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FILTRATION FAMILIES OF SEMIGROUP GENERATORS. THE FEKETE–SZEGÖ PROBLEM

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Dedicated to Professor Simeon Reich on the occasion of his 75th Birthday

Abstract. The paper is devoted to two problems pivotal in the generation theory. The first concerns verifying whether a normalized holomorphic function is an infinitesimal generator. It is also important to trace the influence of properties of a generator on the dynamic behavior of the generated semigroup. Recently, these problems have been studied using the so-called filtrations of generators. In this paper we study three filtrations. The study of the first began earlier, the second is a new one, while the third consists of functions studied previously with no connection to complex dynamics. For all filtration classes studied in the paper, dynamic properties of the corresponding semigroups are established and estimates for the Fekete–Szegö functional are given.
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1. Introduction

Semigroups of non-linear maps is a natural generalization of semigroups of linear operators. As for the one-dimensional case, the breakthrough in the theory of semigroups holomorphic self-mappings of the open unit disk is associated with the work by Berkson and Porta [2]. They proved that every semigroup is generated and the structure of generators was discovered. Later, the coherent generation theory in Banach spaces was established starting from the work [21] by Simeon Reich and the third author. During the last decades, various characterizations of semigroup generators have been found, see details in the monographs [3, 13, 14, 22] and the references therein. It is especially interesting to trace the influence of properties of a generator on the dynamic behavior of the generated semigroup.

Despite of the presence of various criteria for generators, in some specific situations it is difficult to check whether a given function belongs to the set of all generators. It is even harder to check special dynamic properties. At the same time, there are some sufficient conditions providing that a function belongs to *a proper subclass* of all generators the verification of which is easier. This idea served a basis for the so-called filtration theory of generators, beginning from

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the work [4]. Some concrete filtrations have been introduced and studied in [4, 11, 15, 26] and other works.

In this paper, we focus on the one-dimensional case and provide certain sufficient conditions, including a parameter, for a function to be a generator. We study three concrete filtrations with an emphasis on dynamic properties of semigroups generated by elements of different filtration families.

On the other hand, each filtration family is a class of analytic (and sometimes univalent) functions in the open unit disk that is interesting itself. Since evaluation of certain functionals over different classes of analytic (univalent) functions is one of the classical problems in geometric function theory, we investigate this problem for linear and quadratic functionals over all filtration classes we consider.

The structure of the paper is the following. After the preliminaries presented in Section 2, we turn to study filtrations of generators. Section 3 is devoted to the so-called analytic filtration. Families of this filtration are convex sets defined by some linear functionals. It was partially studied in [4, 15, 26]. We supplement the previous results by expanding the range of the parameter from the interval $[0,1]$ to $(-\infty,1]$, give a new characterization of the filtration sets as well as sharp estimates of the generalized Zalcman's functional over these sets. Another filtration, presented in Section 4, is a new one. We call it the pseudo-analytic filtration since its sets are defined by the absolute value of the same linear functional as ones of the analytic filtration. The dynamic and geometric properties of pseudo-analytical filtration are established. The last filtration studied in this paper consists of prestarlike functions (Section 5). This term was introduced by Ruscheweyh [24]. The embedding of the classes of prestarlike functions was established by Suffridge [28]. Paying attention to the fact that these classes form a filtration of generators, we study the dynamic behavior of semigroups generated by such generators. In addition, we give estimates on the Fekete–Szegö functional over the classes of the prestarlike filtration.

It is worth to mention that an analytic function is a semigroup generator if and only if it produces the so-called *resolvent family*. Every its element (non-linear resolvent) is a univalent function. Thus each filtration of generators on a natural way produces parametrical-embedded families of univalent functions (namely, resolvents). For these families sharp bounds on their Taylor coefficients and the Fekete–Szegö functional is established at the end of Section 5. For the class of all generators, this problem was studied earlier in [8, 10].

2. Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Denote by $\mathrm{Hol}(\mathbb{D},\mathbb{C})$ the set of holomorphic functions on $\mathbb{D}$, and by $\mathrm{Hol}(\mathbb{D}) := \mathrm{Hol}(\mathbb{D},\mathbb{C})$, the set of all holomorphic self-mappings of $\mathbb{D}$. Denote by $\mathscr{A}$ the subset of $\mathrm{Hol}(\mathbb{D},\mathbb{C})$ consisting of functions normalized by $f(0) = f'(0)-1 = 0$ and by $\mathscr{P}$ the Carathéodory class, that is, the class of normalized holomorphic functions with positive real part:

$$\mathscr{P} := \{q \in \mathrm{Hol}(\mathbb{D},\mathbb{C}) : \ \mathrm{Re}\, q(z) > 0, \quad z \in \mathbb{D} \quad \text{and} \quad q(0) = 1\}.$$

The Carathéodory class plays an important role in geometric function theory and in adjacent fields. In particular, many famous classes of univalent functions are characterized by constrains on some expression containing a tested function. For instance, the class $\mathscr{S}^*(\alpha)$ of starlike functions of order $\alpha \in [0,1)$ is defined by

$$\mathscr{S}^*(\alpha) = \left\{ f \in \mathscr{A} : \ \operatorname{Re} \frac{zf'(z)}{f(z)} \geq \alpha \right\}.$$

Let $f, g \in \mathscr{A}$ and

$$\Omega = \{\omega \in \operatorname{Hol}(\mathbb{D}) : \ \omega(0) = 0\}.$$

One says that f is subordinate g and write $f \prec g$ if there exists a function $\omega \in \Omega$ such that $f(z) = g(\omega(z))$ for all $z \in \mathbb{D}$. In the case that g is univalent, conditions $f \prec g$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$ are equivalent.

We now turn to main notions and facts from semigroup theory.

Definition 2.1. A family $\{\phi_t\}_{t\geq 0} \subset \operatorname{Hol}(\mathbb{D})$ is called a *one-parameter continuous semigroup* (or just *semigroup*) if

(a) $\phi_{t+s} = \phi_t \circ \phi_s$, $t, s \geq 0$; and

(b) $\lim\limits_{t\to 0^+} \phi_t = \operatorname{Id}$, where Id is the identity map on $\mathbb{D}$ and the limit is taken with respect to the topology of uniform convergence on compact sets in $\mathbb{D}$.

Moreover, according to the fundamental result by Berkson and Porta [2], the family $\{\phi_t\}_{t\geq 0}$ is differentiable with respect to its parameter $t \geq 0$, and the limit

$$f = \lim_{t\to 0^+} \frac{1}{t} \left(\operatorname{Id} - \phi_t \right),$$

exists, and defines a holomorphic function on $\mathbb{D}$. Furthermore, ϕ_t is the solution to the Cauchy problem:

$$\frac{\partial \phi_t(z)}{\partial t} + f(\phi_t(z)) = 0 \quad \text{and} \quad \phi_0(z) = z \in \mathbb{D}.$$

The function f is called the *infinitesimal generator* of the semigroup $\{\phi_t\}_{t\geq 0} \subset \operatorname{Hol}(\mathbb{D})$. The class of all holomorphic generators on $\mathbb{D}$ is denoted by $\mathscr{G}$.

The following assertion contains two characterizations of class $\mathscr{G}$.

Theorem 2.1. *Let $f \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$. Then $f \in \mathscr{G}$ if and only if there are a point $\tau \in \overline{\mathbb{D}}$ and a function $q \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ with $\operatorname{Re} q(z) \geq 0$, $z \in \mathbb{D}$ such that*

$$f(z) = (z - \tau)(1 - z\overline{\tau})\, q(z), \quad z \in \mathbb{D}. \tag{2.1}$$

Moreover, this representation is unique.

