Appl. Set-Valued Anal. Optim. 6 (2024), No. 1, pp. 53-63
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.6.2024.1.05

FRAMES OF QUASI-GEODESICS, VISIBILITY, AND GEODESIC LOOPS

FILIPPO BRACCI

Dipartimento Di Matematica, Università di Roma "Tor Vergata",
Via Della Ricerca Scientifica 1, 00133, Roma, Italy

Dedicated to Professor Simeon Reich on the occasion of his 75th Birthday

Abstract. In this paper, we give a characterization in terms of "quasi-geodesics frames" of visibility and existence of geodesic loops for bounded domains in $\mathbb{C}^d$, which are Kobayashi complete hyperbolic and Gromov hyperbolic.

Keywords. Extension of holomorphic maps; Gromov hyperbolic spaces; Kobayashi hyperbolic spaces; Visibility.

2020 Mathematics Subject Classification. 32F45.

CONTENTS

1. INTRODUCTION

The concept of visibility in several complex variables for bounded and unbounded domains in $\mathbb{C}^d$ has been recently introduced and turned out to be a key notion to study continuous extension of biholomorphisms, estimates for the Kobayashi distance, and iteration theory; see, e.g., [2, 3, 4, 6, 11, 12, 14, 18, 19, 20].

In this paper, we consider $\Omega \subset \mathbb{C}^d$ a bounded domain (with no assumption on the regularity of $\partial\Omega$) and we assume that its Kobayashi distance K_Ω is complete; see, e.g., [17] for the definitions and the properties of the Kobayashi distance.

The domain Ω is *visible* if, roughly speaking, the geodesics which converge to different points in the boundary bend inside the domain. Due to the completeness of K_Ω and Arzelà-Ascoli's theorem, a non-compactly-divergent sequence of geodesics rays or lines admits a subsequence converging *uniformly on compacta* to a geodesics ray or line. One of the basic observations of

E-mail address: fbracci@mat.uniroma2.it.
Received 22 April 2023; Accepted 30 August 2023; Published online 21 December 2023

the present paper is that if Ω is visible, then the convergence is actually *uniform* (not just on compacta)—see Lemma 3.1.

Moreover, visibile domains have a natural family of geodesics which exhibit certain peculiarities (they form a kind of "frame"). Motivated by this, we give the following definition.

Definition 1.1. Let $\Omega \subset\subset \mathbb{C}^d$ be a domain such that (Ω, K_Ω) is complete hyperbolic, and let $A \geq 1$ and $B \geq 0$. A family $\mathscr{F}$ is a *frame of (A,B)-quasi-geodesics* if

(1) for every $\gamma \in \mathscr{F}$, there exists $R_\gamma \in (0, +\infty]$ such that if $R_\gamma < +\infty$, $\gamma : [0, R_\gamma] \to \Omega$ is a (A,B)-quasi-geodesic segment, while, if $R_\gamma = +\infty$, $\gamma : [0, +\infty) \to \Omega$ is a (A,B)-quasi-geodesic ray,
(2) if $\gamma \in \mathscr{F}$ is a quasi-geodesic ray (*i.e.*, $R_\gamma = +\infty$), then there exists $p_\gamma \in \partial\Omega$ such that $\lim_{t\to+\infty} \gamma(t) = p_\gamma$,
(3) there exists a compact subset $K \subset \Omega$ such that $\gamma(0) \in K$ for every $\gamma \in \mathscr{F}$,
(4) for every sequence $\{\gamma_k\} \subset \mathscr{F}$, there exist a subsequence $\{\gamma_{k_m}\}$ and $\gamma \in \mathscr{F}$ such that $\{\gamma_{k_m}\}$ converges uniformly to γ,
(5) there exist $\varepsilon > 0$ and $\delta > 0$ such that, for every $z \in \Omega$ with $\mathrm{dist}(z, \partial\Omega) < \varepsilon$, there exists $\gamma \in \mathscr{F}$ such that $K_\Omega(z, \gamma([0, +\infty]) < \delta$.

It turns out that visibile domains have a frame of geodesics (see Proposition 3.1). One of the main results of this paper is to prove the converse for Gromov hyperbolic domains.

Theorem 1.1. *Let $\Omega \subset\subset \mathbb{C}^d$ be a domain such that (Ω, K_Ω) is complete hyperbolic and Gromov hyperbolic. Assume that $\partial\Omega$ does not contain nontrivial analytic discs. Then Ω is visible if and only if there exists a frame of (A,B)-quasi-geodesics for some $A \geq 1, B \geq 0$.*

The hypothesis on the non-existence of nontrivial analytic discs on the boundary is technical. It is presently not known whether a visible complete hyperbolic and Gromov hyperbolic domain might have nontrivial analytic discs on the boundary (in [4, Theorem 1.2] it is shown that for a visibile complete hyperbolic and Gromov hyperbolic domain with no geodesic loops–see below for the definition—and with C^0-boundary there can not be nontrivial analytic discs on the boundary).

The domain Ω is a *Gromov model domain* provided that the identity map extends as a homeomorphism from the Gromov compactification of Ω to the Euclidean closure $\overline{\Omega}$ (see [6]). Due to the equivalence between Gromov's topology and Carathéodory's prime end topology (see, e.g., [5, Ch. 4]) in dimension one, Gromov model domains play essentially the rôle of simply connected Jordan domains, while, visible domains play the rôle of simply connected domains with locally connected boundary with respect to the extension to the boundary of biholomorphisms (see [6] for a careful explanation of this fact).

