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SOLUTIONS OF ONE CLASS OF SINGULAR TWO-PERSON NASH EQUILIBRIUM GAMES WITH STATE AND CONTROL DELAYS: REGULARIZATION APPROACH

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Abstract. In this paper, a finite-horizon two-person Nash equilibrium game for a linear time-dependent differential system with delays (point-wise and distributed) in the state variable and the players' control variables is considered. The behaviour of each player is evaluated by its own quadratic functional to be minimized by this player. The feature of this game is that a weight matrix of the control in the functional of one player is singular (but, in general, non-zero). Due to this feature, the game itself is singular. Using the regularization method and the asymptotic analysis of the regularized Nash equilibrium game, an open-loop Nash equilibrium solution to the considered singular game is derived. Illustrative example is presented. Along with this example, two examples on non-uniqueness of open-loop Nash equilibrium solutions to the singular game are presented.
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1. INTRODUCTION

Differential game is called singular if first-order solvability conditions are not applicable to its solution. For instance, a zero-sum differential game is called singular if the Isaacs MinMax principle [1, 2] and the Bellman–Isaacs equation method [1, 3] are not applicable to its solution. Similarly, Nash equilibrium set of controls in a singular non zero-sum differential game cannot be obtained by application of the first-order variational method and the generalized Hamilton-Jacobi-Bellman equation method [3, 4]. All this occurs, because the problem of minimization (maximization) of a game's Variational Hamiltonian with respect to a control of at least one player (a singular control) either has no solution or has infinitely many solutions.

Singular differential games appear in various applications. For example, such games appear in pursuit-evasion problems (see, e.g., [5, 6, 7, 8, 9]), in robust controllability problems (see, e.g., [10]), in robust interception problems of maneuvering targets (see, e.g., [11, 12]), in robust tracking problems (see, e.g., [13]), in biology processes (see, e.g., [14]), in robust investment problems (see, e.g., [15]).

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Singular zero-sum differential games were extensively studied in the literature in both, un-delayed and delayed dynamics, settings (see, e.g., [9, 12, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] and references therein). Singular Nash equilibrium differential games also were considered in the literature, but mostly in various stochastic settings (see, e.g., [15, 26, 27, 28, 29] and references therein). Deterministic singular Nash equilibrium differential games were studied only in few works. Thus, in [30], a two-person non zero-sum differential game with a linear second order dynamics and scalar controls of both players was considered. Each player controls one equation of the dynamics. The players' infinite horizon quadratic functionals do not contain control costs at all. The admissible class of controls for both players is the set of linear state-feedbacks. The notion of asymptotic (with respect to time) ε-Nash equilibrium was introduced, and this equilibrium was designed subject to some condition. In [14], a finite-horizon two-person non zero-sum differential game was studied. This game models a biological process. Its fourth-order dynamics is linear with respect to players' scalar controls, which vary in a bounded closed interval. The players' functionals depend only on the state variables, and this dependence is quadratic. For this singular game, a Nash equilibrium set of open-loop controls was obtained in the class of regular functions. In [31] and [32, 33], finite horizon and infinite horizon versions of two-person Nash equilibrium differential game with n-order linear dynamics and vector-valued unconstrained players' controls were considered. The cost functionals of both players in the games are quadratic. The case where both functionals do not contain control costs of one player was studied. The state-feedback solutions of the games were derived by the regularization method. It should be noted that all aforementioned works, dealing with singular Nash equilibrium differential games, studied the games with un-delayed dynamics.

In the present paper, we consider a deterministic finite-horizon two-person Nash equilibrium differential game. The dynamics of this game is linear and time-dependent. Moreover, this dynamics have multiple point-wise and distributed delays in the state variable and controls of both players. The controls of the players are unconstrained. Each player aims to minimize its own quadratic functional. We treat the case where the weight matrix in the control cost of one player (the "singular" player) in its functional is singular but, in general, non-zero. Such a feature means that the game under the consideration is singular. However, if the above mentioned singular weight matrix is non-zero, the control of the "singular" player, appearing in the Nash equilibrium solution, contains both, singular and regular, coordinates. Another feature of the considered game is that the integral parts of the players' functionals do not contain quadratic state costs. By a proper linear transformation of the state variable, the considered differential game is converted to a much simpler equivalent game. This new game also is singular, while its equation of dynamics does not contains state and controls' delays any more. The new singular game with un-delayed dynamics is solved by application of the regularization approach. Namely, this game is associated with a regular differential game. The latter has the same equation of the dynamics and a similar functional of the "singular" player augmented by a finite-horizon integral of the square of its singular control coordinates with a small positive weight (a small parameter). The functional of the other ("regular") player is not changed. Thus, the regular game is a finite-horizon linear-quadratic partial cheap control game. Like in the singular game, the integral parts of the players' functionals in this partial cheap control game do not contain quadratic state costs. This yields inapplicability of the boundary function method [34] to asymptotic analysis and solution of the corresponding singularly perturbed problem, appearing in the solvability conditions of the partial cheap control game. In the present paper, another approach to this asymptotic analysis is proposed, yielding an open-loop Nash equilibrium solution to the singular game with un-delayed dynamics and, therefore, to the initially formulated singular game with state and controls delays.

The paper is organized as follows. In the next section, the initial formulation of the singular differential game with delays in the state and controls variables is presented. The main definitions also are formulated. The transformation of the initially formulated game to the game with un-delayed dynamics is carried out. It is shown that the initially formulated game and the transformed game are equivalent to each other. Due to this equivalence, in the rest of the paper the transformed game is analyzed as an original singular differential game. In Section 3, the regularization of the original singular game is made yielding a partial cheap control regular game. Open-loop Nash equilibrium solution of the latter also is presented in this section. Asymptotic analysis of this solution is carried out in Section 4. In Section 5, based on the results of Sections 3 and 4, an open-loop Nash equilibrium solution to the original singular game is derived. Illustrative example is considered in Section 6. In Section 7, two more examples, illustrating a non-uniqueness of open-loop Nash equilibrium solution to the singular game, are given. Section 8 is devoted to conclusions.

The following main notations are used in the paper:

(1) For an $n \times m$-matrix A, $(n \geq 1,\ m \geq 1)$, its norm is defined as: $\|A\| \stackrel{\triangle}{=} \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|$, where a_{ij}, $(i = 1, ...n;\ j = 1, ..., m)$ are the entries of A.
(2) The upper index $''T''$ denotes the transposition either of a vector x (x^T) or of a matrix A (A^T).
(3) I_n denotes the identity matrix of dimension n.
(4) $O_{n\times m}$ denotes zero matrix of dimension $n \times m$, however, if the dimension of zero matrix is clear it is denoted as 0.
(5) $L^2[t_1, t_2; E^n]$ denotes the space of all functions $x(\cdot) : [t_1, t_2] \to E^n$ square integrable in the interval $[t_1, t_2]$.
(6) $\mathrm{col}(x, y)$, where $x \in E^n$, $y \in E^m$, denotes the column block-vector of the dimension $n+m$ with the upper block x and the lower block y, i.e., $\mathrm{col}(x, y) = (x^T, y^T)^T$.
(7) $\mathrm{diag}(A, B)$, where A and B are matrices of the dimensions $n \times n$ and $m \times m$, is a block-diagonal matrix with the upper left-hand block A and the lower right-hand block B.

2. Problem Formulation

2.1. **Initial game formulation.** The dynamics of the game is described by the following system:

$$\begin{aligned} \frac{dz(t)}{dt} &= \sum_{i=0}^{N_z} A_i(t) z(t - h_{z,i}) + \int_{-h_z}^{0} G(t,\tau) z(t+\tau) d\tau + \sum_{j=0}^{N_u} B_j(t) u(t - h_{u,j}) \\ &+ \int_{-h_u}^{0} P(t,\eta) u(t+\eta) d\eta + \sum_{k=0}^{N_v} C_k(t) v(t - h_{v,k}) + \int_{-h_v}^{0} Q(t,\zeta) v(t+\zeta) d\zeta + f(t) \end{aligned} \tag{2.1}$$

for all $t \in [0, t_f]$, where $z(t) \in E^n$, $u(t) \in E^r$, $(r \leq n)$, $v(t) \in E^s$, (u and v are players' controls); $0 = h_{z,0} < h_{z,1} < ... < h_{z,N_z} = h_z$ and $0 = h_{u,0} < h_{u,1} < ... < h_{u,N_u} = h_u$, $0 = h_{v,0} < h_{v,1} < ... < h_{v,N_v} = h_v$ are given constant time delays in the state and the players' controls, respectively;

$t_f > 0$ is a given time-instant; $A_i(t)$, $(i = 0,1,...,N_z)$, $G(t,\tau)$, $B_j(t)$, $(j = 0,1,...,N_u)$, $P(t,\eta)$, $C_k(t)$, $(k = 0,1,...,N_v)$, and $Q(t,\zeta)$ are given matrices of corresponding dimensions; $f(t) \in E^n$ is a given vector.

The initial conditions for the equation (2.1) are given as:

$$z(\tau) = \varphi_z(\tau), \quad \tau \in [-h_z, 0); \qquad z(0) = \varphi_{0,z}, \tag{2.2}$$

$$u(\eta) = \varphi_u(\eta), \quad \eta \in [-h_u, 0), \tag{2.3}$$

$$v(\zeta) = \varphi_v(\zeta), \quad \zeta \in [-h_v, 0), \tag{2.4}$$

where $\varphi_z(\tau) \in L^2[-h_z, 0; E^n]$, $\varphi_u(\eta) \in L^2[-h_u, 0; E^r]$, $\varphi_v(\zeta) \in L^2[-h_v, 0; E^s]$, and $\varphi_{0,z} \in E^n$ are given functions and vector.

The functionals of the player "u" with the control $u(t)$ and the player "v" with the control $v(t)$ are, respectively,

$$\mathscr{J}_u\big(u(\cdot), v(\cdot)\big) = \frac{1}{2} z^T(t_f) z(t_f) + \frac{1}{2} \int_0^{t_f} u^T(t) R_u(t) u(t) dt, \tag{2.5}$$

and

$$\mathscr{J}_v\big(u(\cdot), v(\cdot)\big) = \frac{1}{2} z^T(t_f) F z(t_f) + \frac{1}{2} \int_0^{t_f} v^T(t) R_v(t) v(t) dt, \tag{2.6}$$

where F is a given symmetric positive semi-definite matrix of corresponding dimension; for any $t \in [0, t_f]$, $R_u(t)$ and $R_v(t)$ are given symmetric matrices of corresponding dimensions.

The player "u" aims to minimize the functional (2.5) along trajectories of the system (2.1)-(2.4) by a proper choice of the control $u(t)$, $t \in [0, t_f]$, while the player "v" aims to minimize the functional (2.6) along trajectories of the system (2.1)-(2.4) by a proper choice of the control $v(t)$, $t \in [0, t_f]$.

We study the game (2.1)-(2.6) with respect to its open-loop Nash equilibrium.

In what follows, we assume:

A1. The matrix-valued functions $A_i(t)$, $(i = 0,1,...,N)$, $B_j(t)$, $(j = 1,...,N_u)$, $C_k(t)$, $(k = 1,...,N_v)$, $R_u(t)$, $R_v(t)$ and the vector-valued function $f(t)$ are continuous in the interval $[0, t_f]$.

A2. For any $t \in [0, t_f]$, the matrix-valued function $G(t,\tau)$ is piece-wise continuous with respect to $\tau \in [-h_z, 0]$, and $G(t,\tau)$ is continuous with respect to $t \in [0, t_f]$ uniformly in $\tau \in [-h_z, 0]$.

A3. For any $t \in [0, t_f]$, the matrix-valued function $P(t,\eta)$ is piece-wise continuous with respect to $\eta \in [-h_u, 0]$, and $P(t,\eta)$ is continuous with respect to $t \in [0, t_f]$ uniformly in $\eta \in [-h_u, 0]$.

A4. For any $t \in [0, t_f]$, the matrix-valued function $Q(t,\zeta)$ is piece-wise continuous with respect to $\zeta \in [-h_v, 0]$, and $Q(t,\zeta)$ is continuous with respect to $t \in [0, t_f]$ uniformly in $\zeta \in [-h_v, 0]$.

A5. The matrix $R_u(t)$ has the block-diagonal form

$$R_u(t) = \operatorname{diag}\big(R_{u,1}(t)\ ,\ O_{(r-q)\times(r-q)}\big), \quad t \in [0, t_f], \tag{2.7}$$

where $0 \le q < r$; if $q > 0$, the $q \times q$-matrix $R_{u,1}(t)$ is positive definite for all $t \in [0, t_f]$.

A6. The matrix $R_v(t)$ is positive definite for all $t \in [0, t_f]$.

Remark 2.1. Due to assumptions A1-A4 and the results of [35], we can conclude that, for any given pair of the players' controls $\big(u(t)\ ,\ v(t)\big)$, $\big(u(t) \in L^2[0, t_f; E^r]\ ,\ v(t) \in L^2[0, t_f; E^s]\big)$, initial-value problem (2.1)-(2.4) has the unique absolutely continuous solution $z = z\big(t; u(\cdot), v(\cdot)\big)$, $t \in [0, t_f]$.

Based on the work [3], we present the following definition of the open-loop Nash equilibrium solution to game (2.1)-(2.6).

Definition 2.1. The pair $\big(u^*(t)\ ,\ v^*(t)\big)$, where $u^*(t)\in L^2[0,t_f;E^r]$ and $v^*(t)\in L^2[0,t_f;E^s]\big)$, is called an open-loop Nash equilibrium solution to game (2.1)-(2.6) if this pair satisfies the following two inequalities:

$$\mathscr{J}_u\big(u^*(t),v^*(t)\big)\le \mathscr{J}_u\big(u(t),v^*(t)\big)\quad \forall\, u(t)\in L^2[0,t_f;E^r], \tag{2.8}$$

$$\mathscr{J}_v\big(u^*(t),v^*(t)\big)\le \mathscr{J}_v\big(u^*(t),v(t)\big)\quad \forall\, v(t)\in L^2[0,t_f;E^s]. \tag{2.9}$$

The values

$$\mathscr{J}_u^* \stackrel{\triangle}{=} \mathscr{J}_u\big(u^*(t),v^*(t)\big) \tag{2.10}$$

and

$$\mathscr{J}_v^* \stackrel{\triangle}{=} \mathscr{J}_v\big(u^*(t),v^*(t)\big) \tag{2.11}$$

are called Nash equilibrium values of functionals (2.5) and (2.6), respectively, in game (2.1)-(2.6). If the open-loop Nash equilibrium solution to game (2.1)-(2.6) exists, this game is called open-loop solvable.

Remark 2.2. Due to assumption A5, matrix $R_u(t)$ is singular, which means that the first-order Nash-equilibrium solvability conditions (see, e.g., [3, 4]) cannot be applied to analysis and the solutions of game (2.1)-(2.6). Thus this game is singular.

2.2. **Transformation of the game (2.1)-(2.6).** Let us consider the following terminal-value problem for $n\times n$-matrix-valued function $\Psi(t)$:

$$\frac{d\Psi(t)}{dt} = -\sum_{i=0}^{N}\Psi(t+h_i)A_i(t+h_i) - \int_{-h}^{0}\Psi(t-\tau)G(t-\tau,\tau)d\tau,\quad t\in[0,t_f],$$
$$\Psi(t_f)=I_n,\qquad \Psi(t)=0,\quad t>t_f. \tag{2.12}$$

In this equation $N=N_z$, $h_i=h_{z,i}$, $h=h_z$. By virtue of the results of [36] (Section 4.3), problem (2.12) has the unique solution $\Psi(t)$, $t\ge 0$. Moreover, matrix-valued function $\Psi(t)$ is continuous in the interval $[0,t_f]$, while its derivative is piece-wise continuous in this interval.

Based on the solution $\Psi(t)$, $t\ge 0$ to terminal-value problem (2.12), we use the Halanay Transformation for a linear system with state delays (see [37]) and the transformation for a linear nonhomogeneous system (see [38, 39]) to make the following change of the state variable in the system (2.1)-(2.4):

$$w(t)=\Psi(t)z(t)+\sum_{i=1}^{N_z}\int_t^{t+h_{z,i}}\Psi(s)A_i(s)z(s-h_{z,i})ds$$
$$+\int_t^{t+h_z}\left(\int_t^s\Psi(\sigma)G\Big(\sigma,s-\sigma-h_z\Big)d\sigma\right)z(s-h)ds+\int_t^{t_f}\Psi(\xi)f(\xi)d\xi,\quad t\in[0,t_f], \tag{2.13}$$

where $w(t)$ is a new state variable.

Let us denote

$$\mathscr{B}(t)\stackrel{\triangle}{=}\Psi(t)B(t),\quad t\in[0,t_f], \tag{2.14}$$

$$\mathscr{P}(t,\eta) \stackrel{\triangle}{=} \Psi(t)P(t,\eta), \quad (t,\eta) \in [0,t_f] \times [-h_u,0], \tag{2.15}$$

$$\mathscr{C}_k(t) \stackrel{\triangle}{=} \Psi(t)C_k(t), \quad t \in [0,t_f], \tag{2.16}$$

$$\mathscr{Q}(t,\zeta) \stackrel{\triangle}{=} \Psi(t)Q(t,\zeta), \quad (t,\zeta) \in [0,t_f] \times [-h_v,0], \tag{2.17}$$

$$w_0 \stackrel{\triangle}{=} \Psi(0)\varphi_{0,z} + \sum_{i=1}^{N_z} \int_{-h_{z,i}}^{0} \Psi(\tau+h_{z,i})A_i(\tau+h_{z,i})\varphi_z(\tau)d\tau$$
$$+ \int_{-h_z}^{0} \left(\int_0^{\tau+h_z} \Psi(\sigma)G\Big(\sigma,\tau-\sigma\Big)d\sigma \right) \varphi_z(\tau)d\tau + \int_0^{t_f} \Psi(\xi)f(\xi)d\xi. \tag{2.18}$$

Direct differentiation of $w(t)$ in (2.13) with respect to t, and use of the problem (2.12) and the expressions (2.14)-(2.18) yield the following assertion.