We notice that formula (2.1) was obtained by Berkson and Porta in [2] and is called the *Berkson–Porta representation*. Let us now return to the case $\tau \in \mathbb{D}$. Up to a Möbius transformation M_τ $\left(M_\tau(z) = \frac{\tau - z}{1 - z\overline{\tau}}\right)$ of the unit disk, one can always consider generators such that $f(0) = 0$, or, what is the same, $\phi_t(0) = 0$ for all $t \geq 0$. Then $f \in \mathscr{A}$ is an infinitesimal generator if and only if $\operatorname{Re} \frac{f(z)}{z} \geq 0, \quad z \in \mathbb{D} \setminus \{0\}$, see Theorem 2.1. We denote

$$\mathscr{G}_0 := \mathscr{A} \cap \mathscr{G} = \left\{ f \in \operatorname{Hol}(\mathbb{D}, \mathbb{C}) : \ \frac{f(z)}{z} \in \mathscr{P} \right\}.$$

The class $\mathscr{G}_0$ is very important for the study of non-autonomous problems and geometric function theory (see, for example, [3, 5, 12, 14, 22]).

Over the years, the study of the asymptotic behavior of semigroups was mainly focused on the local/global rate of convergence and the growth estimates of a semigroup with respect to its parameter. Different estimates of the rate of convergence of semigroups were obtained. As for the one-dimensional case, the reader can be referred to the books [3, 14, 25], the survey [19] and the references therein. Regarding later results, recall that the semigroup $\{\phi_t\}_{t\geq 0}$ is called *exponentially squeezing with squeezing ratio* $k>0$ if it satisfies $|\phi_t(z)|\leq |z|e^{-kt}$ for all $t>0$ and $z\in\mathbb{D}$.

Theorem 2.2 (see [3, 4, 13]). *Let $f\in\mathscr{G}_0$ and $\{\phi_t\}_{t\geq 0}$ be the semigroup generated by f. Then $\{\phi_t\}_{t\geq 0}$ is exponentially squeezing with squeezing ratio $k>0$ if and only if* $\operatorname{Re}\frac{f(z)}{z}>k$ *on* $\mathbb{D}$.

Another direction in the study of the semigroup properties is focused on the possibility of analytic extension with respect to the semigroup parameter into a complex domain. The problem is to find conditions that provide analytic extension of semigroups along with estimates of that sector in $\mathbb{C}$ to which this extension is possible. The progress in this direction presented, for example, in [1, 7, 16]. To be more specific, fix $\theta\in\left(0,\frac{\pi}{2}\right]$ and denote

$$\Lambda(\theta)=\{\zeta\in\mathbb{C}:|\arg\zeta|<\theta\}. \tag{2.2}$$

Theorem 2.3 (Theorem 2.12 in [16]). *Let $\{\phi_t\}_{t\geq 0}$ be a semigroup of holomorphic self-mappings of $\mathbb{D}$ generated by $f\in\mathscr{G}_0$, and let $\alpha\in[0,1)$. Then $\{\phi_t\}_{t\geq 0}$ extends analytically to the sector $\Lambda\left(\frac{\pi(1-\alpha)}{2}\right)$ in $\mathbb{C}$ if and only if $\left|\arg\frac{f(z)}{z}\right|\leq\frac{\pi\alpha}{2}$ on $\mathbb{D}\setminus\{0\}$.*

The above problems show that it is very important to classify generators up to dynamic properties of the semigroups they generate. To this end, we introduce filtration (or parametric embedding) of infinitesimal generators.

Definition 2.2 (see [4]). Let J be a connected subset of $\mathbb{R}$. A *filtration* of $\mathscr{G}_0$ is a family $\mathfrak{F}=\{\mathfrak{F}_s\}_{s\in J}$, $\mathfrak{F}_s\subseteq\mathscr{G}_0$, such that $\mathfrak{F}_s\subseteq\mathfrak{F}_t$ whenever $s,t\in J$ and $s\leq t$.

Let a filtration $\mathfrak{F}=\{\mathfrak{F}_s\}_{s\in J}$ be given. The above problems induce the following questions:

- find the sharp squeezing ratio for all semigroups generated by elements of every filtration set $\mathfrak{F}_s$;
- find the maximal sector into which all these semigroups can be analytically extended.

Obviously, both these quantities are non-increasing with respect to the filtration parameter $s\in J$. We see below that there are filtrations for which the above quantities are decreasing, as well as those for which they are constant on J.

To describe certain filtrations in more details, we introduce additional notions.

Definition 2.3 (see [4, 15]). Let $\mathfrak{F}=\{\mathfrak{F}_s\}_{s\in J}$, $\mathfrak{F}_s\subseteq\mathscr{G}_0$, be a filtration of $\mathscr{G}_0$. We say that the filtration $\{\mathfrak{F}_s\}_{s\in J}$ is *strict* if $\mathfrak{F}_s\subsetneq\mathfrak{F}_t$ for $s<t$ $s,t\in J$. In this case for $t\in J$ we set

$$\partial\mathfrak{F}_t=\mathfrak{F}_t\setminus\bigcup_{s\in J,s<t}\mathfrak{F}_s. \tag{2.3}$$

Note that if the boundaries defined by (2.3) are not empty for every $t\in J$, then filtration $\mathfrak{F}$ is strict although for a strict filtration boundaries of its elements might be empty.

3. Analytic Filtration

The analytic filtration was studied in several works [4, 15]. We extend the range of the parameter from $[0,1]$ as in that works to $(-\infty,1]$ and establish new properties of the analytic filtration.

Definition 3.1. We say that a function $f \in \mathscr{A}$ belongs to the class $\mathfrak{A}_\alpha,\ \alpha \le 1$ if it satisfies the inequality
$$\operatorname{Re}\left[\alpha\frac{f(z)}{z} + (1-\alpha)f'(z)\right] \ge 0, \quad z \in \mathbb{D}\setminus\{0\}. \tag{3.1}$$

We first obtained some criteria of the membership of a function $f \in \mathscr{A}$ in the class $\mathfrak{A}_\alpha$.

Theorem 3.1. *Let $f \in \mathscr{A}$. For $\alpha \in (-\infty, 1]$ denote*
$$F_\alpha(z) = \int_0^1 \frac{1+t^{1-\alpha}z}{1-t^{1-\alpha}z}\,dt. \tag{3.2}$$
The following conditions are equivalent:

(1) *a function f belongs to $\mathfrak{A}_\alpha$;*
(2) *there is a function $q \in \mathscr{P}$ such that*
$$f(z) = z\int_0^1 q\left(t^{1-\alpha}z\right)dt; \tag{3.3}$$
(3) *there is a probability measure μ on the unit circle $\partial\mathbb{D}$ such that*
$$f(z) = z\int_{\partial\mathbb{D}} F_\alpha(z\overline{\zeta})d\mu(\zeta). \tag{3.4}$$

Representation (3.3) for $\alpha \in [0,1]$ was obtained in [4] (see also [15]). We prove it for all $\alpha \le 1$.

Proof. Let $f \in \mathfrak{A}_\alpha$. Then there is a function q of the Carathéodory class such that
$$\alpha\frac{f(z)}{z} + (1-\alpha)f'(z) = q(z). \tag{3.5}$$
Solving this differential equation, we obtain (3.3). Assume now that f can be represented by formula (3.3) with some function $q \in \mathscr{P}$. In turn, q admits the Riesz–Herglotz representation, that is, there is a probability measure μ on the unit circle such that
$$q(z) = \oint_{\partial\mathbb{D}} \frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}d\mu(\zeta).$$
Now, formula (3.3) becomes
$$\begin{aligned} f(z) &= z\int_0^1\left(\oint_{\partial\mathbb{D}}\frac{1+t^{1-\alpha}z\overline{\zeta}}{1-t^{1-\alpha}z\overline{\zeta}}d\mu(\zeta)\right)dt \\ &= z\oint_{\partial\mathbb{D}}\left(\int_0^1\frac{1+t^{1-\alpha}z\overline{\zeta}}{1-t^{1-\alpha}z\overline{\zeta}}dt\right)d\mu(\zeta) = z\int_{\partial\mathbb{D}}F_\alpha(z\overline{\zeta})d\mu(\zeta), \end{aligned}$$
so (3.4) holds. Assume that condition (3) is fulfilled and hence
$$f'(z) = \int_{\partial\mathbb{D}} F_\alpha(z\overline{\zeta})d\mu(\zeta) + z\int_{\partial\mathbb{D}} F'_\alpha(z\overline{\zeta})\overline{\zeta}d\mu(\zeta).$$

