right\rangle \leq 0.$$

Now, using the fact that T is pseudocontractive, we have

$$\left\langle w_t - w_{t_0}, \frac{w_{t_0} - z_{t_0}}{r_{t_0}} - \frac{w_t - z_t}{r_t}\right\rangle \geq 0,$$

and hence

$$\left\langle w_t - w_{t_0}, w_{t_0} - w_t + w_t - z_{t_0} - \frac{r_{t_0}}{r_t}(w_t - z_t)\right\rangle \geq 0. \tag{3.24}$$

By (3.24), we have

$$\begin{aligned}\|w_t - w_{t_0}\|^2 &\leq \left\langle w_t - w_{t_0}, z_t - z_{t_0} + \left(1 - \frac{r_{t_0}}{r_t}\right)(w_t - z_t)\right\rangle \\ &\leq \|w_t - w_{t_0}\|\left(\|z_t - z_{t_0}\| + \frac{1}{r_t}|r_t - r_{t_0}|\|w_t - z_t\|\right),\end{aligned}$$

so

$$\|w_t - w_{t_0}\| \leq \|z_t - z_{t_0}\| + |r_t - r_{t_0}|\frac{M_5}{r}, \tag{3.25}$$

where $M_5 > 0$ is an appropriate constant, Therefore, by (3.9), (3.10), (3.19), and (3.25), we have

$$\begin{aligned}\|w_t - w_{t_0}\| &\leq \|z_t - z_{t_0}\| + |r_t - r_{t_0}|\frac{M_5}{r} \\ &\leq \|u_t - u_{t_0}\| + |\lambda_t - \lambda_{t_0}|\frac{M_2}{\lambda} + |r_t - r_{t_0}|\frac{M_5}{r} \\ &\leq \|v_t - v_{t_0}\| + |\lambda_t - \lambda_{t_0}|\frac{M_2}{\lambda} + |\eta_t - \eta_{t_0}|M_3 \\ &\quad + \frac{1}{L}\|A^*\||\alpha_t - \alpha_{t_0}|\frac{M_4}{\alpha} + |r_t - r_{t_0}|\frac{M_5}{r} \\ &\leq \|x_t - x_{t_0}\| + |\delta_t - \delta_{t_0}|\frac{M_1}{\delta} + |\lambda_t - \lambda_{t_0}|\frac{M_2}{\lambda} + |\eta_t - \eta_{t_0}|M_3 \\ &\quad + \frac{1}{L}\|A^*\||\alpha_t - \alpha_{t_0}|\frac{M_4}{\alpha} + |r_t - r_{t_0}|\frac{M_5}{r}.\end{aligned} \tag{3.26}$$

Again, since $y_t = \theta_t x_t + (1-\theta_t)T_{r_t}z_t$ and $y_{t_0} = \theta_{t_0}x_{t_0} + (1-\theta_{t_0})T_{r_{t_0}}z_{t_0}$, by (3.26), we induce

$$\begin{aligned}
\|y_t - y_{t_0}\| &= \|(\theta_t x_t + (1-\theta_t)T_{r_t}z_t) - (\theta_{t_0}x_{t_0} + (1-\theta_{t_0})T_{r_{t_0}}z_{t_0})\| \\
&= \|(\theta_t - \theta_{t_0})x_t + \theta_{t_0}(x_t - x_{t_0}) - (\theta_t - \theta_{t_0})T_{r_t}z_t + (1-\theta_{t_0})(T_{r_t}z_t - T_{r_{t_0}}z_{t_0})\| \\
&= \|(\theta_t - \theta_{t_0})(x_t - T_{r_t}z_t) + \theta_{t_0}(x_t - x_{t_0}) + (1-\theta_{t_0})(T_{r_t}z_t - T_{r_{t_0}}z_{t_0})\| \\
&\le |\theta_t - \theta_{t_0}|\|x_t - T_{r_t}z_t\| + \theta_{t_0}\|x_t - x_{t_0}\| \\
&\quad + (1-\theta_{t_0})\Big[\|x_t - x_{t_0}\| + |\delta_t - \delta_{t_0}|\frac{M_1}{\delta} + |\lambda_t - \lambda_{t_0}|\frac{M_2}{\lambda} + |\eta_t - \eta_{t_0}|M_3 \\
&\qquad\qquad + \frac{1}{L}\|A^*\||\alpha_t - \alpha_{t_0}|\frac{M_4}{\alpha} + |r_t - r_{t_0}|\frac{M_5}{r}\Big] \\
&= |\theta_t - \theta_{t_0}|\|x_t - T_{r_t}z_t\| + \|x_t - x_{t_0}\| \\
&\quad + (1-\theta_{t_0})\Big[|\delta_t - \delta_{t_0}|\frac{M_1}{\delta} + |\lambda_t - \lambda_{t_0}|\frac{M_2}{\lambda} + |\eta_t - \eta_{t_0}|M_3 \\
&\qquad\qquad + \frac{1}{L}\|A^*\||\alpha_t - \alpha_{t_0}|\frac{M_4}{\alpha} + |r_t - r_{t_0}|\frac{M_5}{r}\Big].
\end{aligned} \tag{3.27}$$

