Appl. Set-Valued Anal. Optim. 6 (2024), No. 1, pp. 1-11
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.6.2024.1.01

HOW TO REDUCE SOME FIXED POINT THEOREMS

J. APPELL[1,*], N. MERENTES[2], S. REINWAND[3], G. VINTI[4]

[1] *Institut für Mathematik, Universität Würzburg, Emil-Fischer-Str. 30, D-97074 Würzburg, Germany*
[2] *Departamento de Matemática, Universidad Central de Venezuela, Caracas 1020 A, Venezuela*
[3] *TNG Technology Consulting GmbH, Beta-Str. 13a, D-85774, Unterföhring, Germany*
[4] *Università di Perugia, Dipartimento di Matematica, Via Vanvitelli 1, I-06123 Perugia, Italy*

Dedicated to Professor Simeon Reich on the occasion of his 75th Birthday

Abstract. We discuss the problem of reducing a fixed point theorem to a previously given simpler one, with a particular emphasis on examples and counterexamples.
Keywords. Contractive maps; Condensing maps; Fixed point theorems; Weakly contractive maps; Weakly condensing maps.
2020 Mathematics Subject Classification. 47H09, 47H10.

1. Introduction

First of all, let us explain the somewhat cryptic title. Suppose that we are given two fixed point theorems A and B, with the property that B is more general than A in the sense that, whenever it is possible to apply A, it is also possible to apply B. Clearly, the new theorem B is only more general if one can find maps which satisfy the hypotheses of B, but not of A. On the other hand, sometimes one may *reduce* the more general theorem to the more special theorem by proving B by means of A. Of course, this does not contradict the fact that B has a wider range of applications than A.

The aim of this simple minded short note (which is actually more "didactical" than "scientific") is to illustrate this phenomenon for several fixed point theorems, with a particular emphasis on examples and counterexamples. A famous example is Sadovskij's fixed point theorem for weakly condensing maps which may be reduced to Darbo's fixed point theorem for condensing maps, as we will show below. This is in contrast to weak contractions which, as we will see, cannot be reduced to classical contractions on compact sets.

2. Metric Fixed Point Theory

During every first year calculus course, the students become acquainted with the following classical fixed point principle:

*Corresponding author.
E-mail address: jurgen@dmuw.de (J. Appell), nmerucv@gmail.com (N. Merentes), simon.reinwand@tngtech.com (S. Reinwand), gianluca.vinti@unipg.it (G. Vinti).
Received 15 October 2022; Accepted 25 October 2022; Published online 21 December 2023

Theorem 2.1 (Banach-Caccioppoli)**.** *Let (X,d) be a complete metric space and $F: X \to X$ a contraction which means that*

$$d(F(x),F(y)) \leq q d(x,y) \qquad (x,y \in X) \tag{2.1}$$

for some $q < 1$. Then f has a unique fixed point in X.

This theorem has the advantage that there are plenty of metric spaces, but the drawback that the contraction condition (2.1) is quite restrictive. Theorem 2.1 has been generalized in many directions over the decades, and in particular there is a wealth of literature on weaker forms of (2.1). An overview of the state of the art of such generalizations by 1976 may be found in the survey [34]. Not all such results seem to be useful, but some generalizations of contraction conditions due to Simeon Reich have turned out to be important in view of applications. In a series of papers published about 50 years ago [13]-[33], Reich has studied generalized contractions, pointwise convergence of their iterations, existence and uniqueness of fixed points, and even some results for multivalued maps which apply to various problems in nonsmooth optimization, complementarity theory, and economic equilibria.

As a sample result, let us only mention the first paper [13] in the series, where the author proves the following

Theorem 2.2 (Reich)**.** *Let (X,d) be a complete metric space and $F: X \to X$ a map satisfying*

$$d(F(x),F(y)) \leq a d(x,F(x)) + b d(y,F(y)) + c d(x,y), \qquad (x,y \in X) \tag{2.2}$$

where $q := a+b+c < 1$. Then f has a unique fixed point in X.