The following are examples of Gromov model domains: C^2-bounded strongly pseudoconvex domains [1], Gromov hyperbolic (bounded or unbounded) convex domain [7, 8, 9], smooth bounded D'Angelo finite type convex domains [21, 22], smooth bounded D'Angelo finite type pseudoconvex domains in $\mathbb{C}^2$ [16], bounded Gromov hyperbolic Lipschitz $\mathbb{C}$-convex domains [12]. It turns out (see [6, 12]) that a bounded Kobayashi complete hyperbolic and Gromov hyperbolic domain Ω is a Gromov model if and only if Ω is visible and has no *geodesic loops*, where a geodesic loop for $\overline{\Omega}$ is a geodesic line $\gamma : (-\infty, +\infty) \to \Omega$ such that the cluster set of γ at $+\infty$ coincides with the cluster set of γ at $-\infty$.

In this direction, we first show (see Proposition 4.1) that if Ω is visible and there exist $p \in \partial\Omega$ and U an open neighborhood of p such that $U \cap \Omega$ has at least two connected components with p on their boundary, then there exists a geodesic loop for $\overline{\Omega}$.

The second main result of the paper is a characterization of (non)existence of geodesic loops in terms of frames of quasi-geodesics.

Definition 1.2. Let $\Omega \subset\subset \mathbb{C}^d$ be a domain such that (Ω, K_Ω) is complete hyperbolic, and let $A \geq 1$ and $B \geq 0$. A frame of (A,B)-quasi-geodesics $\mathscr{F}$ is *looping* if there exist two quasi-geodesic rays $\gamma, \eta \in \mathscr{F}$ such that $\lim_{t\to+\infty}\gamma(t) = \lim_{t\to+\infty}\eta(t)$ and which stay at infinite distance each other. We say that the quasi-geodesic frame $\mathscr{F}$ is *non-looping* otherwise.

Then we prove the following theorem.

Theorem 1.2. *Let $\Omega \subset\subset \mathbb{C}^d$ be a domain such that (Ω, K_Ω) is complete hyperbolic and Gromov hyperbolic. Assume that $\partial\Omega$ does not contain nontrivial analytic discs and that Ω is visible. Then Ω has no geodesic loops if and only if it has a non-looping frame of (A,B)-quasi-geodesics for some $A \geq 1$ and $B \geq 0$.*

As a consequence of Theorem 1.1, Theorem 1.2, and the previous discussion, we have the following theorem.

Theorem 1.3. *Let $\Omega \subset\subset \mathbb{C}^d$ be a domain such that (Ω, K_Ω) is complete hyperbolic and Gromov hyperbolic. Assume that $\partial\Omega$ does not contain nontrivial analytic discs. Then Ω is a Gromov model domain if and only if it has a non-looping frame of (A,B)-quasi-geodesics for some $A \geq 1$ and $B \geq 0$.*

The definition of frames of (non-looping) quasi-geodesics might seem at a first sight rather technical and useless. However, unravelling the proofs in [7, 8, 9, 10, 16, 21, 22, 23] where several kind of (Gromov hyperbolic) bounded domains were proved to be Gromov models, one sees that actually the main work was exactly to construct non-looping quasi-geodesic frames in the sense of our definition. As another example, we sketch an argument for constructing a frame of non-looping quasi geodesics in strongly pseudoconvex domains, taking for granted that a C^2 bounded strongly pseudoconvex domain Ω is Kobayashi hyperbolic and Gromov hyperbolic. Indeed, one can construct easily a frame of non-looping quasi-geodesics as follows. If $p \in \partial\Omega$ and U_p is an open neighborhood of p such that $\Omega \cap U_p$ is biholomorphic to a strongly convex domain V_p, then one can consider the family of all real segments in V_p steaming from a fixed point in V_p. Arguing as in [7, 8] one can see that this family is a frame of non-looping quasi-geodesics in V_p. Hence, its preimage $\mathscr{F}_p$ is a frame of non-looping quasi-geodesics in $\Omega \cap U_p$. Taking $\mathscr{F}$ to be the union of all $\mathscr{F}_p$ and using the localization of the Kobayashi distance at the boundary, one can show that $\mathscr{F}$ is a frame of non-looping quasi-geodesics for Ω. Hence, by Theorem 1.3, Ω is a Gromov model domain (a result well known, as remarked above, after [1]).

The paper is organized as follows. In Section 2, we recall some definitions and state some preliminary results that we need in the proofs. In Section 3, we prove Theorem 1.1 and in Section 4 we prove Theorem 1.2.

2. Notations and Preliminaries

In this section we let $\Omega \subset \mathbb{C}^d$ be a bounded domain. We denote by k_Ω the infinitesimal Kobayashi pseudometric of Ω and by K_Ω the Kobayashi distance of Ω. We refer the reader to [17] for definitions and properties.

If $\Omega \subset\subset \mathbb{C}^d$ is complete hyperbolic, that is, (Ω, K_Ω) is a complete metric space, it follows from the Hopf-Rinow theorem that (Ω, K_Ω) is geodesic and thus, every couple of points in Ω can be joined by a geodesic (*i.e.* length minimizing curve) for K_Ω. If $p, q \in \Omega$, we denote by $[p,q]_\Omega$ any geodesic joining p and q.

A geodesic $\gamma : [a,b] \to \Omega$, $-\infty < a < b < +\infty$ is called a *geodesic segment*. A geodesic $\gamma : [a,+\infty) \to \Omega$, $a \in \mathbb{R}$, is a *geodesic ray* and a geodesic $\gamma : (-\infty,+\infty) \to \Omega$ is a *geodesic line*.

A geodesic triangle T is the union of 3 geodesic segments (called *sides*) $T = [x,y]_\Omega \cup [y,z]_\Omega \cup [z,x]_\Omega$ joining 3 points $x,\ y,z \in X$.