Proposition 2.1. *Let the assumptions A1-A4 be satisfied. Let for given $u(t) \in L^2[0,t_f;E^r]$ and $v(t) \in L^2[0,t_f;E^s]$, the absolutely continuous function $z(t)$, $t \in [0,t_f]$ be the solution of the system (2.1)-(2.4). Then, the function $w(t)$, given by (2.13), is the unique absolutely continuous solution of the system*

$$\frac{dw(t)}{dt} = \sum_{j=0}^{N_u} \mathscr{B}_j(t)u(t-h_{u,j}) + \int_{-h_u}^{0} \mathscr{P}(t,\eta)u(t+\eta)d\eta$$
$$+ \sum_{k=0}^{N_v} \mathscr{C}_k(t)v(t-h_{v,k}) + \int_{-h_v}^{0} \mathscr{Q}(t,\zeta)v(t+\zeta)d\zeta, \quad t \in [0,t_f], \tag{2.19}$$

subject to the initial condition for the state variable

$$w(0) = w_0, \tag{2.20}$$

and the initial conditions (2.3)-(2.4) for the controls $u(\cdot)$ and $v(\cdot)$. Moreover,

$$w(t_f) = z(t_f). \tag{2.21}$$

Thus, by the state transformation (2.13), the functionals (2.5) and (2.6) become as:

$$\mathscr{J}_{u,1}\big(u(\cdot),v(\cdot)\big) = \frac{1}{2}w^T(t_f)w(t_f) + \frac{1}{2}\int_0^{t_f} u^T(t)R_u(t)u(t)dt, \tag{2.22}$$

and

$$\mathscr{J}_{v,1}\big(u(\cdot),v(\cdot)\big) = \frac{1}{2}w^T(t_f)Fw(t_f) + \frac{1}{2}\int_0^{t_f} v^T(t)R_v(t)v(t)dt, \tag{2.23}$$

respectively.

Remark 2.3. The transformation (2.13) converts the originally formulated game (2.1)-(2.6) with the state and controls delays in the dynamics to the new game, consisting of the dynamics equation (2.19), the initial conditions (2.3)-(2.4), (2.20) and the functionals (2.22), (2.23). Due to (2.19), the dynamics equation of this new game does not have state delays any more, while still has the delays in the players' controls.

Remark 2.4. An open-loop Nash equilibrium solution and Nash equilibrium values of the functionals in the new game (2.19), (2.3)-(2.4), (2.20), (2.22), (2.23) are defined quite similarly to such notions in the game (2.1)-(2.6) (see Definition 2.1). Like the game (2.1)-(2.6), the game (2.19), (2.3)-(2.4), (2.20), (2.22), (2.23) also is singular.

Definition 2.2. We say that two two-person Nash equilibrium differential games, considered in the same time-interval, are equivalent to each other, if they have the same open-loop solution and the same Nash equilibrium values of the functionals.

Lemma 2.1. *Let the assumptions A1-A6 be satisfied. Let one of the games (2.1)-(2.6) or (2.19), (2.3)-(2.4), (2.20), (2.22), (2.23) is open-loop solvable. Then, the other game also is open-loop solvable, and these games are equivalent to each other.*

Proof. Let the control $u(t) \in L^2[0,t_f;E^r]$ of the player "u" and the control $v(t) \in L^2[0,t_f;E^s]$ of the player "v" be any prechosen. Then, due to Proposition 2.1,

$$\mathscr{J}_u\big(u(t),v(t)\big) = \mathscr{J}_{u,1}\big(u(t),v(t)\big), \quad \mathscr{J}_v\big(u(t),v(t)\big) = \mathscr{J}_{v,1}\big(u(t),v(t)\big).$$

These two equalities directly yield the statements of the lemma. □

Let us introduce into the consideration the following matrix-valued functions

$$\mathscr{M}_u(t) = \sum_{j=0}^{N_u} \mathscr{B}_j(t+h_{u,j}) + \int_{-h_u}^{0} \mathscr{P}(t-\eta,\eta)d\eta, \quad t \in [0,t_f], \tag{2.24}$$

$$\mathscr{M}_v(t) = \sum_{k=0}^{N_v} \mathscr{C}_k(t+h_{v,k}) + \int_{-h_v}^{0} \mathscr{Q}(t-\zeta,\zeta)d\zeta, \quad t \in [0,t_f], \tag{2.25}$$

where

$$\mathscr{B}_j(t+h_{u,j}) = \begin{cases} \Psi(t+h_{u,j})B_j(t+h_{u,j}), & 0 \le t \le t_f - h_{u,j}, \\ 0, & t > t_f - h_{u,j}, \end{cases} \quad j = 0,1,\dots,N_u, \tag{2.26}$$

$$\mathscr{P}(t-\eta,\eta) = \begin{cases} \Psi(t-\eta)P(t-\eta,\eta), & -h_u \le \eta \le 0, \ 0 \le t-\eta \le t_f, \\ 0, & -h_u \le \eta \le 0, \ t-\eta > t_f, \end{cases} \tag{2.27}$$

$$\mathscr{C}_k(t+h_{v,k}) = \begin{cases} \Psi(t+h_{v,k})C_k(t+h_{v,k}), & 0 \le t \le t_f - h_{v,k}, \\ 0, & t > t_f - h_{v,k}, \end{cases} \quad k = 0,1,\dots,N_v, \tag{2.28}$$

$$\mathscr{Q}(t-\zeta,\zeta) = \begin{cases} \Psi(t-\zeta)Q(t-\zeta,\zeta), & -h_v \le \zeta \le 0, \ 0 \le t-\zeta \le t_f, \\ 0, & -h_v \le \zeta \le 0, \ t-\zeta > t_f. \end{cases} \tag{2.29}$$

Using the Kwon-Pearson-Artstein transformation for a linear system with control delays (see [40, 41]), we make the following change of the state variable in the game (2.19), (2.3)-(2.4), (2.20), (2.22), (2.23):

$$\begin{aligned} x(t) = w(t) &+ \sum_{j=1}^{N_u} \int_{t-h_{u,j}}^{t} \mathscr{B}_j(\sigma+h_{u,j})u(\sigma)d\sigma + \int_{-h_u}^{0} d\eta \left(\int_{t+\eta}^{t} \mathscr{P}(\sigma-\eta,\eta)u(\sigma)d\sigma \right) \\ &+ \sum_{k=1}^{N_v} \int_{t-h_{v,k}}^{t} \mathscr{C}_k(\kappa+h_{v,k})v(\kappa)d\kappa + \int_{-h_v}^{0} d\zeta \left(\int_{t+\zeta}^{t} \mathscr{Q}(\kappa-\zeta,\zeta)v(\kappa)d\kappa \right), \end{aligned} \tag{2.30}$$

where $x(t)$ is a new state variable.

Using the results of [41] and the equations (2.26)-(2.29), we directly have the following assertion.

Proposition 2.2. *Let the assumptions A1-A4 be satisfied. Let for given* $u(t) \in L^2[0,t_f;E^r]$ *and* $v(t) \in L^2[0,t_f;E^s]$*, the absolutely continuous function* $w(t)$*,* $t \in [0,t_f]$ *be the solution of the initial-value problem (2.19)-(2.20), (2.3)-(2.4). Then, the function* $x(t)$*, given by (2.30), is the unique absolutely continuous solution of the initial-value problem*

$$\frac{dx(t)}{dt} = \mathscr{M}_u(t)u(t) + \mathscr{M}_v(t)v(t), \quad t \in [0,t_f], \tag{2.31}$$

$$x(0) = x_0, \tag{2.32}$$

where

$$\begin{aligned} x_0 = w_0 &+ \sum_{j=1}^{N_u} \int_{-h_{u,j}}^{0} \mathscr{B}_j(\eta + h_{u,j})\varphi_u(\eta)d\eta + \int_{-h_u}^{0} d\eta \left(\int_{\eta}^{0} \mathscr{P}(\sigma - \eta, \eta)\varphi_u(\sigma)d\sigma \right) \\ &+ \sum_{k=1}^{N_v} \int_{-h_{v,k}}^{0} \mathscr{C}_k(\kappa + h_{v,k})\varphi_v(\kappa)d\kappa + \int_{-h_v}^{0} d\zeta \left(\int_{\zeta}^{0} \mathscr{Q}(\kappa - \zeta, \zeta)\varphi_v(\kappa)d\kappa \right). \end{aligned} \tag{2.33}$$

Moreover,

$$x(t_f) = w(t_f). \tag{2.34}$$

Thus, by the state transformation (2.30), functionals (2.22) and (2.23) become as:

$$J_u\big(u(\cdot),v(\cdot)\big) = \frac{1}{2}x^T(t_f)x(t_f) + \frac{1}{2}\int_0^{t_f} u^T(t)R_u(t)u(t)dt, \tag{2.35}$$

and

$$J_v\big(u(\cdot),v(\cdot)\big) = \frac{1}{2}x^T(t_f)Fx(t_f) + \frac{1}{2}\int_0^{t_f} v^T(t)R_v(t)v(t)dt, \tag{2.36}$$

respectively.

Remark 2.5. The transformation (2.30) converts the game (2.19), (2.3)-(2.4), (2.20), (2.22), (2.23) with the controls delays in the dynamics to the new game, consisting of the dynamics equation (2.31), the initial condition (2.32) and the functionals (2.35), (2.36). Due to (2.31), the dynamics equation of this new game does not have delays any more.

Remark 2.6. An open-loop Nash equilibrium solution and Nash equilibrium values of the functionals in the new game (2.31), (2.32), (2.35), (2.36) are defined quite similarly to such notions in the game (2.1)-(2.6) (see Definition 2.1). Like the games (2.1)-(2.6) and (2.19), (2.3)-(2.4), (2.20), (2.22), (2.23), the game (2.31), (2.32), (2.35), (2.36) also is singular.

Using Proposition 2.2, we obtain (quite similarly to Lemma 2.1) the following assertion.

Lemma 2.2. *Let the assumptions A1-A6 be satisfied. Let one of the games (2.19), (2.3)-(2.4), (2.20), (2.22), (2.23) or (2.31), (2.32), (2.35), (2.36) is open-loop solvable. Then, the other game also is open-loop solvable, and these games are equivalent to each other.*

Remark 2.7. By virtue of Lemmas 2.1 and 2.2, the initially formulated game (2.1)-(2.6) and the game (2.31), (2.32), (2.35), (2.36) are equivalent to each other. Moreover, due to the equation of dynamics, the later game is much simpler than the former one. Therefore, in what follows of the paper, we deal with the game (2.31), (2.32), (2.35), (2.36) and we call this game the Original Singular Nash Equilibrium Game (OSNEG). In the subsequent sections, we will derive an open-loop solution of the OSNEG valid for all $x_0 \in E^n$.

2.3. **Objectives of the paper.** The objectives of the paper are the following:
(i) to establish the existence of the open-loop solution to the OSNEG;
(ii) to derive this solution;
(iii) to obtain expressions for the Nash equilibrium values of the functionals (2.35) and (2.36) in the OSNEG.

3. Regularization of the OSNEG

3.1. **Partial cheap control Nash equilibrium game.** To establish the existence of the open-loop solution to the OSNEG and to derive this solution, we replace (approximately) this game by a parameter dependent regular Nash equilibrium game. The latter is close in some sense to the OSNEG. Namely, this regular Nash equilibrium game has the same dynamics (2.31), the same initial position (2.32) and the same functional (2.36) of the player ”v” as the OSNEG has. However, the functional of the player ”u” in the new game differs from the one in the OSNEG. Namely, this functional has the form

$$J_{u,\varepsilon}\big(u(\cdot),v(\cdot)\big) \triangleq \frac{1}{2}x^T(t_f)x(t_f) + \frac{1}{2}\int_0^{t_f} u^T(t)\big(R_u(t)+\mathscr{E}\big)u(t)dt, \tag{3.1}$$

where

$$\mathscr{E} = \mathrm{diag}\big(\underbrace{0,...,0}_{q},\underbrace{\varepsilon,...,\varepsilon}_{r-q}\big), \tag{3.2}$$

$\varepsilon > 0$ is a small parameter.

Using (2.7) and (3.2), we have immediately

$$R_u(t)+\mathscr{E} = \mathrm{diag}\big(R_{u,1}(t), \varepsilon I_{r-q}\big). \tag{3.3}$$

Remark 3.1. Due to the assumption A5, the matrix $R_u(t)+\mathscr{E}$ is positive definite for all $t \in [0,t_f]$ and all $\varepsilon > 0$. Thus, the Nash equilibrium game (2.31), (2.32), (3.1), (2.36) is regular for all $\varepsilon > 0$. An open-loop Nash equilibrium solution and Nash equilibrium values of the functionals in this regular game are defined quite similarly to such notions in the game (2.1)-(2.6) (see Definition 2.1).

Remark 3.2. In what follows, we mainly focus on the case $0 < q < r$. In this case, due to (3.3) and the smallness of ε, the Nash equilibrium game (2.31), (2.32), (3.1), (2.36) is a partial cheap control game, i.e., the game where the cost only of some (but not all) control coordinates of at least one player in at least one functional is small. In what follows, we call the game (2.31), (2.32), (3.1), (2.36) the Partial Cheap Control Nash Equilibrium Game (PCCNEG). Zero-sum differential games with a complete/partial cheap control of at least one player were studied in many works (see, e.g., [11, 12, 13, 20, 21, 23, 24, 25, 42, 43]). Non zero-sum differential games with a complete cheap control of one player were considered only in few works (see [4, 31, 32]). However, to the best of our knowledge, a non zero-sum differential game with a partial cheap control of at least one player has been considered only in two works [33, 44] in the literature. It should be noted that the PCCNEG differs considerably from the partial cheap control Nash equilibrium games studied in [33, 44]. Moreover, in the present paper we are going to analyze the open-loop solution of the PCCNEG, while in [33, 44] the feedback solutions of the considered there partial cheap control Nash equilibrium games were analyzed.

3.2. **Solvability condition and solutions of the PCCNEG for a given $\varepsilon > 0$.** Following the results of [3] on the obtaining an open-loop solution to a regular differential Nash equilibrium game, we write down the variational Hamiltonians for the PCCNEG

$$H_u\big(u(t),v(t),t,\varepsilon\big) = \lambda_u^T(t)\big(\mathscr{M}_u(t)u(t)+\mathscr{M}_v(t)v(t)\big)+\frac{1}{2}u^T(t)\big(R_u(t)+\mathscr{E}\big)u(t),\ t\in[0,t_f], \tag{3.4}$$

$$H_v\big(u(t),v(t),t\big) = \lambda_v^T(t)\big(\mathscr{M}_u(t)u(t)+\mathscr{M}_v(t)v(t)\big)+\frac{1}{2}v^T(t)R_v(t)v(t),\quad t\in[0,t_f], \tag{3.5}$$

where $\lambda_u(t)$ and $\lambda_v(t)$ are the corresponding n-dimensional costate variables.

The costate variables satisfy the following terminal-value problems:

$$\frac{d\lambda_u(t)}{dt} = -\frac{\partial H_u\big(u(t),v(t),t,\varepsilon\big)}{\partial x(t)} = 0,\ \ t\in[0,t_f],\ \ \lambda_u(t_f) = \frac{\partial\big(\frac{1}{2}x^T(t_f)x(t_f)\big)}{\partial x(t_f)} = x(t_f), \tag{3.6}$$

$$\frac{d\lambda_v(t)}{dt} = -\frac{\partial H_v\big(u(t),v(t),t\big)}{\partial x(t)} = 0,\ \ t\in[0,t_f],\ \ \lambda_v(t_f) = \frac{\partial\big(\frac{1}{2}x^T(t_f)Fx(t_f)\big)}{\partial x(t_f)} = Fx(t_f). \tag{3.7}$$

The components of the open-loop Nash equilibrium solution to the PCCNEG are

$$u_\varepsilon^*(t) = \arg\min_{u(t)} H_u\big(u(t),v(t),t,\varepsilon\big) = -\big(R_u(t)+\mathscr{E}\big)^{-1}\mathscr{M}_u^T(t)\lambda_u(t),\quad t\in[0,t_f], \tag{3.8}$$

$$v_\varepsilon^*(t) = \arg\min_{v(t)} H_v\big(u(t),v(t),t\big) = -R_v^{-1}(t)\mathscr{M}_v^T(t)\lambda_v(t),\quad t\in[0,t_f]. \tag{3.9}$$

Solving the terminal-value problems (3.6) and (3.7), we obtain

$$\lambda_u(t)\equiv x(t_f),\qquad t\in[0,t_f], \tag{3.10}$$

$$\lambda_v(t)\equiv Fx(t_f),\qquad t\in[0,t_f]. \tag{3.11}$$

Substitution of (3.10) and (3.11) into (3.8) and (3.9), respectively, yields

$$u_\varepsilon^*(t) = -\big(R_u(t)+\mathscr{E}\big)^{-1}\mathscr{M}_u^T(t)x(t_f),\qquad t\in[0,t_f], \tag{3.12}$$

$$v_\varepsilon^*(t) = -R_v^{-1}(t)\mathscr{M}_v^T(t)Fx(t_f),\qquad t\in[0,t_f]. \tag{3.13}$$

Finally, substitution of (3.12) and (3.13) into system (2.31)-(2.32) yields the following initial-value problem for the functional-differential system:

$$\frac{dx(t)}{dt} = -W(t,\varepsilon)x(t_f),\quad t\in[0,t_f],\quad x(0)=x_0, \tag{3.14}$$

where

$$W(t,\varepsilon) = \mathscr{M}_u(t)\big(R_u(t)+\mathscr{E}\big)^{-1}\mathscr{M}_u^T(t)+\mathscr{M}_v(t)R_v^{-1}(t)\mathscr{M}_v^T(t)F. \tag{3.15}$$

The solution of initial-value problem (3.14) is called the Nash equilibrium trajectory of the PCCNEG. Integrating system (3.14), from $t=0$ to $t=t_f$, we have

$$x(t_f)-x_0 = -\int_0^{t_f}W(t,\varepsilon)dt\,x(t_f). \tag{3.16}$$

The latter yields the following algebraic equation for the obtaining the vector $x(t_f)$:

$$\Gamma(\varepsilon)x(t_f) = x_0, \qquad \Gamma(\varepsilon) \stackrel{\triangle}{=} I_n + \int_0^{t_f} W(t,\varepsilon)dt. \tag{3.17}$$

Remark 3.3. If, for a given $\varepsilon = \bar{\varepsilon} > 0$, the matrix $\Gamma(\bar{\varepsilon})$ is invertible, then the algebraic equation (3.17) has the unique solution $x(t_f)$. The latter implies the existence of the unique open-loop solution to the PCCNEG given by (3.8)-(3.9). The invertibility of $\Gamma(\bar{\varepsilon})$ is a particular case of the necessary and sufficient condition for the existence of the unique open-loop solution for a regular differential linear-quadratic Nash equilibrium game (see, e.g., [3] and references therein). However, in the present paper, we are interested in deriving an ε-independent condition which will guarantee the invertibility of $\Gamma(\varepsilon)$ for all sufficiently small $\varepsilon > 0$. Thus, such a condition will guarantee the existence and the uniqueness of the open-loop solution to the PCCNEG for all sufficiently small $\varepsilon > 0$. To derive this condition, an asymptotic behaviour with respect to ε of the matrix $\Gamma(\varepsilon)$ will be studied in the next section. Based on this study, an asymptotic behaviour with respect to ε of the open-loop solution to the PCCNEG will be analyzed.

