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ON A MEAGER FULL MEASURE SUBSET OF N-ARY SEQUENCES

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Abstract. Let $I = \{1, \dots, N\}$ be a finite set of indices and $K = I^{\mathbb{N}}$ the set of all sequences of indices equipped with the product measure and the product topology. Melo, da Cruz Neto, and de Brito [Strong convergence of alternating projections, J. Optim. Theory Appl. 194 (2022), 306-324] defined a family of sequences $\mathscr{N}_0 \subseteq K$ so that whenever one iterates distance minimizing projections on N closed and convex subsets of an Hadamard space, the sequence of projections converges, provided it has at least one accumulation point. They proved that $\mathscr{N}_0$ has full measure, and in the sense of measure almost all iterates of projections converge. We observe that $\mathscr{N}_0$ is meager. The question, which almost all iterates converge in the topological sense, remains open.
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1. Introduction

Given two convex subsets of a Hilbert space with nonempty intersection, we are interested in finding some point within the intersection. We refer to this problem as the *convex feasibility problem*. Von Neumann proposed the method of alternating projections to solve the convex feasibility problem [1]. More broadly speaking, we may consider an Hadamard space (X, ρ), an index set $I := \{1, \dots, N\}$ for $N \in \mathbb{N}$, and a finite number of closed convex sets $(C_n)_{n \in I}$ within X. By P_n we denote the distance minimizing projection to the set C_n, $n \in I$. This projection is well-defined, since C_n is convex and closed. Choose some $\xi_0 \in X$ and a sequence $x = (x_n)_{n \in \mathbb{N}} \in K := I^{\mathbb{N}}$. We iteratively define a projection sequence $(\xi_n)_{n \in \mathbb{N}}$ by

$$\xi_n = P_{x_n}(\xi_{n-1}), \quad n \in \mathbb{N}.$$

This construction is called the method of alternating projections. The intention behind this method is that the sequence $(\xi_n)_{n \in \mathbb{N}}$ converges to a point in the intersection $C_1 \cap \dots \cap C_N$. Indeed, von Neumann proved in [1] that this is the case when X is a Hilbert space, $N = 2$, and C_1 and C_2 are linear subspaces. A simple geometric proof of von Neumann's theorem is provided by Kopecká and Reich in [2]. Halperin showed in [3] that von Neumann's result holds for any finite number of subspaces when x is periodic. Sakai in [4] extended this to quasi-periodic sequences.

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Definition 1.1. A sequence $x = (x_n)_{n\in\mathbb{N}} \in K$ is said to be quasi-periodic if and only if

$$\exists m \in \mathbb{N}\colon \forall k \in \mathbb{N}\colon \{x_k, x_{k+1}, \dots, x_{k+m-1}\} = I.$$

The smallest such m is called the quasi-period of x.

This condition imposes a uniform bound on the distance of occurrences of indices in x, similar to the periodicity.

The authors of [5, 6] proved that not every $x \in K$ induces a converging sequence $(\xi_n)_{n\in\mathbb{N}}$. They provided a counterexample of three closed linear subspaces in an infinite-dimensional Hilbert space such that, for any $0 \neq \xi_0 \in X$, there is a sequence $x \in K$ such that the method of alternating projections does not strongly converge. This motivates the following question:

> How large is the set of sequences $x \in K$ for which $(\xi_n)_{n\in\mathbb{N}}$ is strongly convergent?

Melo, da Cruz Neto, and de Brito studied this question and showed in Proposition 4.3 of [7] that the up until now considered sets of periodic and quasi-periodic sequences are null sets with respect to the Bernoulli measure $\mathbb{P}$ on K. This measure is the product measure of $\mathbb{P}_I$ with $\mathbb{P}_I(\{1\}) = \cdots = \mathbb{P}_I(\{N\}) = \frac{1}{N}$ over the index set $\mathbb{N}$. The authors of [7] introduced a more general notion than quasi-periodic sequences: the notion of quasi-normal sequences.

Definition 1.2 (Definition 4.2 in [7])**.** We call a sequence $(x_n)_{n\in\mathbb{N}} \in K$ quasi-normal if there exists an $L \in \mathbb{N}$ and a sequence of disjoint blocks $(\mathscr{R}_k)_{k\in\mathbb{N}}$ of consecutive elements of $(x_n)_{n\in\mathbb{N}}$ with L terms, where each block $\mathscr{R}_k$ contains every element of I so that there exists a function $f\colon \mathbb{N} \to (0,\infty)$ with $\lim_{n_k\to\infty} f(n_k) = \infty$ such that

$$\sum_{k\in\mathbb{N}} \frac{1}{n_k \cdot f(n_k)} = \infty,$$

where x_{n_k} is the first element of the block $\mathscr{R}_k$ and $(n_k)_{k\in\mathbb{N}}$ is an increasing sequence. We denote the set of quasi-normal sequences by $\mathscr{N}$.