The complete metric space (Ω, K_Ω) is *Gromov hyperbolic* if there exists $\delta > 0$ (the *Gromov constant* of Ω) such that every geodesic triangle T is δ-thin, that is, every point on a side of T has distance from the union of the other two sides less than or equal to δ (see, *e.g.*, [13, 15] for details and further properties).

For an absolutely continuous curve $\gamma : [a,b] \to \Omega$, we denote by $l_{k_\Omega}(\gamma;[s,t])$ the length of the curve γ on $[s,t]$, $a \le s < t \le b$, that is,

$$l_{k_\Omega}(\gamma;[s,t]) := \int_s^t k_\Omega(\gamma(\tau);\gamma'(\tau))d\tau.$$

Let $A > 1$ and $B > 0$. An absolutely continuous curve $\gamma : [a,b] \to \Omega$ is a (A,B)*-quasi-geodesic* if for every $a \le s < t \le b$, we have :

$$l_{k_\Omega}(\gamma;[s,t]) \le AK_\Omega(\gamma(s),\gamma(t)) + B.$$

An (A,B)-quasi-geodesic ray (or line) is an absolutely continuous curve whose restriction to any compact interval in the domain of definition is an (A,B)-quasi-geodesic. A *quasi-geodesic* is just any (A,B)-quasi-geodesic segment/ray/line for some $A \ge 1$ and $B \ge 0$.

One of the main feature of Gromov hyperbolic spaces is the so-called Geodesic Stability Theorem, which says that every (A,B)-quasi-geodesic is shadowed by a geodesic at a distance which depends only on A, B and the Gromov constant of the space. In this paper we do not need it directly, but we will use instead a straightforward consequence for "quasi-geodesic rectangles" (see, *e.g.*, [22, Observation 4.4]):

Lemma 2.1. *Let $\Omega \subset\subset \mathbb{C}^d$ be a Kobayashi complete hyperbolic domain such that (Ω, K_Ω) is Gromov hyperbolic. Let $a,b,c,d \in \Omega$. Let $A \ge 1$ and $B \ge 0$. If Γ_1 is a (A,B)-quasi-geodesic joining a with b, Γ_2 is a (A,B)-quasi-geodesic joining b with c, Γ_3 is a (A,B)-quasi-geodesic joining c with d, Γ_4 is a (A,B)-quasi-geodesic joining a with d (that is, $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is a (A,B)-quasi-geodesic rectangle with sides $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4) then there exists $N > 0$ (which depends only on A, B and the Gromov constant of Ω) such that every point of one side is contained in the N-tubular neighborhood (with respect to K_Ω) of the union of the other three.*

We now turn to the precise definition of visibility.

Let $p, q \in \partial\Omega$, $p \ne q$. We say that the couple (p,q) satisfies the *visibility condition with respect to* K_Ω if there exist a neighborhood V_p of p and a neighborhood V_q of q and a compact subset K of Ω such that $V_p \cap V_q = \emptyset$ and $[x,y]_\Omega \cap K \ne \emptyset$ for every $x \in V_p \cap \Omega$, $y \in V_q \cap \Omega$.

We say that Ω is *visible* if every couple of points $p,q \in \partial\Omega$, $p \neq q$, satisfies the visibility condition with respect to K_Ω.

We say that a geodesic line $\gamma\colon (-\infty,+\infty) \to \Omega$ is a *geodesic loop in* $\overline{D}$ if γ has the same cluster set Γ in $\overline{\Omega}$ at $+\infty$ and $-\infty$. In such a case we say that Γ is the *vertex* of the geodesic loop γ.

By [11, Lemma 3.1], we have:

Lemma 2.2. *Suppose $\Omega \subset \mathbb{C}^d$ is a bounded complete hyperbolic domain. If Ω is visible then every geodesic ray lands, i.e., if $\gamma\colon [0,+\infty) \to \Omega$ is a geodesic then there exists $p \in \partial\Omega$ such that $\lim_{t\to+\infty}\gamma(t) = p$. In particular, the vertex of every geodesic loop in $\overline{\Omega}$ is a point in $\partial\Omega$.*

In the sequel, we will also need the following lemma (see [6, Lemma A.2])

Lemma 2.3 (D'Addezio)**.** *Let $D \subset \mathbb{C}^d$ be a complete hyperbolic bounded domain. Assume that ∂D does not contain non-trivial analytic discs. If $\{z_n\},\{w_n\} \subset D$ are two sequences such that $\lim_{n\to+\infty} z_n = p, \lim_{n\to+\infty} w_n = q$ with $p,q \in \partial D$ and $\sup_n K_D(z_n,w_n) < +\infty$, then $p = q$.*

3. Visible Domains and Frame of Quasi-Geodesics

As a matter of notation, if $\{\gamma_k\}$ is a sequence of curves in $\mathbb{C}^d$ such that, for every k, the curve γ_k is defined on some interval $[0,R_k]$ with $R_k \in (0,+\infty]$, we say that $\{\gamma_k\}$ converges *uniformly* to a curve $\gamma\colon [0,a] \to \Omega$ provided that $a = \lim_{k\to+\infty} R_k$, and for every $\theta > 0$, there exists $k_0 \in \mathbb{N}$ such that, for every $k \geq k_0$ and for every $t \in [0,R_k]$, $|\gamma_k(t) - \gamma(t)| \leq \theta$.

Lemma 3.1. *Let $\Omega \subset\subset \mathbb{C}^d$ be a domain such that (Ω, K_Ω) is complete hyperbolic. Suppose that Ω is visible. If $\{\gamma_k\}$ is a sequence of geodesic (segment or rays) in Ω, which converges uniformly on compacta to a geodesic ray γ, then $\{\gamma_k\}$ converges uniformly to γ.*

Proof. From the hypothesis, for every k, either γ_k is a geodesic segment defined on $[0,R_k]$ for some $R_k > 0$, or γ_k is geodesic ray defined on $[0,+\infty)$.