4. Asymptotic Analysis of the Open-Loop Solution to the PCCNEG

4.1. **Asymptotic analysis of the invertibility of the matrix $\Gamma(\varepsilon)$.** To analyze the invertibility of $\Gamma(\varepsilon)$ for all sufficiently small $\varepsilon > 0$, first we are going to transform equivalently this matrix. For this purpose, we partition the matrix $\mathscr{M}_u(t)$, $t \in [0,t_f]$ into blocks as:

$$\mathscr{M}_u(t) = \Big(\mathscr{M}_{u,1}(t)\ ,\ \mathscr{M}_{u,2}(t)\Big), \quad t \in [0,t_f], \tag{4.1}$$

where the matrices $\mathscr{M}_{u,1}(t)$ and $\mathscr{M}_{u,2}(t)$ have the dimensions $n \times q$ and $n \times (r-q)$, respectively.

In the case $0 < q < r$, the following inequality is valid:

$$\text{rank}\mathscr{M}_{u,2}(t) < n, \quad t \in [0,t_f]. \tag{4.2}$$

Using the equations (3.3), (4.1) and the invertibility property of the matrix $R_{u,1}(t)$, we obtain for all $t \in [0,t_f]$ and $\varepsilon > 0$

$$\mathscr{M}_u(t)\big(R_u(t) + \mathscr{E}\big)^{-1}\mathscr{M}_u^T(t) = \mathscr{M}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t) + \frac{1}{\varepsilon}\mathscr{M}_{u,2}(t)\mathscr{M}_{u,2}^T(t). \tag{4.3}$$

Let us introduce into the consideration the following symmetric and positive semi-definite matrix:

$$K_{u,2} \stackrel{\triangle}{=} \int_0^{t_f} \mathscr{M}_{u,2}(t)\mathscr{M}_{u,2}^T(t)dt. \tag{4.4}$$

Further analysis is based on the following assumption:

A7. The matrix $K_{u,2}$ has zero eigenvalue of the algebraic multiplicity k, where $n-r+q \le k < n$.

This assumption and the results of [45] yield the existence of an orthogonal $n \times n$-matrix L, ($L^T = L^{-1}$), for which the following equality is valid:

$$D_{u,2} \stackrel{\triangle}{=} LK_{u,2}L^T = \begin{pmatrix} O_{k\times k} & O_{k\times(n-k)} \\ O_{(n-k)\times k} & \Theta_{u,2} \end{pmatrix}, \tag{4.5}$$

where the block $\Theta_{u,2}$ has the dimension $(n-k)\times(n-k)$, and it is a positive definite and, therefore, nonsingular matrix.

Furthermore, since the integrand in (4.4) is a positive semi-definite matrix for all $t \in [0, t_f]$, the equations (4.4) and (4.5) imply

$$L\mathscr{M}_{u,2}(t) = \begin{pmatrix} O_{k\times(r-q)} \\ \Lambda_{u,2}(t) \end{pmatrix}, \quad t \in [0, t_f], \tag{4.6}$$

where the block $\Lambda_{u,2}(t)$ is of the dimension $(n-k)\times(r-q)$.

Moreover, due to (4.4)-(4.6),

$$\int_0^{t_f} \Lambda_{u,2}(t)\Lambda_{u,2}^T(t)dt = \Theta_{u,2}. \tag{4.7}$$

Using the equations (3.15), (3.17), (4.3)-(4.5), and the orthogonality of the matrix L, we can represent the matrix $\Gamma(\varepsilon)$ as:

$$\Gamma(\varepsilon) = L^T\left(L\Gamma(\varepsilon)L^T\right)L = L^T\Bigg[I_n + L\int_0^{t_f}\mathscr{M}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)dtL^T + \frac{1}{\varepsilon}D_{u,2} + L\int_0^{t_f}\mathscr{M}_v(t)R_v^{-1}(t)\mathscr{M}_v^T(t)dtFL^T\Bigg]L. \tag{4.8}$$

Let us partition the matrices $L\int_0^{t_f}\mathscr{M}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)dtL^T$ and $L\int_0^{t_f}\mathscr{M}_v(t)R_v^{-1}(t)\mathscr{M}_v^T(t)dtFL^T$ into blocks as:

$$L\int_0^{t_f}\mathscr{M}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)dtL^T = \begin{pmatrix} \Omega_{u,11} & \Omega_{u,12} \\ \Omega_{u,12}^T & \Omega_{u,13} \end{pmatrix},$$
$$L\int_0^{t_f}\mathscr{M}_v(t)R_v^{-1}(t)\mathscr{M}_v^T(t)dtFL^T = \begin{pmatrix} \Omega_{v,1} & \Omega_{v,2} \\ \Omega_{v,3} & \Omega_{v,4} \end{pmatrix}, \tag{4.9}$$

where the blocks $\Omega_{u,11}$ and $\Omega_{v,1}$ are of the dimension $k\times k$; the blocks $\Omega_{u,12}$ and $\Omega_{v,2}$ are of the dimension $k\times(n-k)$; the block $\Omega_{v,3}$ is of the dimension $(n-k)\times k$; the blocks $\Omega_{u,13}$ and $\Omega_{v,4}$ are of the dimension $(n-k)\times(n-k)$; the matrices $\Omega_{u,11}$ and $\Omega_{u,13}$ are symmetric.

Due to (4.5), (4.8), (4.9), the matrix $\Gamma(\varepsilon)$ is invertible if and only if the following matrix is invertible:

$$\Delta(\varepsilon) = \begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_3 & (1/\varepsilon)\Delta_4(\varepsilon) \end{pmatrix},$$
$$\Delta_1 = I_k + \Omega_{u,11} + \Omega_{v,1}, \quad \Delta_2 = \Omega_{u,12} + \Omega_{v,2},$$
$$\Delta_3 = \Omega_{u,12}^T + \Omega_{v,3}, \quad \Delta_4(\varepsilon) = \Theta_{u,2} + \varepsilon(I_{n-k} + \Omega_{u,13} + \Omega_{v,4}). \tag{4.10}$$

If the matrix $\Delta(\varepsilon)$ is invertible, then

$$\Gamma^{-1}(\varepsilon) = L^T\Delta^{-1}(\varepsilon)L. \tag{4.11}$$

Let us derive ε-independent conditions, providing the invertibility of the matrix $\Delta(\varepsilon)$ for all sufficiently small $\varepsilon > 0$.

Since the matrix $\Theta_{u,2}$ is invertible, then there exists a positive number ε_1 such that, for all $\varepsilon \in (0, \varepsilon_1]$, the matrix $\Delta_4(\varepsilon)$ is invertible and the following inequality is valid:

$$\left\|\Delta_4^{-1}(\varepsilon) - \Theta_{u,2}^{-1}\right\| \leq a_1\varepsilon, \tag{4.12}$$

where $a_1 > 0$ is some constant independent of ε. By virtue of this inequality, the matrix $\Delta_4^{-1}(\varepsilon)$ is bounded for all $\varepsilon \in (0, \varepsilon_1]$.

In what follows, we assume

A8. The matrix Δ_1, given in the equation (4.10), is invertible.

Using the above established invertibility of the matrix $\Delta_4(\varepsilon)$, we consider the following matrix:

$$\Upsilon(\varepsilon) \stackrel{\triangle}{=} \Delta_1 - \varepsilon\Delta_2\Delta_4^{-1}(\varepsilon)\Delta_3, \quad \varepsilon \in (0, \varepsilon_1]. \tag{4.13}$$

Using the aforementioned boundedness of the matrix $\Delta_4^{-1}(\varepsilon)$ and the assumption A8, one can conclude that there exists a positive number $\varepsilon_2 \leq \varepsilon_1$ such that, for all $\varepsilon \in (0, \varepsilon_2]$, the matrix $\Upsilon(\varepsilon)$ is invertible and its inverse matrix satisfies the inequality

$$\left\|\Upsilon^{-1}(\varepsilon) - \Delta_1^{-1}\right\| \leq a_2\varepsilon, \tag{4.14}$$

where $a_2 > 0$ is some constant independent of ε.

Now, applying the Frobenius formula (see, e.g., [46]) to the calculation of $\Delta^{-1}(\varepsilon)$ and taking into account the existence of $\Delta_4^{-1}(\varepsilon)$, $\Upsilon^{-1}(\varepsilon)$ for all $\varepsilon \in (0, \varepsilon_2]$, we obtain that $\Delta^{-1}(\varepsilon)$ exists for all $\varepsilon \in (0, \varepsilon_2]$ and has the form

$$\Delta^{-1}(\varepsilon) \stackrel{\triangle}{=} \Phi(\varepsilon) = \begin{pmatrix} \Phi_1(\varepsilon) & \Phi_2(\varepsilon) \\ \Phi_3(\varepsilon) & \Phi_4(\varepsilon) \end{pmatrix}, \tag{4.15}$$

where

$$\begin{aligned} \Phi_1(\varepsilon) = \Upsilon^{-1}(\varepsilon), \quad \Phi_2(\varepsilon) = -\varepsilon\Upsilon^{-1}(\varepsilon)\Delta_2\Delta_4^{-1}(\varepsilon), \\ \Phi_3(\varepsilon) = -\varepsilon\Delta_4^{-1}(\varepsilon)\Delta_3\Upsilon^{-1}(\varepsilon), \quad \Phi_4(\varepsilon) = \varepsilon\Delta_4^{-1}(\varepsilon) + \varepsilon^2\Delta_4^{-1}(\varepsilon)\Delta_3\Upsilon^{-1}(\varepsilon)\Delta_2\Delta_4^{-1}(\varepsilon). \end{aligned} \tag{4.16}$$

For the sake of the further asymptotic analysis of the matrix $\Delta^{-1}(\varepsilon)$, the matrix $\Gamma^{-1}(\varepsilon)$ (see the equation (4.11)) and, therefore, the open-loop solution to the PCCNEG, we are going to estimate the matrices $\Phi_2(\varepsilon)$, $\Phi_3(\varepsilon)$ and $\Phi_4(\varepsilon)$. Using the expressions of these matrices and the inequalities (4.12), (4.14), we obtain the existence of a positive number $\varepsilon_3 \leq \varepsilon_2$ such that. for all $\varepsilon \in (0, \varepsilon_3]$, the following inequalities are valid:

$$\left\|\Phi_2(\varepsilon)\right\| \leq a_3\varepsilon, \quad \left\|\Phi_3(\varepsilon)\right\| \leq a_3\varepsilon, \quad \left\|\Phi_4(\varepsilon)\right\| \leq a_3\varepsilon, \tag{4.17}$$

$$\begin{aligned} \left\|\frac{1}{\varepsilon}\Phi_2(\varepsilon) + \Delta_1^{-1}\Delta_2\Theta_{u,2}^{-1}\right\| \leq a_3\varepsilon, \quad \left\|\frac{1}{\varepsilon}\Phi_3(\varepsilon) + \Theta_{u,2}^{-1}\Delta_3\Delta_1^{-1}\right\| \leq a_3\varepsilon, \\ \left\|\frac{1}{\varepsilon}\Phi_4(\varepsilon) - \Theta_{u,2}^{-1}\right\| \leq a_3\varepsilon. \end{aligned} \tag{4.18}$$

4.2. **Zero-order asymptotic expansion of the open-loop solution to the PCCNEG.** First of all, let us note the following. Due to the results of the previous subsection (see the equations (4.11) and (4.15)) and the results of Subsection 3.2 (see the equation (3.17)), the value $x(t_f)$ on the Nash equilibrium trajectory of the PCCNEG exists, is unique and has the form

$$x(t_f) = L^T\Phi(\varepsilon)Lx_0, \quad \varepsilon \in (0,\varepsilon_2]. \tag{4.19}$$

Therefore, due to the results of Subsection 3.2 (see the equations (3.12)-(3.13)), for all $\varepsilon \in (0,\varepsilon_2]$ the open-loop solution to the PCCNEG exists, is unique and its components have the form

$$u_\varepsilon^*(t) = -\big(R_u(t)+\mathscr{E}\big)^{-1}\mathscr{M}_u^T(t)L^T\Phi(\varepsilon)Lx_0, \quad t\in[0,t_f], \tag{4.20}$$

$$v_\varepsilon^*(t) = -R_v^{-1}(t)\mathscr{M}_v^T(t)FL^T\Phi(\varepsilon)Lx_0, \quad t\in[0,t_f]. \tag{4.21}$$

Based on this observation, we start the construction of the zero-order asymptotic expansion of the open-loop solution to the PCCNEG with its component $u_\varepsilon^*(t)$. Substitution of (3.3) and (4.1) into (4.20) yields for all $t\in[0,t_f]$ and $\varepsilon\in(0,\varepsilon_2]$

$$u_\varepsilon^*(t) = -\begin{pmatrix} R_{u,1}^{-1}(t) & 0\\ 0 & \varepsilon^{-1}I_{r-q}\end{pmatrix}\begin{pmatrix}\mathscr{M}_{u,1}^T(t)\\ \mathscr{M}_{u,2}^T(t)\end{pmatrix}L^T\Phi(\varepsilon)Lx_0 = \begin{pmatrix}u_{\varepsilon,1}^*(t)\\ u_{\varepsilon,2}^*(t)\end{pmatrix}, \tag{4.22}$$

where

$$\begin{aligned} u_{\varepsilon,1}^*(t) &= -R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)L^T\Phi(\varepsilon)Lx_0, \quad t\in[0,t_f], \quad \varepsilon\in(0,\varepsilon_2],\\ u_{\varepsilon,2}^*(t) &= -\varepsilon^{-1}\mathscr{M}_{u,2}^T(t)L^T\Phi(\varepsilon)Lx_0, \quad t\in[0,t_f],, \quad \varepsilon\in(0,\varepsilon_2]. \end{aligned} \tag{4.23}$$

Let us analyze separately the vector-valued functions $u_{\varepsilon,1}^*(t)$ and $u_{\varepsilon,2}^*(t)$. We start with the first one. For the sake of the asymptotic analysis of $u_{\varepsilon,1}^*(t)$, we introduce into the consideration the following $n\times n$-matrix:

$$\Phi_0 \stackrel{\triangle}{=} \begin{pmatrix}\Delta_1^{-1} & O_{k\times(n-k)}\\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)}\end{pmatrix}. \tag{4.24}$$

Due to the equations (4.15), (4.16), (4.24) and the inequalities (4.14), (4.17), we directly have the inequality

$$\big\|\Phi(\varepsilon)-\Phi_0\big\| \le a_4\varepsilon, \quad \forall\, \varepsilon\in(0,\varepsilon_3], \tag{4.25}$$

where $a_4 = a_2 + 3a_3$.

Now, replacing the matrix $\Phi(\varepsilon)$ with the matrix Φ_0 in the expression for $u_{\varepsilon,1}^*(t)$ (see the equation (4.23)), we obtain the following ε-independent vector-valued function:

$$u_{0,1}^*(t) \stackrel{\triangle}{=} -R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)L^T\Phi_0Lx_0, \quad t\in[0,t_f]. \tag{4.26}$$

Due to the inequality (4.25), we immediately have the inequality

$$\big\|u_{\varepsilon,1}^*(t)-u_{0,1}^*(t)\big\| \le c_{u,1}\varepsilon \quad \forall\, t\in[0,t_f],\ \varepsilon\in(0,\varepsilon_3], \tag{4.27}$$

where $c_{u,1} = a_4\max_{t\in[0,t_f]}\big\|R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)\big\|\big\|L\big\|^2\big\|x_0\big\|$.

Proceed to the asymptotic analysis of $u_{\varepsilon,2}^*(t)$. From (4.6), we directly have

$$\mathscr{M}_{u,2}^T(t)L^T = \big(O_{(r-q)\times k}\,,\,\Lambda_{u,2}^T(t)\big), \quad t\in[0,t_f]. \tag{4.28}$$

Substitution of (4.15) and (4.28) into the expression for $u^*_{\varepsilon,2}(t)$ (see the equation (4.23)) yields after a routine matrix algebra the following expression for all $t \in [0,t_f]$ and $\varepsilon \in (0,\varepsilon_2]$:

$$u_2^*(t,\varepsilon) = -\Big(\varepsilon^{-1}\Lambda^T_{u,2}(t)\Phi_3(\varepsilon)\ ,\ \varepsilon^{-1}\Lambda^T_{u,2}(t)\Phi_4(\varepsilon)\Big)Lx_0. \tag{4.29}$$

Replacing $\varepsilon^{-1}\Phi_3(\varepsilon)$ and $\varepsilon^{-1}\Phi_4(\varepsilon)$ with $-\Theta^{-1}_{u,2}\Delta_3\Delta_1^{-1}$ and $\Theta^{-1}_{u,2}$, respectively, in the expression (4.29) for $u_2^*(t,\varepsilon)$, we obtain the following ε-independent vector-valued function:

$$u^*_{0,2}(t) \stackrel{\triangle}{=} \Big(\Lambda^T_{u,2}(t)\Theta^{-1}_{u,2}\Delta_3\Delta_1^{-1}\ ,\ -\Lambda^T_{u,2}(t)\Theta^{-1}_{u,2}\Big)Lx_0, \quad t \in [0,t_f]. \tag{4.30}$$

Due to the estimates for $\varepsilon^{-1}\Phi_3(\varepsilon)$ and $\varepsilon^{-1}\Phi_4(\varepsilon)$ in (4.18), we obtain the inequality

$$\big\|u^*_{\varepsilon,2}(t) - u^*_{0,2}(t)\big\| \le c_{u,2}\varepsilon \quad \forall\, t \in [0,t_f],\ \varepsilon \in (0,\varepsilon_3], \tag{4.31}$$

where $c_{u,2} = 2a_3 \max_{t\in[0,t_f]} \big\|\Lambda_{u,2}(t)\big\|\big\|L\big\|\big\|x_0\big\|$.