A still more general condition due to Hardy and Rogers [10] reads

$$d(F(x),F(y)) \leq a d(x,F(x)) + b d(y,F(y)) + c d(x,y) + e d(y,F(x)) + f d(x,F(y)), \tag{2.3}$$

where $q := a+b+c+e+f < 1$. The authors of [10] proved a corresponding existence and uniqueness theorem for fixed points under condition (2.3) doing some algebraic manipulation and showing that

$$\min\left\{\frac{a+c+f}{1-b-f}, \frac{b+c+e}{1-a-e}\right\} < 1.$$

Since (2.2) reduces to (2.1) for $a = b = 0$, and (2.3) reduces to (2.2) for $e = f = 0$, one can expect that the range of applications of Reich's result is larger than that of Banach, and that of Hardy-Rogers is larger than that of Reich. In fact, here are two very simple scalar examples to illustrate this fact.

Example 2.1. Let $F: [0,1] \to [0,1]$ be defined by $F(x) := x/3$ for $0 \leq x < 1$ and $F(1) := 1/6$. This map has the unique fixed point $x_* = 0$ in $[0,1]$ and satisfies (2.2) with $a = b := 1/5$ and $c := 1/3$, but of course not (2.1) since F is discontinuous.

Example 2.2. Let $F: [0,1] \to [0,1]$ be defined by $F(x) := 0$ for $0 \leq x < 1$ and $F(1) := 1/2$. This map has the unique fixed point $x_* = 0$. For $0 \leq x < 1$ and $y := 1$, we have

$$d(F(x),F(y)) = \frac{1}{2}, \quad d(x,F(x)) = x, \quad d(y,F(y)) = \frac{1}{2},$$

$$d(x,y) = 1-x, \quad d(y,F(x)) = 1, \quad d(x,F(y)) = \left|x - \frac{1}{2}\right|.$$

Consequently, F satisfies (2.3) with $e := 1/2$ and a, b, c, f small enough. To show that F does not satisfy the hypothesis of Theorem 2.2, one supposes that (2.2) holds with $a+b+c<1$. Choosing $x := 1/2$ and $y := 1$, we obtain

$$ad(x,F(x))+bd(y,F(y))+cd(x,y) = \frac{1}{2}(a+b+c) < \frac{1}{2} = d(F(x),F(y)),$$

contradicting (2.2).

A whole group of fixed point theorems which is of interest in this connection is due to R. Kannan [11]. One of the simplest result in this spirit reads as follows.

Theorem 2.3 (Kannan)**.** *Let (X,d) be a complete metric space, and let $F : X \to X$ be a map satisfying*

$$d(F(x),F(y)) \leq q[d(x,F(x))+d(y,F(y))] \qquad (x,y \in X) \tag{2.4}$$

for some $q < 1/2$. Then F has a unique fixed point in X.

Although Kannan's condition (2.4) is similar to the classical contraction condition (2.1), one can easily show that Kannan's and Banach's fixed point theorems are independent. The following example was given in [14].

Example 2.3. Let $F : [0,1] \to [0,1]$ be defined by $F(x) := x/8$ for $0 \leq x \leq 1/2$ and $F(x) := x/4$ for $1/2 < x \leq 1$. This map is discontinuous and has the unique fixed point $x_* = 0$. For $0 \leq x \leq 1/2$ and $y > 1/2$, we have

$$d(F(x),F(y)) = \frac{1}{8}(2y-x), \quad d(x,F(x)) = \frac{7}{8}x, \quad d(y,F(y)) = \frac{3}{4}y.$$

Consequently, F satisfies Kannan's condition (2.4) with $q := 1/3$, but of course not condition (2.1).

Example 2.4. Let $F : [0,1] \to [0,1]$ be defined by $F(x) := x/3$. This map is a contraction and has the unique fixed point $x_* = 0$. On the other hand, Kannan's theorem does not apply since (2.4) fails for $x := 1/3$ and $y := 0$.

Observe that choosing $a = b$ and $c = 0$ in Reich's fixed point theorem yields Kannan's Theorem 2.3. Interestingly, as was observed by Reich himself [14], the map F in Example 2.1 does not satisfy the hypotheses of Kannan's fixed point theorem because (2.4) fails for $x := 1/3$ and $y := 0$.

The following remarkable theorem given by Subrahmanyam [37] illustrates the importance of Kannan maps because it shows that Theorem 2.3 characterizes completeness of the underlying space. We include the short proof here since it provides interesting insight into fixed point theory.

Theorem 2.4. *The following two properties of a metric space (X,d) are equivalent*

(a) *(X,d) is complete.*
(b) *Every map $F : X \to X$ satisfying (2.4) has a fixed point.*

Proof. We only have to prove that (b) implies (a). Assume that we find a Cauchy sequence $(x_n)_n$ in X which does not converge. Since any Cauchy sequence which contains a convergent subsequence, must converge itself, we conclude that no element x_n occurs infinitely often in the sequence. Passing, if necessary, to a subsequence, we may therefore assume that all terms in the sequence are pairwise distinct and $A := \{x_n : n \in \mathbb{N}\}$ is closed.