Therefore, by (3.1) and (3.27), we derive

$$\begin{aligned}
\|x_t - x_{t_0}\| &= \|t\gamma Vx_t + (I - t\mu G)y_t - (t_0\gamma Vx_{t_0} + (I - t_0\mu G)y_{t_0})\| \\
&\le \|(t-t_0)\gamma Vx_t + t_0(\gamma Vx_t - \gamma Vx_{t_0})\| + \|(I - t\mu G)y_t - (I - t_0\mu G)y_t\| \\
&\quad + \|(I - t_0\mu G)y_t - (I - t_0\mu G)y_{t_0}\| \\
&\le |t-t_0|\|\gamma Vx_t\| + t_0\gamma l\|x_t - x_{t_0}\| + |t-t_0|\|\mu Gy_t\| + (1-t_0\tau)\|y_t - y_{t_0}\| \\
&\le |t-t_0|(\gamma\|Vx_t\| + \mu\|Gy_t\|) + t_0\gamma l\|x_t - x_{t_0}\| \\
&\quad + (1-t_0\tau)[|\theta_t - \theta_{t_0}|\|x_t - T_{r_t}z_t\| + \|x_t - x_{t_0}\|] \\
&\quad + (1-t_0\tau)(1-\theta_{t_0})\Big[|\delta_t - \delta_{t_0}|\frac{M_1}{\delta} + |\lambda_t - \lambda_{t_0}|\frac{M_2}{\lambda} + |\eta_t - \eta_{t_0}|M_3 \\
&\qquad\qquad + \frac{1}{L}\|A^*\||\alpha_t - \alpha_{t_0}|\frac{M_4}{\alpha} + |r_t - r_{t_0}|\frac{M_5}{r}\Big].
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\|x_t - x_{t_0}\| &\le \frac{\gamma\|Vx_t\| + \mu\|Gy_t\|}{t_0(\tau - \gamma l)}|t - t_0| + \frac{(1-t_0\tau)\|x_t - T_{r_t}z_t\|}{t_0(\tau - \gamma l)}|\theta_t - \theta_{t_0}| \\
&\quad + \frac{(1-t_0\tau)(1-\theta_{t_0})}{t_0(\tau - \gamma l)}\Big[|\delta_t - \delta_{t_0}|\frac{M_1}{\delta} + |\lambda_t - \lambda_{t_0}|\frac{M_2}{\lambda} + |\eta_t - \eta_{t_0}|M_3 \\
&\qquad\qquad + \frac{1}{L}\|A^*\||\alpha_t - \alpha_{t_0}|\frac{M_4}{\alpha} + |r_t - r_{t_0}|\frac{M_5}{r}\Big].
\end{aligned} \tag{3.28}$$

Since $\theta_t : (0,1) \to (0,1)$ is continuous, δ_t, λ_t, α_t, $r_t : (0,1) \to (0,\infty)$ are continuous and $\eta_t : (0,1) \to (0, \frac{1}{L})$ is continuous, from (3.28), we conclude that x_t is continuous. Also, it follows from (3.27) that y_t is continuous.

(3) Since

$$\begin{aligned} x_t &= t\gamma V x_t + (I - t\mu G)y_t \\ &= t\gamma V x_t + (I - t\mu G)(\theta_t x_t + (1-\theta_t)T_{r_t} z_t) \\ &= t\gamma V x_t + \theta_t x_t + (1-\theta_t)T_{r_t} z_t - t\mu G y_t, \end{aligned}$$

we have

$$\|x_t - T_{r_t} z_t\| \leq \frac{t}{1-\theta_t}(\gamma\|Vx_t\| + \mu\|Gy_t\|) \to 0 \ \text{ as } t \to 0.$$

(4) Let $p \in \Omega$. Using $v_t = J^F_{\delta_t} x_t$, $J^F_{\delta_t} p = p$ and firmly nonexpansivity of $J^F_{\delta_t}$, we derive from (2.1) and (2.3) that

$$\begin{aligned} \|v_t - p\|^2 &= \|J^F_{\delta_t} x_t - p\|^2 \\ &\leq \langle J^F_{\delta_t} x_t - J^F_{\delta_t} p, x_t - p\rangle \\ &= \frac{1}{2}(\|v_t - p\|^2 + \|x_t - p\|^2 - \|v_t - x_t\|^2). \end{aligned} \tag{3.29}$$