Consider the following block-form vector-valued function:

$$u_0^*(t) \stackrel{\triangle}{=} \mathrm{col}\big(u^*_{0,1}(t)\ ,\ u^*_{0,2}(t)\big), \quad t \in [0,t_f]. \tag{4.32}$$

Using the equations (4.22), (4.32) and the inequalities (4.27), (4.31), we directly have the following inequality:

$$\big\|u^*_\varepsilon(t) - u^*_0(t)\big\| \le c_u\varepsilon \quad \forall\, t \in [0,t_f],\ \varepsilon \in (0,\varepsilon_3], \tag{4.33}$$

where $c_u = c_{u,1} + c_{u,2}$.

The inequality (4.33) means that $u_0^*(t)$ is the zero-order asymptotic expansion with respect to $\varepsilon > 0$ of $u^*_\varepsilon(t)$, and this expansion is uniform in $t \in [0,t_f]$.

Proceed to the constructing the zero-order asymptotic expansion of the component $v^*_\varepsilon(t)$ of the open-loop solution to the PCCNEG. Replacing the matrix $\Phi(\varepsilon)$ with the matrix Φ_0 in the expression for $v^*_\varepsilon(t)$ (see the equation (4.21)), we obtain the following ε-independent vector-valued function:

$$v_0^*(t) \stackrel{\triangle}{=} -R_v^{-1}(t)\mathscr{M}_v^T(t)FL^T\Phi_0Lx_0, \quad t \in [0,t_f]. \tag{4.34}$$

Using the inequality (4.25) directly yields the inequality

$$\big\|v^*_\varepsilon(t) - v^*_0(t)\big\| \le c_v\varepsilon \quad \forall\, t \in [0,t_f],\ \varepsilon \in (0,\varepsilon_3], \tag{4.35}$$

where $c_v = a_4 \max_{t\in[0,t_f]} \big\|R_v^{-1}(t)\mathscr{M}_v^T(t)\big\|\big\|F\big\|\big\|L\big\|^2\big\|x_0\big\|$.

The inequality (4.35) means that $v_0^*(t)$ is the zero-order asymptotic expansion with respect to $\varepsilon > 0$ of $v^*_\varepsilon(t)$, and this expansion is uniform in $t \in [0,t_f]$.

4.3. **Zero-order asymptotic expansion of the Nash equilibrium values of the functionals in the PCCNEG.** We start with the Nash equilibrium value $J^*_{u,\varepsilon}$ of the functional $J_{u,\varepsilon}\big(u(\cdot),v(\cdot)\big)$ in the PCCNEG (see the equation (3.1)). By virtue of Remark 3.1 and Definition 2.1, we have

$$J^*_{u,\varepsilon} = J_{u,\varepsilon}\big(u^*_\varepsilon(t),v^*_\varepsilon(t)\big) = \frac{1}{2}x^T(t_f)x(t_f) + \frac{1}{2}\int_0^{t_f}\big(u^*_\varepsilon(t)\big)^T\big(R_u(t)+\mathscr{E}\big)u^*_\varepsilon(t)dt. \tag{4.36}$$

Substitution of (4.19) and (4.20) into (4.36) yields after some routine rearrangement the following expression for $J^*_{u,\varepsilon}$ valid for all $\varepsilon \in (0,\varepsilon_2]$:

$$J^*_{u,\varepsilon} = \frac{1}{2}x_0^T L^T \Phi^T(\varepsilon)\left[I_n + L\int_0^{t_f} \mathscr{M}_u(t)\big(R_u(t)+\mathscr{E}\big)^{-1}\mathscr{M}_u^T(t)dtL^T\right]\Phi(\varepsilon)Lx_0. \tag{4.37}$$

Using the equations (4.1)-(4.5), we can rewrite (4.37) as:

$$J^*_{u,\varepsilon} = \frac{1}{2}x_0^T L^T \Phi^T(\varepsilon)\left[I_n + L\int_0^{t_f} \mathscr{M}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)dtL^T + \frac{1}{\varepsilon}D_{u,2}\right]\Phi(\varepsilon)Lx_0. \tag{4.38}$$

Calculating the product $\frac{1}{\varepsilon}\Phi^T(\varepsilon)D_{u,2}\Phi(\varepsilon)$ and using the block representations of the matrices $D_{u,2}$ and $\Phi(\varepsilon)$ (see the equations (4.5) and (4.15), we obtain

$$\frac{1}{\varepsilon}\Phi^T(\varepsilon)D_{u,2}\Phi(\varepsilon) = \frac{1}{\varepsilon}\begin{pmatrix} \Phi_3^T(\varepsilon)\Theta_{u,2}\Phi_3(\varepsilon) & \Phi_3^T(\varepsilon)\Theta_{u,2}\Phi_4(\varepsilon) \\ \Phi_4^T(\varepsilon)\Theta_{u,2}\Phi_3(\varepsilon) & \Phi_4^T(\varepsilon)\Theta_{u,2}\Phi_4(\varepsilon) \end{pmatrix}. \tag{4.39}$$

Due to the inequalities for $\Phi_3(\varepsilon)$ and $\Phi_4(\varepsilon)$ (see the equation (4.17)), the equation (4.39) yields the inequality

$$\left\|\frac{1}{\varepsilon}\Phi^T(\varepsilon)D_{u,2}\Phi(\varepsilon)\right\| \le a_5\varepsilon \quad \forall\, \varepsilon \in (0,\varepsilon_3], \tag{4.40}$$

where $a_5 = 4a_3^2\|\Theta_{u,2}\|$.

Consider the value

$$J^*_{u,0} \triangleq \frac{1}{2}x_0^T L^T \Phi_0^T\left[I_n + L\int_0^{t_f} \mathscr{M}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)dtL^T\right]\Phi_0 Lx_0. \tag{4.41}$$

Note that the value $J^*_{u,0}$ is obtained from the value $J^*_{u,\varepsilon}$ (see the equation (4.38)) by the replacement of the matrix $\frac{1}{\varepsilon}\Phi^T(\varepsilon)D_{u,2}\Phi(\varepsilon)$ with zero matrix and the matrix $\Phi(\varepsilon)$ with the matrix Φ_0.

Using the block representations of the matrices $L\int_0^{t_f}\mathscr{M}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)dtL^T$ and Φ_0 (see the equations (4.9) and (4.24)), we can rewrite the value $J^*_{u,0}$ as:

$$J^*_{u,0} = \frac{1}{2}x_0^T L^T \begin{pmatrix} (\Delta_1^{-1})^T(I_k+\Omega_{u,11})\Delta_1^{-1} & O_{k\times(n-k)} \\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)} \end{pmatrix} Lx_0. \tag{4.42}$$

Due to the equations (4.38), (4.41) and the inequalities (4.25), (4.40), we have the inequality

$$\left|J^*_{u,\varepsilon} - J^*_{u,0}\right| \le \alpha_u\varepsilon \quad \forall\, \varepsilon \in (0,\varepsilon_3], \tag{4.43}$$

where $\alpha_u = \left(a_4\left\|I_n + L\int_0^{t_f}\mathscr{M}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)dtL^T\right\|\left(\|\Delta_1^{-1}\|+0.5\varepsilon_3\right)+0.5a_5\right)\|L\|^2\|x_0\|^2$.

The inequality (4.43) means that $J^*_{u,0}$ is the zero-order asymptotic expansion with respect to $\varepsilon > 0$ of $J^*_{u,\varepsilon}$.

Proceed to constructing the zero-order asymptotic expansion of the Nash equilibrium value $J^*_{v,\varepsilon}$ of the functional $J_v\big(u(\cdot),v(\cdot)\big)$ in the PCCNEG (see the equation (2.36)). Due to Remark 3.1 and Definition 2.1, we have

$$J^*_{v,\varepsilon} = J_v\big(u^*_\varepsilon(t),v^*_\varepsilon(t)\big) = \frac{1}{2}x^T(t_f)Fx(t_f) + \frac{1}{2}\int_0^{t_f}\big(v^*_\varepsilon(t)\big)^T R_v(t)v^*_\varepsilon(t)dt. \tag{4.44}$$

Substituting (4.19) and (4.21) into (4.44), we obtain after some routine rearrangement the following expression for $J^*_{v,\varepsilon}$ valid for all $\varepsilon \in (0,\varepsilon_2]$:

$$J^*_{v,\varepsilon} = \frac{1}{2}x_0^T L^T \Phi^T(\varepsilon) L \left[F + F\int_0^{t_f} \mathscr{M}_v(t) R_v^{-1}(t) \mathscr{M}_v^T(t) dt F \right] L^T \Phi(\varepsilon) L x_0. \tag{4.45}$$

Replacing the matrix $\Phi(\varepsilon)$ with the matrix Φ_0 in (4.45), we obtain the value

$$J^*_{v,0} = \frac{1}{2}x_0^T L^T \Phi_0^T L \left[F + F\int_0^{t_f} \mathscr{M}_v(t) R_v^{-1}(t) \mathscr{M}_v^T(t) dt F \right] L^T \Phi_0 L x_0. \tag{4.46}$$

Using the equations (4.45), (4.46) and the inequality (4.25), we directly obtain the inequality

$$\left| J^*_{v,\varepsilon} - J^*_{v,0} \right| \le \alpha_v \varepsilon \quad \forall\, \varepsilon \in (0,\varepsilon_3], \tag{4.47}$$

where $\alpha_v = a_4 \left\| F + LF\int_0^{t_f} \mathscr{M}_{u,1}(t) R_{u,1}^{-1}(t) \mathscr{M}_{u,1}^T(t) dt F L^T \right\| \left(\|\Delta_1^{-1}\| + 0.5\varepsilon_3 \right) \|L\|^2 \|x_0\|^2$.

The inequality (4.47) means that $J^*_{v,0}$ is the zero-order asymptotic expansion with respect to $\varepsilon > 0$ of $J^*_{v,\varepsilon}$.

4.4. **Game-theoretic interpretation of the values $J^*_{u,0}$ and $J^*_{v,0}$.** In this subsection, we are going to show that the pair $\left(J^*_{u,0}, J^*_{v,0}\right)$ is the outcome of the OSNEG generated by the pair of the controls $\left(u_0^*(t), v_0^*(t)\right)$, $t \in [0,t_f]$, i.e.,

$$J^*_{u,0} = J_u\left(u_0^*(t), v_0^*(t)\right), \tag{4.48}$$

$$J^*_{v,0} = J_v\left(u_0^*(t), v_0^*(t)\right), \tag{4.49}$$

where the functionals $J_u\left(u(\cdot), v(\cdot)\right)$ and $J_v\left(u(\cdot), v(\cdot)\right)$ are given by (2.35) and (2.36), respectively.

Let $x_0^*(t)$, $t \in [0,t_f]$ be the solution of the initial-value problem (2.31)-(2.32) generated by the pair of the controls $\left(u_0^*(t), v_0^*(t)\right)$, $t \in [0,t_f]$. Thus,

$$x_0^*(t_f) = x_0 + \int_0^{t_f} \left[\mathscr{M}_u(t) u_0^*(t) + \mathscr{M}_v(t) v_0^*(t) \right] dt. \tag{4.50}$$

Substituting (4.1) and (4.32) into (4.50), we obtain after some routine algebra

$$x_0^*(t_f) = x_0 + \int_0^{t_f} \mathscr{M}_{u,1}(t) u_{0,1}^*(t) dt + \int_0^{t_f} \mathscr{M}_{u,2}(t) u_{0,2}^*(t) dt + \int_0^{t_f} \mathscr{M}_v(t) v_0^*(t) dt. \tag{4.51}$$

Let us treat separately the first, second, third and forth addends in the right-hand side of (4.51). We start with the first addend. Since the matrix L is an orthogonal $n \times n$-matrix, then

$$x_0 = L^T L x_0. \tag{4.52}$$

Continue with the second addend. Substitution of (4.26) into this addend yields

$$\int_0^{t_f} \mathscr{M}_{u,1}(t) u_{0,1}^*(t) dt = -\int_0^{t_f} \mathscr{M}_{u,1}(t) R_{u,1}^{-1}(t) \mathscr{M}_{u,1}^T(t) dt L^T \Phi_0 L x_0. \tag{4.53}$$

Since $L^T L = I_n$, we can multiply both sides of (4.53) from the left by $L^T L$, which yields the equivalent equality

$$\int_0^{t_f} \mathscr{M}_{u,1}(t) u_{0,1}^*(t) dt = -L^T \left(L \int_0^{t_f} \mathscr{M}_{u,1}(t) R_{u,1}^{-1}(t) \mathscr{M}_{u,1}^T(t) dt L^T \right) \Phi_0 L x_0. \tag{4.54}$$

Using the block representations of the matrices $L\int_0^{t_f}\mathscr{M}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)dtL^T$ and Φ_0 (see the equations (4.9) and (4.24)), we can rewrite the equation (4.54) as:

$$\int_0^{t_f}\mathscr{M}_{u,1}(\sigma)u_{0,1}^*(\sigma)d\sigma = -L^T\begin{pmatrix}\Omega_{u,11} & \Omega_{u,12}\\ \Omega_{u,12}^T & \Omega_{u,13}\end{pmatrix}\begin{pmatrix}\Delta_1^{-1} & O_{k\times(n-k)}\\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)}\end{pmatrix}Lx_0 = -L^T\begin{pmatrix}\Omega_{u,11}\Delta_1^{-1} & O_{k\times(n-k)}\\ \Omega_{u,12}^T\Delta_1^{-1} & O_{(n-k)\times(n-k)}\end{pmatrix}Lx_0. \tag{4.55}$$

Proceed to the analysis of the third addend in the right-hand side of (4.51). Substitution of (4.30) into this addend yields after some rearrangement

$$\int_0^{t_f}\mathscr{M}_{u,2}(t)u_{0,2}^*(t)dt = \int_0^{t_f}\mathscr{M}_{u,2}(t)\Lambda_{u,2}^T(t)dt\Theta_{u,2}^{-1}\Big(\Delta_3\Delta_1^{-1}\ ,\ -I_{n-k}\Big)Lx_0. \tag{4.56}$$

Multiplying both sides of this equality from the left by $L^TL = I_n$, we obtain the equivalent equality

$$\int_0^{t_f}\mathscr{M}_{u,2}(t)u_{0,2}^*(t)dt = L^T\int_0^{t_f}\big(L\mathscr{M}_{u,2}(t)\big)\Lambda_{u,2}^T(t)dt\Theta_{u,2}^{-1}\Big(\Delta_3\Delta_1^{-1}\ ,\ -I_{n-k}\Big)Lx_0. \tag{4.57}$$

By virtue of the equations (4.6) and (4.7), we can rewrite the equation (4.57) as:

$$\begin{aligned}\int_0^{t_f}\mathscr{M}_{u,2}(t)u_{0,2}^*(t)dt &= L^T\int_0^{t_f}\begin{pmatrix}O_{k\times(r-k)}\\ \Lambda_{u,2}(t)\end{pmatrix}\Lambda_{u,2}^T(t)dt\Theta_{u,2}^{-1}\Big(\Delta_3\Delta_1^{-1}\ ,\ -I_{n-k}\Big)Lx_0 =\\ &L^T\int_0^{t_f}\begin{pmatrix}O_{k\times(n-k)}\\ \Lambda_{u,2}(t)\Lambda_{u,2}^T(t)\end{pmatrix}dt\Theta_{u,2}^{-1}\Big(\Delta_3\Delta_1^{-1}\ ,\ -I_{n-k}\Big)Lx_0 =\\ &L^T\begin{pmatrix}O_{k\times(n-k)}\\ \int_0^{t_f}\Lambda_{u,2}(t)\Lambda_{u,2}^T(t)dt\end{pmatrix}\Theta_{u,2}^{-1}\Big(\Delta_3\Delta_1^{-1}\ ,\ -I_{n-k}\Big)Lx_0 =\\ &L^T\begin{pmatrix}O_{k\times(n-k)}\\ \Theta_{u,2}\end{pmatrix}\Theta_{u,2}^{-1}\Big(\Delta_3\Delta_1^{-1}\ ,\ -I_{n-k}\Big)Lx_0 =\\ L^T\begin{pmatrix}O_{k\times(n-k)}\\ I_{n-k}\end{pmatrix}\Big(\Delta_3\Delta_1^{-1}\ ,\ -I_{n-k}\Big)Lx_0 &= L^T\begin{pmatrix}O_{k\times k} & O_{k\times(n-k)}\\ \Delta_3\Delta_1^{-1} & -I_{n-k}\end{pmatrix}Lx_0.\end{aligned} \tag{4.58}$$

Finally, let us treat the fourth addend in the right-hand side of (4.51). Substituting (4.34) into this addend, we obtain

$$\int_0^{t_f}\mathscr{M}_v(t)v_0^*(t)dt = -\int_0^{t_f}\mathscr{M}_v(t)R_v^{-1}(t)\mathscr{M}_v^T(t)dtFL^T\Phi_0Lx_0. \tag{4.59}$$

Similarly to (4.54), we multiply both sides of (4.59) from the left by $L^TL = I_n$, which yields the equivalent equality

$$\int_0^{t_f}\mathscr{M}_v(t)v_0^*(t)dt = -L^T\left(L\int_0^{t_f}\mathscr{M}_v(t)R_v^{-1}(t)\mathscr{M}_v^T(t)dtFL^T\right)\Phi_0Lx_0. \tag{4.60}$$

Using the block representations of the matrices $L\int_0^{t_f}\mathscr{M}_v(t)R_v^{-1}(t)\mathscr{M}_v^T(t)dtFL^T$ and Φ_0 (see the equations (4.9) and (4.24)), we can rewrite the equation (4.60) as:

$$\int_0^{t_f}\mathscr{M}_v(t)v_0^*(t)dt = -L^T\begin{pmatrix}\Omega_{v,1} & \Omega_{v,2}\\ \Omega_{v,3} & \Omega_{v,4}\end{pmatrix}\begin{pmatrix}\Delta_1^{-1} & O_{k\times(n-k)}\\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)}\end{pmatrix}Lx_0 = -L^T\begin{pmatrix}\Omega_{v,1}\Delta_1^{-1} & O_{k\times(n-k)}\\ \Omega_{v,3}\Delta_1^{-1} & O_{(n-k)\times(n-k)}\end{pmatrix}Lx_0. \tag{4.61}$$

Using (4.52), (4.55), (4.58) and (4.61), we can rewrite the equality (4.51) as:

$$x_0^*(t_f) = L^T\begin{pmatrix}I_k-(\Omega_{u,11}+\Omega_{v,1})\Delta_1^{-1} & O_{k\times(n-k)}\\ (\Delta_3-\Omega_{u,12}^T-\Omega_{v,3})\Delta_1^{-1} & O_{(n-k)\times(n-k)}\end{pmatrix}Lx_0. \tag{4.62}$$

Due to (4.10),

$$I_k-(\Omega_{u,11}+\Omega_{v,1})\Delta_1^{-1} = I_k-(\Delta_1-I_k)\Delta_1^{-1} = \Delta_1^{-1},$$
$$(\Delta_3-\Omega_{u,12}^T-\Omega_{v,3})\Delta_1^{-1} = O_{(n-k)\times k}. \tag{4.63}$$

Substitution of (4.63) into (4.62) and use of Φ_0, yield immediately

$$x_0^*(t_f) = L^T\begin{pmatrix}\Delta_1^{-1} & O_{k\times(n-k)}\\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)}\end{pmatrix}Lx_0 = L^T\Phi_0Lx_0. \tag{4.64}$$

Now, substituting $x_0^*(t_f)$ and $u_0^*(t)$, $t\in[0,t_f]$ (see the equations (4.64) and (4.32)) into (2.35) instead of $x(t_f)$ and $u(t)$, $t\in[0,t_f]$, respectively, as well as using the equations (2.7), (4.26), (4.41) and taking into account that the matrix L is orthogonal, we directly obtain the equality (4.48). Similarly, substituting $x_0^*(t_f)$ and $v_0^*(t)$, $t\in[0,t_f]$ (see the equations (4.64) and (4.34)) into (2.36) instead of $x(t_f)$ and $v(t)$, $t\in[0,t_f]$, respectively, and using the equation (4.46), we immediately obtain the equality (4.49).