Proposition 4.2 of [7] states that quasi-normal sequences form a subset of K of full measure and Theorem 4.1 of [7] guarantees strong convergence if the sequence $(\xi_n)_{n\in\mathbb{N}}$ has at least one accumulation point.

Melo, da Cruz Neto, and de Brito [7] showed that $\mathscr{N}$ is of full measure by defining a stronger condition resulting in a subset $\mathscr{N}_0$ of the quasi-normal sequences and proved that this set has full measure. More precisely, the set $\mathscr{N}_0$ is defined as the set of all sequences satisfying the conditions of the following proposition.

Proposition 1.1 (Proposition 4.1 in [7])**.** *Let $(x_n)_{n\in\mathbb{N}} \in K$. Suppose that there exists an $L \in \mathbb{N}$, a sequence of disjoint blocks $(\mathscr{R}_k)_{k\in\mathbb{N}}$ of consecutive elements of $(x_n)_{n\in\mathbb{N}}$ with L terms, where each block $\mathscr{R}_k$ contains every element of I. For each $k \in \mathbb{N}$, let $\mathscr{S}_k$ be the block formed by the elements between $\mathscr{R}_{k-1}$ and $\mathscr{R}_k$, which may eventually be empty. Thus, $(x_n)_{n\in\mathbb{N}}$ can be seen as follows:*

$$\mathscr{S}_1\mathscr{R}_1\mathscr{S}_2\mathscr{R}_2\dots\mathscr{R}_{k-1}\mathscr{S}_k\mathscr{R}_k\dots.$$

Let $|\mathscr{S}_k|$ be the number of elements of this block, and let c be a constant. If, for all $k \in \mathbb{N}$, we have

$$\sum_{i=1}^{k} |\mathscr{S}_i| \leq ck,$$

then sequence $(x_n)_{n\in\mathbb{N}}$ is quasi-normal.

Instead of using a measure theoretic notion of large and small subsets, we deal with topological or metric notions. We are interested in whether the set of sequences $x \in K$ leading to a strongly convergent projection sequence $(\xi_n)_{n\in\mathbb{N}}$ is a large subset in a topological and metric sense and, in particular, if the quasi-normal sequences already form such a large subset. More specifically, we are interested in the topological notions of meager and dense G_δ subsets and the stronger notions of σ-porous and co-σ-porous subsets as a metric notion of small and large subsets, respectively; see, e.g., [8].

Definition 1.3 ((σ-)porous subset)**.** A subset A of a metric space (X,d) is called *porous at* $x \in A$ if there are $r_0 > 0$ and $\alpha > 0$ such that for every $r \in (0,r_0)$ there is a point $y \in X \setminus A$ with $\rho(x,y) < r$ and $B(y,\alpha r) \cap A = \emptyset$ or, put differently,

$$A \text{ porous at } x :\Leftrightarrow \exists r_0 > 0 : \exists \alpha > 0 : \forall r \in (0,r_0) : \exists y \in X \setminus A : \rho(x,y) < r \wedge B(y,\alpha r) \cap A = \emptyset.$$

The set A is called *porous* if it is porous at all its points. A subset of X is called σ-*porous* if it is a countable union of porous sets. We call a set *co-porous* or *co-*σ*-porous* if its complement is porous or σ-porous respectively.

For $N \in \mathbb{N}$, we equip $I = \{1,\dots,N\}$ with the discrete topology and $K := I^{\mathbb{N}}$ with the product topology. It is well known that the topology on K is induced by the complete metric

$$d(x,y) := \max\{2^{-j} d_0(x_j,y_j) \colon j \in \mathbb{N}\},$$

where d_0 denotes the discrete metric on I. Note that, for $x \in K$ and $j \in \mathbb{N}$, we have that

$$B(x,2^{-j}) = \{y \in K \colon y_1 = x_1 \wedge \cdots \wedge y_j = x_j\}.$$

2. RESULTS

In the following, we show that the subset $\mathscr{N}_0$ of the quasi-normal sequences $\mathscr{N}$ is meager, and hence, a small subset in a topological sense. This stands in contrast to the result of [7] where $\mathscr{N}_0$ is shown to be of full measure. We begin by extracting the condition of Proposition 1.1 by defining properties in the following definition.