Again, noting that $x_t = t\gamma V x_t + (I - t\mu G)y_t$, $y_t = \theta_t x_t + (1-\theta_t)T_{r_t} z_t$, and $T_{r_t} p = p$, from (3.29), we induce that

$$\begin{aligned} \|x_t - p\|^2 &= \|t(\gamma V x_t - \mu G y_t) + (y_t - p)\|^2 \\ &= \|t(\gamma V x_t - \mu G y_t) + \theta_t(x_t - T_{r_t} z_t) + (T_{r_t} z_t - p)\|^2 \\ &\leq [(\|t(\gamma V x_t - \mu G y_t)\| + \|z_t - p\|) + \theta_t\|x_t - T_{r_t} z_t\|]^2 \\ &= t^2\|\gamma V x_t - \mu G y_t\|^2 + 2t\|\gamma V x_t - \mu G y_t\|\|z_t - p\| + \|z_t - p\|^2 \\ &\quad + \theta_t\|x_t - T_{r_t} z_t\|[2(t\|\gamma V x_t - \mu G y_t\| + \|z_t - p\|) + \theta_t\|x_t - T_{r_t} z_t\|] \\ &\leq t\|\gamma V x_t - \mu G y_t\|^2 + \|z_t - p\|^2 + M_t \\ &\leq t\|\gamma V x_t - \mu G y_t\|^2 + \|u_t - p\|^2 + M_t \\ &\leq t\|\gamma V x_t - \mu G y_t\|^2 + \|v_t - p\|^2 + M_t \\ &\leq t\|\gamma V x_t - \mu G y_t\|^2 + (\|x_t - p\|^2 - \|x_t - v_t\|^2) + M_t, \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} M_t = {} & \theta_t\|x_t - T_{r_t} z_t\|[2(t\|\gamma V x_t - \mu G y_t\| + \|z_t - p\|) + \theta_t\|x_t - T_{r_t} z_t\|] \\ & + 2t\|\gamma V x_t - \mu G y_t\|\|z_t - p\|. \end{aligned} \tag{3.31}$$

By (3.30), we obtain

$$\|v_t - x_t\|^2 \leq t\|\gamma V x_t - \mu G y_t\|^2 + M_t. \tag{3.32}$$

Noting $\lim_{t\to 0} M_t = 0$ by (3), it follows from (3.32) that

$$\lim_{t\to 0}\|v_t - x_t\| = \lim_{t\to 0}\|J^F_{\delta_t} x_t - x_t\| = 0.$$

(5) By (3.6), we see that

$$\begin{aligned} \|u_t - p\|^2 &= \|v_t + \eta_t A^*(R_{\alpha_t} - I)Av_t - p\|^2 \\ &\leq \|v_t - p\|^2 + \eta_t(L\eta_t - 1)\|(R_{\alpha_t} - I)Av_t\|^2 \\ &\leq \|v_t - p\|^2 \ \ (\text{by } \eta_t \in (0, \frac{1}{L})). \end{aligned} \tag{3.33}$$

Again, since $J^B_{\lambda_t}$ is firmly nonexpansive, by (2.1) and (2.3), we have

$$\|z_t - p\|^2 \leq \langle J^B_{\lambda_t} u_t - J^B_{\lambda_t} p, u_t - p\rangle = \frac{1}{2}[\|u_t - p\|^2 + \|z_t - p\|^2 - \|u_t - z_t\|^2],$$

and hence

$$\begin{aligned} \|z_t - p\|^2 &\leq \|u_t - p\|^2 - \|u_t - z_t\|^2 \\ &\leq \|v_t - p\|^2 - \|u_t - z_t\|^2 \\ &\leq \|x_t - p\|^2 - \|u_t - z_t\|^2. \end{aligned} \tag{3.34}$$

Thus, as in (3.30), we derive from(3.34) that

$$\begin{aligned} \|x_t - p\|^2 &\leq t\|\gamma V x_t - \mu G y_t\|^2 + \|z_t - p\|^2 + M_t \\ &\leq t\|\gamma V x_t - \mu G y_t\|^2 + \|u_t - p\|^2 - \|u_t - z_t\|^2 + M_t \\ &\leq t\|\gamma V x_t - \mu G y_t\|^2 + \|v_t - p\|^2 - \|u_t - z_t\|^2 + M_t \\ &\leq t\|\gamma V x_t - \mu G y_t\|^2 + \|x_t - p\|^2 - \|u_t - z_t\|^2 + M_t, \end{aligned}$$

where M_t is of (3.31). Hence

$$\|u_t - z_t\|^2 \leq t\|\gamma V x)t - \mu G y_t\|^2 + M_t. \tag{3.35}$$

Therefore, by (3.35), we have

$$\lim_{t\to 0}\|u_t - z_t\| = \lim_{t\to 0}\|u_t - J^B_{\lambda_t} u_t\| = 0.$$

(6) In fact, from (3.6), we know that

$$\begin{aligned} \|z_t - p\|^2 &\leq \|v_t + \eta_t A^*(R_{\delta_t} - I)Av_t - p\|^2 \\ &\leq \|v_t - p\|^2 + \eta_t(L\eta_t - 1)\|(R_{\alpha_t} - I)Av_t\|^2 \\ &\leq \|x_t - p\|^2 + \eta_t(L\eta_t - 1)\|(R_{\alpha_t} - I)Av_t\|^2. \end{aligned} \tag{3.36}$$