In case that γ_k is a geodesic ray, since Ω is visible, by Lemma 2.2, one sees that there exists $\gamma_k(+\infty) := \lim_{t\to+\infty}\gamma_k(t)$. Hence, in this case, in order to unify notations, we can consider γ_k to be defined on $[0,+\infty]$ and set $R_k = +\infty$.

Now, we assume by contradiction that there exists $\theta > 0$ and a sequence $\{t_k\}$ such that $t_k \in [0,R_k]$ and
$$|\gamma_k(t_k) - \gamma(t_k)| \geq \theta,$$
where, as before, if $t_k = +\infty$, we set $\gamma(+\infty) := \lim_{t\to+\infty}\gamma(t)$.

Since $\{\gamma_k\}$ converges uniformly on compacta to γ, it follows that either $t_k = +\infty$ or $\{t_k\}$ converges to $+\infty$. Hence, up to subsequences, we can assume that $\{\gamma_k(t_k)\}$ converges to a point $q \in \partial\Omega$ such that
$$|q - \gamma(+\infty)| \geq \theta.$$
Fix an open neighborhood U of $\gamma(+\infty)$ and an open neighborhood V of q such that $\overline{U} \cap \overline{V} = \emptyset$. By visibility hypothesis, there exists a compact subset $K \subset \Omega$ such that every geodesic in Ω joining a point of $U \cap \Omega$ to a point in $V \cap \Omega$ has to intersect K.

Since $\gamma(t)$ tends to $\gamma(+\infty)$ as $t \to +\infty$, there exists $R > 0$ such that $\gamma(r) \in U \cap \Omega$ for all $r \geq R$. Hence, for every $r \geq R$, there exists $k_r \in \mathbb{N}$ such that $\gamma_k(r) \in U \cap \Omega$ for $k \geq k_r$. Up to take k_r larger, we can also assume that $\gamma_k(t_k) \in V \cap \overline{\Omega}$ for $k \geq k_r$. Thus, $\gamma_k([r,t_k))$ is a geodesic from $U\cap\Omega$ to $V\cap\Omega$ for $k\geq k_r$. Therefore, for every $k\geq k_r$, there exists $s_k\in(r,t_k)$ such that $\gamma_k(s_k)\in K$, but

$$\begin{aligned}+\infty>\max_{\xi\in K}K_\Omega(\gamma(0),\xi))&\geq K_\Omega(\gamma(0),\gamma_k(s_k))\geq K_\Omega(\gamma_k(0),\gamma_k(s_k))-K_\Omega(\gamma(0),\gamma_k(0))\\&=s_k-K_\Omega(\gamma(0),\gamma_k(0))>r-K_\Omega(\gamma(0),\gamma_k(0)).\end{aligned}$$

Since $K_\Omega(\gamma(0),\gamma_k(0))$ tends to zero and thus it is bounded, we have a contradiction for r large enough. □

Proposition 3.1. *Let $\Omega\subset\subset\mathbb{C}^d$ be a domain such that (Ω,K_Ω) is complete hyperbolic. If Ω is visible, then there exists a frame of geodesics.*

Proof. Fix $z_0\in\Omega$ and let $\mathscr{F}$ be the set of all geodesic steaming from z_0, *i.e.*, $\gamma\in\mathscr{F}$ if either there exists $R_\gamma\in(0,+\infty)$ such that $\gamma\colon[0,R_\gamma]\to\Omega$ is a geodesic segment or $\gamma\colon[0,+\infty)\to\Omega$ is a geodesic ray, and (in both cases) $\gamma(0)=z_0$. By definition, $\mathscr{F}$ satisfies the conditions (1) and (3) of Definition 1.1. Also, since the distance K_Ω is complete, by Hopf-Rinow's theorem, for every $w\in\Omega$, there exists a geodesic segment γ joining z_0 and w. Hence, (5) is trivially satisfied for any $\varepsilon>0$ and $\delta>0$. Moreover, since Ω is visible, it follows from Lemma 2.2 that $\mathscr{F}$ enjoys also (2).

Now, we show that $\mathscr{F}$ satisfies (4). Assume that $\{\gamma_k\}$ is a sequence of geodesic steaming from z_0. Note that, for every k, either γ_k is a geodesic segment defined on $[0,R_k]$ for some $R_k>0$ or γ_k is geodesic ray defined on $[0,+\infty)$. However, as remarked above, in this case that there exists $\gamma_k(+\infty):=\lim_{t\to+\infty}\gamma_k$, we can consider γ_k to be defined on $[0,+\infty]$ and set $R_k=+\infty$. By Arzelà-Ascoli's theorem, it follows that, up to subsequences, $\{\gamma_k\}$ converges *uniformly on compacta* to some $\gamma\in\mathscr{F}$. By Lemma 3.1, the convergence is actually uniform. This completes the proof. □

Now we are in a good shape to prove Theorem 1.1:

Proof of Theorem 1.1. One direction follows from Proposition 3.1.

Conversely, assume that Ω has a family $\mathscr{F}$ of (A,B)-quasi-geodesics. Hence, there exists a compact subset $K\subset\Omega$ such that every quasi-geodesic in $\mathscr{F}$ steams from K. Assume that $\{z_k^j\}\subset\Omega$ is a sequence converging to $p_j\in\partial\Omega$, $j=1,2$ with $p_1\neq p_2$.