Therefore

$$\begin{aligned}\alpha\frac{f(z)}{z}+(1-\alpha)f'(z) &= \int\limits_{\partial\mathbb{D}} F_\alpha(z\overline{\zeta})d\mu(\zeta)+(1-\alpha)\int\limits_{\partial\mathbb{D}} F'_\alpha(z\overline{\zeta})z\overline{\zeta}d\mu(\zeta)\\ &= \int\limits_{\partial\mathbb{D}} \frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}d\mu(\zeta)\end{aligned}$$

by the definition of functions F_α in formula (3.2), which means that $f\in\mathfrak{A}_\alpha$. The proof is complete. $\square$

It is worth mentioning that the functions F_α defined by formula (3.2) have important analytic properties. In particular, they have been demonstrated in [4, 15] that F_α is a univalent function. In addition, the condition (3) of Theorem 3.1 immediately implies

Corollary 3.1. *Let $\alpha\le 1$. If $f\in\mathfrak{A}_\alpha$, then, for every $\lambda\in\mathbb{C}$,*

$$\max_{|z|=r}\operatorname{Re}\left(\lambda\frac{f(z)}{z}\right)\le\max_{|z|=r}\operatorname{Re}\left(\lambda F_\alpha(z)\right). \tag{3.6}$$

Our next aim is to prove that the sets $\mathfrak{A}_\alpha$ form a filtration of $\mathscr{G}_0$ and to study properties of this filtration. Moreover, the membership of a function f to a set of the analytic filtration means that the semigroup generated by f owns certain dynamical properties, and we describe these properties. To this end we following [4] denote

$$\varkappa(\alpha):=\int_0^1\frac{1-t^{1-\alpha}}{1+t^{1-\alpha}}dt=\inf_{z\in\mathbb{D}}\operatorname{Re}F_\alpha(z). \tag{3.7}$$

Theorem 3.2. *The following assertions hold:*

(a) *The family $\mathfrak{A}=\{\mathfrak{A}_\alpha\}_{\alpha\le1}$ is a filtration with $\mathfrak{A}_{-\infty}:=\bigcap_{\alpha<1}\mathfrak{A}_\alpha=\{\mathrm{Id}\}$ and $\mathfrak{A}_1=\mathscr{G}_0$.*
(b) *For $\alpha\le1$ denote $f_\alpha=zF_\alpha(z)$. Then $f_\alpha\in\partial\mathfrak{A}_\alpha$. Consequently, the filtration $\mathfrak{A}$ is strict.*
(c) *Let $\alpha<1$, $f\in\mathfrak{A}_\alpha$ and $\{\phi_t(z)\}_{t\ge0}$ be the semigroup generated by f. Then*
  (i) *this semigroup is exponentially squeezing with squeezing ratio $\varkappa(\alpha)$;*
  (ii) *this semigroup extends analytically to the sector $\Lambda\left(\frac{\pi(1-\delta(\alpha))}{2}\right)$, where*

$$\delta(\alpha):=\frac{2}{\pi}\cdot\max_{0<\theta<\pi}\arg F_\alpha(e^{i\theta})$$

and F_α is defined by (3.2).

Note meanwhile that function $\varkappa$ is decreasing with $\lim\limits_{\alpha\to-\infty}\varkappa(\alpha)=1$ and $\varkappa(1)=0$, see Fig. 1. Function δ is increasing with $\lim\limits_{\alpha\to-\infty}\delta(\alpha)=0$ and $\delta(1)=1$, see Fig. 2. Numerical calculations using Maple give $\varkappa(-5)\approx0.8075$ and $\delta(-5)\approx0.2024$.

Proof. First we prove that $\mathfrak{A}$ is a filtration of $\mathscr{G}_0$. Indeed, let $\alpha<\beta\le1$ and $f\in\mathfrak{A}_\alpha$. Write f in the form $f(z)=zp(z)$. Then inequality (3.1) means that

$$\operatorname{Re}\left[\frac{1}{1-\alpha}p(z)+zp'(z)\right]\ge0,\quad z\in\mathbb{D}. \tag{3.8}$$

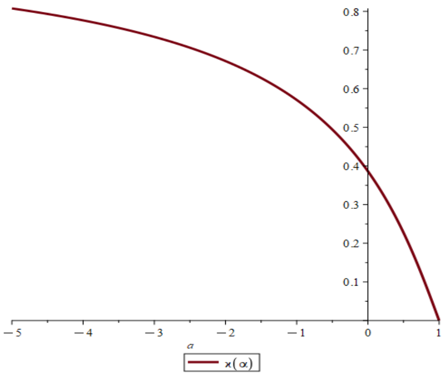

FIGURE 1. The graph of $\varkappa(\alpha)$.

It follows from Lemma 3.5.3 in [25] that $\operatorname{Re} p(z) \geq 0$ on $\mathbb{D}$, and hence $p \in \mathscr{P}$, that is, $f \in \mathscr{G}_0$. Moreover,

$$\operatorname{Re}\left[\frac{1}{1-\beta}p(z)+zp'(z)\right] \geq \operatorname{Re}\left[\frac{1}{1-\alpha}p(z)+zp'(z)\right] \geq 0.$$

So $f \in \mathfrak{A}_\beta$. Relation $\mathfrak{A}_1 = \mathscr{G}_0$ follows directly.

To find the intersection $\bigcap_{\alpha<1}\mathfrak{A}_\alpha$, we substitute $f(z) = zp(z)$ in (3.1) and rewrite it in the form (3.8). Letting $\alpha \to -\infty$, we obtain the inequality $\operatorname{Re} zp'(z) \geq 0$ for all $z \in \mathbb{D}$, which is possible only in the case $p' \equiv 0$. Thus $\mathfrak{A}_{-\infty} = \{\mathrm{Id}\}$, which completes the proof of assertion (a).

It follows from formula (3.2) that $\alpha\frac{f_\alpha(z)}{z} + (1-\alpha)f'_\alpha(z) = \frac{1+z}{1-z}$. Therefore $f_\alpha \in \mathfrak{A}_\alpha$. Now, we prove that $f_\alpha \notin \mathfrak{A}_\beta$ whenever $\beta < \alpha \leq 1$. Indeed, formula (3.7) implies that $\inf_{z\in\mathbb{D}} \operatorname{Re}\frac{f_\alpha(z)}{z} = \varkappa(\alpha) < \varkappa(\beta)$, while $\operatorname{Re}\frac{f(z)}{z} \geq \varkappa(\beta)$ for any $f \in \mathfrak{A}_\beta$ by formula (3.6) with $\lambda = -1$. So, assertion (b) is proven.

Assertion (c) follows from Theorems 2.2–2.3 by Corollary 3.1. □

Now we turn to one of the classical problems in geometric function theory, namely, to estimation of different functionals over various classes of analytic functions. Let $f \in \operatorname{Hol}(\mathbb{D},\mathbb{C})$ have Taylor expansion $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. Estimates of the Taylor's coefficients a_n including the

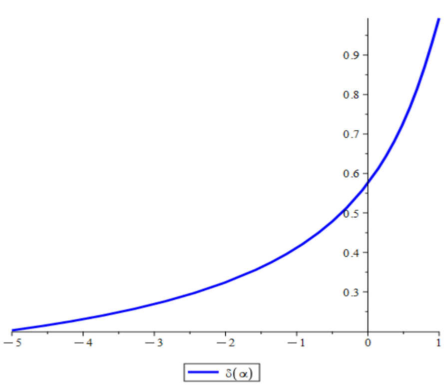

FIGURE 2. The graph of $\delta(\alpha)$.