5. Open-Loop Solution of the OSNEG

Lemma 5.1. *Let the assumptions A1-A8 be satisfied. Then, the following inequality is valid in the OSNEG:*

$$J_v\big(u_0^*(t),v_0^*(t)\big)\le J_v\big(u_0^*(t),v(t)\big)\quad \forall\, v(t)\in L^2[0,t_f;E^s], \tag{5.1}$$

where the functional $J_v\big(u(\cdot),v(\cdot)\big)$ is given by (2.36), the control $u_0^*(t)$, $t\in[0,t_f]$ is given by (4.26), (4.30), (4.32), the control $v_0^*(t)$, $t\in[0,t_f]$ is given by (4.34).

Proof. First of all, let us note that the inequality (5.1) is valid if and only if the control $v_0^*(t)$, $t\in[0,t_f]$ is an open-loop solution (an open-loop optimal control) of the optimal control problem consisting of the dynamics

$$\frac{d\widetilde{x}(t)}{dt} = \mathscr{M}_u(t)u_0^*(t)+\mathscr{M}_v(t)\widetilde{v}(t),\quad t\in[0,t_f],\quad \widetilde{x}(0)=x_0 \tag{5.2}$$

and the performance index

$$\widetilde{J}_v\big(\widetilde{v}(\cdot)\big) = \frac{1}{2}\widetilde{x}^T(t_f)F\widetilde{x}(t_f)+\frac{1}{2}\int_0^{t_f}\widetilde{v}^T(t)R_v(t)\widetilde{v}(t)dt\to\min_{\widetilde{v}(t)\in L^2[0,t_f;E^s]}. \tag{5.3}$$

Let us derive the solution of the optimal control problem (5.2)-(5.3). Following the results of [2], we write down the variational Hamiltonian for this problem

$$\widetilde{H}\big(\widetilde{v}(t),t\big) = \widetilde{\lambda}^T(t)\big(\mathscr{M}_u(t)u_0^*(t) + \mathscr{M}_v(t)\widetilde{v}(t)\big) + \frac{1}{2}\widetilde{v}^T(t)R_v(t)\widetilde{v}(t), \;\; t\in[0,t_f], \tag{5.4}$$

where $\widetilde{\lambda}(t)$ is the n-dimensional costate variable.

The costate variable $\widetilde{\lambda}(t)$ satisfies the following terminal-value problem:

$$\frac{d\widetilde{\lambda}(t)}{dt} = -\frac{\partial \widetilde{H}\big(\widetilde{v}(t),t\big)}{\partial \widetilde{x}(t)} = 0, \;\; t\in[0,t_f], \;\; \widetilde{\lambda}(t_f) = \frac{\partial\big(\frac{1}{2}\widetilde{x}^T(t_f)F\widetilde{x}(t_f)\big)}{\partial \widetilde{x}(t_f)} = F\widetilde{x}(t_f). \tag{5.5}$$

The open-loop optimal control of the problem (5.2)-(5.3) is

$$\widetilde{v}^*(t) = \arg\min_{\widetilde{v}(t)} \widetilde{H}\big(\widetilde{v}(t),t\big) = -R_v^{-1}(t)\mathscr{M}_v^T(t)\widetilde{\lambda}(t), \quad t\in[0,t_f]. \tag{5.6}$$

Solving the terminal-value problem (5.5), we obtain

$$\widetilde{\lambda}(t) \equiv F\widetilde{x}(t_f), \quad t\in[0,t_f]. \tag{5.7}$$

Substitution of (5.7) into (5.6) yields

$$\widetilde{v}^*(t) = -R_v^{-1}(t)\mathscr{M}_v^T(t)F\widetilde{x}(t_f), \quad t\in[0,t_f]. \tag{5.8}$$

Finally, substitution of (5.8) into the system (5.2) instead of $\widetilde{v}(t)$ yields the following initial-value problem for the functional-differential system:

$$\frac{d\widetilde{x}(t)}{dt} = \mathscr{M}_u(t)u_0^*(t) - \mathscr{M}_v(t)R_v^{-1}(t)\mathscr{M}_v^T(t)F\widetilde{x}(t_f), \;\; t\in[0,t_f], \;\; \widetilde{x}(0)=x_0. \tag{5.9}$$

Integrating the system (5.9) from $t=0$ to $t=t_f$, we directly obtain the following linear algebraic equation with respect to $\widetilde{x}(t_f)$:

$$\widetilde{x}(t_f) = \mathscr{H}\big(\widetilde{x}(t_f)\big) \stackrel{\triangle}{=} x_0 + \int_0^{t_f}\big[\mathscr{M}_u(t)u_0^*(t) - \mathscr{M}_v R_v^{-1}(t)\mathscr{M}_v^T(t)F\widetilde{x}(t_f)\big]dt. \tag{5.10}$$

Let us show that $\widetilde{x}(t_f) = L^T\Phi_0 L x_0$ is a solution of the equation (5.10). Substituting this expression into $\mathscr{H}\big(\widetilde{x}(t_f)\big)$ and using (4.34), (4.50), (4.64), we have

$$\begin{aligned}\mathscr{H}\big(L^T\Phi_0 L x_0\big) &= x_0 + \int_0^{t_f}\big[\mathscr{M}_u(t)u_0^*(t) - \mathscr{M}_v R_v^{-1}(t)\mathscr{M}_v^T(t)FL^T\Phi_0 L x_0\big]dt \\ &= x_0 + \int_0^{t_f}\big[\mathscr{M}_u(t)u_0^*(t) + \mathscr{M}_v(t)v_0^*(t)\big]dt = L^T\Phi_0 L x_0,\end{aligned} \tag{5.11}$$

meaning that $\widetilde{x}(t_f) = L^T\Phi_0 L x_0$ satisfies the equation (5.10).

Now, let us show that $\widetilde{x}(t_f) = L^T\Phi_0 L x_0$ is the unique solution of the equation (5.10). For this purpose, it is necessary and sufficient to show that the matrix

$$\mathscr{G} \stackrel{\triangle}{=} I_n + \int_0^{t_f}\mathscr{M}_v R_v^{-1}(t)\mathscr{M}_v^T(t)dtF \tag{5.12}$$

is invertible. Let us observe that the matrices $\int_0^{t_f}\mathscr{M}_v R_v^{-1}(t)\mathscr{M}_v^T(t)dt$ and F are symmetric and positive semi-definite. Let the matrix $\int_0^{t_f}\mathscr{M}_v R_v^{-1}(t)\mathscr{M}_v^T(t)dt$ has zero eigenvalue of the algebraic multiplicity l, where $0\le l\le n$. If $l=n$, then $\int_0^{t_f}\mathscr{M}_v R_v^{-1}(t)\mathscr{M}_v^T(t)dt = O_{n\times n}$, meaning that $\mathscr{G}$ is invertible. Proceed with the case $l < n$. Since $\int_0^{t_f} \mathscr{M}_v R_v^{-1}(t)\mathscr{M}_v^T(t)dt$ is a symmetric matrix, then, due to the work [45], there exists the orthogonal matrix $\mathscr{L}$ ($\mathscr{L}\mathscr{L}^T = I_n$) such that

$$\mathscr{L}\left(\int_0^{t_f} \mathscr{M}_v R_v^{-1}(t)\mathscr{M}_v^T(t)dt\right)\mathscr{L}^T = \mathrm{diag}\big(O_{l\times l}, \Theta_v\big), \tag{5.13}$$

where $\Theta_v = \mathrm{diag}\big(\nu_1, ..., \nu_{n-l}\big)$. Moreover, since $\int_0^{t_f} \mathscr{M}_v R_v^{-1}(t)\mathscr{M}_v^T(t)dt$ is a positive semi-definite matrix, then $\nu_j > 0$, $(j = 1, ..., n-l)$.

Let

$$\Theta_v^{1/2} \triangleq \mathrm{diag}\big(\sqrt{\nu_1}, ..., \sqrt{\nu_{n-l}}\big), \quad \Theta_v^{-1/2} \triangleq \mathrm{diag}\left(\frac{1}{\sqrt{\nu_1}}, ..., \frac{1}{\sqrt{\nu_{n-l}}}\right), \tag{5.14}$$

$$\Upsilon_v \triangleq \mathrm{diag}\big(I_l, \Theta_v^{-1/2}\big). \tag{5.15}$$

Then,

$$\Upsilon_v^{-1} = \mathrm{diag}\big(I_l, \Theta_v^{1/2}\big). \tag{5.16}$$

Multiplying $\mathscr{G}$ by $\Upsilon_v\mathscr{L}$ from the left and by $\mathscr{L}^T\Upsilon_v^{-1}$ from the right, as well as using the equations (5.12)-(5.16) and that $\mathscr{L}$ is an orthogonal matrix, we obtain

$$\begin{aligned}\Upsilon_v\mathscr{L}\mathscr{G}\mathscr{L}^T\Upsilon_v^{-1} &= \Upsilon_v\mathscr{L}\left(I_n + \int_0^{t_f} \mathscr{M}_v R_v^{-1}(t)\mathscr{M}_v^T(t)dtF\right)\mathscr{L}^T\Upsilon_v^{-1} \\ &= I_n + \left[\Upsilon_v\mathscr{L}\left(\int_0^{t_f} \mathscr{M}_v R_v^{-1}(t)\mathscr{M}_v^T(t)dt\right)\mathscr{L}^T\Upsilon_v\right]\Upsilon_v^{-1}\mathscr{L}F\mathscr{L}^T\Upsilon_v^{-1} \\ &= I_n + \mathrm{diag}\big(O_{l\times l}, I_{n-l}\big)\Upsilon_v^{-1}\mathscr{L}F\mathscr{L}^T\Upsilon_v^{-1}.\end{aligned} \tag{5.17}$$

Let us note the following. Since the matrix F is symmetric and positive semi-definite, then the matrix $\Upsilon_v^{-1}\mathscr{L}F\mathscr{L}^T\Upsilon_v^{-1}$ is symmetric and positive semi-definite.

To continue the proof of the invertibility of the matrix $\mathscr{G}$, we consider two cases with respect to the number l: (i) $l = 0$; (ii) $0 < l < n$.

In the case (i), we have from (5.17) that $\Upsilon_v\mathscr{L}\mathscr{G}\mathscr{L}^T\Upsilon_v^{-1} = I_n + \Upsilon_v^{-1}\mathscr{L}F\mathscr{L}^T\Upsilon_v^{-1}$. Thus, in this case the matrix $\Upsilon_v\mathscr{L}\mathscr{G}\mathscr{L}^T\Upsilon_v^{-1}$ is invertible, meaning that the matrix $\mathscr{G}$ is invertible.

Proceed to the case (ii). Let us partition the matrix $\Upsilon_v^{-1}\mathscr{L}F\mathscr{L}^T\Upsilon_v^{-1}$ into blocks as:

$$\Upsilon_v^{-1}\mathscr{L}F\mathscr{L}^T\Upsilon_v^{-1} = \begin{pmatrix} \mathscr{F}_1 & \mathscr{F}_2 \\ \mathscr{F}_2^T & \mathscr{F}_3 \end{pmatrix}, \tag{5.18}$$

where the blocks $\mathscr{F}_1$, $\mathscr{F}_2$ and $\mathscr{F}_3$ have the dimensions $l\times l$, $l\times(n-l)$ and $(n-l)\times(n-l)$, respectively; the matrices $\mathscr{F}_1$ and $\mathscr{F}_3$ are symmetric and positive semi-definite.

Using the equations (5.17) and (5.18), we obtain

$$\Upsilon_v\mathscr{L}\mathscr{G}\mathscr{L}^T\Upsilon_v^{-1} = I_n + \mathrm{diag}\big(O_{l\times l}, I_{n-l}\big)\begin{pmatrix} \mathscr{F}_1 & \mathscr{F}_2 \\ \mathscr{F}_2^T & \mathscr{F}_3 \end{pmatrix} = \begin{pmatrix} I_l & O_{l\times(n-l)} \\ \mathscr{F}_2^T & I_{n-l}+\mathscr{F}_3 \end{pmatrix}, \tag{5.19}$$

implying the invertibility of the matrix $\Upsilon_v\mathscr{L}\mathscr{G}\mathscr{L}^T\Upsilon_v^{-1}$ and, therefore, the invertibility of the matrix $\mathscr{G}$.

Thus, for any $0 \le l \le n$, the matrix $\mathscr{G}$ is invertible, yielding the uniqueness of the solution $\widetilde{x}(t_f) = L^T\Phi_0 L x_0$ to the equation (5.10).

Substitution of $\widetilde{x}(t_f) = L^T \Phi_0 L x_0$ into (5.8) yields the following final expression for the open-loop optimal control of the problem (5.2)-(5.3):

$$\widetilde{v}^*(t) = -R_v^{-1}(t)\mathscr{M}_v^T(t)FL^T\Phi_0 Lx_0, \quad t \in [0,t_f]. \tag{5.20}$$

Comparing (5.20) with (4.34), one can directly conclude that $\widetilde{v}^*(t) = v_0^*(t)$, $t \in [0,t_f]$, meaning that the control $v_0^*(t)$, $t \in [0,t_f]$ is the open-loop solution (the open-loop control) of the optimal control problem (5.2)-(5.3). This yields the validity of the inequality (5.1), which completes the proof of the lemma. □

Lemma 5.2. *Let the assumptions A1-A8 be satisfied. Then, the following inequality is valid in the OSNEG:*

$$J_u\big(u_0^*(t), v_0^*(t)\big) \le J_u\big(u(t), v_0^*(t)\big) \quad \forall\, u(t) \in L^2[0,t_f;E^r], \tag{5.21}$$

where the functional $J_u\big(u(\cdot),v(\cdot)\big)$ is given by (2.35), the control $u_0^*(t)$, $t \in [0,t_f]$ is given by (4.26), (4.30), (4.32), the control $v_0^*(t)$, $t \in [0,t_f]$ is given by (4.34).

Proof. First of all, let us note that the inequality (5.21) is valid if and only if the control $u_0^*(t)$, $t \in [0,t_f]$ is an open-loop solution (an open-loop optimal control) of the optimal control problem consisting of the dynamics

$$\frac{d\widehat{x}(t)}{dt} = \mathscr{M}_u(t)\widehat{u}(t) + \mathscr{M}_v(t)v_0^*(t), \quad t \in [0,t_f], \quad \widehat{x}(0) = x_0 \tag{5.22}$$

and the performance index

$$\widehat{J}_u\big(\widehat{u}(\cdot)\big) \triangleq \frac{1}{2}\widehat{x}^T(t_f)\widehat{x}(t_f) + \frac{1}{2}\int_0^{t_f} \widehat{u}^T(t)R_u(t)\widehat{u}(t)dt \to \min_{\widehat{u}(t)\in L^2[0,t_f;E^r]}. \tag{5.23}$$

Let us observe that, due to the assumption A5 (see the equation (2.7)), the matrix $R_u(t)$, $t \in [0,t_f]$ is singular. Therefore, the optimal control problem (5.22)-(5.23) is singular (see, e.g., [47]). To show that $\widehat{u}(t) = u_0^*(t)$, $t \in [0,t_f]$ is an open-loop optimal control of this problem, it is necessary and sufficient to prove the equality

$$\inf_{\widehat{u}(t)\in L^2[0,t_f;E^r]} \widehat{J}_u\big(\widehat{u}(t)\big) = \widehat{J}_u\big(u_0^*(t)\big) \tag{5.24}$$

along trajectories of (5.22).

To show the validity of the equality (5.24), we apply the regularization method, i.e., we replace approximately the singular optimal control problem (5.22)-(5.23) with the regular one consisting of the system (5.22) and the new performance index with the new (regular) functional

$$\widehat{J}_{u,\varepsilon}\big(u(\cdot)\big) \triangleq \frac{1}{2}\widehat{x}^T(t_f)\widehat{x}(t_f) + \frac{1}{2}\int_0^{t_f} \widehat{u}(t)^T\big(R_u(t)+\mathscr{E}\big)\widehat{u}(t)dt \to \min_{\widehat{u}(t)\in L^2[0,t_f;E^r]}, \tag{5.25}$$

where the $r\times r$-matrix $\mathscr{E}$ is given by (3.2) and $\varepsilon > 0$ is a small parameter of the regularization.