For $q \in (0, 1/2)$ and $x \in X \setminus A$, we find an index $n(x) \in \mathbb{N}$ such that

$$d(x_m, x_n) < q \operatorname{dist}(x, A) \leq q d(x, x_k) \qquad (m, n \geq n(x), k \in \mathbb{N}),$$

since $(x_n)_n$ is Cauchy. In particular,

$$d(x_m, x_{n(x)}) < q d(x, x_k) \qquad (m \geq n(x), k \in \mathbb{N}).$$

For fixed $n \in \mathbb{N}$, the infimum

$$m := \inf\{d(x_n, x_k) : k \in \mathbb{N}\}$$

is positive since otherwise a subsequence of $(x_k)_k$ would converge to x_n. We may choose an index $\hat{n} = \hat{n}(n) > n$ such that

$$d(x_m, x_p) < qm \leq q d(x_n, x_{\hat{n}}) \qquad (m, p \geq \hat{n}).$$

In particular,

$$d(x_m, x_{\hat{n}}) < q d(x_n, x_{\hat{n}}) \qquad (m \geq \hat{n}).$$

Clearly, the map $F : X \to X$ defined by

$$F(x) := \begin{cases} x_{\hat{n}} & \text{for } x = x_n, \\ x_{n(x)} & \text{for } x \in X \setminus A \end{cases}$$

has no fixed point. Now, writing $F(x) = x_m$ and $F(y) = x_n$, we first assume $m \geq n$. In case $y \in X \setminus A$, we have $x_n = x_{n(y)}$, so by what we have shown above,

$$d(F(x), F(y)) = d(x_m, x_n) = d(x_m, x_{n(y)}) < q d(y, x_n) = q d(y, F(y)).$$

On the other hand, in case $y \in A$, i.e., $y = x_p$ for some $p \in \mathbb{N}$, we have $x_n = x_{\hat{p}}$, so by what we have shown above,

$$d(F(x), F(y)) = d(x_m, x_n) = d(x_m, x_{\hat{p}}) < q d(x_p, x_n) = q d(y, F(y)).$$

Similarly, for $m < n$, we obtain $d(F(x), F(y)) \leq q d(x, F(x))$ by interchanging the roles of x and y. This proves that F satisfies (2.4) and proves the assertion. □

The independence of Banach's and Kannan's fixed point theorems is illustrated by the fact that there exist non-complete metric spaces X with the property that every contraction $F : X \to X$ has a fixed point; an example of such a space has been given by Connell [7]. This is in sharp contrast to Theorem 2.4. We now prove a parallel result to Theorem 2.4 which shows that, although we cannot characterize completeness of a metric space X by means of contractions on X, we may characterize it by means on contractions *on its closed subspaces*. Since this result seems to be unknown in the literature, we state it here with proof as follows.

Theorem 2.5. *The following two properties of a metric space (X, d) are equivalent*

(a) *(X, d) is complete.*
(b) *For each closed subset $A \subseteq X$, every contraction $F : A \to A$ has a fixed point.*

Proof. Again, we only have to prove that (b) implies (a). Assume that we find a Cauchy sequence $(x_n)_n$ in X which does not converge. As before, we may assume that all terms in the sequence are pairwise different. Define an auxiliary function $p : X \to [0,\infty)$ by

$$p(x) := \inf\{d(x,x_n) : x_n \neq x \text{ for all } n \in \mathbb{N}\}.$$

The fact that $(x_n)_n$ has no convergent subsequences implies that $p(x) > 0$. Fix $q \in (0,1)$ and let $n_1 := 1$. Choose $n_2 > n_1$ in such a way that $d(x_i,x_j) \leq qp(x_1)$ for all $i,j \geq n_2$. Having selected indices $n_2,\ldots,n_{k-1}$ successively for some $k \in \mathbb{N}$, we choose $n_k > n_{k-1}$ in such a way that $d(x_i,x_j) \leq qp(x_{n_{k-1}})$ for all $i,j \geq n_k$. The subsequence $(x_{n_k})_k$ has then still pairwise distinct terms and does not contain any convergent subsequence. Consequently, the set $A := \{x_{n_k} : k \in \mathbb{N}\}$ is closed, and the map $F : A \to A$ defined by $F(x_{n_k}) := x_{n_{k+1}}$ has no fixed point. To finish the proof, we observe that for $i,j \in \mathbb{N}$ with $i < j$,