Definition 2.1. Let $L \in \mathbb{N}$ and $c > 0$. We then define

$$P_{L,c}(x) :\longleftrightarrow \exists \text{ representation } \mathscr{S}_1\mathscr{R}_1\mathscr{S}_2\mathscr{R}_2\cdots = x \text{ such that}$$

$$\wedge \begin{cases} \forall k \in \mathbb{N} \colon |\mathscr{R}_k| = L \\ \forall k \in \mathbb{N} \colon \mathscr{R}_k \text{ contains all elements of } I \\ \forall k \in \mathbb{N} \colon \sum_{i=1}^{k} |\mathscr{S}_i(x)| \leq ck \end{cases}$$

and

$$P(x) :\longleftrightarrow \exists L \in \mathbb{N} : \exists c > 0 : P_{L,c}(x).$$

We may now write $\mathscr{N}_0 = \{x \in K : P(x)\}$. Let us for $L \in \mathbb{N}$ and $c > 0$ define the set

$$\mathscr{N}_{L,c} := \{x \in K : P_{L,c}(x)\}.$$

Apparently,

$$\mathscr{N}_0 = \bigcup_{L\in\mathbb{N}} \bigcup_{c\in\mathbb{Q}_+} \mathscr{N}_{L,c}.$$

Theorem 2.1. *For every $L \in \mathbb{N}$ and $c \in \mathbb{Q}_+$, the set $\mathscr{N}_{L,c}$ is nowhere dense.*

The proof of this theorem is provided in Section 3.1.

Corollary 2.1. *The set $\mathscr{N}_0$ is meager.*

Unfortunately, the strategy used by Melo, da Cruz Neto, and de Brito [7] for proving that $\mathscr{N}$ is a large subset by showing that $\mathscr{N}_0$ is large does not work in the context of the product topology. As of now, the question whether the quasi-normal sequences $\mathscr{N}$ themselves are a large subset or not in a topological or metric sense remains an open question.

3. Proofs of Statements

As a warm up for the proof of $\mathscr{N}_0$ being meager, we first show that the set of quasi-periodic sequences Q is σ-porous. The fact that the periodic sequences are σ-porous too, is clear since they are countable. Note that this is in line with the results by Melo, da Cruz Neto, and de Brito [7], where these sets were shown to be null sets; see [7, Proposition 4.3].

Proposition 3.1. *The set Q_m of quasi-periodic sequences with quasi-period $m \in \mathbb{N}$ is porous.*

The proof of this proposition follows after some lemmas.

Corollary 3.1. *The set Q of quasi-periodic sequences in K is σ-porous.*

Proof. Since, for every $m \in \mathbb{N}$, the set Q_m of quasi-periodic sequences with quasi-period m is porous and since $Q = \bigcup_{m\in\mathbb{N}} Q_m$, we have that Q is σ-porous. □

We show that for every $m \in \mathbb{N}$ the set Q_m is porous. Let us fix some arbitrary point $x \in Q_m$. Let

$$r_0 := \frac{1}{2}, \quad \alpha := 2^{-m-1},$$

and $r \in (0, r_0)$. Choose $j \in \mathbb{N}$ such that $2^{-j} \leq r < 2^{-j+1}$ and set

$$y := (x_1, x_2, \dots, x_j, 1, 1, \dots).$$

Lemma 3.1. *$y \notin Q_m$ and $d(x,y) < r$.*

Proof. By construction, in y, there are m many consecutive elements in which not all elements of I appear. Therefore, we have $y \notin Q_m$. Since $x_1 = y_1 \wedge \cdots \wedge x_j = y_j$, we have that $d(x,y) < 2^{-j} \leq r$. □

Lemma 3.2. *$B(y, \alpha r) \cap Q_m = \emptyset$.*

Proof. Let $z \in B(y, \alpha r) = B(y, 2^{-m-1}r)$. Since $2^{-m-1}r < 2^{-m-j}$, we have

$$z_1 = y_1 \wedge \cdots \wedge z_j = y_j \quad \text{and}$$
$$z_{j+1} = 1 \wedge \cdots \wedge z_{j+m} = 1.$$

Since apparently in z, there are m consecutive elements in which not all elements of I appear we have that $z \notin Q_m$. □

Proof of Proposition 3.1. For every $x \in Q_m$, we have chosen an $r_0 > 0$ and an $\alpha > 0$. For every $r \in (0, r_0)$, we construct a $y \in K$, which by Lemma 3.1 is not contained in Q_m and fulfills $d(x,y) < r$. By Lemma 3.2, we have that $B(y, \alpha r) \cap Q_m = \emptyset$. Hence, we prove that Q_m is porous. □