Let us solve the optimal control problem (5.22), (5.25). By virtue of the results of [2], we write down the variational Hamiltonian for this problem

$$\widehat{H}\big(\widehat{u}(t),t,\varepsilon\big) = \widehat{\lambda}^T(t)\big(\mathscr{M}_u(t)\widehat{u}(t) + \mathscr{M}_v(t)v_0^*(t)\big) + \frac{1}{2}\widehat{u}^T(t)\big(R_u(t)+\mathscr{E}\big)\widehat{u}(t), \ \ t \in [0,t_f], \tag{5.26}$$

where $\widehat{\lambda}(t)$ is the n-dimensional costate variable.

The costate variable $\widehat{\lambda}(t)$ satisfies the following terminal-value problem:

$$\frac{d\widehat{\lambda}(t)}{dt} = -\frac{\partial \widehat{H}\big(\widehat{u}(t),t,\varepsilon\big)}{\partial \widehat{x}(t)} = 0, \ \ t \in [0,t_f], \ \ \widehat{\lambda}(t_f) = \frac{\partial\big(\frac{1}{2}\widehat{x}^T(t_f)\widehat{x}(t_f)\big)}{\partial \widehat{x}(t_f)} = \widehat{x}(t_f). \tag{5.27}$$

The open-loop optimal control of the problem (5.22), (5.25) is

$$\widehat{u}_{\varepsilon}^{*}(t) = \arg\min_{\widehat{u}(t)} \widehat{H}\big(\widehat{u}(t),t,\varepsilon\big) = -\big(R_u(t)+\mathscr{E}\big)^{-1}\mathscr{M}_u^T(t)\widehat{\lambda}(t), \quad t \in [0,t_f]. \tag{5.28}$$

The solution of the terminal-value problem (5.27) is

$$\widehat{\lambda}(t) \equiv \widehat{x}(t_f), \quad t \in [0,t_f]. \tag{5.29}$$

Substituting (5.29) into (5.28), we have

$$\widehat{u}_{\varepsilon}^{*}(t) = -\big(R_u(t)+\mathscr{E}\big)^{-1}\mathscr{M}_u^T(t)\widehat{x}(t_f), \quad t \in [0,t_f]. \tag{5.30}$$

Finally, substituting (5.30) into the system (5.22) instead of $\widehat{u}(t)$, we obtain the following initial-value problem for the functional-differential system:

$$\frac{d\widehat{x}(t)}{dt} = -\mathscr{M}_u(t)\big(R_u(t)+\mathscr{E}\big)^{-1}\mathscr{M}_u^T(t)\widehat{x}(t_f) + \mathscr{M}_v(t)v_0^*(t), \ \ t \in [0,t_f], \ \ \widehat{x}(0) = x_0. \tag{5.31}$$

Integration of the system (5.31) from $t=0$ to $t=t_f$ yields after a routine algebra the following linear algebraic equation with respect to $\widehat{x}(t_f)$:

$$\widehat{\Gamma}(\varepsilon)\widehat{x}(t_f) = \widehat{x}_0, \quad \widehat{\Gamma}(\varepsilon) = I_n + \int_0^{t_f}\mathscr{M}_u(t)\big(R_u(t)+\mathscr{E}\big)^{-1}\mathscr{M}_u^T(t)dt, \tag{5.32}$$

where

$$\widehat{x}_0 = x_0 + \int_0^{t_f}\mathscr{M}_v(t)v_0^*(t)dt. \tag{5.33}$$

Let us show the existence and the uniqueness of the solution to the equation (5.32) for all sufficiently small $\varepsilon > 0$ and derive this solution.

Quite similarly to the results of Subsection 4.1, we obtain the existence of a number $\widehat{\varepsilon}_1 > 0$ such that, for all $\varepsilon \in (0,\widehat{\varepsilon}_1]$, the matrix $\widehat{\Gamma}(\varepsilon)$ is invertible and $\widehat{\Gamma}^{-1}(\varepsilon)$ has the form

$$\widehat{\Gamma}^{-1}(\varepsilon) = L^T\widehat{\Phi}(\varepsilon)L, \tag{5.34}$$

where the matrix L is the orthogonal matrix appearing for the first time in the equation (4.5); the matrix $\widehat{\Phi}(\varepsilon)$ has the form

$$\begin{aligned}
\widehat{\Phi}(\varepsilon) &= \begin{pmatrix} \widehat{\Phi}_1(\varepsilon) & \widehat{\Phi}_2(\varepsilon) \\ \widehat{\Phi}_2^T(\varepsilon) & \widehat{\Phi}_3(\varepsilon) \end{pmatrix}, \\
\widehat{\Phi}_1(\varepsilon) &= \Big[I_k + \Omega_{u,11} - \varepsilon\Omega_{u,12}\big(\Theta_{u,2}+\varepsilon(I_{n-k}+\Omega_{u,13})\big)^{-1}\Omega_{u,12}^T\Big]^{-1}, \\
\widehat{\Phi}_2(\varepsilon) &= -\varepsilon\widehat{\Phi}_1(\varepsilon)\Omega_{u,12}\big(\Theta_{u,2}+\varepsilon(I_{n-k}+\Omega_{u,13})\big)^{-1}, \\
\widehat{\Phi}_3(\varepsilon) &= \varepsilon\big(\Theta_{u,2}+\varepsilon(I_{n-k}+\Omega_{u,13})\big)^{-1}\Big[I_{n-k} \\
&\quad + \varepsilon\Omega_{u,12}^T\widehat{\Phi}_1(\varepsilon)\Omega_{u,12}\big(\Theta_{u,2}+\varepsilon(I_{n-k}+\Omega_{u,13})\big)^{-1}\Big].
\end{aligned} \tag{5.35}$$

Remember that the matrix $\Theta_{u,2}$ is given by the equation (4.5); the matrices $\Omega_{u,11}$, $\Omega_{u,12}$ and $\Omega_{u,13}$ are given in the equation (4.9); the number k is given in the assumption A7.

Similarly to inequalities (4.14), (4.17), and (4.18), we have the following inequalities for $\widehat{\Phi}_1(\varepsilon)$, $\widehat{\Phi}_2(\varepsilon)$ and $\widehat{\Phi}_3(\varepsilon)$:

$$\left\|\widehat{\Phi}_2(\varepsilon)\right\| \le \widehat{a}_1\varepsilon, \quad \left\|\widehat{\Phi}_3(\varepsilon)\right\| \le \widehat{a}_1\varepsilon \quad \forall\varepsilon \in (0,\widehat{\varepsilon}_2], \tag{5.36}$$

$$\begin{gathered}\left\|\widehat{\Phi}_1(\varepsilon) - (I_k+\Omega_{u,11})^{-1}\right\| \le \widehat{a}_1\varepsilon \quad \forall\varepsilon \in (0,\widehat{\varepsilon}_2],\\ \left\|\frac{1}{\varepsilon}\widehat{\Phi}_2(\varepsilon) + (I_k+\Omega_{u,11})^{-1}\Omega_{u,12}\Theta_{u,2}^{-1}\right\| \le \widehat{a}_1\varepsilon, \quad \left\|\frac{1}{\varepsilon}\widehat{\Phi}_3(\varepsilon) - \Theta_{u,2}^{-1}\right\| \le \widehat{a}_1\varepsilon \quad \forall\varepsilon \in (0,\widehat{\varepsilon}_2],\end{gathered} \tag{5.37}$$

where $0 < \widehat{\varepsilon}_2 \le \widehat{\varepsilon}_1$ is some number; $\widehat{a}_1 > 0$ is some number independent of ε.

Using the above established invertibility of the matrix $\widehat{\Gamma}(\varepsilon)$ and the equation (5.34), we obtain the unique solution of the equation (5.32) as: $\widehat{x}(t_f) = L^T\widehat{\Phi}(\varepsilon)L\widehat{x}_0$, $\varepsilon \in (0,\widehat{\varepsilon}_1]$, yielding, along with the equation (5.30), the following expression for the open-loop optimal control of the problem (5.22), (5.25):

$$\widehat{u}_\varepsilon^*(t) = -\big(R_u(t)+\mathscr{E}\big)^{-1}\mathscr{M}_u^T(t)L^T\widehat{\Phi}(\varepsilon)L\widehat{x}_0, \quad t\in[0,t_f], \quad \varepsilon \in (0,\widehat{\varepsilon}_1]. \tag{5.38}$$

Let us derive the expression for the optimal value $\widehat{J}_{u,\varepsilon}^*$ of the functional in the problem (5.22), (5.25). Using the symmetry of the matrix $\widehat{\Phi}(\varepsilon)$, we obtain similarly to the results of Subsection 4.3 (see the equations (4.36)-(4.38)) the following expression for $\widehat{J}_{u,\varepsilon}^*$:

$$\widehat{J}_{u,\varepsilon}^* = \frac{1}{2}\widehat{x}_0^T L^T\widehat{\Phi}(\varepsilon)\left[I_n + L\int_0^{t_f}\mathscr{M}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)dtL^T + \frac{1}{\varepsilon}D_{u,2}\right]\widehat{\Phi}(\varepsilon)L\widehat{x}_0, \tag{5.39}$$

where the matrix $\mathscr{M}_{u,1}(t)$ is defined in (4.1); the matrix $D_{u,2}$ is defined by (4.5).

Consider the following $n\times n$-matrix and value:

$$\widehat{\Phi}_0 \triangleq \begin{pmatrix} (I_k+\Omega_{u,11})^{-1} & O_{k\times(n-k)}\\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)}\end{pmatrix}, \tag{5.40}$$

$$\widehat{J}_{u,0}^* \triangleq \frac{1}{2}\widehat{x}_0^T L^T\widehat{\Phi}_0\left[I_n + L\int_0^{t_f}\mathscr{M}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{M}_{u,1}^T(t)dtL^T\right]\widehat{\Phi}_0L\widehat{x}_0. \tag{5.41}$$

Using the inequalities in (5.36) and the first inequality in (5.37), we obtain quite similarly to the inequality (4.43), the following inequality:

$$\left|\widehat{J}_{u,\varepsilon}^* - \widehat{J}_{u,0}^*\right| \le \widehat{a}_2\varepsilon \quad \forall\, \varepsilon \in (0,\widehat{\varepsilon}_2], \tag{5.42}$$

where $\widehat{a}_2 > 0$ is some constant independent of ε.

Based on the inequality (5.42), we are going to calculate the value

$$\widehat{J}_u^* \triangleq \inf_{\widehat{u}(t)\in L^2[0,t_f;E^r]}\widehat{J}_u\big(\widehat{u}(t)\big) \tag{5.43}$$

along trajectories of (5.22).

Using the equations (5.23), (5.25), (5.43) and the inequality (5.42), we have the following chain of the inequalities and equality:

$$0 \le \widehat{J}_u^* \le \widehat{J}_u\big(\widehat{u}_\varepsilon^*(t)\big) \le \widehat{J}_{u,\varepsilon}\big(\widehat{u}_\varepsilon^*(t)\big) = \widehat{J}_{u,\varepsilon}^* \le \widehat{J}_{u,0}^* + \widehat{a}_2\varepsilon \;\; \forall\, \varepsilon \in (0,\widehat{\varepsilon}_2]. \tag{5.44}$$

Since the values $\widehat{J}_u^*$ and $\widehat{J}_{u,0}^*$ are independent of ε, then (5.44) yields the inequality

$$\widehat{J}_u^* \le \widehat{J}_{u,0}^*. \tag{5.45}$$

Let us show the validity of the equality

$$\widehat{J}_u^* = \widehat{J}_{u,0}^*. \tag{5.46}$$

To prove (5.46), we assume the opposite which, due to (5.45), is

$$\widehat{J}_u^* < \widehat{J}_{u,0}^*. \tag{5.47}$$

This strong inequality implies the existence of the control $\widehat{u}(t) \in L^2[0,t_f;E^r]$ such that

$$\widehat{J}_u^* < \widehat{J}_u\big(\widehat{u}(t)\big) < \widehat{J}_{u,0}^*. \tag{5.48}$$

Using the inequalities (5.42) and that $\widehat{u}_\varepsilon^*(t)$, $t \in [0,t_f]$ is the open-loop optimal control of the problem (5.22), (5.25), we obtain

$$\widehat{J}_{u,0}^* - \widehat{a}_2\varepsilon \le \widehat{J}_{u,\varepsilon}^* = \widehat{J}_{u,\varepsilon}\big(\widehat{u}_\varepsilon^*(t)\big) \le \widehat{J}_{u,\varepsilon}\big(\widehat{u}(t)\big) = \widehat{J}_u\big(\widehat{u}(t)\big) + b\varepsilon \quad \forall\, \varepsilon \in (0,\widehat{\varepsilon}_2], \tag{5.49}$$

where $b = 0.5\int_0^{t_f} \widehat{u}_{\text{low}}^T(t)\widehat{u}_{\text{low}}(t)dt$ and $\widehat{u}_{\text{low}}(t)$ is the lower block of the vector $\widehat{u}(t)$ of the dimension $r-q$.

The chain of the equalities and the inequalities (5.49) implies the inequality $\widehat{J}_{u,0}^* \le \widehat{J}_u\big(\widehat{u}(t)\big) + (b+\widehat{a}_2)\varepsilon$ valid for all $\varepsilon \in (0,\varepsilon_2]$. This inequality yields the inequality $\widehat{J}_{u,0}^* \le \widehat{J}_u\big(\widehat{u}(t)\big)$, which contradicts the right-hand side inequality in (5.48). This contradiction means that the inequality (5.47) is wrong, meaning the validity of the equality (5.46).

Now, we are going to show the validity of the equality

$$\widehat{J}_{u,0}^* = J_{u,0}^*, \tag{5.50}$$

where $\widehat{J}_{u,0}^*$ is given by (5.41); $J_{u,0}^*$ is given by (4.41).

Comparing (5.41) with (4.41), one can conclude the following. To show the validity of the equality (5.50), it is sufficient to prove the validity of the equality

$$\widehat{\Phi}_0 L\widehat{x}_0 = \Phi_0 L x_0. \tag{5.51}$$

Using equations (4.9), (4.24), (4.34), (5.33), and (5.40), we obtain

$$\begin{aligned}
\widehat{\Phi}_0 L\widehat{x}_0 &= \widehat{\Phi}_0\left[I_n - \left(L\int_0^{t_f}\mathscr{M}_v(t)R_v^{-1}(t)\mathscr{M}_v^T\,dtFL^T\right)\Phi_0\right]Lx_0\\
&= \widehat{\Phi}_0\left[\begin{pmatrix} I_k & O_{k\times(n-k)}\\ O_{(n-k)\times k} & I_{n-k}\end{pmatrix} - \begin{pmatrix}\Omega_{v,1} & \Omega_{v,2}\\ \Omega_{v,3} & \Omega_{v,4}\end{pmatrix}\begin{pmatrix}\Delta_1^{-1} & O_{k\times(n-k)}\\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)}\end{pmatrix}\right]Lx_0\\
&= \begin{pmatrix}(I_k+\Omega_{u,11})^{-1} & O_{k\times(n-k)}\\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)}\end{pmatrix}\begin{pmatrix} I_k - \Omega_{v,1}\Delta_1^{-1} & O_{k\times(n-k)}\\ -\Omega_{v,3}\Delta_1^{-1} & I_{n-k}\end{pmatrix}Lx_0\\
&= \begin{pmatrix}(I_k+\Omega_{u,11})^{-1}(I_k-\Omega_{v,1}\Delta_1^{-1}) & O_{k\times(n-k)}\\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)}\end{pmatrix}Lx_0.
\end{aligned} \tag{5.52}$$

Let us treat the upper left-hand block of the matrix in the right-hand side of (5.52). Using the expression for Δ_1 (see the equation (4.10)), we have

$$\begin{aligned}
(I_k+\Omega_{u,11})^{-1}(I_k-\Omega_{v,1}\Delta_1^{-1}) &= (I_k+\Omega_{u,11})^{-1}[I_k - (\Delta_1 - I_k - \Omega_{u,11})\Delta_1^{-1}]\\
&= (I_k+\Omega_{u,11})^{-1}[I_k - I_k + (I_k+\Omega_{u,11})\Delta_1^{-1}] = \Delta_1^{-1}.
\end{aligned} \tag{5.53}$$

Taking into account (4.24) and (5.53), the expression for $\widehat{\Phi}_0 L\widehat{x}_0$ in (5.52) can be rewritten as:

$$\widehat{\Phi}_0 L\widehat{x}_0 = \begin{pmatrix} \Delta_1^{-1} & O_{k\times(n-k)} \\ O_{(n-k)\times k} & O_{(n-k)\times(n-k)} \end{pmatrix} Lx_0 = \Phi_0 Lx_0. \tag{5.54}$$

Thus, equality (5.51) is valid and, therefore, equality (5.50) is valid. The latter, along with the equations (4.48), (5.23), (5.43), (5.46), yields the validity of the equation (5.24) along trajectories of (5.22). This completes the proof of the lemma. □

Based on Definition 2.1, Remark 2.6, equations (4.48)-(4.49), and Lemmas 5.1-5.2, we directly obtain the following assertion.

Theorem 5.1. *Let assumptions A1-A8 be satisfied. Then the pair of the controls* $\left(u_0^*(t), v_0^*(t)\right)$, $t \in [0,t_f]$ *(*$u_0^*(t)$ *and* $v_0^*(t)$ *are given by (4.26), (4.30), (4.32), and (4.34), respectively) is the open-loop solution of the OSNEG. Moreover, the values* $J_{u,0}^*$ *and* $J_{v,0}^*$*, given by (4.41) and (4.46), are the Nash equilibrium values of the corresponding functionals in the OSNEG.*

6. EXAMPLES

Consider the following particular case of system (2.1):

$$\begin{aligned}\frac{dz_1(t)}{dt} &= z_1(t-2) + 2u_1(t) - 2u_2(t) - 4u_1(t-2.5) + 2u_2(t-2.5) \\ &\quad -2v_1(t) + 2v_2(t) + 2v_1(t-2) - 2v_2(t-2), \quad t \in [0,4],\end{aligned} \tag{6.1}$$

$$\begin{aligned}\frac{dz_2(t)}{dt} &= z_2(t-2) + 4u_1(t) - 2u_2(t) + 2u_1(t-2.5) + 2u_2(t-2.5) \\ &\quad +2v_1(t) + 2v_2(t) - 2v_1(t-2) - 2v_2(t-2), \quad t \in [0,4].\end{aligned} \tag{6.2}$$

where $z_1(t)$, $z_2(t)$, $u_1(t)$, $u_2(t)$, $v_1(t)$, and $v_2(t)$ are scalar variables.