We argue by contradiction and we assume that $\{[z_k^1,z_k^2]_\Omega\}$ eventually escapes every compact subset of Ω. By hypothesis, for k large enough, we can find $\gamma_k^1,\gamma_k^2\in\mathscr{F}$ and $w_k^j\in\gamma_k^j$ such that $K_\Omega(z_k^j,w_k^j)<\delta$, $j=1,2$ (here, we a slight abuse of notation, we denote by γ_k^j also the image of the quasi-geodesic γ_k^j). By Lemma 2.3, since $\partial\Omega$ does not contain nontrivial analytic discs, it follows that $\{w_k^j\}$ converges to p_j, $j=1,2$ for $k\to+\infty$. By the condition (4) of Definition 1.1, we can assume, up to subsequences, that $\{\gamma_k^j\}$ converges uniformly to some $\gamma^j\in\mathscr{F}$, $j=1,2$. Hence,

$$\lim_{t\to+\infty}\gamma^j(t)=p_j,\quad j=1,2.\tag{3.1}$$

We claim now that $\{[w_k^1,w_k^2]_\Omega\}$ eventually escapes every compact subset of Ω, if so does $\{[z_k^1,z_k^2]_\Omega\}$. Indeed, assume by contradiction that there exists a compact set $Q\subset\subset\Omega$ and $c_k\in[w_k^1,w_k^2]_\Omega$ such that $c_k\in Q$ for every k. Since

$$[w_k^1,w_k^2]_\Omega\cup[z_k^1,z_k^2]_\Omega\cup[z_k^1,w_k^1]_\Omega\cup[z_k^2,w_k^2]_\Omega$$

is a geodesic rectangle, by Lemma 2.1, it follows that there exist $N > 0$ and

$$t_k \in [z_k^1, z_k^2]_\Omega \cup [z_k^1, w_k^1]_\Omega \cup [z_k^2, w_k^2]_\Omega$$

such that $K_\Omega(t_k, c_k) \leq N$. Since for every $s \in [z_k^j, w_k^j]_\Omega$ we have that $K_\Omega(s, z_k^j) \leq K_\Omega(w_k^j, z_k^j) < \delta$, $j = 1, 2$, it follows from the completeness of K_Ω that $t_k \in [z_k^1, z_k^2]_\Omega$. But then t_k belongs to a N-tubular neighborhood of Q, which is as well a relatively compact subset of Ω, and we get a contradiction. Therefore, $\{[w_k^1, w_k^2]_\Omega\}$ eventually escapes every compact subset of Ω. Let $a_k^j > 0$ be such that $\gamma_k^j(a_k^j) = w_k^j$, $j = 0, 1$ and consider now the quasi-geodesic rectangle

$$[w_k^1, w_k^2]_\Omega \cup \gamma_k^1([0, a_k^1]) \cup \gamma_k^2([0, a_k^1]) \cup [\gamma_k^1(0), \gamma_k^2(0)]_\Omega.$$

Note that $\{[\gamma_k^1(0), \gamma_k^2(0)]_\Omega\}$ is relatively compact in Ω. Hence, by Lemma 2.1, there exists $N > 0$ such that for every point $s_k \in [w_k^1, w_k^2]_\Omega$ there exists $\zeta(s_k) \in \gamma_k^1([0, a_k^1]) \cup \gamma_k^2([0, a_k^1])$ such that

$$K_\Omega(s_k, \zeta(s_k)) \leq N. \tag{3.2}$$

Since $\{s_k\}$ is compactly divergent, it follows by the completeness of K_Ω that $\{\zeta(s_k)\}$ is compactly divergent as well. By (3.1), the cluster set of $\{\zeta(s_k)\}$ is contained in $\{p_1\} \cup \{p_2\}$ and, by (3.2) and Lemma 2.3, so is the cluster set of $\{s_k\}$. But this is a contradiction because the cluster set of $[w_k^1, w_k^2]_\Omega$ has to be connected and contains both p_1 and p_2. Hence, Ω is visible. □

4. Geodesic Loops in Visible Domains

Definition 4.1. Let $\Omega \subset\subset \mathbb{C}^d$ be a domain such that (Ω, K_Ω) is complete hyperbolic. We say that two quasi-geodesic rays γ, η *stay at infinite distance each other* provided that for every $C > 0$ there exists $t_C > 0$ such that

$$\min_{t \geq t_C} K_\Omega(\gamma(t), \eta([0, +\infty)) > C, \quad \min_{t \geq t_C} K_\Omega(\eta(t), \gamma([0, +\infty)) > C.$$

Lemma 4.1. *Let $\Omega \subset\subset \mathbb{C}^d$ be a domain such that (Ω, K_Ω) is complete hyperbolic. Assume that $\gamma : \mathbb{R} \to \Omega$ is a geodesic loop for $\overline{D}$. Let $\gamma^+ : [0, +\infty) \to \Omega$ be the geodesic ray defined by $\gamma^+(t) := \gamma(t)$, and let $\gamma^- : [0, +\infty) \to \Omega$ be the geodesic ray defined by $\gamma^-(t) := \gamma^-(t)$. Then γ^+ and γ^- stay at infinite distance each other.*

Proof. Assuming by contradiction that γ^+, γ^- do not stay at infinite distance each other, we can find $C > 0$ such that, for every $t \in [0, +\infty)$, there exists $s_t \in [0, +\infty)$ such that

$$K_\Omega(\gamma^+(t), \gamma^-(s_t)) \leq C.$$

Since γ is a geodesic,

$$t + s_t = K_\Omega(\gamma(t), \gamma(-s_t)) = K_\Omega(\gamma^+(t), \gamma^-(s_t)) \leq C,$$

and, for $t \to +\infty$ we have a contradiction. □

Proposition 4.1. *Let $\Omega \subset\subset \mathbb{C}^d$ be a domain such that (Ω, K_Ω) is complete hyperbolic. Assume that Ω is visible. If there exists $p \in \partial\Omega$ and an open neighborhood U of p such that $U \cap \Omega$ has at least two connected components such that p belongs to their closure, then there exists a geodesic loop for $\overline{\Omega}$ with vertex p.*

Proof. Let p, U as in the statement, and let V_1, V_2 be two connected components of $\Omega \cap U$ such that $p \in \overline{V_j}$, $j = 1, 2$. For $j = 1, 2$, let $\mathscr{V}_j \subset\subset V_j$ be an open set such that $p \in \overline{\mathscr{V}_j}$ and $p \notin \partial \mathscr{V}_j$. Moreover, for $j = 1, 2$, let $\{z_n^j\} \subset \mathscr{V}_j$ be a sequence converging to p. For every n, let $\gamma_n : [a_n, b_n] :\to \Omega$ be a geodesic such that $\gamma_n(a_n) = z_n^1$ and $\gamma_n(b_n) = z_n^2$, $a_n < b_n$.