Again, as in (3.30), we induce from (3.36) that

$$\begin{aligned} \|x_t - p\|^2 &\leq t\|\gamma V x_t - \mu G y_t\|^2 + \|z_t - p\|^2 + M_t \\ &\leq t\|\gamma V x_t - \mu G y_t\|^2 + \|x_t - p\|^2 + \eta_t(L\eta_t - 1)\|(R_{\alpha_t} - I)Av_t\|^2 + M_t. \end{aligned}$$

where M_t is of (3.31). So, we have

$$\eta_n(1 - L\eta_t)\|(R_{\alpha_t} - I)Av_t\|^2 \leq t\|\gamma V x_t - \mu G y_t\|^2 + M_t.$$

Since $1 - L\eta_t > 0$ and $0 < \eta \leq \eta_t$ for $t > 0$, we obtain

$$\|(R_{\alpha_t} - I)Av_t\| \to 0 \ \text{ as } t \to 0. \tag{3.37}$$

Therefore, we derive from (3.37) that

$$\lim_{t\to 0}\|u_t - v_t\| = \lim_{t\to 0}\|\eta_t A^*(R_{\alpha_t} - I)Av_t\| \leq \lim_{t\to 0}\frac{1}{L}\|A^*\|\|(R_{\alpha_t} - I)Av_t\| = 0.$$

(7) In fact, by (3), (4), (5), and (6), we have

$$\|z_t - T_{r_t} z_t\| \leq \|z_t - u_t\| + \|u_t - v_t\| + \|v_t - x_t\| + \|x_t - T_{r_t} z_t\| \to 0 \text{ as } t \to 0.$$

(8) By (3) and (7), we have

$$\|x_t - z_t\| \leq \|x_t - T_{r_t} z_t\| + \|T_{r_t} z_t - z_t\| \to 0 \text{ as } t \to 0.$$

(9) By (4) and (6), we have

$$\|x_t - u_t\| \le \|x_t - v_t\| + \|v_t - u_t\| \to 0 \text{ as } t \to 0.$$

(10) By (7) and (8), we have

$$\begin{aligned}\|x_t - T_{r_t}x_t\| \le& \|x_t - z_t\| + \|z_t - T_{r_t}z_t\| + \|T_{r_t}z_t - T_{r_t}x_t\| \\ &\le 2\|x_t - z_t\| + \|z_t - T_{r_t}z_t\| \to 0 \text{ as } t \to 0.\end{aligned}$$

(11) By (4), (5), (6), and (9), we have

$$\begin{aligned}\|x_t - J^B_{\lambda_t}x_t\| \le& \|x_t - v_t\| + \|v_t - u_t\| + \|u_t - J^B_{\lambda_t}u_t\| + \|J^B_{\lambda_t}u_t - J^B_{\lambda_t}x_t\| \\ &\le \|x_t - v_t\| + \|v_t - u_t\| + \|u_t - J^B_{\lambda_t}u_t\| + \|u_t - x_t\| \to 0 \ \text{ as } t \to 0.\end{aligned}$$

□

By using Proposition 3.1, we establish strong convergence of the path $\{x_t\}$ to a point of Ω, which guarantees the existence of solutions of the variational inequality (3.38) below.

Theorem 3.1. *Let the path $\{x_t\}$ be defined by (3.1). Let $\delta_t, \lambda_t, \alpha_t, r_t : (0,1) \to (0,\infty)$ be continuous with $0 < \delta \le \delta_t$, $0 < \lambda \le \lambda_t$, $0 < \alpha \le \alpha_t$, $0 < r \le r_t$ for $t \in (0,1)$, and let $\theta_t : (0,1) \to (0,1)$ be continuous with $0 < \theta \le \theta_t < 1$. Let $\eta_t : (0,1) \to (0, \frac{1}{L})$ be continuous with $0 < \eta \le \eta_t$ for $t \in (0,1)$. Then $\{x_t\}$ converges strongly, as $t \to 0$, to a point $q \in \Omega$, which is the unique solution to the variational inequality:*

$$\langle (\mu G - \gamma V)q, p - q\rangle \ge 0 \ \text{ for all } p \in \Omega. \tag{3.38}$$

Proof. We first note that the uniqueness of a solution of variational inequality (3.38) is a direct consequence of the strong monotonicity of $\mu G - \gamma V$ (see Lemma 2.5).

From now, let $v_t = J^F_{\delta_t}x_t$, $u_t = v_t + \eta_t A^*(R_{\alpha_t} - I)Av_t = K_t v_t$, $z_t = J^B_{\lambda_t}u_t$ and $y_t = \theta_t x_t + (1 - \theta_t)T_{r_t}z_t$ for $t \in (0,1)$. Let $\{t_n\} \subset (0,1)$ be a sequence with $t_n \to 0$ as $n \to \infty$. Put $\delta_n := \delta_{t_n}$, $\lambda_n := \lambda_{t_n}$, $\alpha_n := \alpha_{t_n}$, $r_n := r_{t_n}$, $\eta_n := \eta_{t_n}$, $\theta_n := \theta_{t_n}$, $x_n := x_{t_n}$, $y_n := y_{t_n}$, $v_n := v_{t_n}$, $u_n := u_{t_n}$, $z_n := z_{t_n}$ and $w_n := T_{r_n}z_n$. Since $\{x_n\}$ is bounded by Proposition 3.1, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $q \in H_1$. First of all, we demonstrate that $q \in \Omega$. To this end, we divide its proof into four steps.