System (6.1)-(6.2) is subject to the initial conditions

$$z_1(\tau) = 0, \quad \tau \in [-2,0), \quad z_1(0) = 8, \tag{6.3}$$

$$z_2(\tau) = 0, \quad \tau \in [-2,0), \quad z_2(0) = 2, \tag{6.4}$$

$$u_1(\eta) = 0, \quad u_2(\eta) = 0, \quad \eta \in [-2.5,0), \tag{6.5}$$

$$v_1(\zeta) = 0, \quad v_2(\zeta) = 0, \quad \zeta \in [-2,0). \tag{6.6}$$

Comparison of the systems (6.1)-(6.2) and (2.1) yields that in (6.1)-(6.2) $n = 2$, $r = 2$, $s = 2$,

$$\begin{aligned} N_z = 1, \quad N_u = 1, \quad N_v = 1, \quad h_{z,1} = 2, \quad h_{u,1} = 2.5, \quad h_{v,1} = 2, \\ t_f = 4, \quad f(t) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \in [0,4],\end{aligned} \tag{6.7}$$

and the matrices of the coefficients have the form

$$A_0(t) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in [0,4], \tag{6.8}$$

$$G(t,\tau) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (t,\tau) \in [0,4] \times [-2,0], \tag{6.9}$$

$$B_0(t) \equiv \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}, \quad B_1(t) \equiv \begin{pmatrix} -4 & 2 \\ 2 & 2 \end{pmatrix}, \quad t \in [0,4], \tag{6.10}$$

$$P(t,\eta) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (t,\eta) \in [0,4] \times [-2.5,0], \tag{6.11}$$

$$C_0(t) \equiv \begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix}, \quad C_1(t) \equiv \begin{pmatrix} 2 & -2 \\ -2 & -2 \end{pmatrix}, \quad t \in [0,4], \tag{6.12}$$

$$Q(t,\zeta) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (t,\zeta) \in [0,4] \times [-2,0]. \tag{6.13}$$

Comparing the initial conditions (6.3)-(6.6) with the initial conditions (2.2)-(2.4), we can see that in (6.3)-(6.6)

$$\varphi_z(\tau) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \tau \in [-2,0); \quad \varphi_{0,z} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \tag{6.14}$$

$$\varphi_u(\eta) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \eta \in [-2.5,0); \quad \varphi_v(\zeta) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \zeta \in [-2,0). \tag{6.15}$$

In this example, the functional to be minimized by the player "u" with the control $u(t) = \mathrm{col}\big(u_1(t),u_2(t)\big)$, $t \in [0,4]$ and the functional to be minimized by the player "v" with the control $v(t) = \mathrm{col}\big(v_1(t),v_2(t)\big)$, $t \in [0,4]$ are, respectively,

$$\mathcal{J}_u\big(u(\cdot),v(\cdot)\big) = \frac{1}{2}z^T(4)z(4) + \frac{1}{2}\int_0^4 u_1^2(t)dt, \tag{6.16}$$

and

$$\mathcal{J}_v\big(u(\cdot),v(\cdot)\big) = \frac{1}{2}z^T(4)Fz(4) + \frac{1}{2}\int_0^4 \big[v_1^2(t) + v_2^2(t)\big]dt, \tag{6.17}$$

where $z(t) = \mathrm{col}\big(z_1(t),z_2(t)\big)$, $t \in [0,4]$;

$$F = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}. \tag{6.18}$$

Comparing the functionals (6.16) and (6.17) with the functionals (2.5) and (2.6), respectively, and taking into account the assumption A5 (see the equation (2.7)), we have that $q = 1$ and

$$R_{u,1}(t) \equiv 1, \quad R_v(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in [0,4]. \tag{6.19}$$

Thus Nash equilibrium differential game (6.1)-(6.6), (6.16)-(6.17) is a particular case of Nash equilibrium differential game (2.1)-(2.6). Let us convert the game (6.1)-(6.6), (6.16)-(6.17) to the form of the game (2.31), (2.32), (2.35), (2.36). For this purpose, first, we should obtain the matrix-valued function $\Psi(t)$, defined by the terminal-value problem (2.12). In this example (see the equations (6.8)-(6.9)), the problem (2.12) becomes as:

$$\frac{d\Psi(t)}{dt} = -\Psi(t+2), \quad t \in [0,4]; \quad \Psi(4) = I_2; \quad \Psi(t) = O_{2\times 2}, \quad t > 4.$$

Solving this problem, we directly obtain

$$\Psi(t) = \psi(t)I_2, \quad t \in [0,4], \tag{6.20}$$

where the scalar function $\psi(t)$ has the form

$$\psi(t) = \begin{cases} 3-t, & t \in [0,2], \\ 1, & t \in (2,4]. \end{cases} \tag{6.21}$$

Using equations (6.7), (6.10)-(6.13), and (6.20-(6.21), we can calculate the matrix-valued functions $\mathscr{M}_u(t)$ and $\mathscr{M}_v(t)$, defined by (2.24)-(2.29), as follows:

$$\mathscr{M}_u(t) = \begin{cases} \begin{pmatrix} 2-2t & 2t-4 \\ 14-4t & 2t-4 \end{pmatrix}, & 0 \le t \le 1.5, \\ \\ \begin{pmatrix} 6-2t & 2t-6 \\ 12-4t & 2t-6 \end{pmatrix}, & 1.5 < t \le 2, \\ \\ \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix}, & 2 < t \le 4, \end{cases} \tag{6.22}$$

$$\mathscr{M}_v(t) = \begin{cases} \begin{pmatrix} 2t-4 & 4-2t \\ 4-2t & 4-2t \end{pmatrix}, & 0 \le t \le 2, \\ \\ \begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix}, & 2 < t \le 4. \end{cases} \tag{6.23}$$

Furthermore, using equations (2.18), (2.33), (6.14)-(6.15), and (6.20)-(6.21), we can directly calculate the vector x_0

$$x_0 = \begin{pmatrix} 24 \\ 6 \end{pmatrix}. \tag{6.24}$$

Remark 6.1. Thus, the Nash equilibrium differential game (6.1)-(6.6), (6.16)-(6.17) is converted to the equivalent Nash equilibrium differential game (2.31)-(2.32), (2.35), (2.36), where the state vector $x(t) \in E^2$, the coefficients $\mathscr{M}_u(t)$ and $\mathscr{M}_v(t)$ in the equation of dynamics are given by (6.22) and (6.23), the initial state vector of the game is given by (6.24), the final time instant $t_f = 4$, the coefficients in the functionals are given by (6.18), (6.19). Due to Remark 2.7, we call this game the Original Singular Nash Equilibrium Game (OSNEG) in the example.

Due to equation (4.1), let us partition the matrix $\mathscr{M}_u(t)$ into two blocks $\mathscr{M}_{u,1}(t)$ and $\mathscr{M}_{u,2}(t)$. Using (6.22), we obtain

$$\mathscr{M}_{u,1}(t) = \begin{cases} \begin{pmatrix} 2-2t \\ 14-4t \end{pmatrix}, & 0 \le t \le 1.5, \\ \\ \begin{pmatrix} 6-2t \\ 12-4t \end{pmatrix}, & 1.5 < t \le 2, \\ \\ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, & 2 < t \le 4, \end{cases} \tag{6.25}$$

$$\mathscr{M}_{u,2}(t) = \begin{cases} \begin{pmatrix} 2t-4 \\ 2t-4 \end{pmatrix}, & 0 \leq t \leq 1.5, \\ \\ \begin{pmatrix} 2t-6 \\ 2t-6 \end{pmatrix}, & 1.5 < t \leq 2, \\ \\ \begin{pmatrix} -2 \\ -2 \end{pmatrix}, & 2 < t \leq 4, \end{cases} \tag{6.26}$$

Continue with the calculation of the matrix $K_{u,2}$, given by the equation (4.4). Using (6.26), we directly have

$$K_{u,2} = \begin{pmatrix} \frac{65}{3} & \frac{65}{3} \\ \\ \frac{65}{3} & \frac{65}{3} \end{pmatrix}. \tag{6.27}$$

This matrix has zero eigenvalue of the algebraic multiplicity 1, meaning that the assumption A7 is fulfilled.

Let us choose the orthogonal matrix L, appearing in the equation (4.5), as follows:

$$L = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{6.28}$$

Thus, using the matrices $K_{u,2}$, L, given by the equations (6.27), (6.28), and calculating the matrix $D_{u,2}$, defined by the equation (4.5), we obtain

$$D_{u,2} = \begin{pmatrix} 0 & 0 \\ \\ 0 & \frac{130}{3} \end{pmatrix}. \tag{6.29}$$

The latter, along with (4.5), yields

$$\Theta_{u,2} = \frac{130}{3}. \tag{6.30}$$

Calculating $\Lambda_{u,2}(t)$, defined by the equation (4.6), we obtain

$$\Lambda_{u,2}(t) = \begin{cases} (2t-4)\sqrt{2}, & t \in [0, 1.5], \\ (2t-6)\sqrt{2}, & t \in (1.5, 2], \\ -2\sqrt{2}, & t \in (2, 4]. \end{cases} \tag{6.31}$$

Now, we are going to calculate the block-form matrices, given in (4.9). Using (6.19), (6.23), (6.25) and (6.28), we obtain by a routine algebra

$$\begin{pmatrix} \Omega_{u,11} & \Omega_{u,12} \\ \Omega_{u,12}^T & \Omega_{u,13} \end{pmatrix} = \begin{pmatrix} \frac{533}{6} & -\frac{654}{6} \\ \\ -\frac{654}{6} & \frac{927}{6} \end{pmatrix},$$
$$\begin{pmatrix} \Omega_{v,1} & \Omega_{v,2} \\ \Omega_{v,3} & \Omega_{v,4} \end{pmatrix} = \begin{pmatrix} \frac{56}{3} & 56 \\ \\ 56 & 168 \end{pmatrix}. \tag{6.32}$$

Furthermore, using (4.10) and (6.32), we have

$$\Delta_1 = \frac{217}{2}, \quad \Delta_3 = -53. \tag{6.33}$$

Thus, $\Delta_1 \neq 0$, i.e., the assumption A8 is fulfilled.

Now, based on the equations (4.24), (4.26), (4.30), (4.32), (4.34) and using the equations (6.18), (6.19), (6.23)-(6.26), (6.28), (6.30), (6.31), (6.33), we can derive the components $u_{0,1}^*(t)$, $u_{0,2}^*(t)$ and $v_0^*(t)$ of the open-loop solution to the OSNEG in the example. Namely,

$$u_{0,1}^*(t) = 0.1659 \begin{cases} 6-t, & 0 \le t \le 1.5, \\ 3-t, & 1.5 < t \le 2, \\ 1 & 2 < t \le 4, \end{cases} \tag{6.34}$$

$$u_{0,2}^*(t) = 1.7904 \begin{cases} 2-t, & 0 \le t \le 1.5, \\ 3-t, & 1.5 < t \le 2, \\ 1 & 2 < t \le 4, \end{cases} \tag{6.35}$$

$$v_0^*(t) = 0.1659 \begin{cases} \begin{pmatrix} 2-t \\ 3t-6 \end{pmatrix}, & 0 \le t \le 2, \\ \\ \begin{pmatrix} 1 \\ -3 \end{pmatrix}, & 2 < t \le 4. \end{cases} \tag{6.36}$$

Furthermore, based on the equations (4.24), (4.41), (4.46) and using the equations (6.18), (6.19), (6.23)-(6.25), (6.28), (6.33), we can calculate the Nash equilibrium values $J_{u,0}^*$ and $J_{v,0}^*$ of the functionals of the OSNEG in the example. Namely,

$$J_{u,0}^* = 0.6181, \quad J_{v,0}^* = 0.6456. \tag{6.37}$$

7. EXAMPLES ON NON-UNIQUENESS OF OPEN-LOOP SOLUTION TO THE OSNEG

7.1. **Example 1.** Consider the following particular case of the initial-value problem (2.31)-(2.32):

$$\frac{dx(t)}{dt} = (t+2)u(t) + (t^2-1)v(t), \quad t \in [0,2], \quad x(0) = x_0, \tag{7.1}$$

where $x(t)$, $u(t)$, $v(t)$ are scalar variables; x_0 is any given scalar value.

In this example, the functional to be minimized by the player "u" with the control $u(t)$, $t \in [0,2]$ and the functional to be minimized by the player "v" with the control $v(t)$, $t \in [0,2]$ are, respectively,

$$J_u\big(u(\cdot), v(\cdot)\big) = \frac{1}{2}x^2(2), \tag{7.2}$$

and

$$J_v\big(u(\cdot), v(\cdot)\big) = \frac{5}{2}x^2(2) + \frac{1}{2}\int_0^2 v^2(t)dt. \tag{7.3}$$

The differential game (7.1)-(7.3) is a particular case of the OSNEG (2.31), (2.32), (2.35), (2.36), where

$$n = 1, \quad r = 1, \quad s = 1, \quad q = 0, \quad t_f = 2, \tag{7.4}$$

$$\mathscr{M}_u(t) = t+2, \ \mathscr{M}_v(t) = t^2-1, \ R_u(t) \equiv 0, \ R_v(t) \equiv 1, \ t \in [0,2], \ F = 5. \tag{7.5}$$

Based on the results of Sections 4,5 and taking into account that in the present example $q=0$, let us derive the open-loop Nash equilibrium solution $\big(u_0^*(t), v_0^*(t)\big) \in L^2[0,2]\times L^2[0,2]$ to the game (7.1)-(7.3).

First of all let us observe that, due to $q=0$ and the equation (4.1), $\mathscr{M}_{u,2}(t) = \mathscr{M}_u(t)$, $t\in[0,2]$, while $\mathscr{M}_{u,1}(t)$ disappears. Therefore, due to the equations (4.26), (4.32), $u_0^*(t) = u_{0,2}^*(t)$, $t\in[0,2]$, where $u_{0,2}^*(t)$ is given by (4.30). Using this observation, as well as the equations (4.4)-(4.6), (7.4)-(7.5) and the assumption A7, we directly have

$$K_{u,2} = \frac{56}{3},\quad k=0,\quad L=1,\quad \Theta_{u,2} = \frac{56}{3},\quad \Lambda_{u,2}(t) = t+2,\ \ t\in[0,2]. \tag{7.6}$$

Since $k=0$, then the block-form matrices of the equation (4.9) become the scalar values $\Omega_{u,13}$ and $\Omega_{v,4}$ respectively, while the other blocks disappear. Therefore, the block-form matrix $\Delta(\varepsilon)$ (see the equation (4.10)) becomes the scalar value $\Delta_4(\varepsilon) = \frac{56}{3} + \varepsilon(1+\Omega_{u,13}+\Omega_{v,4})$, while the other blocks disappear. Since the block Δ_1 disappears, the assumption A8 becomes irrelevant. The matrix Φ_0 (see the equation (4.24)) becomes the following scalar value: $\Phi_0 = 0$.

Since the blocks Δ_1 and Δ_3 of the matrix $\Delta(\varepsilon)$ disappear, the control $u_{0,2}^*(t)$ (see the equation (4.30)) becomes as follows: $u_{0,2}^*(t) = -\Lambda_{u,2}^T(t)\Theta_{u,2}^{-1}Lx_0$, $t\in[0,t_f]$. Hence, using the aforementioned equality $u_0^*(t) = u_{0,2}^*(t)$, $t\in[0,2]$ and the equation (7.6), we obtain

$$u_0^*(t) = -\frac{3x_0}{56}(t+2),\quad t\in[0,2]. \tag{7.7}$$

Furthermore, using the equations (4.34), (7.4)-(7.5) and the equality $\Phi_0=0$, we have

$$v_0^*(t) = 0,\quad t\in[0,2]. \tag{7.8}$$

Thus, the pair

$$\big(u_0^*(t), v_0^*(t)\big) = \left(-\frac{3x_0}{56}(t+2), 0\right) \in L^2[0,2]\times L^2[0,2]$$

is the open-loop Nash equilibrium solution to the differential game (7.1)-(7.3). Using the equations (4.41) and (4.46), as well as the equality $\Phi_0=0$, we obtain the Nash equilibrium values of the functionals in this differential game

$$J_{u,0}^* = 0,\quad J_{v,0}^* = 0. \tag{7.9}$$

It should be noted that any pair of controls $\big(\widetilde{u}(t), \widetilde{v}(t)\big) = \big(\widetilde{u}(t), 0\big) \in L^2[0,2]\times L^2[0,2]$, where $\widetilde{u}(t)$ satisfies the Fredholm integral equation of the first kind

$$\int_0^2 (t+2)\widetilde{u}(t)dt = -x_0,$$

is an open-loop Nash equilibrium solution to the differential game (7.1)-(7.3). The corresponding Nash equilibrium values of the functionals also equal zero.

Proceed to the derivation of another solution to the differential game (7.1)-(7.3). For this purpose, we extend the set $L^2[0,2]\times L^2[0,2]$ of admissible open-loop solutions in this game. Namely, the extended set of admissible solutions in the game (7.1)-(7.3) is

$$\mathscr{S} \triangleq \left(U_\delta \bigcup L^2[0,2]\right)\times L^2[0,2], \tag{7.10}$$

where $U_\delta \stackrel{\triangle}{=} \{\beta\delta(t-\bar{t})\}$, β is any real number; $\bar{t} \in [0,2]$ is any time instant; $\delta(t-\bar{t})$, $t \in [0,2]$ is the δ-function of Dirac with the impulse at $t = \bar{t}$.

Let us show that, for any $\bar{t} \in [0,2]$, the pair of controls $\big(\bar{u}(t), \bar{v}(t)\big) = \big(\beta(\bar{t}, x_0)\delta(t-\bar{t}), 0\big) \in \mathscr{S}$ with a properly chosen number $\beta(\bar{t}, x_0)$ is an open-loop Nash equilibrium solution to the differential game (7.1)-(7.3).