We claim that there exists a compact subset $K \subset \Omega$ such that $\gamma_n([a_n, b_n]) \cap K \neq \emptyset$ for all n. Indeed, if this were not the case, we can assume that $\{\gamma_n([a_n, b_n])\}$ escapes every compact subset of Ω for n large. Let $r_n \in (a_n, b_n)$ be such that $\gamma_n(r_n) \in \partial \mathscr{V}_2$ for all t. Then, up to subsequences, $\{\gamma_n(r_n)\}$ converges, for $n \to \infty$, to a point $q \in \partial\Omega \setminus \{p\}$. Since Ω is visible, it follows that there exists a compact subset $K' \subset \Omega$ such that $\gamma_n([r_n, b_n]) \cap K' \neq \emptyset$, reaching a contradiction.

Up to an affine change of parameterization, we can thus assume that $\{\gamma_n(0)\}$ is relatively compact in Ω and $a_n < 0 < b_n$ for all n. We claim that $\{\gamma_n\}$ (under the assumption that $\{\gamma_n(0)\}$ is relatively compact in Ω) converges, up to subsequences, to a geodesic loop in $\overline{\Omega}$ with vertex p.

Indeed, since (Ω, K_Ω) is complete and $\gamma_n(0) \in K$,

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} K_\Omega(\gamma_n(0), z_n^1) = +\infty,$$

and thus $a_n \to -\infty$ for $n \to \infty$. Similarly, $b_n \to +\infty$ for $n \to \infty$. Thus, for every $S, T \in \mathbb{R}$, there exists n_0 such that $S, T \in (a_n, b_n)$ for $n > n_0$. Moreover, setting for $R > 0$,

$$d_K(R) := \max\{K_\Omega(z, w) : K_\Omega(z, K) \leq R, K_\Omega(w, K) \leq R\} < +\infty,$$

we have, for all $n > n_0$,

$$K_\Omega(\gamma_n(S), \gamma_n(T)) = K_\Omega(\gamma_n(0), \gamma_n(S)) + K_\Omega(\gamma_n(0), \gamma_n(T)) \leq d_K(|S|) + d_K(|T|).$$

Hence $\{\gamma_n\}$ are locally equibounded and locally equi-continuous and by Arzelà-Ascoli's theorem, and taking into account that $K_D(\gamma_n(T), \gamma_n(S)) = |T - S|$ for all n, we can extract a subsequence converging on compacta of $\mathbb{R}$ to a geodesic line γ such that $\gamma(0) \in K$. By Lemma 2.2, $\lim_{t\to\pm\infty} \gamma(t) = p$. Hence, γ is a geodesic loop in $\overline{\Omega}$ with vertex p, and we are done. □

For Gromov hyperbolic domains, the existence of a geodesic loop is equivalent to the existence of two geodesic rays landing at the same point, which stay at infinite distance each other.

Proposition 4.2. *Let $\Omega \subset\subset \mathbb{C}^d$ be a domain such that (Ω, K_Ω) is complete hyperbolic and Gromov hyperbolic. Assume that Ω is visible. Then there exists a geodesic loop for $\overline{\Omega}$ with vertex $p \in \partial\Omega$ if and only if there exists two geodesic rays in Ω landing at p, which stay at infinite distance each other.*

Proof. If γ is a geodesic loop for $\overline{\Omega}$ with vertex p, then γ^+ and γ^- are geodesic rays at infinite distance each other, landing at p (see Lemma 4.1).

Conversely, if $\alpha, \beta : [0, +\infty) \to \Omega$ are two geodesic rays, which stay at infinite distance each other, for each $T > 0$, we consider the geodesic rectangle given by

$$[\alpha(0), \beta(0)]_\Omega \cup \alpha([0, T]) \cup [\alpha(T), \beta(T)]_\Omega \cup \beta([0, T]).$$

Since (Ω, K_Ω) is Gromov hyperbolic, by Lemma 2.1, there exists $\delta > 0$ such that $[\alpha(T), \beta(T)]_\Omega$ is contained in the δ-tubular neighborhood (with respect to K_Ω) of $[\alpha(0), \beta(0)]_\Omega \cup \alpha([0, T]) \cup \beta([0, T])$. Since α and β stay at infinite distance each other, it follows that there exists $T_0 > 0$ such that $K_\Omega(\alpha(t), \beta(s)) > 4\delta$ for all $t, s \geq T_0$.

We claim that for all $T > T_0$ there exists $z_T \in [\alpha(T), \beta(T)]_\Omega$ such that z_T belongs to the δ-tubular neighborhood $\mathcal{N}$ of $[\alpha(0), \beta(0)]_\Omega \cup \alpha([0, T_0)] \cup \beta([0, T_0])$.