Bieberbach conjecture (see for example, [5]) have a long history. Among other significant functionals we emphasize the generalized Zalcman functional $\Phi_{m,n}(f,\lambda)=a_1a_{m+n-1}-\lambda a_m a_n$ and its particular case the Fekete–Szegö functional $\Phi(f,\lambda):=\Phi_{2,2}(f,\lambda)=a_1a_3-\lambda a_2^2$. The Fekete–Szegö problem for a class of analytic functions is to find the sharp estimate on $|\Phi(f,\lambda)|$ over this class.

In this section we present estimates on the coefficient functionals over the class $\mathfrak{A}_\alpha,\ \alpha\le 1$.

Theorem 3.3. *Let $f\in\mathfrak{A}_\alpha$ have the Taylor expansion $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$. For $2\le k\le m-1$ denote*

$$\Lambda_\alpha(m,k):=\frac{(\alpha+(1-\alpha)k)\,(\alpha+(1-\alpha)(m-k+1))}{2\,(\alpha+(1-\alpha)m)}. \tag{3.9}$$

Then

$$|a_m-\Lambda_\alpha(m,k)a_ka_{m-k+1}|+\Lambda_\alpha(m,k)\,|a_ka_{m-k+1}|\le\frac{2}{\alpha+(1-\alpha)m}. \tag{3.10}$$

Consequently, $|a_m|\le\dfrac{2}{\alpha+(1-\alpha)m}$ and for any $\nu\in\mathbb{C}$,

$$|a_m-\nu a_ka_{m-k+1}|\le\frac{2}{\alpha+(1-\alpha)m}\max\left\{1,\left|1-\frac{\nu}{\Lambda_\alpha(m,k)}\right|\right\}. \tag{3.11}$$

Inequalities (3.10)–(3.11) are sharp.

Proof. Since $f\in\mathfrak{A}_\alpha$, the function q defined by $q(z)=\alpha\frac{f(z)}{z}+(1-\alpha)f'(z)$ has positive real part. Its Taylor coefficients are $p_n=[\alpha+(1-\alpha)(n+1)]\,a_{n+1},\ n\ge1$. According to [6, Proposition 2.2], we have $\left|p_n-\frac12p_kp_{n-k}\right|+\frac12\left|p_kp_{n-k}\right|\le2,\ 1\le k\le n-1$, or, which is the same,

$$\begin{aligned}&\left|a_{n+1}-\frac{(\alpha+(1-\alpha)(k+1))\,(\alpha+(1-\alpha)(n-k+1))}{2\,(\alpha+(1-\alpha)(n+1))}a_{k+1}a_{n-k+1}\right|\\&+\frac{(\alpha+(1-\alpha)(k+1))\,(\alpha+(1-\alpha)(n-k+1))}{2\,(\alpha+(1-\alpha)(n+1))}\,|a_{k+1}a_{n-k+1}|\\&\le\frac{2}{(\alpha+(1-\alpha)(n+1))}.\end{aligned}$$

This inequality is equivalent to (3.10) by (3.9). In its turn, (3.10) implies $|a_m|\le\frac{2}{\alpha+(1-\alpha)m}$ by the triangle inequality. To proceed, we note that Lemma 2.1 in [6] can be reformulated as follows:

Let $a,b\in\mathbb{C}$ and $D>0$. Then $|a|+|b|\le D$ if and only if $|a+\mu b|\le D\max\{1,|\mu|\}$ for all $\mu\in\mathbb{C}$.

Applying this conclusion to inequality (3.10), we obtain

$$|a_m-\Lambda_\alpha(m,k)(1-\mu)a_ka_{m-k+1}|\le\frac{2}{\alpha+(1-\alpha)m}\max\{1,|\mu|\}.$$

Denote $\nu:=\Lambda_\alpha(m,k)(1-\mu)$. Then the last displayed formula is equivalent to (3.11). The sharpness follows from the sharpness of the used estimates from [6, Proposition 2.2]. $\square$

Corollary 3.2. *Let $f \in \mathfrak{A}_\alpha$ have the Taylor expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then, for any $\nu \in \mathbb{C}$,*

$$|\Phi(f,\nu)| = \left|a_3 - \nu a_2^2\right| \le \frac{2}{3-2\alpha} \max\left\{1, \left|1 - \frac{2\nu(3-2\alpha)}{(2-\alpha)^2}\right|\right\}.$$

4. Pseudo-Analytic Filtration

Consider the classes of functions defined as follows:

$$\mathfrak{A}_\alpha^1 := \left\{ f \in \mathscr{A} : \left| \alpha \frac{f(z)}{z} + (1-\alpha) f'(z) - 1 \right| \le 2 - \alpha \right\}, \quad \alpha \le 1. \tag{4.1}$$

Note in passing that a similar classes were considered in a different context in [29].

For each $\alpha \in (-\infty, 1]$, the set $\mathfrak{A}_\alpha^1$ is non-empty as the following example demonstrates.

Example 4.1. Let $\alpha \le 1$. For any $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$, the function

$$f_{\theta,n,\alpha}(z) = z + \frac{2-\alpha}{1+n(1-\alpha)} e^{i\theta} z^{n+1}$$

belongs to the class $\mathfrak{A}_\alpha^1$. Indeed,

$$\begin{aligned} & \left| \alpha \frac{f_{\theta,n,\alpha}(z)}{z} + (1-\alpha) f'_{\theta,n,\alpha}(z) - 1 \right| \\ = & \left| \alpha \left(1 + \frac{(2-\alpha)e^{i\theta} z^n}{1+n(1-\alpha)} \right) + (1-\alpha) \left(1 + \frac{(n+1)(2-\alpha)e^{i\theta} z^n}{1+n(1-\alpha)} \right) - 1 \right| \\ = & \left| (2-\alpha) z e^{i\theta} \right| \le 2 - \alpha. \end{aligned}$$

Moreover, it follows immediately from (4.1) and the Schwarz lemma that $f \in \mathfrak{A}_1^1$ if and only if it admits the representation $f(z) = z(1+\omega(z))$, $\omega \in \Omega$. Our first result presents a one-to-one correspondence between the sets $\mathfrak{A}_\alpha^1$ and Ω.

Theorem 4.1. *Let $\alpha < 1$. For each $f \in \mathfrak{A}_\alpha^1$, there is the unique $\omega \in \Omega$ such that*

$$f(z) = z \left(1 + \frac{2-\alpha}{1-\alpha} \cdot \int_0^1 s^{\frac{\alpha}{1-\alpha}} \omega(sz) ds \right). \tag{4.2}$$

Consequently, if $f \in \mathfrak{A}_\alpha^1$, then $\frac{f(z)}{z} - 1 \in \Omega$. Conversely, for any $\omega \in \Omega$, the function f defined by (4.2) belongs to $\mathfrak{A}_\alpha^1$.

Proof. Let $f \in \mathfrak{A}_\alpha^1$. Denote $g(z) := \alpha \frac{f(z)}{z} + (1-\alpha) f'(z) - 1$. Since $g(0) = 0$, the Schwarz Lemma implies that $|g(z)| \le (2-\alpha)|z|$, which is equivalent to $g(z) \prec (2-\alpha)z$. Therefore there exists $\omega \in \Omega$ such that

$$\alpha \frac{f(z)}{z} + (1-\alpha) f'(z) - 1 = (2-\alpha)\omega(z). \tag{4.3}$$

Solving this differential equation, we obtain (4.2). Therefore,

$$\begin{aligned} \left| \frac{f(z)}{z} - 1 \right| &= \left| \frac{2-\alpha}{1-\alpha} \cdot \int_0^1 s^{\frac{\alpha}{1-\alpha}} \omega(sz) ds \right| \le \frac{2-\alpha}{1-\alpha} \cdot \int_0^1 s^{\frac{\alpha}{1-\alpha}} |\omega(sz)| ds \\ &\le \frac{2-\alpha}{1-\alpha} \cdot \int_0^1 s^{\frac{\alpha}{1-\alpha}} ds = 1. \end{aligned}$$

The converse assertion follows from the equivalence of (4.2) and (4.3). $\square$

Corollary 4.1. *$\mathfrak{A}^1_\alpha \subseteq \mathfrak{A}^1_1 \subsetneq \mathscr{G}_0$ for any $\alpha \le 1$.*

Now we are ready to show that these sets form a filtration and to establish its main properties.