Substituting $u(t) = \bar{u}(t) = \beta(\bar{t}, x_0)\delta(t-\bar{t})$, $v(t) = \bar{v}(t) = 0$ into the initial-value problem (7.1), we have

$$\frac{dx(t)}{dt} = (t+2)\beta(\bar{t}, x_0)\delta(t-\bar{t}), \quad t \in [0,2], \quad x(0) = x_0. \tag{7.11}$$

Integrating (7.11) from $t = 0$ to $t = 2$, we obtain

$$x(2) = x_0 + (\bar{t}+2)\beta(\bar{t}, x_0). \tag{7.12}$$

Let us observe that $t + 2 \neq 0$ for any $t \in [0,2]$. Therefore, the choice of $\beta(\bar{t}, x_0) = -x_0/(\bar{t}+2)$ yields $x(2) = 0$ in (7.12). Thus, the values of the functionals in the differential game (7.1)-(7.3), corresponding to the pair of controls $\big(\bar{u}(t), \bar{v}(t)\big) = \big(-x_0\delta(t-\bar{t})/(\bar{t}+2), 0\big)$, are $J_u\big(\bar{u}(t), \bar{v}(t)\big) = 0$, $J_v\big(\bar{u}(t), \bar{v}(t)\big) = 0$. Since $J_u\big(u(t), v(t)\big) \geq 0$, $J_v\big(u(t), v(t)\big) \geq 0$ for any pair $\big(u(t), v(t)\big) \in \mathscr{S}$ then the pair $\big(\bar{u}(t), \bar{v}(t)\big) = \big(-x_0\delta(t-\bar{t})/(\bar{t}+2), 0\big)$ is the open-loop Nash equilibrium solution to the differential game (7.1)-(7.3) for any $\bar{t} \in [0,2]$.

Thus, along with the above derived open-loop Nash equilibrium solution $\big(u_0^*(t), v_0^*(t)\big) \in L^2[0,2] \times L^2[0,2] \subset \mathscr{S}$ to the game (7.1)-(7.3), we have obtained infinitely many its open-loop Nash equilibrium solutions, namely, $\big(\widetilde{u}(t), 0\big) \in L^2[0,2] \times L^2[0,2] \subset \mathscr{S}$ and $\big(\bar{u}(t), \bar{v}(t)\big) \in \mathscr{S}$. The Nash equilibrium values of the functionals in the game (7.1)-(7.3), corresponding to all these solutions are the same, namely, zero. It should be noted that, due to the presence of the Dirac δ-function in $\bar{u}(t)$, the solution $\big(\bar{u}(t), \bar{v}(t)\big)$ can be useful rather for a theoretical analysis of the game, while the solutions $\big(u_0^*(t), v_0^*(t)\big)$ and $\big(\widetilde{u}(t), 0\big)$ can be useful for both purposes, theoretical analysis of the game and practical implementation of these solutions.

7.2. **Example 2.** Consider the following particular case of the initial-value problem (2.31)-(2.32):

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= (t-2)u_1(t) + (2t-4)u_2(t) + v_1(t) - v_2(t), \quad t \in [0,2], \quad x_1(0) = x_{01}, \\
\frac{dx_2(t)}{dt} &= (4-2t)u_1(t) + (2-t)u_2(t) - v_1(t) + v_2(t), \quad t \in [0,2], \quad x_2(0) = x_{02},
\end{aligned} \tag{7.13}$$

where $x_1(t)$, $x_2(t)$, $u_1(t)$, $u_2(t)$, $v_1(t)$, $v_2(t)$ are scalar variables; x_{01} and x_{02} are some given scalar values.

In this example, the functional to be minimized by the player "u" with the control $u(t) = \mathrm{col}\big(u_1(t), u_2(t)\big)$, $t \in [0,2]$ and the functional to be minimized by the player "v" with the control $v(t) = \mathrm{col}\big(v_1(t), v_2(t)\big)$, $t \in [0,2]$ are, respectively,

$$J_u\big(u(\cdot), v(\cdot)\big) = \frac{1}{2}x^T(2)x(2) + \frac{1}{2}\int_0^2 u_1^2(t)dt, \tag{7.14}$$

and

$$J_v\big(u(\cdot), v(\cdot)\big) = \frac{1}{2}x^T(2)Fx(2) + \frac{1}{2}\int_0^2 \big[v_1^2(t) + v_2^2(t)\big]dt, \tag{7.15}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad t \in [0,2]; \qquad F = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}. \tag{7.16}$$

The differential game (7.13)-(7.15) is a particular case of the OSNEG (2.31), (2.32), (2.35), (2.36), where

$$n = 2, \quad r = 2, \quad s = 2, \quad q = 1, \quad t_f = 2, \tag{7.17}$$

$$\mathscr{M}_u(t) = \begin{pmatrix} t-2 & 2t-4 \\ 4-2t & 2-t \end{pmatrix}, \quad \mathscr{M}_v(t) \equiv \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad t \in [0,2],$$
$$R_{u,1}(t) \equiv 1, \qquad R_v(t) \equiv I_2, \quad t \in [0,2]. \tag{7.18}$$

Based on the results of Sections 4 and 5, let us derive the open-loop Nash equilibrium solution $\big(u_0^*(t), v_0^*(t)\big) \in L^2[0,2;E^2] \times L^2[0,2;E^2]$ to the game (7.13)-(7.15).

Due to the equation (4.1), let us partition the matrix $\mathscr{M}_u(t)$ into two blocks $\mathscr{M}_{u,1}(t)$ and $\mathscr{M}_{u,2}(t)$. Using the expression for $\mathscr{M}_u(t)$ (see the equation (7.18)), we obtain

$$\mathscr{M}_{u,1}(t) = \begin{pmatrix} t-2 \\ 4-2t \end{pmatrix}, \quad \mathscr{M}_{u,2}(t) = \begin{pmatrix} 2t-4 \\ 2-t \end{pmatrix}, \quad t \in [0,2]. \tag{7.19}$$

Furthermore, let us calculate the matrix $K_{u,2}$, given by the equation (4.4). Using (7.19), we directly have

$$K_{u,2} = \begin{pmatrix} \frac{32}{3} & -\frac{16}{3} \\ -\frac{16}{3} & \frac{8}{3} \end{pmatrix}. \tag{7.20}$$

This matrix has zero eigenvalue of the algebraic multiplicity $k = 1$, meaning that the assumption A7 is fulfilled.

Let us choose the orthogonal matrix L, appearing in the equation (4.5), as follows:

$$L = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}. \tag{7.21}$$

Thus, using the matrices $K_{u,2}$, L (see the equations (7.20), (7.21)) and calculating the matrix $D_{u,2}$, defined by the equation (4.5), we obtain

$$D_{u,2} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{40}{3} \end{pmatrix}. \tag{7.22}$$

The latter, along with (4.5), yields

$$\Theta_{u,2} = \frac{40}{3}. \tag{7.23}$$

Calculating $\Lambda_{u,2}(t)$, defined by the equation (4.6), we obtain

$$\Lambda_{u,2}(t) = \sqrt{5}(2-t), \quad t \in [0,2]. \tag{7.24}$$

Now, calculating the block-form matrices, given in (4.9), and using (7.18), (7.19) and (7.21), we obtain by a routine algebra

$$\begin{pmatrix} \Omega_{u,11} & \Omega_{u,12} \\ \Omega_{u,12}^T & \Omega_{u,13} \end{pmatrix} = \begin{pmatrix} \frac{72}{15} & \frac{96}{15} \\ \\ \frac{96}{15} & \frac{128}{15} \end{pmatrix},$$
$$\begin{pmatrix} \Omega_{v,1} & \Omega_{v,2} \\ \Omega_{v,3} & \Omega_{v,4} \end{pmatrix} = \begin{pmatrix} -\frac{16}{5} & \frac{12}{5} \\ \\ -\frac{48}{5} & \frac{36}{5} \end{pmatrix}. \tag{7.25}$$

Furthermore, using (4.10) and (7.25), we obtain

$$\Delta_1 = \frac{13}{5}, \quad \Delta_3 = -\frac{16}{5}. \tag{7.26}$$

Thus, $\Delta_1 \neq 0$, i.e., the assumption A8 is fulfilled.

Now, based on the equations (4.24), (4.26), (4.30), (4.32), (4.34) and using the equations (7.16), (7.18), (7.19), (7.21), (7.23), (7.24), (7.26), we can derive the components $u_{0,1}^*(t)$, $u_{0,2}^*(t)$ and $v_0^*(t)$ of the open-loop Nash equilibrium solution to the differential game (7.13)-(7.15). Namely,

$$u_{0,1}^*(t) = \frac{3}{13}(x_{01} + 2x_{02})(t-2), \quad t \in [0,2],$$
$$u_{0,2}^*(t) = \frac{3}{104}(2x_{01} - 9x_{02})(2-t), \quad t \in [0,2],$$
$$v_0^*(t) = \frac{4}{13}(x_{01} + 2x_{02})\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad t \in [0,2]. \tag{7.27}$$

Furthermore, based on the equations (4.24), (4.41), (4.46) and using the equations (7.16), (7.18), (7.19), (7.21), (7.26), we can calculate the Nash equilibrium values $J_{u,0}^*$ and $J_{v,0}^*$ of the functionals of the differential game (7.13)-(7.15). Namely,

$$J_{u,0}^* = \frac{29}{338}(x_{01} + 2x_{02})^2, \quad J_{v,0}^* = \frac{40}{169}(x_{01} + 2x_{02})^2. \tag{7.28}$$

Proceed to the derivation of another solution to the differential game (7.13)-(7.15). For this purpose, we extend the set $L^2[0,2;E^2] \times L^2[0,2;E^2]$ of admissible open-loop solutions in this game. Namely, the extended set of admissible solutions in the game (7.13)-(7.15) is

$$\mathscr{S} \triangleq \mathscr{S}_u \times L^2[0,2;E^2], \tag{7.29}$$

where

$$\mathscr{S}_u \triangleq \left\{ \mathrm{col}\big(u_1(t), u_2(t)\big) : u_1(t) \in L^2[0,2],\ u_2(t) \in \left(U_\delta \bigcup L^2[0,2]\right) \right\}, \quad U_\delta \triangleq \{\beta\delta(t-\bar{t})\},$$

β is any real number; $\bar{t} \in [0,2]$ is any time instant; $\delta(t-\bar{t})$, $t \in [0,2]$ is the δ-function of Dirac with the impulse at $t = \bar{t}$.

Let us calculate the outcome of the game (7.13)-(7.15), corresponding to the pair of controls $\big(\bar{u}(t),\bar{v}(t)\big)$, $t\in[0,2]$, where

$$\bar{u}(t)=\mathrm{col}\big(0,\beta\delta(t-\bar{t})\big),\quad \bar{v}(t)\equiv 0,\quad t\in[0,2]. \tag{7.30}$$

Substituting $u(t)=\bar{u}(t)$, $v(t)=\bar{v}(t)$ into the initial-value problem (7.13), we have

$$\begin{aligned}\frac{dx_1(t)}{dt}&=(2t-4)\beta\delta(t-\bar{t}),\quad t\in[0,2],\quad x_1(0)=x_{01},\\ \frac{dx_2(t)}{dt}&=(2-t)\beta\delta(t-\bar{t}),\quad t\in[0,2],\quad x_2(0)=x_{02}.\end{aligned} \tag{7.31}$$

Integration of (7.31) from $t=0$ to $t=2$ yields

$$\begin{aligned}x_1(2)&=x_{01}+(2\bar{t}-4)\beta,\\ x_2(2)&=x_{02}+(2-\bar{t})\beta.\end{aligned} \tag{7.32}$$

Using (7.30) and (7.32), we directly obtain the following outcome of the game (7.13)-(7.15), corresponding to the pair of controls $\big(\bar{u}(t),\bar{v}(t)\big)$, $t\in[0,2]$:

$$\begin{aligned}J_u\big(\bar{u}(t),\bar{v}(t)\big)&=\frac{1}{2}\Big[\big(x_{01}+(2\bar{t}-4)\beta\big)^2+\big(x_{02}+(2-\bar{t})\beta\big)^2\Big],\\ J_v\big(\bar{u}(t),\bar{v}(t)\big)&=\frac{1}{2}\Big[2\big(x_{01}+(2\bar{t}-4)\beta\big)+\big(x_{02}+(2-\bar{t})\beta\big)\Big]^2.\end{aligned} \tag{7.33}$$

The sufficient condition for the pair of controls $\big(\bar{u}(t),\bar{v}(t)\big)$, $t\in[0,2]$ to be an open-loop Nash equilibrium solution of the game (7.13)-(7.15) is

$$J_u\big(\bar{u}(t),\bar{v}(t)\big)=0,\quad J_v\big(\bar{u}(t),\bar{v}(t)\big)=0, \tag{7.34}$$

which, along with (7.33), yields

$$\begin{cases}x_{01}+(2\bar{t}-4)\beta=0,\\ x_{02}+(2-\bar{t})\beta=0.\end{cases} \tag{7.35}$$

This set of equations with respect to the unknown β has a solution if and only if

$$x_{01}=-2x_{02}, \tag{7.36}$$

and this solution is $\beta=\beta(\bar{t},x_{02})\overset{\triangle}{=}x_{02}/(\bar{t}-2)$ for any $\bar{t}\in[0,2)$.

Hence, subject to the condition (7.36), the pair of controls

$$\big(\bar{u}(t),\bar{v}(t)\big)=\Big(\mathrm{col}\big(0,\beta(\bar{t},x_{02})\delta(t-\bar{t})\big),0\Big)\in\mathscr{S}$$

is an open-loop Nash equilibrium solution to the differential game (7.13)-(7.15) for any $\bar{t}\in[0,2)$, and this solution yields zero Nash equilibrium values of the functionals.

It should be noted that, subject to the condition (7.36), the component $u^*_{0,1}(t)$ of the above derived open-loop Nash equilibrium solution to the differential game (7.13)-(7.15) becomes zero ($u^*_{0,1}(t)\equiv 0$, $t\in[0,2]$), the component $u^*_{0,2}(t)$ of this solution becomes as $u^*_{0,2}(t)=\frac{3}{8}x_{0,2}(t-2)$, $t\in[0,2]$, the component $v^*_0(t)$ of this solution becomes zero ($v^*_0(t)\equiv 0$, $t\in[0,2]$). Also, due to (7.28) and (7.36), the Nash equilibrium values $J^*_{u,0}$ and $J^*_{v,0}$ of the functionals of the differential game (7.13)-(7.15), corresponding to the above mentioned solution $\Big(\mathrm{col}\big(u_{0,1}^*(t), u_{0,2}^*(t)\big), v_0^*(t)\Big)$ $\in L^2[0,2;E^2] \times L^2[0,2;E^2]$, become zero. Moreover, subject to the condition (7.36), any pair of controls $\big(\widetilde{u}(t), \widetilde{v}(t)\big) = \Big(\mathrm{col}\big(\widetilde{u}_1(t), \widetilde{u}_2(t)\big), \widetilde{v}(t)\Big) = \Big(\mathrm{col}\big(0, \widetilde{u}_2(t)\big), 0\Big) \in L^2[0,2;E^2] \times L^2[0,2;E^2]$, where $\widetilde{u}_2(t)$ satisfies the Fredholm integral equation of the first kind

$$\int_0^2 (t-2)\widetilde{u}_2(t)dt = x_{0,2},$$

is an open-loop Nash equilibrium solution to the differential game (7.13)-(7.15). The corresponding Nash equilibrium values of the functionals also equal zero. Thus, subject to the condition (7.36), we have derived infinitely many open-loop Nash equilibrium solutions to the differential game (7.13)-(7.15). Among these solutions are $\Big(\mathrm{col}\big(u_{0,1}^*(t), u_{0,2}^*(t)\big), v_0^*(t)\Big) \in L^2[0,2;E^2] \times L^2[0,2;E^2] \subset \mathscr{S}$ and $\Big(\mathrm{col}\big(0, \widetilde{u}_2(t)\big), 0\Big) \in L^2[0,2;E^2] \times L^2[0,2;E^2] \subset \mathscr{S}$, while the other solutions are $\big(\bar{u}(t), \bar{v}(t)\big) \in \mathscr{S}$ valid for any $\bar{t} \in [0,2)$. The Nash equilibrium values of the functionals in the game, corresponding to these solutions are the same, namely, zero. As in the previous example, it should be noted that, due to the presence of the Dirac δ-function in $\bar{u}(t)$, the solution $\big(\bar{u}(t), \bar{v}(t)\big)$ can be useful rather for a theoretical analysis of the game, while each of the solutions $\Big(\mathrm{col}\big(u_{0,1}^*(t), u_{0,2}^*(t)\big), v_0^*(t)\Big)$ and $\Big(\mathrm{col}\big(0, \widetilde{u}_2(t)\big), 0\Big)$ can be useful for both purposes, theoretical analysis of the game and practical implementation of this solution.

8. Conclusions

In this paper, the deterministic finite-horizon two-person linear-quadratic Nash equilibrium differential game was considered. The dynamics equation of the game has multiple point-wise and distributed delays in the state variable and control variables of both players. The weight matrix of the control cost of one player (the "singular" player) in its own functional is block-diagonal with two blocks on the main diagonal. One of these blocks is a positive definite matrix, while the other is zero matrix. Thus, the aforementioned weight matrix of the control cost is singular, meaning that the considered game is singular. The control coordinates, which are present in the functional are regular, while the other control coordinates are singular. The weight matrix of the control cost of the other player (the "regular" player) in its own functional is positive definite. A set of admissible solutions (the pairs of the players' controls) in the game is chosen as the set of all pairs of square integrable control functions.

By two consecutive linear changes of the state variable, the initially formulated differential game was transformed equivalently to a much simpler one. This new game also is a singular Nash equilibrium game, while its equation of dynamics does not have delays any more. In the sequel of the paper, this new un-delayed game was considered as an original Nash equilibrium game. To solve this game, the regularization method was applied. Namely, the original game was replaced by a regular Nash equilibrium game, which depends on a small positive parameter ε. This new game has the same dynamics, the same functional of the "regular" player and the same set of admissible controls as the original game has. However, the functional of the "singular" player in the new game differs from the one in the original problem. The new functional is the sum of the original functional and the finite-horizon integral of the squares of the singular control coordinates with the small positive weight ε. Thus the obtained parameter dependent

Nash equilibrium game is a partial cheap control game, and it becomes the original game for $\varepsilon = 0$. Asymptotic analysis with respect to ε of the solution to this partial cheap control game was carried out. Based on this analysis, the open-loop solution of the original singular Nash equilibrium game was derived. The corresponding values of the players' functionals also were obtained.

It was demonstrated by examples that, subject to some additional conditions, the singular Nash equilibrium game can have infinitely many solutions in the original set and in some extended set of admissible pairs of the players' controls.

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