Indeed, if this were not the case, then every $\xi \in [\alpha(T), \beta(T)]_\Omega$ would belong to a δ-tubular neighborhood U of $\alpha([T_0, T]) \cup \beta([T_0, T])$. However, since $K_\Omega(\alpha(t), \beta(s)) > 4\delta$ for all $t, s \geq T_0$, it follows that U is the disjoint union of two open sets. But then $[\alpha(T), \beta(T)]_\Omega$ is the disjoint union of the two open sets given by $[\alpha(T), \beta(T)]_\Omega \cap U$, hence, not connected, a contradiction and the claim follows.

Now, since K_Ω is a complete distance, $K := \overline{\mathcal{N}}$ is a compact subset of Ω and

$$[\alpha(T), \beta(T)]_\Omega \cap K \neq \emptyset \quad \forall T \geq 0.$$

Choosing a parametrization γ_T of $[\alpha(T), \beta(T)]_\Omega$ such that $\gamma_T(0) \in K$ and arguing as in the proof of Proposition 4.1, we see that we can extract a sequence $\{\gamma_{T_n}\}$ converging to a geodesic loop for $\overline{\Omega}$ with vertex p. □

Now we are in good shape to prove Theorem 1.2:

Proof of Theorem 1.2. First, assume that Ω has no geodesic loops. Then by Proposition 3.1 (and its proof), the family $\mathcal{F}$ of all geodesics steaming from a given point $z_0 \in \Omega$ is a geodesic frame. Such a geodesic frame is non-looping by Proposition 4.2.

Suppose now that Ω has a non-looping frame of quasi-geodesics $\mathcal{F}$. Assume by contradiction that $\gamma : \mathbb{R} \to \Omega$ is a geodesic loop. Since Ω is visible, it follows that there exists $p \in \partial\Omega$ such that $\lim_{t\to\pm\infty} \gamma(t) = p$. Let $\delta > 0, \varepsilon > 0$ be given as in (5) of Definition 1.1 and let

$$V := \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) < \varepsilon\}.$$

Let $\gamma^+(t) := \gamma(t)$ for $t \geq 0$ and $\gamma^-(t) := \gamma(-t)$ for $t \geq 0$. Hence, there exists $t_0 > 0$ such that for $t \geq t_0$ sufficiently large, $\gamma^\pm(t) \in V$. By hypothesis, for each $k \geq t_0$, $k \in \mathbb{N}$, there exist $\gamma_k^\pm \in \mathcal{F}$ and some $s_k^\pm$ in the domain of definition of $\gamma_k^\pm$ such that

$$K_\Omega(\gamma^\pm(k), \gamma_k^\pm(s_k^\pm)) \leq \varepsilon. \tag{4.1}$$

As $k \to +\infty$, since K_Ω is a complete distance and by Lemma 2.3, it follows that $s_k^\pm \to +\infty$ and $\gamma_k^\pm(s_k^\pm) \to p$. By definition of the quasi-geodesic frame, up to subsequences, $\{\gamma_k^\pm\}$ converges uniformly to some quasi-geodesic ray $\gamma_\infty^\pm \in \mathcal{F}$ such that $\lim_{t\to+\infty} \gamma_\infty^\pm(t) = p$. Since $\mathcal{F}$ is non-looping, γ_∞^+ and γ_∞^- stay at finite distance each other.

We claim that this implies that there exists $C > 0$ such that for all $T \geq 0$,

$$K_\Omega(\gamma^+(T), \gamma^-(T)) \leq C, \tag{4.2}$$

reaching a contradiction to Lemma 4.1. In order to prove (4.2), for every $k \geq t_0$, consider the quasi-geodesic rectangle given by

$$[\gamma^+(k), \gamma_k^+(s_k)]_\Omega \cup \gamma^+([0,k]) \cup \gamma_k^+([0, s_k^+]) \cup [\gamma(0), \gamma_k^+(0)]_\Omega.$$

Fix $T \geq t_0$. Hence, by Lemma 2.1, there exists some $R > 0$ such that for every $k > T$ there exists $z_k \in [\gamma^+(k), \gamma_k^+(s_k)]_\Omega \cup \gamma_k^+([0, s_k^+]) \cup [\gamma(0), \gamma_k^+(0)]_\Omega$ such that

$$K_\Omega(z_k, \gamma^+(T)) \leq R. \tag{4.3}$$

Since by hypothesis $\{\gamma_k^+(0)\}$ is relatively compact in Ω—and so is $\{[\gamma(0), \gamma_k^+(0)]_\Omega\}$—and $\gamma^+(t) \to p \in \partial\Omega$ as $t \to +\infty$, it follows that, if t_0 (and hence T) is large enough, $z_k \notin [\gamma(0), \gamma_k^+(0)]_\Omega$.

Moreover, $\{[\gamma^+(k), \gamma_k^+(s_k)]_\Omega\}$ is compactly divergent. Indeed, if this is not the case, there is $y_k \in [\gamma^+(k), \gamma_k^+(s_k)]_\Omega$ such that $\{y_k\}$ is relatively compact, thus

$$K_\Omega(\gamma^+(k), \gamma_k^+(s_k)) = K_\Omega(\gamma^+(k), y_k) + K_\Omega(y_k, \gamma_k^+(s_k)),$$

and the latter quantity tends to $+\infty$ as $k \to +\infty$ because K_Ω is a complete distance, against (4.1). Hence, if k is sufficiently large, $z_k \in \gamma_k^+([0, s_k^+])$. By (4.3), $\{z_k\}$ is relatively compact in Ω. Since γ_∞^- and γ_∞^+ stay at finite distance each other and $\{\gamma_k^\pm\}$ converges uniformly to $\gamma_\infty^\pm$, it follows that there exist $D > 0$ and $w_k \in \gamma_k^-$ for k sufficiently large, so $K_\Omega(z_k, w_k) \leq D$. In particular, this implies that also $\{w_k\}$ is relatively compact in Ω. Using again Lemma 2.1 and arguing as before with the quasi-geodesic rectangle