Step 1. We prove that $q \in \arg\min_{y \in H_1} F(y)$. For this purpose, let $d > 0$. Using $v_n = J^F_{\delta_n}x_n$ and (2.7), we derive that

$$\begin{aligned}\|x_n - J^F_d x_n\| &\le \|v_n - J^F_d x_n\| + \|x_n - v_n\| \\ &= \|v_n - x_n\| + \left\| J^F_d\left(\left(1 - \frac{d}{\delta_n}\right)J^F_{\delta_n}x_n + \frac{d}{\delta_n}x_n\right) - J^F_d x_n\right\| \\ &\le \|v_n - x_n\| + \left\|\left(1 - \frac{d}{\delta_n}\right)J^F_{\delta_n}x_n + \frac{d}{\delta_n}x_n - x_n\right\| \\ &\le \|v_n - x_n\| + \left|1 - \frac{d}{\delta_n}\right| \|v_n - x_n\| \\ &= \left(1 + \left|1 - \frac{d}{\delta_n}\right|\right)\|v_n - x_n\| \le K\|v_n - x_n\|\end{aligned}$$

for some $K > 0$. Hence it follows from Proposition 3.1 (4) that

$$\|x_n - J^F_d x_n\| \to 0 \ \text{ as } n \to \infty. \tag{3.39}$$

Since J_d^F is single-valued and nonexpansive, using (3.39) and Lemma 2.7, we obtain

$$q \in Fix(J_d^F) = \arg\min_{y \in H_1} F(y).$$

Step 2. We prove that $q \in Fix(T)$. To demonstrate this, we put $w_n = T_{r_n} z_n$. Then, by Lemma 2.3, we have

$$\langle y - w_n, Tw_n \rangle - \frac{1}{r_n} \langle y - w_n, (1+r)n)w_n - z_n \rangle \leq 0 \text{ for all } y \in H_1. \tag{3.40}$$

Put $v_\varepsilon = \varepsilon v + (1-\varepsilon)q$ for $\varepsilon \in (0,1]$ and $v \in H_1$. Then $v_\varepsilon \in H_1$. From (3.40) and pseudocontracttivity of T, it follows that

$$\begin{aligned} \langle w_n - v_\varepsilon, Tv_\varepsilon \rangle &\geq \langle w_n - v_\varepsilon, Tv_\varepsilon \rangle + \langle v_\varepsilon - w_n, Tw_n \rangle - \frac{1}{r_n} \langle v_\varepsilon - w_n, (1+r_n)w_n - z_n \rangle \\ &= -\langle v_\varepsilon - w_n, Tv_\varepsilon - Tw_n \rangle - \frac{1}{r_n} \langle v_\varepsilon - w_n, w_n - z_n \rangle - \langle v_\varepsilon - w_n, w_n \rangle \\ &\geq -\|v_\varepsilon - w_n\|^2 - \frac{1}{r_n} \langle v_\varepsilon - w_n, w_n - z_n \rangle - \langle v_\varepsilon - w_n, w_n \rangle \\ &= -\langle v_\varepsilon - w_n, v_\varepsilon \rangle - \langle v_\varepsilon - w_n, \frac{w_n - z_n}{r_n} \rangle. \end{aligned} \tag{3.41}$$

Since $\{x_n\}$, $\{z_n\}$ and $\{w_n\}$ have the same asymptotical behavior (due to Proposition 3.1 (7) and (8)), $w_{n_i} \rightharpoonup q$ as $i \to \infty$. Also, by Proposition 3.1 (7), we have $\frac{\|w_n - z_n\|}{r_n} \leq \frac{\|w_n - z_n\|}{r} \to 0$ as $n \to \infty$. So, replacing n by n_i and letting $i \to \infty$, we derive from (3.41) that $\langle q - v_\varepsilon, Tv_\varepsilon \rangle \geq \langle q - v_\varepsilon, v_\varepsilon \rangle$ and $-\langle v - q, Tv_\varepsilon \rangle \geq -\langle v - q, v_\varepsilon \rangle$ for all $v \in H_1$. Letting $\varepsilon \to 0$ and using the fact that T is continuous, we obtain

$$-\langle v - q, Tq \rangle \geq -\langle v - q, q \rangle \text{ for all } v \in H_1. \tag{3.42}$$

Let $v = Tq$ in (3.42). Then we have $q = Tq$, that is, $q \in Fix(T)$.