$$d(F(x_{n_i}),F(x_{n_j})) = d(x_{n_{i+1}},x_{n_{j+1}}) \leq qp(x_{n_i}) \leq qd(x_{n_i},x_{n_j}),$$

which shows that F is a contraction on A. □

Although Banach's and Kannan's fixed point theorems are independent, it is interesting to note that Theorem 2.1 can be reduced in a certain sense to Theorem 2.3. To see this, suppose that X is a metric space, and $F : X \to X$ is contracting. We claim that some iterate F^n of F with sufficiently large n satisfies Kannan's condition (2.4). In fact, if F satisfies (2.1) with $q < 1$, then

$$d(F^n(x),F^n(y)) \leq q^n d(x,y) \leq q^n\left[d(x,F^n(x)) + d(F^n(x),F^n(y)) + d(F^n(y),y)\right].$$

Hence

$$d(F^n(x),F^n(y)) \leq \frac{q^n}{1-q^n}\left[d(x,F^n(x)) + d(y,F^n(y))\right].$$

Choosing n large enough, we have

$$q^n < \frac{1}{3}, \qquad \frac{q^n}{1-q^n} < \frac{1}{2},$$

which proves the claim.

Now, if $x_* \in X$ is the unique fixed point of F^n, then $F^n(F(x_*)) = F^{n+1}(x_*) = F(F^n(x_*)) = F(x_*)$. The uniqueness implies that $F(x_*) = x_*$, so x_* is also the (unique) fixed point of F. For instance, in Example 2.4, the choice $n = 2$ does the job. It is a very useful device in fixed point theory that existence and uniqueness of fixed points for some iterate F^n implies existence and uniqueness of fixed points for F itself.

One could ask if an analogous reduction of Theorem 2.3 to Theorem 2.1 is possible by considering appropriate powers of the map. The next example shows that this is not possible.

Example 2.5. Define $F : [0,1] \to [0,1]$ as in Example 2.3. We already know that F is a Kannan map. On the other hand, all iterates F^n are discontinuous at $x = 1/2$, so there is no hope to apply Banach's theorem to the iterates.

Observe that the map F from Example 2.3 also satisfies the hypotheses of Reich's fixed point theorem. So a reduction of Theorem 2.2 to Theorem 2.1 is not possible either.

3. Topological Fixed Point Theory

The most important topological fixed point principle is without any doubt the following

Theorem 3.1 (Schauder)**.** *Let $(X, \| \|)$ be a Banach space, and let $M \subset X$ be a bounded, closed, and convex subset. Let $F : M \to M$ be a continuous compact operator which means that $\overline{F(M)}$ is compact. Then F has a fixed point in M.*

In contrast to Theorem 2.1, we cannot expect uniqueness of fixed points in Theorem 3.1. Schauder's celebrated fixed point principle applies to a large number of nonlinear problems. However, since many operators occurring in applications are not compact, various attempts have been made to relax the compactness requirement. Here the notion of a *measure of noncompactness*is crucial. Roughly speaking, a measure of noncompactness associates to every bounded set M in a Banach space a nonnegative real number which is zero precisely if M is precompact (i.e., $\overline{M}$ is compact), which motivates the name. The most important example is the *Kuratowski measure of noncompactness* [12]

$$\alpha(M) = \inf\{\varepsilon > 0 : M \text{ may be covered by finitely many sets of diameter } \leq \varepsilon\}. \tag{3.1}$$

In some cases, the calculation of the measure of noncompactness (3.1) is easy. For example, denoting the closed ball around zero of radius $r > 0$ in a Banach space X by $B_r(X)$, we have $\alpha(B_r(X)) = 0$ if X is finite dimensional, by the classical Heine-Borel theorem. On the other hand, $\alpha(B_r(X)) = 2r$ if X is infinite dimensional, by the well-known Riesz theorem.