3.1. **Proof of Theorem 2.1.**

Lemma 3.3. *Let $L \in \mathbb{N}, c > 0, x \in \mathscr{N}_{L,c}$, and $n \in \mathbb{N}$. Furthermore, let*

$$y := (x_1, \dots, x_n, 1, 1, \dots)$$

and

$$m := \lceil c \rceil \left(L \left\lfloor \frac{n}{L} \right\rfloor + 1 \right) + 2L + 2.$$

Then

$$B(y, 2^{-n-m}) \cap \mathscr{N}_{L,c} = \emptyset.$$

Proof. Let us fix some arbitrary $z \in B(y, 2^{-n-m})$ and assume that $z \in \mathscr{N}_{L,c}$. Note that the ball may be written as

$$B(z, 2^{-n-m}) = \{x_1\} \times \cdots \times \{x_n\} \times \underbrace{\{1\} \times \cdots \times \{1\}}_{m} \times \prod_{k=n+m+1}^{\infty} K.$$

Since $z \in \mathscr{N}_{L,c}$, we can find a partition $z = \mathscr{S}_1 \mathscr{R}_1 \mathscr{S}_2 \mathscr{R}_2 \dots$ satisfying all conditions in the definition of $\mathscr{N}_{L,c}$. Let us fix this partition. Now, let s_k denote the index in z of the leftmost entry of $\mathscr{S}_k$. By i, we denote the smallest index j such that $s_j \geq n+1$, i.e.,

$$i := \min\{j \in \mathbb{N} : s_j \geq n+1\}.$$

This retrieves the beginning of the first block $\mathscr{S}_i$ that starts in or after the section of z with the consecutive ones. Since $\mathscr{R}_{i-1}$ has to contain all elements of I and has a length of L, it can at most extend $L - (N-1)$ places into the section of consecutive ones. Since $m > L - (N-1)$, we conclude that in fact i is the index of the leftmost index of the first $\mathscr{S}_k$ to start within the section of consecutive ones, and not after.

Next, we find an upper bound on i. All $\mathscr{R}_k$ have the same length L, but the length of the $\mathscr{S}_k$ is arbitrary. Hence, i will be the largest if $\mathscr{S}_1, \dots, \mathscr{S}_{i-1} = ()$. For this reason, it follows that

$$i \leq L \left\lfloor \frac{n}{L} \right\rfloor + 1. \tag{3.1}$$

We now establish a lower bound on $|\mathscr{S}_i|$. The block $\mathscr{S}_i$ ends one place before $\mathscr{R}_i$ begins. Since $\mathscr{R}_i$ is of length L and contains all elements of I, it can extend at most $L - (N-1)$ places into the section of consecutive ones from the right. This together with the identical condition at the beginning of the section of m consecutive ones gives us that

$$|\mathscr{S}_i| \geq m - 2(L - (N-1)).$$

By the definition of m and by (3.1), we have

$$\begin{aligned} |\mathscr{S}_i| &\geq m - 2(L-(N-1)) \\ &= \lceil c \rceil \left(L \left\lfloor \frac{n}{L} \right\rfloor + 1 \right) + 2L + 2 - 2(L-(N-1)) \\ &= \lceil c \rceil \left(L \left\lfloor \frac{n}{L} \right\rfloor + 1 \right) + 2N \\ &\geq \lceil c \rceil i + 2N \\ &> ci. \end{aligned}$$

Hence,

$$\sum_{k=1}^{i} |\mathscr{S}_k| \geq |\mathscr{S}_i| > ci.$$

Since the partition $\mathscr{S}_1\mathscr{R}_1\mathscr{S}_2\mathscr{R}_2\dots$ is arbitrary, we have $z \notin \mathscr{N}_{L,c}$. Since $z \in B(y, 2^{-n-m})$ is arbitrary, we prove the assertion. □

Proof of Theorem 2.1. Let $\varepsilon > 0$ and choose some $x \in \mathscr{N}_{L,c}$. Choose $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$. Define

$$m := \lceil c \rceil \left(L \left\lfloor \frac{n}{L} \right\rfloor + 1 \right) + 2L + 2$$

and

$$y := (x_1, \dots, x_n, 1, 1, \dots).$$

Note that $y \in B(x, 2^{-n}) \subseteq B(x, \varepsilon)$. Furthermore, we easily see that

$$B(y, 2^{-n-m}) \subseteq B(x, 2^{-n}),$$

since all points in the former ball agree with x in the first n entries. By Lemma 3.3, we have that

$$B(y, 2^{-m-n}) \cap \mathscr{N}_{L,c} = \emptyset.$$

Hence, $\mathscr{N}_{L,c}$ is nowhere dense. □

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