Step 3. We prove that $q \in B^{-1}0$. To this end, let $z_n = J_{\lambda_n}^B u_n$. Then it follows that

$$u_n \in (I + \lambda_n B) z_n, \text{ that is, } \frac{u_n - z_n}{\lambda_n} \in Bz_n.$$

Since B is monotone, we know that, for any $v \in Bu$,

$$\langle z_n - u, \frac{u_n - z_n}{\lambda_n} - v \rangle \geq 0. \tag{3.43}$$

Since $\frac{\|u_n - z_n\|}{\lambda_n} \leq \frac{\|u_n - J_{\lambda_n}^{B_1} u_n\|}{\lambda} \to 0$ as $n \to \infty$ by Proposition 3.1(5), we have $u_{n_i} \rightharpoonup q$ and $z_{n_i} \rightharpoonup q$ as $i \to \infty$ by Proposition 3.1 (5) and (9). By replacing n by n_i in (3.43) and letting $i \to \infty$, we obtain $\langle q - u, -v \rangle \geq 0$. Since B is maximal monotone, we conclude $0 \in Bq$, that is, $q \in B^{-1}0$.

Step 4. We prove that $Aq \in Fix(R)$. In fact, since $\{x_n\}$, $\{v_n\}$, $\{u_n\}$, and $\{z_n\}$ have the same asymptotical behavior (due to Proposition 3.1 (4), (5), and (6)), $\{Ax_{n_i}\}$ converges weakly to Aq. Again, let $\hat{\alpha} > \alpha > 0$. Then, using (3.18) and (3.37), we obtain

$$\|R_{\alpha_{n_i}}(Av_{n_i}) - R_{\hat{\alpha}}(Av_{n_i})\| \leq \frac{|\alpha_{n_i} - \hat{\alpha}|}{\alpha} \|(R_{\alpha_{n_i}} - I)Av_{n_i}\| \to 0 \text{ as } i \to \infty. \tag{3.44}$$

Hence, from (3.44), it follows that

$$\lim_{i \to \infty} \|(R_{\hat{\alpha}} - I)Av_{n_i}\| = \lim_{i \to \infty} \|(R_{\alpha_{n_i}} - I)Av_{n_i}\| = 0. \tag{3.45}$$

Since $R_{\hat{\alpha}}$ is nonexpansive, by (3.45) and Lemma 2.7, we obtain $Aq = R_{\hat{\alpha}}(Aq)$, that is, $Aq \in Fix(R_{\hat{\alpha}}) = Fix(R)$, which means that $q \in A^{-1}(Fix(R))$. This along with Steps 1 – 3 obtains $q \in \Omega$.

Next, we prove that q is a solution to the variational inequality (3.38). In fact, observe

$$\begin{aligned}
&\|x_t - p\|^2 \\
&= \|(I - t\mu G)y_t - (I - t\mu G)p - t(\mu G - \gamma V)p + t\gamma(Vx_t - Vp)\|^2 \\
&= \|(I - t\mu G)y_t - (I - t\mu G)p\|^2 \\
&\quad - 2t[\langle(\mu G - \gamma V)p, y_t - p\rangle - t\langle(\mu G - \gamma V)p, \mu Gy_t - \mu Gp\rangle] \\
&\quad + 2t\gamma[\langle Vx_t - Vp, y_t - p\rangle - t\langle Vx_t - Vp, \mu Gy_t - \mu Gp\rangle] \\
&\quad - 2t^2\gamma\langle(\mu G - \gamma V)p, Vx_t - Vp\rangle + t^2\|(\mu G - \gamma V)p\|^2 + t^2\gamma^2\|Vx_t - Vp\|^2 \\
&\le (1 - t\tau)^2\|y_t - p\|^2 - 2t\langle(\mu G - \gamma V)p, y_t - p\rangle + 2t\gamma l\|x_t - p\|\|y_t - p\| \\
&\quad + 2t^2\|(\mu G - \gamma V)p\|(\|\mu Gy_t\| + \|\mu Gp\|) \\
&\quad + 2t^2\gamma l\|x_t - p\|(\|\mu Gy_t\| + \|\mu Gp\|) + 2t^2\gamma l\|x_t - p\|\|(\mu G - \gamma V)p\| \\
&\quad + t^2(\|(\mu G - \gamma V)p\|^2 + \gamma^2 l^2\|x_t - p\|^2) \\
&= (1 - 2t\tau + t^2\tau^2)\|y_t - p\|^2 - 2t\langle(\mu G - \gamma V)p, y_t - p\rangle + 2t\gamma l\|x_t - p\|\|y_t - p\| \\
&\quad + 2t^2\|(\mu G - \gamma V)p\|(\|\mu Gy_t\| + \|\mu Gp\|) + 2t^2\gamma l\|x_t - p\|(\|\mu Gy_t\| + \|\mu Gp\|) \\
&\quad + 2t^2\gamma l\|(\mu G - \gamma V)p\|\|x_t - p\| + t^2(\|(\mu G - \gamma V)p\|^2 + \gamma^2 l^2\|x_t - p\|^2) \\
&\le (1 - 2t\tau)\|y_t - p\|^2 + 2t\langle(\mu G - \gamma V)p, p - y_t\rangle + t\gamma l(\|x_t - p\|^2 + \|y_t - p\|^2) + t^2 M,
\end{aligned} \tag{3.46}$$

where

$$\begin{aligned}
M = \ &\sup\{\tau^2\|y_t - p\|^2 + 2(\|(\mu G - \gamma V)p\| + \gamma l\|x_t - p\|)(\|\mu Gy_t\| + \|\mu Gp\|) \\
&+ 2\gamma l\|(\mu G - \gamma V)p\|\|x_t - p\| + \|(\mu G - \gamma V)p\|^2 + \gamma^2 l^2\|x_t - p\|^2\}.
\end{aligned}$$