Theorem 4.2. *The following assertions hold:*

(a) *The family $\mathfrak{A}^1 = \{\mathfrak{A}^1_\alpha\}_{\alpha\le 1}$ is a filtration of $\mathscr{G}_0$.*
(b) *For every $\theta \in \mathbb{R}$ and $n \in \mathbb{N}\setminus\{1\}$, the function $f_{\theta,n,\alpha}$, defined in Example 4.1, belongs to $\partial\mathfrak{A}^1_\alpha$. Hence the filtration $\mathfrak{A}^1$ is strict.*
(c) *For any $\alpha \le 1$, the semigroup generated by $f_{\theta,1,\alpha} \in \mathfrak{A}^1_\alpha$ is not exponentially squeezing and cannot be analytically extended to a domain in $\mathbb{C}$.*

Proof. Note that $f \in \mathfrak{A}^1_\alpha$ if and only if, for every fixed $z \in \mathbb{D}$, the value $f'(z)$ belongs to the disk of radius $\frac{2-\alpha}{1-\alpha}$ and centered at $\frac{1}{1-\alpha}\left(1-\alpha\frac{f(z)}{z}\right)$. Hence, in order to prove that $\mathfrak{A}^1_\alpha \subseteq \mathfrak{A}_{1,\beta}$ whenever $\alpha \le \beta \le 1$, we need to ensure that such disks are included one into another. To this end, we denote $w = \frac{1}{1-\alpha}\left(1-\alpha\frac{f(z)}{z}\right) + e^{i\theta}\frac{2-\alpha}{1-\alpha}$ and show that $\left|w - \frac{1}{1-\beta}\left(1-\beta\frac{f(z)}{z}\right)\right| \le \frac{2-\beta}{1-\beta}$. Indeed,

$$\begin{aligned}
&\left|\frac{1}{1-\alpha}\left(1-\alpha\frac{f(z)}{z}\right) + e^{i\theta}\frac{2-\alpha}{1-\alpha} - \frac{1}{1-\beta}\left(1-\beta\frac{f(z)}{z}\right)\right| \\
= &\left|\left(\frac{f(z)}{z}-1\right)\frac{\alpha-\beta}{(1-\alpha)(1-\beta)} + e^{i\theta}\frac{2-\alpha}{1-\alpha}\right| \\
\le &\left|\frac{f(z)}{z}-1\right|\cdot\frac{\beta-\alpha}{(1-\alpha)(1-\beta)} + \frac{2-\alpha}{1-\alpha}.
\end{aligned}$$

By Theorem 4.1, we have

$$\left|\frac{f(z)}{z}-1\right|\cdot\frac{\beta-\alpha}{(1-\alpha)(1-\beta)} + \frac{2-\alpha}{1-\alpha} \le \frac{\beta-\alpha}{(1-\alpha)(1-\beta)} + \frac{2-\alpha}{1-\alpha} = \frac{2-\beta}{1-\beta},$$

which proves assertion (a).

Assume now that $\alpha < \beta \le 1$ and calculate

$$\begin{aligned}
&\left|\alpha\frac{f_{\theta,n,\beta}(z)}{z} + (1-\alpha)f'_{\theta,n,\beta}(z) - 1\right| \\
= &\left|\alpha\left(1+\frac{(2-\beta)e^{i\theta}z^n}{1+n(1-\beta)}\right) + (1-\alpha)\left(1+\frac{(n+1)(2-\beta)e^{i\theta}z^n}{1+n(1-\beta)}\right) - 1\right| \\
= &(2-\beta)\frac{1+n(1-\alpha)}{1+n(1-\beta)}|z|^n.
\end{aligned}$$

Since $(2-\beta)\frac{1+n(1-\alpha)}{1+n(1-\beta)} > 2-\alpha$ for every natural $n > 1$, then there exists $z \in \mathbb{D}$ such that

$$\left|\alpha\frac{f_{\theta,n,\beta}(z)}{z} + (1-\alpha)f'_{\theta,n,\beta}(z) - 1\right| > 2-\alpha.$$

Thus $f_{\theta,n,\beta} \in \mathfrak{A}^1_\beta \setminus \mathfrak{A}^1_\alpha$, which implies $f_{\theta,n,\beta} \in \partial\mathfrak{A}^1_\beta$. So, assertion (b) follows.

Since $f_{\theta,1,\alpha}(z) = z + e^{i\theta}z^2 \in \mathfrak{A}^1_\alpha$ for every $\alpha \le 1$ by Exaqmple 4.1, one concludes

$$\inf_{z\in\mathbb{R}} \operatorname{Re} \frac{f_{\theta,1,\alpha}(z)}{z} = \inf_{z\in\mathbb{R}} \operatorname{Re}(1+e^{i\theta}) = 0$$

and

$$\sup_{z\in\mathbb{R}} \arg \frac{f_{\theta,1,\alpha}(z)}{z} = \sup_{z\in\mathbb{R}} \left|\arg(1+e^{i\theta})\right| = \frac{\pi}{2}.$$

This implies assertion (c) by Theorems 2.2–2.3. The proof is complete. □

Definition 4.1. We denote $\mathfrak{A}^1 = \left\{\mathfrak{A}^1_\alpha\right\}_{\alpha\in(-\infty,1]}$ and call it the *pseudo-analytic filtration*.

Corollary 4.2. *The set* $\mathfrak{A}^1_{-\infty} := \bigcap_{\alpha<1} \mathfrak{A}^1_\alpha = \{f \in \mathscr{A} : |f(z) - zf'(z)| \le |z|\}$ *is not a singleton since it containes al the convex combinations of functions* $f_{\theta,n} = z + \frac{1}{n}e^{i\theta}z^{n+1}$, $n \in \mathbb{N}$.

We now establish estimates for coefficient functionals related to the pseudo-analytic filtration.

Theorem 4.3. *Let* $f \in \mathfrak{A}^1_\alpha$ *have the Taylor expansion* $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$. *The following assertions hold:*

(i) $|a_{k+1}| \le \dfrac{2-\alpha}{k(1-\alpha)+1}$ *for all* $k \in \mathbb{N}$. *Moreover,* $|a_3| \le \dfrac{2-\alpha}{3-2\alpha} \cdot (1-|a_2|^2)$ *and* $|a_4| \le \dfrac{2-\alpha}{4-3\alpha} \cdot (1-|a_2|^2)$.

(ii) *For every* $\lambda \in \mathbb{C}$ *we have* $|\Phi(f,\lambda)| \le \max\left(\dfrac{2-\alpha}{3-2\alpha}, |\lambda|\right)$.