Now, we given two Banach spaces X and Y, a set $M \subseteq X$, and a (linear or nonlinear) operator $F : M \to Y$, the characteristic

$$[F]_\alpha = \inf\{k > 0 : \ \alpha(F(N)) \leq k\alpha(N) \text{ for bounded } N \subseteq M\} \tag{3.2}$$

is called the *α-norm* of F. The α-norm (3.2) has the property that $[F]_\alpha = 0$ if and only if F is a compact operator; for this reason (3.2) is often called measure of noncompactness of the operator A. If $F : X \to Y$ satisfies a Lipschitz condition on $M \subseteq X$, i.e.,

$$Lip(F) := \inf\{k > 0 : \|F(x) - F(y)\| \leq k\|x - y\| \ \text{ for all } x, y \in M\} < \infty, \tag{3.3}$$

then it is obvious that

$$[F]_\alpha \leq Lip(F). \tag{3.4}$$

So the crucial condition in Theorem 2.1 is $Lip(F) < 1$. Clearly, if $L : X \to Y$ is a bounded *linear* operator, then $Lip(L) = \|L\|$. Operators F satisfying $[F]_\alpha < 1$ are usually called *condensing*. Compact operators are trivially condensing, and (3.4) shows that contractions in Banach spaces are also condensing. So the following fixed point principle [8] generalizes both Theorem 3.1 and (a special version of) Theorem 2.1.

Theorem 3.2 (Darbo)**.** *Let $(X, \| \|)$ be a Banach space, $M \subset X$ a bounded, closed, convex subset, and $F : M \to M$ a continuous condensing operator. Then F has a fixed point in M.*

In applications of Theorem (3.2), it is important whether one has equality or strict inequality in (3.4). Darbo's theorem applies only if $[F]_\alpha < 1$, but in case $[F]_\alpha = Lip(F)$, one can then also apply Banach's theorem, which means that Darbo's theorem does not provide any more information than Banach's theorem. So only the case $0 < [F]_\alpha < 1 \leq Lip(F)$ is interesting because Theorem 2.1 and Theorem 3.1 do not apply, but Theorem 3.2 does. Many examples for this situation can be found in the survey paper [3].

In metric fixed point theory, one of the most natural generalizations of the contraction condition (2.1) is

$$d(F(x),F(y)) < d(x,y) \qquad (x \neq y). \tag{3.5}$$

The operators satisfying (3.5) are called *weakly contracting*. Recall that a weakly contracting operator in a complete metric space need not have fixed points (consider $F(x) := \log(1+e^x)$ on $X := \mathbb{R}$). However, the following fixed point result by [9] for weak contractions on a very restricted class of spaces is true.

Theorem 3.3 (Edelstein)**.** *Let (X,d) be a compact metric space, and let $F : X \to X$ be a weak contraction. Then f has a unique fixed point in X.*

Motivated by condition (3.5), Sadovskij [35, 36] calls an operator F on a Banach space X *weakly condensing* if

$$\alpha(F(M)) < \alpha(M) \qquad (\alpha(M) > 0) \tag{3.6}$$

and proves the following fixed point theorem for such operators.

Theorem 3.4 (Sadovskij)**.** *Let $(X,\|\,\|)$ be a Banach space, $M \subset X$ a bounded, closed, and convex subset, and $F : M \to M$ a continuous weakly condensing operator. Then F has a fixed point in M.*

We remark that the nonlinear community remained unaware of Theorem 3.2 and Theorem 3.4 for many years, perhaps because the paper [8] was written in Italian and the paper [35] in Russian. However, after the publication of the monographs [1, 2, 4, 5] measures of noncompactness and condensing operators received much more attention and became an important and quite fashionable field of nonlinear functional analysis.

To show that Theorem 3.4 covers in fact a wider range of applications than Theorem 3.2, let us give an example. This example is based on the elementary geometric fact that the scalar function $\phi(t) := (1-|t|)t$ maps $[-1,1]$ into $[-1/4,1/4]$ and has maximal slope $\phi'(0) = 1$.

Example 3.1. Let X be an infinite dimensional Banach space, and define F on the closed unit ball $B_1(X)$ of X by $F(x) := (1-\|x\|)x$. We claim that F is weakly condensing on $B_1(X)$. To this end, we use the fact that the measure of noncompactness (3.1) is invariant under passing to the convex closed hull of a set, i.e., $\alpha(\overline{co}\,M) = \alpha(M)$. For any $M \subseteq B_1(X)$ with $\alpha(M) > 0$ and $0 < r < \alpha(M)/2$, we have

$$\alpha(F(M \cap B_r(X))) \leq \alpha(M \cap B_r(X)) \leq \alpha(B_r(X)) = 2r < \alpha(M),$$

on the one hand, and

$$\alpha(F(M \setminus B_r(X))) \leq \alpha(\overline{co}\,[(1-r)M \cup \{o\}]) = (1-r)\alpha(M) < \alpha(M),$$

on the other. Using the fact that α is subadditive, we obtain

$$\alpha(F(M)) = \max\{\alpha(F(M \cap B_r(X))), \alpha(F(M \setminus B_r(X)))\} < \alpha(M),$$

which shows that F satisfies (3.6) as claimed.