Hence, for small enough t, by (3.8) and (3.46), we obtain

$$\begin{aligned}
\|x_t - p\|^2 &\le \frac{1 - 2t\tau + t\gamma l}{1 - t\gamma l}\|y_t - p\|^2 + \frac{2t}{1 - t\gamma l}\langle(\mu G - \gamma V)p, p - y_t\rangle + \frac{t^2}{1 - t\gamma l}M \\
&\le \frac{1 - 2t\tau + t\gamma l}{1 - t\gamma l}\|x_t - p\|^2 + \frac{2t}{1 - t\gamma l}\langle(\mu G - \gamma V)p, p - y_t\rangle + \frac{t^2}{1 - t\gamma l}M.
\end{aligned} \tag{3.47}$$

Observe that

$$\begin{aligned}
\langle(\mu G - \gamma V)p, p - y_t\rangle &= \langle(\mu G - \gamma V)p, p - (\theta_t x_t + (1 - \theta_t)T_{r_t}z_t)\rangle \\
&= \langle(\mu G - \gamma V)p, p - T_{r_t}z_t\rangle + \theta_t\langle(\mu G - \gamma V)p, T_{r_t}z_t - x_t\rangle \\
&= \langle(\mu G - \gamma V)p, p - z_t\rangle + \langle(\mu G - \gamma V)p, z_t - T_{r_t}z_t\rangle \\
&\quad + \theta_t\langle(\mu G - \gamma V)p, T_{r_t}z_t - x_t\rangle \\
&\le \langle(\mu G - \gamma V)p, p - z_t\rangle + \|(\mu G - \gamma V)p\|\|z_t - T_{r_t}z_t\| \\
&\quad + \theta_t\|(\mu G - \gamma V)p\|\|T_{r_t}z_t - x_t\| \\
&\le \langle(\mu G - \gamma V)p, p - z_t\rangle + L_t,
\end{aligned} \tag{3.48}$$

where $L_t = \|(\mu G - \gamma V)p\|\|z_t - T_{r_t}z_t\| + \|(\mu G - \gamma V)p\|\|T_{r_t}z_t - x_t\|$. Then, from (3.47) and (3.48), we derive that

$$\|x_t - p\|^2 \le \frac{1}{\tau - \gamma l}\langle(\mu G - \gamma V)p, p - z_t\rangle + \frac{t}{2(\tau - \gamma l)}M + \frac{L_t}{\tau - \gamma l}.$$

In particular,

$$\|x_{n_i} - p\|^2 \le \frac{1}{\tau - \gamma l}\langle(\mu G - \gamma V)p, p - z_{n_i}\rangle + \frac{t_{n_i}}{2(\tau - \gamma l)}M + \frac{L_{t_{n_i}}}{\tau - \gamma l}. \tag{3.49}$$

Note that $z_{n_i} \rightharpoonup q$ by Proposition 3.1 (8) and $\lim_{t\to 0} L_t = 0$ by Proposition 3.1 (3) and (9). This fact and the inequality (3.49) with q instead of p imply that $x_{n_i} \to q$ strongly. Moreover, by taking the limit as $i \to \infty$ in (3.49), we see that

$$\|q - p\|^2 \le \frac{1}{\tau - \gamma l}\langle(\mu G - \gamma V)p, p - q\rangle.$$

In particular, q solves the following variational inequality

$$q \in \Omega, \quad \langle(\mu G - \gamma V)p, p - q\rangle \ge 0, \quad p \in \Omega,$$

or the equivalent dual variational inequality (see Lemma 2.4)

$$q \in \Omega, \quad \langle(\mu G - \gamma V)q, p - q\rangle \ge 0, \quad p \in \Omega.$$

Finally, we prove that the net $\{x_t\}$ converges strongly, as $t \to 0$, to q. For this purpose, let $\{s_k\} \subset (0,1)$ be another sequence such that $s_k \to 0$ as $k \to \infty$. Put $x_k := x_{s_k}$ and $z_k := z_{s_k}$. Let $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ and assume that $x_{k_j} \rightharpoonup \overline{q}$. Then, by the same proof as the one above, we have $\overline{q} \in \Omega$. Moreover, from strong monotonicity of $\mu G - \gamma V$, it follows that $q = \overline{q}$. Therefore, we conclude that $x_t \to q \in \Omega$ as $t \to 0$, which is the unique solution to variational inequality (3.38). This completes the proof. □

By taking $V \equiv 0$, $G \equiv I$, $\mu = 1$ in Theorem 3.1, we obtain the following result.