Proof. Let $f \in \mathfrak{A}^1_\alpha$. Then, by Theorem 4.1, there exists a unique function $\omega \in \Omega$, $\omega(z) = \sum\limits_{k=1}^{\infty} \beta_k z^k$ such that (4.2) holds. Then, for any fixed $z \in \mathbb{D}$,

$$\begin{aligned} f(z) &= z\left(1 + \frac{2-\alpha}{1-\alpha} \cdot \int_0^1 s^{\frac{\alpha}{1-\alpha}} \left(\sum_{k=1}^{\infty} \beta_k z^k s^k\right) ds\right) \\ &= z + (2-\alpha) \cdot \sum_{k=1}^{\infty} \frac{\beta_k}{k+1-k\alpha} z^{k+1}. \end{aligned}$$

Thus

$$a_{k+1} = \frac{(2-\alpha)\beta_k}{k+1-k\alpha}, \qquad k \in \mathbb{N}. \tag{4.4}$$

Since $|\beta_k| \le 1$ for all $k \in \mathbb{N}$, the estimates on $|a_{k+1}|$ follow. Assume that γ_k, $k \in \mathbb{N}$, with $|\gamma_k| \le 1$ are the Schur parameters of ω, then by Schur's recurrence relation (see for details [27], see also [8]) we have

$$\left\{ \begin{array}{rl} |\beta_1| = & |\gamma_1|; \\ |\beta_2| = & (1-|\gamma_1|^2)|\gamma_2| \le (1-|\gamma_1|^2); \\ |\beta_3| = & (1-|\gamma_1|^2)\left|(1-|\gamma_2|^2)\gamma_3 - \gamma_1\gamma_2^2\right| \\ \le & (1-|\gamma_1|^2)\max(|\gamma_3|,|\gamma_1|) \le (1-|\gamma_1|^2). \end{array} \right. \tag{4.5}$$

For $k=1$, formula (4.4) implies $a_2 = \gamma_1$. For $k=2$, it follows from (4.4)–(4.5) that

$$|a_3| \le \frac{2-\alpha}{3-2\alpha} \cdot (1-|a_2|^2).$$

For $k=3$, formula (4.5) implies $|\beta_3| \le (1-|a_2|^2)$. Similarly to the above,

$$|a_4| \le \frac{2-\alpha}{4-3\alpha} \cdot (1-|a_2|^2).$$

Thus assertion (i) is proven.

Next, we use formulas (4.4) and (4.5) to estimate the Fekete–Szegö functional over $\mathfrak{A}^1_\alpha$ and obtain

$$\begin{aligned} |a_3 - \lambda a_2^2| &= \left| \frac{(2-\alpha)\beta_2}{3-2\alpha} - \lambda \frac{(2-\alpha)^2\beta_1^2}{(2-\alpha)^2} \right| \le \frac{2-\alpha}{3-2\alpha} \left|(1-|\gamma_1|^2)\gamma_2\right| + |\lambda| \cdot |\gamma_1|^2 \\ &\le (1-|\gamma_1|^2) \cdot \frac{2-\alpha}{3-2\alpha} + |\gamma_1|^2 \cdot |\lambda| \le \max\left(\frac{2-\alpha}{3-2\alpha}, |\lambda| \right). \end{aligned}$$

This proves assertion (ii). □

5. Prestarlike Filtration

The term 'prestarlike function' was introduce by Ruscheweyh in [24] after the previous works [23] and [28]. To define it, recall that the Hadamard product of two functions $f, g \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ with Taylor expansions $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$, respectively, is defined by

$$f * g(z) := \sum_{n=0}^{\infty} a_n b_n z^n, \qquad z \in \mathbb{D}. \tag{5.1}$$

Definition 5.1. One says that a function $f \in \mathscr{A}$ is prestarlike of order $\alpha \in [0,1)$ if

$$f(z) * \frac{z}{(1-z)^{2-2\alpha}} \in \mathscr{S}^*(\alpha).$$

We denote by $\mathfrak{R}_\alpha$ the class of all prestarlike functions of order α. In addition, we set

$$\mathfrak{R}_1 = \left\{ f \in \mathscr{A} : \ \mathrm{Re}\, \frac{f(z)}{z} > \frac{1}{2}, \quad z \in \mathbb{D} \setminus \{0\} \right\}.$$

The following description of prestarlike functions partially follows from [24]. We provide its proof for the sake of completeness.

Proposition 5.1. *Let $f \in \mathscr{A}$ and $\alpha \in [0,1]$. The following conditions are equivalent:*

(1) *function f belongs to $\mathfrak{R}_\alpha$;*
(2) $\mathrm{Re}\left(\dfrac{f(z) * \frac{z^2}{(1-z)^{3-2\alpha}}}{f(z) * \frac{z}{(1-z)^{2-2\alpha}}} \right) > -\dfrac{1}{2}$ *for all $z \in \mathbb{D} \setminus \{0\}$;*
(3) $f(z) * \dfrac{z(1-z\zeta)}{(1-z)^{3-2\alpha}} \ne 0$ *for all $\zeta \in \partial\mathbb{D}$ and $z \in \mathbb{D} \setminus \{0\}$.*

Proof. According to the definition, $f \in \mathfrak{R}_\alpha$ if and only if

$$\operatorname{Re}\left(\frac{z\left(f(z) * \frac{z}{(1-z)^{2-2\alpha}}\right)'}{f(z) * \frac{z}{(1-z)^{2-2\alpha}}}\right) > \alpha \Longleftrightarrow \operatorname{Re}\left(\frac{f(z) * z\left(\frac{z}{(1-z)^{2-2\alpha}}\right)'}{f(z) * \frac{z}{(1-z)^{2-2\alpha}}}\right) > \alpha$$

$$\Longleftrightarrow \operatorname{Re}\left(\frac{f(z) * \left(\frac{z}{(1-z)^{2-2\alpha}} + \frac{(2-2\alpha)z^2}{(1-z)^{3-2\alpha}}\right)}{f(z) * \frac{z}{(1-z)^{2-2\alpha}}}\right) > \alpha$$

$$\Longleftrightarrow \frac{1}{2} + \operatorname{Re}\left(\frac{f(z) * \frac{z^2}{(1-z)^{3-2\alpha}}}{f(z) * \frac{z}{(1-z)^{2-2\alpha}}}\right) > 0, \tag{5.2}$$

which implies the first condition. As for the second one, we note that

$$\frac{z(1-z\zeta)}{(1-z)^{3-2\alpha}} = \frac{z(1-\zeta)}{(1-z)^{3-2\alpha}} + \frac{z\zeta}{(1-z)^{2-2\alpha}}.$$

Therefore $f(z) * \frac{z(1-z\zeta)}{(1-z)^{3-2\alpha}} \neq 0$ if and only if

$$f(z) * \frac{z(1-\zeta)}{(1-z)^{3-2\alpha}} \neq -f(z) * \frac{z\zeta}{(1-z)^{2-2\alpha}} \Longleftrightarrow \frac{f(z) * \frac{z}{(1-z)^{3-2\alpha}}}{f(z) * \frac{z}{(1-z)^{2-2\alpha}}} \neq \frac{-\zeta}{1-\zeta}.$$

This is equivalent to the last inequality in (5.2). □

In the next theorem, we present properties of the family of sets $\mathfrak{R} = \{\mathfrak{R}_\alpha\}_{\alpha\in[0,1]}$.

Theorem 5.1. *The following assertion hold:*

(a) *The family $\mathfrak{R}$ is a strict filtration of $\mathscr{G}_0$;*
(b) *Let f_α, $\alpha \in (0,1]$, be defined by $f_\alpha(z) = z + \frac{1}{2(2-\alpha)}z^2$, then $f_\alpha \in \partial\mathfrak{R}_\alpha$;*
(c) *For any $\alpha \in [0,1]$, the semigroup generated by $f \in \mathfrak{R}_\alpha$ is exponentially squeezing with squeezing ratio at least $\frac{1}{2}$. At the same time, there is $f \in \mathfrak{R}_\alpha$ that generates the semigroup with sharp squeezing ratio $\frac{1}{2}$ and cannot be analytically extended to a domain in $\mathbb{C}$.*

Proof. Let $0 \le \beta < \alpha \le 1$. The inclusion $\mathfrak{R}_\beta \subseteq \mathfrak{R}_\alpha$ was proved by Suffridge in [28] (see also [24]). Furthermore, $\mathfrak{R}_1 \subset \mathscr{G}_0$ by definition. Thus $\mathfrak{R}$ is a filtration of $\mathscr{G}_0$.