$$[\gamma^-(k), \gamma_k^-(s_k)]_\Omega \cup \gamma^-([0,k]) \cup \gamma_k^-([0, s_k^-]) \cup [\gamma(0), \gamma_k^-(0)]_\Omega,$$

we can find $S \geq 0$ and $E > 0$ such that for all k sufficiently large

$$K_\Omega(\gamma^-(S), w_k) \leq E.$$

Then, by the triangle inequality, $K_\Omega(\gamma^+(T), \gamma^-(S)) \leq C := R + D + E$. Therefore,

$$\begin{aligned}
K_\Omega(\gamma^+(T), \gamma^-(T)) &\leq K_\Omega(\gamma^+(T), \gamma^-(S)) + K_\Omega(\gamma^-(S), \gamma^-(T)) \\
&= K_\Omega(\gamma^+(T), \gamma^-(S)) + |T - S| \\
&= K_\Omega(\gamma^+(T), \gamma^-(S)) + |K_\Omega(\gamma^+(T), \gamma(0)) - K_\Omega(\gamma^-(S), \gamma(0))| \\
&\leq 2K_\Omega(\gamma^+(T), \gamma^-(S)) \leq C.
\end{aligned}$$

Thus, (4.2) follows, and we are done. □

Acknowledgements

This paper was partially supported by PRIN Real and Complex Manifolds: Topology, Geometry and holomorphic dynamics n.2017JZ2SW5, by GNSAGA of INdAM, and by the MUR Excellence Department Project MatMod@TOV CUP:E83C23000330006 awarded to the Department of Mathematics, University of Rome Tor Vergata.

REFERENCES

[1] Z. M. Balogh, M. Bonk, Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains, Comment. Math. Helv. 75 (2000), 504-533.

[2] G. Bharali, A. Maitra, A weak notion of visibility, a family of examples, and Wolff-Denjoy theorems, Ann. Scuola Norm. Sup. di Pisa, Cl. Sci. XXII (2021), 195–240.

[3] G. Bharali, A. Zimmer, Goldilocks domains, a weak notion of visibility, and applications, Adv. Math. 310 (2017), 377-425.

[4] G. Bharali, A. Zimmer, Unbounded visibility domains, the end compactification, and applications, Trans. Amer. Math. Soc. 376 (2023), 5949-5988.

[5] F. Bracci, M. D. Contreras, S. Díaz-Madrigal, Continuous semigroups of holomorphic self-maps of the unit disc, Springer Monographs in Mathematics. Springer, Cham, 2020.

[6] F. Bracci, H. Gaussier, Abstract boundaries and continuous extension of biholomorphisms, Anal. Math. 48 (2022), 393-409.

[7] F. Bracci, H. Gaussier, Horosphere topology, Ann. Scuola Norm. Sup. di Pisa, Cl. Sci. XX (2020), 239-289.

[8] F. Bracci, H. Gaussier, A proof of the Muir-Suffridge conjecture for convex maps of the unit ball in $\mathbb{C}^n$, Math. Ann. 372 (2018), 845-858.

[9] F. Bracci, H. Gaussier, A. Zimmer, Homeomorphic extension of quasi-isometries for convex domains in $\mathbb{C}^d$ and iteration theory, Math. Ann. 379 (2021), 691-718.

[10] F. Bracci, H. Gaussier, A. Zimmer, The geometry of domains with negatively pinched Kaehler metrics, arXiv:1810.11389, 2018. To appear in J. Differential Geom.

[11] F. Bracci, N. Nikolov, P.J. Thomas, Visibility of Kobayashi geodesics in convex domains and related properties, Math. Z. 301 (2022), 2011-2035.

[12] F. Bracci, H. Gaussier, N. Nikolov, P. J. Thomas, Local and global visibility and Gromov hyperbolicity of domains with respect to the Kobayashi distance, arXiv:2201.03070, 2022. To appear in Trans. Amer. Math. Soc.

[13] M. Bridson, A. Haefliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin, 1999.

[14] V. S. Chandel, A. Maitra, A. D. Sarkar, Notions of visibility with respect to the Kobayashi distance: comparison and applications, arXiv:2111.00549, 2021.

[15] M. Coornaert, T. Delzant, A. Papadopoulos, Géométrie et théorie des groupes. Les groupes hyperboliques de Gromov, Lecture Notes in Mathematics, 1441. Springer-Verlag, Berlin, 1990.

[16] M. Fiacchi, Gromov hyperbolicity of pseudoconvex finite type domains in $\mathbb{C}^2$, Math. Ann. 382 (2022), 37-68.

[17] S. Kobayashi, Hyperbolic Complex Spaces, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 318. Springer-Verlag, Berlin, 1998.

[18] A. Maitra, On the continuous extension of Kobayashi isometries, Proc. Amer. Math. Soc. 148 (2020), 3437-3451.

[19] N. Nikolov, A. Y. Ökten, P. J. Thomas, Local and global notions of visibility with respect to Kobayashi distance, a comparison, arXiv:2210.10007, 2022.

[20] N. Nikolov, A. Y. Ökten, Strong Localizations of the Kobayashi Distance, arXiv:2211.15488, 2022.

[21] A.M. Zimmer, Gromov hyperbolicity and the Kobayashi metric on convex domains of finite type, Math. Ann. 365 (2016), 1425-1498.

[22] A.M. Zimmer, Gromov hyperbolicity, the Kobayashi metric, and $\mathbb{C}$-convex sets, Trans. Amer. Math. Soc. 369 (2017), 8437-8456.

[23] A.M. Zimmer, Subelliptic estimates from Gromov hyperbolicity, Adv. Math. 402 (2022), 108334.