Corollary 3.1. *Let the path $\{x_t\}$ be defined by*

$$\begin{cases} v_t = \arg\min_{y\in H_1}[F(y) + \frac{1}{2\delta_t}\|x_t - y\|^2] \\ z_t = J^B_{\lambda_t}(v_t + \eta_t A^*(R_{\alpha_t} - I)Av_t), \\ x_t \ =(1-t)(\theta_t x_t + (1-\theta_t)T_{r_t}z_t),, \ t \in (0,1). \end{cases}$$

Let $\delta_t, \lambda_t, \alpha_t, r_t : (0,1) \to (0,\infty)$ be continuous with $0 < \delta \le \delta_t$, $0 < \lambda \le \lambda_t$, $0 < \alpha \le \alpha_t$, $0 < r \le r_t$ for $t \in (0,1)$, and let $\theta_t : (0,1) \to (0,1)$ be continuous with $0 < \theta \le \theta_t < 1$. Let $\eta_t : (0,1) \to (0, \frac{1}{L})$ be continuous with $0 < \eta \le \eta_t$ for $t \in (0,1)$. Then $\{x_t\}$ converges strongly, as $t \to 0$, to $q \in \Omega$, which solves the following minimum-norm problem : find $q \in \Omega$ such that $\|q\| = \min_{x\in\Omega}\|x\|$.

Proof. From (3.38) with $V \equiv 0$, $G \equiv I$ and $\mu = 1$, we derive $0 \le \langle q, p - q\rangle$ for all $p \in \Omega$. This obviously implies that

$$\|q\|^2 \le \langle p, q\rangle \le \|p\|\|q\| \ \text{ for all } p \in \Omega.$$

It turns out that $\|q\| \le \|p\|$ for all $p \in \Omega$. Therefore, q is the minimum-norm point of Ω. □

By taking $T \equiv I$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *Let the path $\{x_t\}$ be defined by*

$$\begin{cases} v_t = \arg\min_{y\in H_1}[F(y)+\frac{1}{2\delta_t}\|x_t-y\|^2], \\ z_t = J^B_{\lambda_t}(v_t+\eta_t A^*(R_{\alpha_t}-I)Av_t) \\ x_t = t\gamma V x_t+(I-t\mu G)(\theta_t x_t+(1-\theta_t)z_t),\ t\in(0,1). \end{cases}$$

Let $\delta_t,\lambda_t,\alpha_t:(0,1)\to(0,\infty)$ be continuous with $0<\delta\le\delta_t$, $0<\lambda\le\lambda_t$, $0<\alpha\le\alpha_t$ for $t\in(0,1)$, and let $\theta_t:(0,1)\to(0,1)$ be continuous with $0<\theta\le\theta_t<1$. Let $\eta_t:(0,1)\to(0,\frac{1}{L})$ be continuous with $0<\eta\le\eta_t$ for $t\in(0,1)$. Then $\{x_t\}$ converges strongly, as $t\to 0$, to $q\in\Gamma:=\arg\min_{y\in H_1}F(y)\cap B^{-1}0\cap A^{-1}(Fix(R))$, which is the unique solution of the variational inequality:

$$\langle(\mu G-\gamma V)q,p-q\rangle\ge 0 \text{ for all } p\in\Gamma.$$

By taking $V\equiv 0$, $G\equiv I$, $\mu=1$ in Corollary 3.2, we also obtain the following result.

Corollary 3.3. *Let the path $\{x_t\}$ be defined by*

$$\begin{cases} v_t = \arg\min_{y\in H_1}[F(y)+\frac{1}{2\delta_t}\|x_t-y\|^2], \\ z_t = J^B_{\lambda_t}(v_t+\eta_t A^*(R_{\alpha_t}-I)Av_t) \\ x_t = (1-t)(\theta_t x_t+(1-\theta_t)z_t),\ t\in(0,1). \end{cases}$$

Let $\delta_t,\lambda_t,\alpha_t:(0,1)\to(0,\infty)$ be continuous with $0<\delta\le\delta_t$, $0<\lambda\le\lambda_t$, $0<\alpha\le\alpha_t$ for $t\in(0,1)$, and let $\theta_t:(0,1)\to(0,1)$ be continuous with $0<\theta\le\theta_t<1$. Let $\eta_t:(0,1)\to(0,\frac{1}{L})$ be continuous with $0<\eta\le\eta_t$ for $t\in(0,1)$. Then x_t converges strongly, as $t\to 0$, to $q\in\Gamma$, which is the minimum-norm point of Γ.

Remark 3.1. 1) It is worth pointing out that our path is a new ones different from those announced by several authors. In particular, Theorem 3.1 is a new result which guarantees the existence of solutions for variational inequality (3.38) along with utilizing the more general classes of κ-Lipschitzian and ρ-strongly monotone mappings, continuous pseudocontractive mappings and Lipschizian mappings in comparison with [28, 29, 30].

2) Corollary 3.1 is also a new result for finding a minimum-norm point of Ω

3) Corollary 3.2 and Corollary 3.3 are new results for finding a point of Γ and a minimum-norm point of Γ, respectively.

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