Consider the functions f_α, $\alpha \in (0,1]$, defined by $f_\alpha(z) = z + \frac{1}{2(2-\alpha)}z^2$. If $\alpha = 1$, then $\operatorname{Re} f_1(z) = \operatorname{Re}\left(z + \frac{z^2}{2}\right) > \frac{1}{2}$ as $z \in \mathbb{D}$. So, $f_1 \in \mathfrak{R}_1$. If $\alpha < 1$, then $f_\alpha(z) * \frac{z}{(1-z)^{2-2\alpha}} = z + \frac{1-\alpha}{2-\alpha}z^2$. It is easy to see that the last function belongs to $\mathscr{S}^*(\alpha)$, hence $f_\alpha \in \mathfrak{R}_\alpha$.

We show now that $f_\alpha \notin \mathfrak{R}_\beta$ for $\beta < \alpha$. Indeed, $h(z) := f_\alpha(z) * \frac{z}{(1-z)^{2-2\beta}} = z + \frac{1-\beta}{2-\alpha}z^2$. Then the value of $\frac{zh'(z)}{h(z)}$ at the point $z = -1$ is $\frac{2\beta-\alpha}{1+\beta-\alpha} < \beta$ since $\beta < \alpha$. So, $h \notin \mathscr{S}^*(\beta)$ and $f_\alpha \notin \mathfrak{R}_\beta$. Therefore $f_\alpha \in \partial\mathfrak{R}_\alpha$ and the filtration is strict. This proves assertion (a) and (b). To proceed, let us remind that, for any $\alpha \in [0,1]$

$$\inf_{f\in\mathfrak{R}_\alpha}\inf_{z\in\mathbb{D}} \operatorname{Re}\frac{f(z)}{z} \ge \inf_{f\in\mathfrak{R}_1}\inf_{z\in\mathbb{D}} \operatorname{Re}\frac{f(z)}{z} = \frac{1}{2}$$

and
$$\frac{2}{\pi}\sup_{f\in\mathfrak{R}_\alpha}\sup_{z\in\mathbb{D}}\left|\arg\frac{f(z)}{z}\right| \le \frac{2}{\pi}\sup_{f\in\mathfrak{R}_1}\sup_{z\in\mathbb{D}}\left|\arg\frac{f(z)}{z}\right| = 1.$$
On the other hand,
$$\frac{z}{1-z} * \frac{z}{(1-z)^{2(1-\alpha)}} = \frac{z}{(1-z)^{2(1-\alpha)}} \in \mathscr{S}^*(\alpha).$$
Hence $\frac{z}{1-z} \in \mathfrak{R}_\alpha$ for any $\alpha \in [0,1]$. Therefore
$$\inf_{f\in\mathfrak{R}_\alpha}\inf_{z\in\mathbb{D}}\operatorname{Re}\frac{f(z)}{z} \le \inf_{z\in\mathbb{D}}\operatorname{Re}\frac{1}{1-z} = \frac{1}{2}$$
and
$$\frac{2}{\pi}\sup_{f\in\mathfrak{R}_\alpha}\sup_{z\in\mathbb{D}}\left|\arg\frac{f(z)}{z}\right| \ge \frac{2}{\pi}\sup_{z\in\mathbb{D}}\left|\arg\frac{1}{1-z}\right| = 1.$$
This completes our proof. □

Further, we present the solution of the Fekete–Szegö problem for the classes $\mathfrak{R}_\alpha$.

Theorem 5.2. *For every* $\alpha \in [0,1]$, *we have* $\max_{f\in\mathfrak{R}_\alpha}|\Phi(f,\lambda)| = \max\left\{\frac{1}{3-2\alpha}, |\lambda-1|\right\}$.

Proof. Let $f \in \mathfrak{R}_\alpha$ have the Taylor expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then
$$\mathscr{S}^*(\alpha) \ni \frac{z}{(1-z)^{2(1-\alpha)}} * f(z) = z + \sum_{n=2}^{\infty} c_n z^n,$$
where
$$c_2 = 2(1-\alpha)a_2 \qquad \text{and} \qquad c_3 = (1-\alpha)(3-2\alpha)a_3.$$
It follows from [20, Theorem 1] that $\left|c_3 - \mu c_2^2\right| \le (1-\alpha)\max\{1, |2(1-\alpha)(2\mu-1)-1|\}$ for any $\mu \in \mathbb{C}$ or, equivalently,
$$\left|(3-2\alpha)a_3 - 4\mu(1-\alpha)a_2^2\right| \le \max\{1, |4\mu(1-\alpha)-(3-2\alpha)|\}.$$
Denote $\lambda = \dfrac{4\mu(1-\alpha)}{3-2\alpha}$. The last displayed inequality means that
$$|\Phi(f,\lambda)| \le \max\left\{\frac{1}{3-2\alpha}, |\lambda-1|\right\}.$$
It remains to show that this estimate is sharp, more precisely, that for every λ there is a function $f \in \mathfrak{R}_\alpha$ for which it becomes equality.

First assume that $|\lambda-1| \ge \frac{1}{3-2\alpha}$. As we already saw $\frac{1}{1-z} \in \mathfrak{R}_\alpha$ for every α. Since the Taylor coefficients of this functions are $a_2 = a_3 = 1$, we have $a_3 - \lambda a_2^2 = 1-\lambda$, then the equality holds. Otherwise, in tne case where $|\lambda-1| < \frac{1}{3-2\alpha}$, let consider the function $f_\alpha \in \mathscr{A}$ defined by its Taylor expansion:
$$f_\alpha(z) = z + \sum_{k=1}^{\infty}\frac{(1-\alpha)_k(2k)!}{(2-2\alpha)_{2k}k!}z^{2k+1}.$$

It follows from (5.1) that

$$f_\alpha(z) * \frac{z}{(1-z)^{2-2\alpha}} = z + \sum_{k=1}^{\infty} \frac{(1-\alpha)_k}{k!} z^{2k+1} = \frac{z}{(1-z^2)^{1-\alpha}}.$$

Since the last function is starlike of order α, we conclude that $f_\alpha \in \mathfrak{R}_\alpha$. In this case $a_2 = 0$ and $a_3 = \frac{1}{3-2\alpha}$. Thus $a_3 - \lambda a_2^2 = \frac{1}{3-2\alpha}$ which completes the proof. □

Note that for $\alpha = 1$ this result is known (see, for example, Theorem 2.2 in [10]).

To complete the paper, we recall that one of the main properties of generators is the following result by Simeon Reich and the third author (see [21], see also [13, 22, 25]):

Theorem 5.3. *Let $f \in \mathrm{Hol}(\mathbb{D},\mathbb{C})$. Then $f \in \mathcal{G}$ if and only if for every $z \in \mathbb{D}$ and every $r > 0$ the functional equation $w + rf(w) = z$ has the unique solution $w = G_r(z)$ such that $G_r \in \mathrm{Hol}(\mathbb{D})$.*

The functions $G_r,\ r > 0$, are called the *non-linear resolvents* of $f \in \mathcal{G}$. Coefficient inequalities over the classes of nonlinear resolvents were recently studied in [8, 10, 15] (and also in multi-dimensional settings in [9, 17, 18]). Here we complete the previous result by a more qualified estimates for generators belonging to the filtration sets studied in the paper.