On the other hand, F is not condensing on $B_1(X)$ because, for the sphere $S_r := \{x \in X : \|x\| = r\}$ with $0 < r < 1$, we have $\alpha(S_r) = 2r$ and $\alpha(F(S_r)) = 2(1-r)r$. So assumption $[F]_\alpha =: q < 1$ would lead to $r - r^2 \leq qr$, a contradiction for $r < 1-q$.

We point out that Theorem 3.1 is usually proved by considering projections to a sequence of finite dimensional subspaces, and Theorem 3.2 may be proved by an iterated application of a condensing operator until reaching compactness. In this way, Schauder's theorem may be reduced to Brouwer's theorem, and Darbo's theorem to Schauder's theorem. However, Sadovskij's proof of Theorem 3.4 in [36] uses transfinite induction, without recurring to previous fixed point theorems. It is therefore interesting to note that Theorem 3.4 can be reduced to Theorem 3.2.

To show this, we suppose that X, M, and F satisfy the hypotheses of Theorem 3.4, where without loss of generality M contains the zero vector of X. For $0 < q < 1$, the operator qF is then condensing and maps M into itself since M is convex. By Theorem 3.2, there exists x_q such that $x_q = qF(x_q)$. Hence,

$$\|x_q - F(x_q)\| \leq \|x_q - qF(x_q)\| + (1-q)\|F(x_q)\| = (1-q)\|F(x_q)\|,$$

which implies that $\inf\{\|x - F(x)\| : x \in M\} = 0$ since q may be taken arbitrarily close to 1. Choose a sequence $(x_n)_n$ in M such that $x_n - F(x_n)$ converges to zero as $n \to \infty$. Then the set $N := \{x_1, x_2, x_3, \ldots\}$ is precompact because $\alpha(F(N)) = \alpha(N)$ and F is weakly condensing. Therefore $(x_n)_n$ admits a convergent subsequence, and the limit of this subsequence is a fixed point of F.

It is a tempting idea to consider that the above mentioned fixed point theorem for weakly contracting maps might be directly reduced to Banach's Theorem 2.1 by a compactness argument, similarly as we just reduced Theorem 3.4 to Theorem 3.2. However, the following example shows that a map may be a weak contraction on a compact set without being a contraction.

Example 3.2. Consider the function $F : [0,1] \to [0,1]$ defined by $F(x) := x - x^2$. The unique fixed point of F is $x_* = 0$. It is easy to see that F is not a contraction. However, F is a weak contraction because

$$|F(x) - F(y)| = |x - y - (x+y)(x-y)| = |1 - x - y|\,|x - y|$$

and the term $|1 - x - y| \leq 1$ can become equal to 1 only for $x = y = 0$ or $x = y = 1$, which is excluded by condition (3.5).

The reason for Example 3.2 is that the condition $|F'(x)| \leq q < 1$ is sufficient for F to be contracting, but not necessary, and the condition $|F'(x)| < 1$ is sufficient for F to be weakly contracting, but not necessary. Of course, if $F \in C^1(I)$ and I is a compact interval, then both conditions $|F'(x)| \leq q < 1$ and $|F'(x)| < 1$ are equivalent, by the continuity of F'. Here is another example of a differentiable map F on a compact interval with discontinuous derivative F'.

Example 3.3. The function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} (1-x)\sin\dfrac{1}{x} & \text{for } \ 0 < x \leq 1, \\ 0 & \text{for } \ x = 0 \end{cases}$$

is discontinuous at $x = 0$, but has the primitive

$$F(x) := x^2 \cos\frac{1}{x} - 2\int_0^x t\cos\frac{1}{t}\,dt - \int_0^x t\sin\frac{1}{t}\,dt$$

on $[0,1]$ with $F(0)=0$, which is the unique fixed point of F in $[0,1]$. Since

$$|f(x)|<1, \quad \limsup_{x\to 0+} f(x)=1,$$

the map F is a weak contraction, but not a contraction.

Although this is quite elementary, we collect for the reader's ease the various conditions on a map F on an interval I which ensure the existence and uniqueness of a fixed point of F.

$F'(x)\le q<1$	$\Rightarrow$	$F'(x)<1$
$\Downarrow$		$\Downarrow$
F contraction	$\Rightarrow$	F weak contraction

The upper arrow can be inverted if I is compact and $F\in C^1(I)$. On the other hand, none of the other arrows can be inverted, even if I is compact, which is shown by our examples.

The estimate (3.4) shows that a contraction is always condensing. One might think that similarly the weak contraction condition (3.5) implies that (3.6) holds. Surprisingly enough, this is not true: The following Example 3.4 shows that a weak contraction is not necessarily weakly condensing, although a contraction is condensing.

Example 3.4. Let $X=c_0$ be the Banach space of all sequences $x=(\xi_n)_n$ converging to zero with the supremum norm. It is not hard to prove that the map F defined by

$$F(x)=F(\xi_1,\xi_2,\xi_3,\ldots,\xi_n,\ldots)=\left(\tfrac{1}{2}(1+||x||),\tfrac{3}{4}\xi_1,\tfrac{7}{8}\xi_2,\ldots,(1-2^{-n})\xi_{n-1},\ldots\right) \tag{3.7}$$

maps the unit ball $B_1(c_0)$ into itself and is a continuous weak contraction in the sense of (3.5). However, F has no fixed point. Indeed, if $\hat{x}=(\hat{\xi}_1,\hat{\xi}_2,\hat{\xi}_3,\ldots)$ were a fixed point of F in $B_1(c_0)$, then

$$\hat{\xi}_1=\tfrac{1}{2}(1+||\hat{x}||),\ \hat{\xi}_2=\tfrac{1}{2}\frac{3}{4}(1+||\hat{x}||),\ \ldots,\ \hat{\xi}_n=(1+\|\hat{x}\|)\prod_{j=1}^{n}\left(1-2^{-j}\right),\ \ldots.$$

So we have the estimate

$$\hat{\xi}_n\ge(1+\|\hat{x}\|)\prod_{j=1}^{\infty}\left(1-2^{-j}\right). \tag{3.8}$$

Since

$$\sum_{j=1}^{\infty}\log\left(1-2^{-j}\right)\ge-2\sum_{j=1}^{\infty}2^{-j}=-2,$$

the product in (3.8) converges to some value $\ge 1/e^2$. So the point $\hat{x}$ cannot belong to X. From Theorem 3.4, it follows that operator (3.7) cannot be weakly condensing. This may also be verified directly by observing that $F(e_n)=e_1+(1-2^{-(n+1)})e_{n+1}$ with $(e_n)_n$ being the canonical basis in X.

The hierarchy Brouwer-Schauder-Darbo-Sadovskij is one of the best known examples in nonlinear analysis for reducing a result to a simpler one which has been previously proved. In this connection, we cannot resist citing a fictious funny story invented by the authors of the beautiful recent Lecture Notes [6] which reads as follows.

When Brouwer claimed that his fixed point theorem could not be improved, Schauder came up and told him "Please hold my Goldwasser glass a moment and look at this!", and showed him our Theorem 3.1. Afterwards, when Schauder claimed that his fixed point theorem could not be improved, Darbo came up and told him "Please hold my Wine glass a moment and look at this!", and showed him our Theorem 3.2. Finally, when Darbo claimed that his fixed point theorem could not be improved, Sadovskij came up and told him "Please hold my Vodka glass a moment and look at this!", and showed him our Theorem 3.4.

REFERENCES

[1] R. R. Akhmerov, M. I. Kamenskij, A. S. Potapov, B. N. Sadovskij, Condensing Operators, Itogi Nauki Tekhniki 18 (1980), 185–250. [in Russian]

[2] R. R. Akhmerov, M. I. Kamenskij, A. S. Potapov, A. E. Rodkina, B. N. Sadovskij, Measures of Noncompactness and Condensing Operators, Nauka, Novosibirsk 1986; Engl. transl.: Birkhäuser, Basel 1992. [in Russian].

[3] J. Appell, Measures of noncompactness, condensing operators and fixed points: An application-oriented survey, Fixed Point Theory 6 (2005), 157–229.