Theorem 5.4. *Let $\lambda \in \mathbb{C}$, $f \in \mathcal{G}_0$ and let G_r be its non-linear resolvent with $r > 0$. Then*

$$|\Phi(G_r,\lambda)| = \frac{r}{(1+r)^5} |\Phi(f,\nu)| \tag{5.3}$$

with $\nu = (2-\lambda)\frac{r}{1+r}$. Consequently,

(i) *If $f \in \mathfrak{A}_\alpha$ with $\alpha \le 1$, then*

$$|\Phi(G_r,\lambda)| \le \frac{2r}{(3-2\alpha)(1+r)^5} \max\left\{1, \left|1 - \frac{2(3-2\alpha)}{(2-\alpha)^2} \cdot \frac{r}{1+r}(2-\lambda)\right|\right\}.$$

(ii) *If $f \in \mathfrak{A}^1_\alpha$ with $\alpha \le 1$, then*

$$|\Phi(G_r,\lambda)| \le \frac{r}{(1+r)^5} \max\left(\frac{2-\alpha}{3-2\alpha},\ \frac{r}{1+r}|2-\lambda|\right).$$

(iii) *If $f \in \mathfrak{R}_\alpha$ with $\alpha \le 1$, then*

$$|\Phi(G_r,\lambda)| \le \frac{r}{(3-2\alpha)(1+r)^5} \max\left\{1, \left|3 - 2\alpha - \frac{r}{1+r}(2-\lambda)\right|\right\}.$$

Proof. Let f have the Taylor expansion $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$. Assume that for some $r > 0$ its resolvent has the Taylor expansion $G_r(z) = \sum\limits_{n=1}^{\infty} b_n z^n$. Differentiating the equality $G_r(z) + rf(G_r(z)) = z$ at $z = 0$, we find $b_1 = \frac{1}{1+r}$, $b_2 = -\frac{ra_2}{(1+r)^3}$ and $b_3 = \frac{2r^2a_2^2}{(1+r)^5} - \frac{ra_3}{(1+r)^4}$ (see also [8, 10]). Therefore

$$b_1b_3 - \lambda b_2^2 = -\frac{r}{(1+r)^5}\left[a_3 - (2-\lambda)\frac{r}{1+r}a_2^2\right],$$

which proves (5.3).

To verify assertion (i), let $f \in \mathfrak{A}_\alpha$. Then $|\Phi(f,\nu)| \le \frac{2}{3-2\alpha} \max\left\{1, \left|1-\frac{2\nu(3-2\alpha)}{(2-\alpha)^2}\right|\right\}$ by Corollary 3.2. Thus formula (5.3) implies

$$|\Phi(G_r,\lambda)| \le \frac{2r}{(3-2\alpha)(1+r)^5} \max\left\{1, \left|1-\frac{2\nu(3-2\alpha)}{(2-\alpha)^2}\right|\right\},$$

which is equivalent to the required inequality in assertion (i).

Assertion (ii) follows directly from assertion (ii) in Theorem 4.3 by the formula (5.3).

Finally, let $f \in \mathfrak{R}_\alpha$. Then $|\Phi(f,\nu)| = \max\left\{\frac{1}{3-2\alpha}, |\nu-1|\right\}$ by Theorem 5.2. Similarly to the above, we apply this inequality to relation (5.3) and obtain

$$\begin{aligned}|\Phi(G_r,\lambda)| \le &\frac{r}{(1+r)^5} \max\left\{\frac{1}{3-2\alpha}, |1-\nu|\right\}\\ = &\frac{r}{(1+r)^5} \max\left\{\frac{1}{3-2\alpha}, \left|1-(2-\lambda)\frac{r}{1+r}\right|\right\}.\end{aligned}$$

This completes our proof. □

References

[1] C. Avicou, I. Chalendar, J. R. Partington, Analyticity and compactness of semigroups of composition operators, J. Math. Anal. Appl. 437 (2016), 545–560.

[2] E. Berkson, H. Porta, Semigroups of analytic functions and composition operators, Michigan Math. J. 25 (1978), 101–115.

[3] F. Bracci, M. D. Contreras, S. Díaz-Madrigal, Continuous Semigroups of holomorphic self-maps of the unit disc, Springer Monographs in Mathematics, Springer, 2020.

[4] F. Bracci, M. D. Contreras, S. Díaz-Madrigal, M. Elin, D. Shoikhet, Filtrations of infinitesimal generators, Funct. Approx. Comment. Math. 59 (2018), 99–115.

[5] P. Duren, Univalent Functions, Springer, New York, 1983.

[6] I. Efraimidis, D. Vukotić, Applications of Livingston-type inequalities to the generalized Zalcman functional, Math. Nach. 291 (2018), 1502–1513.

[7] M. Elin, F. Jacobzon, Analyticity of semigroups in the right half-plane, J. Math. Anal. Appl. 448 (2017), 750–766.

[8] M. Elin, F. Jacobzon, Coefficient body for nonlinear resolvents, Annales UMCS, 2 (2020), 41–53.

[9] M. Elin, F. Jacobzon, The Fekete–Szegö problem for spirallike mappings and non-linear resolvents in Banach spaces, Stud. Univ. Babeş–Bolyai Math. 67 (2022), 329–344.

[10] M. Elin, F. Jacobzon, Estimates on some functionals over non-linear resolvents, Filomat 37 (2023), 797–808.

[11] M.Elin, F. Jacobzon, N. Tuneski, The Fekete–Szego functional and filtration of generators, Rend. Circ. Mat. Palermo, II. Ser (2022). https://doi.org/10.1007/s12215-022-00824-w

[12] M. Elin, S. Reich, D. Shoikhet, Complex Dynamical Systems and the Geometry of Domains in Banach Spaces, Dissertationes Math. (Rozprawy Mat.) 427 (2004), 62 pp.

[13] M. Elin, S. Reich, D. Shoikhet, Numerical range of holomorphic mappings and applications, Birkhäuser, Cham, 2019.

[14] M. Elin, D. Shoikhet, Linearization Models for Complex Dynamical Systems. Topics in Univalent Functions, Functional Equations and Semigroup Theory, Birkhäuser Basel, 2010.

[15] M. Elin, D. Shoikhet, T. Sugawa, Filtration of semi-complete vector fields revisited, In: Complex Analysis and Dynamical Systems. New Trends and Open Problems, pp. 93–102, Trends in Math., Birkhäuser/Springer, Cham, 2018.

[16] M. Elin, D. Shoikhet, N. Tarkhanov, Analytic extension of semigroups of holomorphic mappings and composition operators, Comput. Methods Function Theory 18 (2018), 269–294.

[17] I. Graham, H. Hamada, G. Kohr, Loewner chains and nonlinear resolvents of the Carathéodory family on the unit ball in $\mathbb{C}^n$, J. Math. Anal. Appl. 491 (2020), 124289.
[18] H. Hamada, G. Kohr, M. Kohr, The Fekete–Szegö problem for starlike mappings and nonlinear resolvents of the Carathéodory family on the unit balls of complex Banach spaces, Anal. Math. Phys. 11 (2021), 115.
[19] F. Jacobzon, M. Levenshtein, S. Reich, Convergence characteristics of one-parameter continuous semigroups, Anal. Math. Phys. 1 (2011), 311–335.
[20] F. R. Keogh, E. P. Merkes, A coefficient inequality for certain classes of analytic functions, Proc Amer. Math. Soc. 20 (1969), 8–12.
[21] S. Reich, D. Shoikhet, Generation theory for semigroups of holomorphic mappings in Banach spaces, Abstr. Appl. Anal. 1 (1996), 1–44.
[22] S. Reich, D. Shoikhet, Nonlinear Semigroups, Fixed Points, and the Geometry of Domains in Banach Spaces, World Scientific Publisher, Imperial College Press, London, 2005.
[23] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975), 109–115.
[24] St. Ruscheweyh, Linear operators between classes of prestarlike functions, Comment. Math. Helvetici 52 (1977), 497–509.
[25] D. Shoikhet, Semigroups in Geometrical Function Theory, Kluwer Academic Publishers, Dordrecht, 2001.
[26] D. Shoikhet, Rigidity and parametric embedding of semi-complete vector fields on the unit disk, Milan J. Math. 84 (2016), 159–202.
[27] B. Simon, Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, Colloquium Publications, Amer. Math. Society, 2005.
[28] T. J. Suffridge, Starlike functions as limits of polynomials, Advances in complex function theory, In: Lecture Notes in Maths. vol. 505, pp. 164—202, Springer, Berlin–Heidelberg–New York, 1976.
[29] N. Tuneski, Some simple sufficient conditions for starlikeness and convexity, Appl. Math. Lett. 22 (2009), 693–697.