[4] J. M. Ayerbe Toledano, T. Domínguez Benavides, G. López Azedo, Measures of Noncompactness in Metric Fixed Point Theory, Birkhäuser, Basel 1997.

[5] J. Banaś, K. Goebel, Measures of Noncompactness in Banach Spaces, M. Dekker, New York 1980.

[6] M. Borkowski, D. Bugajewska, P. Kasprzak, Selected Topics in Nonlinear Analysis, Lect. Notes Nonlinear Anal. 19, Toruń 2021.

[7] E. H. Connell, Properties of fixed point spaces, Proc. Amer. Math. Soc. 10 (1959), 974–979.

[8] G. Darbo, Punti uniti in trasformazioni a codominio non compatto, Rend. Sem. Mat. Univ. Padova 24 (1955), 84–92.

[9] M. Edelstein, On fixed and peridoc points under contractive mappings, J. London Math. Soc. 37 (1962), 74–79.

[10] G. E. Hardy, T. D. Rogers, A generalization of a fixed point theorem of Reich, Canadian Math. Bull. 16 (1973), 201–206.

[11] R. Kannan, Some results on fixed points, Bull. Calcutta Math. 60 (1969), 71–76.

[12] K. Kuratowski, Sur les espaces complètes, Fund. Math. 15 (1934), 301-335.

[13] S. Reich, Some remarks concerning contraction mappings, Canad. Math. Bull. 14 (1971), 121–124.

[14] S. Reich, Kannan's fixed point theorem, Boll. Unione Mat. Ital. 4 (1971), 1–11.

[15] S. Reich, A fixed point theorem, Atti Accad. Naz. Lincei 51 (1971), 26–28.

[16] S. Reich, Fixed points of multivalued functions, Atti Accad. Naz. Lincei 51 (1971), 32–35.

[17] S. Reich, A fixed point theorem in locally convex spaces, Bull. Calcutta Math. Soc. 63 (1971), 199–200.

[18] S. Reich, Fixed points in complete metric spaces, Atti Accad. Naz. Lincei 51 (1971), 270–273.

[19] S. Reich, Remarks on fixed points, Atti Accad. Naz. Lincei 52 (1971), 689–697.

[20] S. Reich, Fixed points in locally convex spaces, Math. Zeitschr. 125 (1972), 17–31.

[21] S. Reich, A fixed point theorem for locally contractive multi-valued functions, Rev. Roum. Math. Pures Appl. 17 (1972), 569–572.

[22] S. Reich, m Some remarks on fixed point sets, Delta 3 (1972), 38–43.

[23] S. Reich, Fixed points of contractive functions, Boll. Unione Mat. Ital. 5 (1972), 26–42.

[24] S. Reich, Remarks on fixed points II, Atti Accad. Naz. Lincei 53 (1972), 250–254.

[25] S. Reich, Fixed points of condensing functions, J. Math. Anal. Appl. 41 (1973), 460–467.

[26] S. Reich, Fixed points of non-expansive functions, J. London Math. Soc. 7 (1973), 5–10.

[27] S. Reich, Asymptotic behavior of contractions in Banach spaces, J. Math. Anal. Appl. 44 (1973), 57–70.

[28] S. Reich, Fixed points via Toeplitz iteration, Bull. Calcutta Math. Soc. 65 (1973), 203–207.

[29] S. Reich, Fixed point iterations of nonexpansive mappings, Pacific J. Math. 60 (1975), 195–198.

[30] S. Reich, Some fixed point problems, Atti Accad. Naz. Lincei 57 (1975), 194—198.

[31] S. Reich, On fixed point theorems obtained from existence theorems for differential equations, J. Math. Anal. Appl. 54 (1976), 26–36.

[32] S. Reich, The fixed point property for non-expansive mappings, Amer. Math. Monthly 83 (1976), 266–268.

[33] S. Reich, Asymptotic behavior of semigroups on nonlinear contractions in Banach spaces, J. Math. Anal. Appl. 53 (1976), 277–290.

[34] B. E. Rhoades, A comparison of various definition of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257–289.

[35] B. N. Sadovskij, On a fixed point principle, Funkts. Analiz Prilozh. 1 (1967), 74–76. [in Russian]

[36] B. N. Sadovskij, Measures of noncompactness and condensing operators, Problemy Mat. Anal. Slozhnykh Sistem 2 (1968), 89–119.[in Russian]

[37] P. V. Subrahmanyam, Completeness and fixed points, Monatsh. Math. 80 (1975), 325–330.